

Discrete Applied Mathematics 45 (1993) 197–204
North-Holland

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Two-dimensional rectangle packing: on-line methods and results

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Received 17 August 1990

Revised 16 October 1991

Abstract

Csirik, J., J.B.G. Frenk and M. Labbé, Two-dimensional rectangle packing: on-line methods and results, *Discrete Applied Mathematics* 45 (1993) 197–204.

The first algorithms for the on-line two-dimensional rectangle packing problem were introduced by Coppersmith and Raghavan. They showed that for a family of heuristics $13/4$ is an upper bound for the asymptotic worst-case ratios. We have investigated the Next Fit and the First Fit variants of their method. We proved that the asymptotic worst-case ratio equals $13/4$ for the Next Fit variant and that $49/16$ is an upper bound of the asymptotic worst-case ratio for the First Fit variant.

1. Introduction

We consider the following problem: let

$$L = (r_1, r_2, \dots, r_n)$$

be a list of rectangles, each rectangle r having height $h(r)$ ($\leq H$) and width $w(r)$ ($\leq W$). A packing P of L into a collection $\{B_1, B_2, \dots, B_m\}$ of rectangular bins of size $H \times W$ is an assignment of each rectangle to a bin in such a way that

- each rectangle is contained entirely within its bin with its sides parallel to the sides of the bin,
- no two rectangles in a bin overlap,

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- the orientations of the rectangles cannot be changed, i.e., the width of a rectangle must be aligned with the width of the bin.

In the two-dimensional rectangle packing problem the number of bins used should be minimized. As the problem is clearly a generalization of “classical” one-dimensional bin packing [4], it is NP-hard. So analyzing fast heuristics for approximate solutions is important. It is easy to see that without loss of generality we can normalize the problem with $H = W = 1$.

The two-dimensional rectangle packing problem was analyzed for the first time by Chung et al. [1]. They defined the asymptotic worst-case ratio to measure the “goodness” of a heuristic A . To give this ratio let us first denote the number of nonempty bins used in an optimal packing of L by $OPT(L)$, and the number used by a heuristic A by $A(L)$. Let

$$R_A^n = \max \left\{ \frac{A(L)}{OPT(L)} \mid OPT(L) = n \right\}.$$

Then the asymptotic worst-case ratio of A is given by

$$R_A^\infty = \limsup_{n \rightarrow \infty} R_A^n.$$

Chung et al. [1] proved that for an adapted mixture of the one-dimensional First Fit and First Fit Decreasing heuristic (named Hybrid First Fit, *HFF*),

$$2.022... \leq R_{HFF}^\infty \leq 2.125.$$

On the other hand, Liang [6] has shown that for classical one-dimensional bin packing problem no on-line algorithm has an asymptotic worst-case ratio better than 1.5364.... (In on-line packing, items are given to the algorithm sequentially; each item must be packed before the next item is seen.)

Coppersmith and Raghavan gave the first results for the on-line two-dimensional rectangle packing problem [2]. In this paper we improve some bounds given in their paper.

2. The algorithm of Coppersmith and Raghavan

We will now describe the algorithm of Coppersmith and Raghavan. We present the algorithm only for those rectangles for which $h(r) \leq w(r)$. For rectangles with $w(r) < h(r)$ we can use the same algorithm, interchanging the interpretation of h - and w -sizes and packing these items in separate bins. For this second class of rectangles exactly the same arguments can be used, and so we do not distinguish these classes in the following description.

Given an item $(h(r), w(r))$, we round its height $h(r)$ up to the smallest number $\bar{h} \geq h(r)$ belonging to the set

$$\bigcup_{k \geq 0} \left\{ 2^{-k}, \frac{1}{3} \cdot 2^{-k} \right\},$$

and replace this item by a “dummy” item with sizes $(\bar{h}, w(r))$. Introduce now among these newly created items the following types:

- If $\bar{h} = 1$ call the corresponding item a type-1 item,
- if $\bar{h} \in \{1/2, 1/4, \dots, 1/2^k, \dots\}$ call this item a type-2 item,
- if $\bar{h} \in \{1/3, 1/6, 1/12, \dots, 1/3 \cdot 2^{-k}, \dots\}$ call this item a type-3 item.

