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Anatomy of production functions: a technological menu and a choice of the best technology

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Abstract

Jones (2005) proposed microfoundations for the Cobb-Douglas production function. We show that Jones' technological menu is a special case of a concept of support set discussed by Matveenko (1997) and Rubinov, Glover (1998) by use of a duality approach. We use this approach to clarify the relation between different production functions and technological menus. Also we construct an "ideas model" generating CES production function.

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1. Introduction

An important topic in macroeconomics and growth theory concerns microfoundations of basic classes of production functions. A number of authors argue that production function (which has received an attribute 'global' in recent publications) is not a primary economic object but a result of an optimal choice of a 'local' technology from a given technological menu (Matveenko 1997, Rubinov and Glover 1998, Jones 2005, Caselli and Coleman 2006, Growiec 2008). This approach perfectly matches with a view that, given a combination of production factors, only one local technology can be used efficiently[1].

In particular, Matveenko (1997) and Rubinov and Glover (1998), by use of a duality approach, showed that each global *n*-factor production function, *F*, with constant returns to scale (CRS) can be represented as an optimal choice of a local Leontief technology from a menu (a set of technologies) corresponding to the function *F*.

Later Jones (2005) indicated a similar representation of a 2-factor CRS global production function:

$$F(K,L,N) = \max_{a,b:H(a,b)=N} \widetilde{F}(bK,aL).$$

Here $\widetilde{F}$ is a local production function with an elasticity of substitution less than one, *N* is a parameter characterizing available technologies, and the set

$$\{(a,b): H(a,b) = N\}$$

is a technological menu where technological parameters are chosen from. Under the following technological menu:

$$H(a,b) = a^{\alpha} b^{\beta} = N, \tag{1}$$

where $\alpha > 0, \beta > 0$, Jones has received the global Cobb-Douglas function:

$$F(K,L,N) = N^{\frac{1}{\alpha+\beta}} K^{\frac{\beta}{\alpha+\beta}} L^{\frac{\alpha}{\alpha+\beta}}.$$

The present paper develops this approach in several directions. We prove that for each *n*-factor neoclassical global production function F there exists a unique technological menu consisting of Leontief local technologies and generating F. Basic properties of technological menus are studied. A simple method for indicating technological menus is proposed. As examples, technological menus for the Cobb-Douglas and the CES global production functions are constructed. A case of local CES functions is also considered.

One more result of the paper concerns Jones (2005) "ideas model" based on the Pareto probability distribution and considered as a microfoundation for the global Cobb-Douglas function; its modification was recently constructed by Growiec (2008). We propose a simpler modification leading to the CES global production function. Different microfoundations also leading to the CES function have been proposed by Acemoglu (2003).

2. Technological menus and their properties

Let $i = 1,...,n$ be factors of production. We will consider a family of local production functions $\varphi(l,x)$; each of them is characterized by fixed technological coefficients (factor efficiencies) $l_i, i = 1,...,n$. A basic case is the Leontief local production function

[1] This view is distinctly formulated by (Basu and Weil 1998) who argue that "each technology is… appropriate for one and only one capital-labor ratio". This idea is close to a concept of localized technological change (Atkinson and Stiglitz 1969, Nelson and Winter 1982, ch. 9, Stiglitz 1989, Antonelli 1995, 2008).

$\varphi(l,x) = \min_{i=1,...,n} l_i x_i$. Let $F(x)$ be a global neoclassical production function[2]. A set $\Lambda = \{l = (l_1,...,l_n)\}$ is called a technological menu generating the global production function $F(x)$ if

$$F(x) = \max_{l \in \Lambda} \varphi(l,x). \tag{2}$$

An economic meaning of this notion is quite transparent. A firm (or a country) has available a set of local technologies Λ. Given a vector of production factors $x = (x_1,...,x_n)$ it chooses a technology l from Λ to achieve the maximum[3] output $F(x)$. In result the global production function $F(x)$ is formed by use of the family of local production functions.

A number of natural questions arise. Is each global production function generated by a technological menu? Is the technological menu, generating a concrete global production function, unique? If yes, what is the structure of the menu? The following Theorem 1 provides an exhaustive answer to these questions for the case of local Leontief technologies.

Let M_1 be a unit level surface of the function $F(x)$:

$$M_1 = \{x : F(x) = 1\},$$

i.e. the set of all vectors of production factors which provide a unit output.

We will narrow the domain of production functions in some way. We will consider production functions defined on the space R^n_{++} which consists of positive n-dimensional vectors and the origin[4].

