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Abstract

This paper provides an elementary characterization of a family of multi-dimensional development indices that allows introducing distributive considerations. It consists of the generalized mean of the egalitarian equivalent values of the different dimensions. The key property that defines that family of indices is that of separability.

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1. Introduction

The need of multidimensional indicators for the assessment of economic development is already well established. The recent report by Stiglitz, Sen and Fitoussi (2009) is one of the last attempts to transform such a need into an institutional commitment that should lead to a change in our national accounting systems. That report discusses the limits of the standard growth indicators and suggests some sensible ways of introducing additional variables that capture the relevant dimensions of economic development.

Once those relevant dimensions have been identified and the variables that approximate them agreed upon, there is always the problem of how to aggregate this information into a single indicator. This paper aims at contributing to that discussion by providing a characterization of a well defined family of indices: the generalized means. Our approach responds to two main concerns. One is that of building indicators that are distribution sensitive, as we believe distributive considerations are part of the basic features of the economic performance. The other is to provide a theoretical support to those indicators in a relatively simple way.

Let us take as reference the Human Development Index (HDI), perhaps the most popular multidimensional socio-economic indicator. It consists of the average of three normalized variables that approach the social achievements in health, education and material well-being (see UNDP (2009)). In spite of its popularity, the HDI has a number of well known shortcomings (e.g. Sagar & Najam (1999)), among which there is the lack of theoretical justification of the additive structure and the absence of distributive considerations. Hicks (1997), Chakravarty (2003), Foster, López-Calva & Székely (2005), and Herrero, Martínez & Villar (2009), among others, are contributions that propose some improvements of the HDI on those respects and are closely related to our aim.

Hicks (1997) and Foster, López-Calva & Székely (2005) introduce distributive considerations into the HDI, following a constructive approach (incorporating the Gini index on the additive structure, in the first case, and in terms of a generalized mean in the second one). Chakravarty (2003) provides a simple characterization of the additive aggregation formula when average values (rather than individual values) are taken as the inputs of the social evaluation index. Finally, Herrero, Martínez & Villar (2009) characterize a multiplicative version of the HDI that is flexible enough to admit incorporating distributive considerations.

We provide here a unified treatment of all those indices following an axiomatic approach. Extending the notion of *additive separability*, we characterize a family of development indices that corresponds to the generalized mean of order α of the egalitarian equivalent values of the different dimensions considered. We call this property *separability of degree* α (see below). This result can be regarded as giving support to the proposal in Foster, López-Calva & Székely (2005). Yet it is more flexible concerning distributive aspects. It includes as particular cases the standard HDI (the arithmetic mean) or the multiplicative index in Herrero, Martínez & Villar (2009) (the geometric mean), among others.

2. The model

Let *N* = {1,2,…,*n*} denote a society consisting of *n* individuals and *K* = {1,2,…,*k*} a set of characteristics. Each characteristic corresponds to a variable that approximates one relevant dimension of social development. A **social state** is a matrix *Y* with *n* rows (one for each individual) and *k* columns (one for each characteristic). The element $y_{ij} \in [c,1]$ of matrix *Y* describes the value of the variable *j* for individual *i*, where *c* > 0 is an arbitrarily small scalar. That is, we assume that the values of each of those characteristics are already normalized and bounded above from zero.[1] Therefore, $\Omega = [c,1]^{nk}$ is the space of admissible social state matrices. We denote by *Y** the matrix all whose

[1] We can think that the original variable is $z_{ij} \in [z_j^{\min}, z_j^{\max}] \subset \mathbb{R}_{++}$ and then simply define $y_{ij} = \frac{z_{ij}}{z_j^{\max}}$, with $c = \min_j \left\{ \frac{z_j^{\min}}{z_j^{\max}} \right\}$.

elements are equal to 1. A bold letter $\mathbf{y}_j$ indicates the *j*th column of matrix *Y*. It describes the distribution of the *j*th characteristic in the population. The vector $\mathbf{1}_n(j)$ corresponds to the *j*th column of matrix *Y**. We denote by Y_{-j} the $n \times (k-1)$ matrix obtained from *Y* by deleting its *j*-th column. We can therefore write $Y = (Y_{-j}, \mathbf{y}_j)$, in the understanding that $\mathbf{y}_j$ actually occupies the *j*-th position in the array of columns.