We extend the definition of the types to the original elements of the list giving them the same type values as the corresponding “dummy” items.

In the used heuristic only type- i items for $1 \leq i \leq 3$ are packed together in so-called type- i bins. This (on-line) heuristic is now defined as follows:

- If a type-1 item is the next item to be packed we open a new type-1 bin and put this item in it.
- If a type-2 item with rounded sizes $(2^{-m}, w(r))$ is the next item to be packed we consider the set of opened type-2 bins. Each of these bins contains a set S_B of used strips with width 1 and height 2^{-k_i} , $k_i \geq 1$, where for each k_i the height 2^{-k_i} corresponds to the (rounded) height of an already packed type-2 item. Moreover, it also contains a set S_E of empty strips with width 1 and height 2^{-m_j} , m_j different, $m_j \geq 1$ satisfying

$$\sum_i 2^{-k_i} + \sum_j 2^{-m_j} = 1.$$

For the above item we now do the following: check whether there is a used strip in one of these bins which has an unused width of at least $w(r)$ and a height of 2^{-m} . If this holds, pack this item into *one such strip*. Otherwise, verify whether in one of these bins there is an unused strip with height 2^{-m} . If so, pack it into this strip. Failing again we consider an empty strip with smallest height $2^{-M} > \bar{h} = 2^{-m}$ (opening a new bin with $M=0$ if necessary), and break this empty strip into empty strips of height $2^{-M-1}, 2^{-M-2}, \dots, 2^{-m+1}$ and two new empty strips of height 2^{-m} . (Observe $\sum_{M+1}^m 2^{-k} + 2^{-m} = 2^{-M}$.) The second empty strip of height 2^{-m} will now contain the item and we start the procedure again for the next type-2 item.

- If a type-3 item is the next item to be packed we apply a similar procedure to type-3 bins as for the above case.

Let us call this heuristic *CRA*. Coppersmith and Raghavan proved for this procedure *CRA* the following results.

- For every list L of rectangles

$$CRA(L) \leq 3.25 \cdot OPT(L) + 8,$$

- for every list L of squares

$$CRA(L) \leq \frac{43}{16} \cdot OPT(L) + 8.$$

In the next section we present results concerning two heuristics which can be viewed

as special cases of the *CRA* method. The reason for studying such heuristics is that Coppersmith and Raghavan did not specify the method for choosing “one such strip”. Precisely, we shall investigate the First Fit method, i.e., we pack an item into the first strip which has enough room for it. We name this algorithm *CRFF*. In a second heuristic, only the last opened strip will be checked to see, whether it has enough place for the current item (Next Fit type packing). If this fails, we open a new strip for this item, as described above. We name this algorithm *CRNF*. We shall see that there is a difference in the worst-case bound of these two heuristics. Finally, our analysis concerns lists of rectangles. A similar treatment of lists of squares can be made.

3. Results

Lemma 3.1.

$$R_{CRNF}^{\infty} = 3.25.$$

Proof. From the proof given in [2] it follows that

$$R_{CRNF}^{\infty} \leq 3.25,$$

using the simple fact that in Next Fit packing in two consecutive strips of the same height the sum of the widths of items is at least 1.