This narrowing allows us to consider for each vector of factors, $x \in M_1$, a vector of inverse elements:

$$x^- = (x_1^{-1}, x_2^{-1},...,x_n^{-1}).$$

Its economic meaning is that x_i^{-1} is an average product of the i-th factor. (Evidently, $x_i^{-1} = F(x)/x_i$ as soon as $F(x) = 1$ for $x \in M_1$).

We will see that the set

$$\Lambda_1 = \{l : l = x^-,\ x \in M_1\} \tag{3}$$

(known as a support set – see (Matveenko, 1997, Rubinov and Glover, 1998)) is a unique technological menu generating the global production function $F(x)$ under Leontief local technologies.

There is an equivalent way to describe the technological menu. For the global production function $F(x)$ let us define an auxiliary function

$$F^\circ(l) = \frac{1}{F\left(\frac{1}{l_1},...,\frac{1}{l_n}\right)}, \tag{4}$$

so called conjugate function. An advantage of its use is that it is easily computable (see examples in Section 3). The technological menu (3) can be found as a unit level surface of the conjugate function:

$$\Lambda_1 = \{l : F^\circ(l) = 1\}.$$

Both the technological menu and the conjugate functions have simple economic interpretations. It is easy to verify that the technological menu Λ_1 generating the global

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[2] The functions are assumed to be non-negative, continuous, increasing and possessing CRS.

[3] This corresponds in full to the concept of (Basu and Weil 1998) – see footnote 1.

[4] Thereby, we will not consider points where at least one factor is not used. This does not contract the class of production functions.

production function F consists of all vectors l with coordinates equal to average products of factors that are possible given F. For example, for a 2-factor global production function $F(K,L)$ the technological menu Λ_1 consists of all admissible pairs $(F(K,L)/K, F(K,L)/L)$ of average capital and labor productivities.

An economic interpretation of the conjugate function $F^{\circ}(l)$ is the following. For each $l=(l_1,...,l_n)$ it shows a minimum value of a total factor productivity (TFP) A such that the function $AF(.)$ makes admissible average products $l_1,...,l_n$.

THEOREM 1. *The set* Λ_1 *is a unique technological menu generating the global production function F.*

Proves of theorems are provided in Section 6.

One more question: is it possible, knowing a form of a technological menu, to predetermine properties of the global production function generated by this menu? A partial answer is given by the following Theorem 2, where, for the sake of simplicity, only a 2-dimensional case is considered.

Let us define a set of all available technologies, $\tilde{\Lambda}$, which includes the technological menu Λ_1 as well as all worse technologies:

$$\tilde{\Lambda} = \{l : l_K > 0, l_L > 0, F^{\circ}(l) \le 1\}$$

THEOREM 2. *If the set* $\tilde{\Lambda}$ of *available technologies is convex then the elasticity of substitution* σ *of the global production function* $F(K,L)$ *in any point* $x=(K,L)$ *is less than* ½.

Notice that here the production function can possess different elasticities of substitution in different points, nevertheless they all have to be less than ½.

On an intuitive level the link between a form of the set of available technologies and a size of the elasticity of substitution of the global production function can be explained as follows. A low elasticity of substitution means a limited possibility to change technologies. A convexity of the set $\tilde{\Lambda}$ just restricts a possibility of changing technologies: a technology $l \in \Lambda_1$ may be changed for a technology $\tilde{l} \in \Lambda_1$ if and only if there exists a chord connecting l and $\tilde{l}$ and situated in interior of the set $\tilde{\Lambda}$.

3. Examples

For the global 2-factor Cobb-Douglas production function,

$$F(K,L,\overline{N}) = \overline{N}K^{\overline{\beta}}L^{\overline{\alpha}} \text{ (where } \overline{\alpha}+\overline{\beta}=1, 0<\overline{\alpha}<1),$$

its conjugate function is

$$F^{\circ}(l_K,l_L) = \frac{1}{\overline{N}l_K^{-\overline{\beta}}l_L^{-\overline{\alpha}}},$$

hence the technological menu is

$$\Lambda_1 = \left\{l : \overline{N}l_K^{-\overline{\beta}}l_L^{-\overline{\alpha}} = 1\right\} = \left\{l : l_K^{\overline{\beta}}l_L^{\overline{\alpha}} = \overline{N}\right\};$$

this coincides with Jones' menu (1).

For the global 2-factor CES production function,

$$F(K,L,N) = N\left[AL^{-r}+BK^{-r}\right]^{-1/r}$$

with $A,B>0, -1 \le r, r \ne 0$, the conjugate function is:

$$F^{\circ}(l_K,l_L) = N^{-1}\left[Al_L^{\ r}+Bl_K^{\ r}\right]^{1/r},$$

and the technological menu is:

$$\Lambda_1 = \{l : (Al_L^{\ r} + Bl_K^{\ r})^{1/r} = N\}.$$

4. Local CES function

Similar results concerning the technological menus are also true in case of local CES function:

$$\varphi(l,x) = (l_1 x_1^p + \ldots + l_n x_n^p)^{\frac{1}{p}},$$

where $p < 0$ is a fixed parameter[5].

THEOREM 3. *With local CES function,*

(i) *A global production function generated by the technological menu*

$$\Lambda = \left\{ \sum_{i=1}^{n} l_i^{\alpha} a_i^{1-\alpha} = 1 \right\} \ (\alpha < 1),$$

has a CES form.

(ii) *A global production function generated by the technological menu*

$$\Lambda = \left\{ l : l_1^{\theta_1} \ldots l_n^{\theta_n} = B \right\} \ (\text{where } B > 0, 0 < \theta_i < 1, i = 1,\ldots,n, \sum_{i=1}^{n} \theta_i = 1)$$

has a Cobb-Douglas form.

5. Technological ideas model

Jones (2005), looking for microfoundations of global production functions, proposed a model of technological ideas. An idea *i* means the use of Leontief technological coefficients a_i, b_i which are random and independent; precisely, they are described by independent Pareto distributions:

$$P\{a_i \le a\} = 1 - \left(\frac{a}{\gamma_a}\right)^{-\alpha}, \ P\{b_i \le b\} = 1 - \left(\frac{b}{\gamma_b}\right)^{-\beta},$$

where $a \ge \gamma_a > 0, b \ge \gamma_b > 0, \alpha > 0, \beta > 0, \alpha + \beta > 1$, and their joint distribution is:

$$G_1(b,a) = P\{b_i > b, a_i > a\} = \left(\frac{b}{\gamma_b}\right)^{-\beta} \left(\frac{a}{\gamma_a}\right)^{-\alpha}.$$

However, the independency assumption is not motivated at all. Let us make an alternative assumption: an idea is a pair of interdependent technological coefficients a_i, b_i. The following joint probability distribution can be used as a simple model:

$$G_2(b,a) = P\{b_i > b, a_i > a\} = \left[\lambda\left(\frac{b}{\gamma_b}\right)^{-h} + (1-\lambda)\left(\frac{a}{\gamma_a}\right)^{-h} \right]^s,$$

where $a \ge \gamma_a > 0, b \ge \gamma_b > 0, 0 < \lambda < 1, h < 0$ or $0 < h < 1$, $hs > 1$.[6]

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[5] The condition on *p* makes possible the part (ii) of Theorem 3.

[6] The conditions on *h* and *s* are imposed to provide appropriate properties of a production function and a probability distribution below.

The functions G_1, G_2 have resembling properties, moreover, under $h \to 0$ and $hs = const$, a conversion $G_2(b,a) \to G_1(b,a)$ takes place, where parameters of the functions G_1, G_2 are linked by relations $\beta = \lambda hs, \alpha = (1-\lambda)hs$.

Assuming the distribution G_2,

$$P\{Y_i > \tilde{y}\} = P\{b_i K > \tilde{y}, a_i L > \tilde{y}\} = \tilde{y}^{-hs}\left[\lambda(K\gamma_b)^h + (1-\lambda)(L\gamma_a)^h\right]^s.$$

With N ideas,

$$P\{Y \le \tilde{y}\} = \left[1 - \tilde{y}^{-hs}\left[\lambda(K\gamma_b)^h + (1-\lambda)(L\gamma_a)^h\right]^s\right]^N.$$

By using a normalization,

$$z_N = \left[\lambda(K\gamma_b)^h + (1-\lambda)(L\gamma_a)^h\right]^{\frac{1}{h}} N^{\frac{1}{hs}},$$

it is easy to receive:

$$P\{Y \le z_N \tilde{y}\} = \left(1 - \frac{\tilde{y}^{-hs}}{N}\right)^N \to \exp(-\tilde{y}^{-hs}).$$

As well as in the Jones' case, with large N,

$$Y \approx z_N \varepsilon,$$

where ε is a random variable described by the Frechet distribution.

Thus, when the number of ideas is great, we come to a CES production function.

6. Proves of theorems

For n-dimensional vectors, $x \ge y$ means that $x_i \ge y_i$; $x > y$ means that $x_i > y_i$ ($i = 1,...,n$). A function f is called increasing if $x > y$ implies $f(x) > f(y)$.

As a preliminary we prove the following Lemma.

LEMMA. *If $F(x)$ is an increasing function homogeneous of the power α then*

$$\varphi(l,x) \le F(x) = 1$$

for each $x \in M_1, l \in \Lambda_1$.