To show that the bound is tight we shall give a series $L_1, L_2, \dots, L_n, \dots$ of lists so that $OPT(L_n) \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} \frac{CRNF(L_n)}{OPT(L_n)} = 3.25. \quad (1)$$

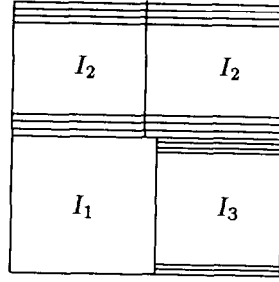
We give the lists by the optimal packing, and then we compute $CRNF(L_n)$ too. Let $i = n \cdot 2^n$, where n is a positive integer ($n \geq 2$) and

$$\varepsilon_n = \frac{1}{2(3 \cdot 2^{n-1} - 1)}.$$

We note that a rectangle of height ε_n will have its height rounded up to $1/2^{(n+1)}$. In the optimal packing of L_n the first i bins have the same structure (see Fig. 1). We give a detailed definition of items in Table 1. Then, in the optimal packing of L_n we have $2n$ further filled bins with items of sizes $(\varepsilon_n, 2\varepsilon_n)$ (I_4 items), altogether

$$2n \cdot 2 \cdot (3 \cdot 2^{n-1} - 1)(3 \cdot 2^{n-1} - 1)$$

such pieces.

Fig. 1. The first i bins in the optimal packing of L_n .

Now let

$$\begin{aligned}
 L_n = & \underbrace{(I_1, I_1, \dots, I_1)}_{n \cdot 2^n \text{ times}} \\
 & \underbrace{I_2, I_4, I_3, I_4, \dots, I_2, I_4, I_3, I_4}_{3 \cdot n \cdot 2^n \cdot 2^{n-1} \text{ times}} \\
 & \underbrace{I_2, I_4, \dots, I_2, I_4}_{n \cdot 2^n (3 \cdot 2^{n-1} - 4) \text{ times}} \\
 & \underbrace{I_4, I_4, \dots, I_4}_{9n \cdot 2^{2n-1} - n \cdot 2^{n+3} + 4n \text{ times}}.
 \end{aligned}$$

Then

$$OPT(L_n) = n \cdot 2^n + 2n$$

and

$$CRNF(L_n) \geq n \cdot 2^n + \frac{3}{2} n \cdot 2^n + \frac{3}{4} n \cdot 2^n - 2n = \frac{13}{4} n \cdot 2^n - 2n,$$

and so (1) holds. (Here, $n \cdot 2^n$ bins are required for the I_1 items, $\frac{3}{2} n \cdot 2^n$ bins are needed for the I_2, I_4, I_3, I_4 section of the list, and $\frac{3}{4} n \cdot 2^n - 2$ bins are needed for the I_2, I_4 section of the list.) \square

Table 1
Detailed definition of items

Number of items	Sizes	Name
i	$(1/2 + \varepsilon_n, 1/2 + \varepsilon_n)$	I_1 item
$i \cdot 2(3 \cdot 2^{n-1} - 2)$	$(\varepsilon_n, 1/2)$	I_2 item
$i \cdot 3 \cdot 2^{n-1}$	$(\varepsilon_n, 1/2 - \varepsilon_n)$	I_3 item
$2n \cdot 2(3 \cdot 2^{n-1} - 1)^2$	$(\varepsilon_n, 2\varepsilon_n)$	I_4 item

Lemma 3.2.

$$R_{CRFF}^{\infty} \leq \frac{49}{16}.$$

Proof. Let us consider strips of height H used by the *CRFF* heuristic for packing type-2 or type-3 items and denote the set of such strips by

$$S_H = \{s_1, s_2, \dots, s_{n_H}\}.$$

We divide this set into two disjoint subsets S'_H and S''_H where S'_H consists of strips containing an item r with $w(r) > 1/2$. Clearly all strips in the set S'_H are packed at least half of their width.