Proof. Let $x \in M_1$. Let us prove that $\varphi(l,x) \le 1$ for any $l \in \Lambda_1$. Assume the opposite: $\varphi(l,x) > 1$ for some $l \in \Lambda_1$. Then

$$l_i x_i > 1,\ i = 1,\dots,n,$$

and hence $x > l^-$. A number $\lambda > 1$ can be picked up such that $x > \lambda l^-$. Then

$$F(x) > F(\lambda l^-) = \lambda^\alpha F(l^-) > F(l^-),$$

which contradicts to the belonging $x \in M_1, l^- \in M_1$.

Q.E.D

Proof of Theorem 1. Each vector $x \in R^n_{++}$ can be represented in the form $x = F(x)\bar{x}$ where $\bar{x} \in M_1$. For any $l \in \Lambda_1$ it follows from Lemma that

$$\varphi(l,\bar{x}) \le F(\bar{x}) = 1 = \varphi(\bar{x}^-,\bar{x}).$$

We use the Lemma again to receive

$$\varphi(l,x) = F(x)\varphi(l,\bar{x}) \le F(x),\ l \in \Lambda_1;$$

$$\varphi(\bar{x}^-,x) = F(x)\varphi(\bar{x}^-,\bar{x}) = F(x).$$

This means the validity of (2).

To prove the uniqueness, let $\overline{\Lambda}$ be another technological menu generating the same global production function $F(x)$. Let us consider in $\overline{\Lambda}$ points $\arg\max_{l\in\Lambda}\varphi(l,x)$ for all $x>0$. No one of these points can lie below Λ_1 (i.e. inside $\widetilde{\Lambda}$). On the other hand, if any of these points $\bar{l}$ lies above Λ, then there exists a point $l\in\Lambda$ such that $l<\bar{l}$. For the point $x=l^{-}=\left(l_1^{-1},...,l_n^{-1}\right)$,

$$F(x)=\varphi(l,x)<\varphi(\bar{l},x)=F(x).$$

This contradiction implies $\overline{\Lambda}=\Lambda$.

Q.E.D.

Proof of Theorem 2. Convexity of $\widetilde{\Lambda}$ is equivalent to concavity of the function $l_L(l_K)$ and, hence, to the inequality:

$$\frac{d^2 l_L}{d{l_K}^2}<0.$$

By use of representation $F(K,L)=Lf(k),\ l_L=f(k), l_K=f(k)/k$ where $k=K/L$, we find

$$\frac{d^2 l_L}{d{l_K}^2}=\frac{2k(f'(k))^2-kf(k)f''(k)-2f(k)f'(k)}{(f'(k)k-f(k))^3}k^3.$$

The denominator is negative, so the sign of the fraction coincides with the sign of the expression

$$\Phi=2f'(k)(f(k)-kf'(k))+kf(k)f''(k)$$

Recalling the well-known formula for the elasticity of substitution of production function $F(K,L)$,

$$\sigma=\frac{f'(k)(f(k)-kf'(k))}{-kf(k)f''(k)},$$

we can rewrite Φ as

$$\Phi=-kf(k)f''(k)(2\sigma-1).$$

As soon as $kf(k)f''(k)<0$ for neoclassical production functions, the sign of Φ coincides with the sign of $(2\sigma-1)$, so the convexity of $\widetilde{\Lambda}$ implies $2\sigma-1<0$.

Q.E.D.

Proof of Theorem 3. (i) First let us prove the validity of the equation (2) for any $x\in M_1$. The Lagrange multipliers method provides the first order conditions:

$$a_1^{\frac{1}{q-p}}l_1^{-\frac{1}{q-p}}x_1=...=a_n^{\frac{1}{q-p}}l_n^{-\frac{1}{q-p}}x_n, \tag{5}$$

$$a_1^{-\frac{p}{q-p}}l_1^{\frac{q}{q-p}}+...+a_n^{-\frac{p}{q=p}}l_n^{\frac{q}{q-p}}=1. \tag{6}$$

In (5) denote $a_i^{\frac{1}{q-p}}l_i^{-\frac{1}{q-p}}x_i=c\ (i=1,...,n)$. It follows from (6) that $c=1$. Hence, in the maximum point:

$$l_i=a_i x^{q-p},\ i=1,...,n,$$

and (2) is true.

Each vector $x>0$ can be represented as $x=\mu\bar{x}$, where $\bar{x}\in M_1$. Because of the homogeneity,

$$F(x)=F(\mu\bar{x})=\mu F(\bar{x})\ =\mu\max_{l\in\Lambda}\varphi(l,\bar{x})=\max_{l\in\Lambda}\varphi(l,x).$$

(ii) is proved in a similar way.
Q.E.D.

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