On the other hand, all except at most two strips in S''_H are packed at least two-thirds of their width. This can be shown as follows. Let

$$S''_H = \{s''_1, s''_2, \dots, s''_{n''_H}\}$$

and denote by i^* the largest index for which s''_i contains an element, say r_l , with width smaller than or equal to $1/3$. Since all the items in strips of height H are packed according to the First Fit heuristic and r_l was not packed in the strips s''_j , $j < i^*$, we obtain that the sum of widths of items in s''_j , $j < i^*$, is bounded from below by $1 - w(r_l) \geq 2/3$. Moreover, by the definition of i^* , only items with width greater than $1/3$ are packed in $s''_{i^*+1}, s''_{i^*+2}, \dots, s''_{n''_H-1}, s''_{n''_H}$. This implies since the widths of all these (type-2 and type-3) items are bounded from above by $1/2$ that the strips $s''_{i^*+1}, s''_{i^*+2}, \dots, s''_{n''_H-1}$ contain exactly two items with total width greater than $2/3$. Hence our claim is proved and we are now ready to verify the stated inequality.

Let b denote the number of type-1 items in L . Clearly in the optimal packing of this list L all type-1 items are contained in different bins. Hence

$$OPT(L) = b + c, \tag{2}$$

where c denotes the number of bins not containing type-1 items.

Define now

$$M(r) := \max(h(r), w(r))$$

and divide the type-2 and type-3 items into the disjoint sets L_M and L_S with

$$L_M = \{r \mid r \text{ is a type-2 or type-3 item with } M(r) > 1/2\}.$$

All strips containing an item from this set L_M are covered at least $1/3$ of their area since the heights of these items are at least $2/3$ of the strip height, and the strips are packed at least half of their width. Also by the above claim and the definition of L_S one can easily verify that almost all strips containing items from L_S are covered at least $4/9$ of their area. Introduce now

$$h(L_M) = \sum_{r \in L_M} \min(h(r), w(r))$$

and denote by A_{23} the sum of areas of all rectangles from L_S , i.e.,

$$A_{23} = \sum_{r \in L_S} h(r) \cdot w(r).$$

It is not difficult to verify that

$$A_{23} \leq OPT(L) - \left(\frac{b}{4} + \frac{1}{2} h(L_M) \right). \quad (3)$$

From (2) and the previous observations we have

$$CRFF(L) \leq b + \frac{3}{2} h(L_M) + \frac{9}{4} A_{23} + const \quad (4)$$

where *const* denotes the sum of areas of “exceptional” strips, i.e., the last strips from all heights and the only strips from packing of L_S items with a total width of less than $2/3$.

Finally, we derive an upper bound on $h(L_M)$. Observe in the optimal packing that a bin B containing a type-1 item might also contain type-2 and type-3 items from L_M . Due to the definition of L_M and a type-1 item we always have

$$\sum_{r \in L_M \subset B} \min(h(r), w(r)) \leq 1.$$

Moreover, for a bin B in the optimal packing containing only type-2 and type-3 items it follows that

$$\sum_{r \in L_M \subset B} \min(h(r), w(r)) \leq 3/2$$

and hence combining the above inequalities yields

$$h(L_M) \leq b + \frac{3}{2} c. \quad (5)$$

Combining (2), (3), (4) and (5) we finally obtain

$$\begin{aligned} CRFF(L) &\leq b + \frac{9}{4} \left(b + c - \frac{b}{4} \right) + \frac{3}{8} h(L_M) + const \\ &\leq \frac{49}{16} b + \frac{45}{16} c + const \leq \frac{49}{16} OPT(L) + const. \quad \square \end{aligned}$$

4. Open questions

Very recently, Galambos proved a nontrivial lower bound for on-line rectangle packing [3]. He showed that

$$R_A^\infty \geq 1.6$$

for every on-line algorithm A . However, the difference between this bound and that given in Lemma 3.2 is surprisingly large. We think that the bound given in Lemma 3.2 is close to the tight bound of *CRFF*. An adaptation of the Harmonic Fit heuristic to the two-dimensional rectangle packing problem will probably give a slight better worst-case bound than *CRFF*. In the meantime it was shown by Li and Cheng [5] that a generalized version of Harmonic Fit has a worst-case bound which can be made arbitrarily close to 2.86. However, the *CRA* algorithms are conceptually simpler than the Harmonic family of algorithms, and yield simple analyses.

Finally, it would be interesting to know something about on-line algorithms for vector packing.

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