A **Social Evaluation Index** is a continuous single-valued mapping $I : \Omega \to \mathbb{R}$ that provides a numerical evaluation of social states.[2]

For a given evaluation index *I* and a given social state matrix *Y*, we define the **egalitarian equivalent value** of the *j*th characteristic as the number $\xi(Y_{-j}, \mathbf{y}_j) \in (0,1]$ such that $I(Y) = I\left(Y_{-j}, \mathbf{1}_n(j)\xi(Y_{-j}, \mathbf{y}_j)\right)$. When $\xi(Y_{-j}, \mathbf{y}_j)$ does not depend on Y_{-j} we shall simply write $\xi(\mathbf{y}_j)$.

We first introduce two basic requirements on the social evaluation index: neutrality and normalization. *Neutrality* makes it explicit that all characteristics enter the evaluation function on an equal foot. That can be formalized by requiring that a permutation of the characteristics does not affect the social evaluation (recall that all variables vary in the interval [c, 1], so that their mean differences have already been neutralized). *Normalization* fixes the scale of the index. It requires that when the matrix is uniform (i.e. all entries are identical), the index takes on the very same value. Formally:

- **Neutrality**. For each $Y \in \Omega$, if $\pi_C(Y)$ denotes a permutation of the columns of *Y*, then: $I(\pi_C(Y)) = I(Y)$.

- **Normalization**. Let $Y(q) = q\left[\mathbf{1}_n(1), \ldots, \mathbf{1}_n(k)\right]$, for some $q \in [c,1]$. Then, $I(Y(q)) = q$.

[2] Note that we introduce the requirement of continuity in the very definition of the index. That is, we focus on those mappings for which small changes in the variables imply small changes in the index.

We now establish conditions on the behaviour of the index when the social state matrix changes. Suppose that the *j*th column of matrix *Y* changes from $\mathbf{y}_j$ to $\mathbf{y}'_j = \mathbf{y}_j + \mathbf{a}$, for some $\mathbf{a} \in \mathbb{R}^n$. We say that index *I* is ***additively separable*** if there exists $s_j : \mathbb{R}^{2n} \to \mathbb{R}$ such that:

$$I(Y_{-j}, \mathbf{y}_j + \mathbf{a}) = I(Y) + s_j(\mathbf{y}_j, \mathbf{a})$$

This property says that when the change in the *j*th column of social state matrix *Y* is the result of an additive composition, then its social evaluation index also corresponds to an additive composition of the original index and a real-valued function that depends on the change experienced in $\mathbf{y}_j$ (more specifically, it depends on the particular characteristic –hence the subindex in the function, the original value $\mathbf{y}_j$, and the perturbation term $\mathbf{a} \in \mathbb{R}^n$).[3]

The notion of *separability of degree* α extends this idea as follows:

- **Separability of degree α.** Let $Y, Y' \in \Omega$ be such that $Y' = (Y_{-j}, \mathbf{y}_j + \mathbf{a})$, for some admissible $\mathbf{a} \in \mathbb{R}^n$, and let $\alpha \in \mathbb{R}$ be given. Then,

$$\left[I(Y')\right]^{\alpha} = \left[I(Y)\right]^{\alpha} + \left[s_j(\mathbf{y}_j, \mathbf{a})\right]^{\alpha}.$$

According to this notion, the index associated with a matrix that results from an additive change in its *j*th column is an additive transformation of order α of the original index and a function that depends on the original value and the perturbation term $\mathbf{a}$. The coefficient α parameterizes impact of the change on index and is related to the elasticity of substitution of the different characteristics (see below for a discussion).

[3] This notion is related to that of "consistency in aggregation" introduced in Chakravarty (2003). Note that the very definition of a social evaluation index implies that all tose functions s_j are to be continuous.

The following result is obtained:

Theorem 1: *A social evaluation index I satisfies neutrality, normalization and separablity of degree* α*, if an only if it corresponds to the generalized mean of order* $\alpha \in \mathbb{R}$ *of the egalitarian equivalent values. That is:*

$$I(Y) = \begin{cases} \left[\dfrac{1}{k} \sum_{j \in K} \left[\xi(\mathbf{y}_j) \right]^{\alpha} \right]^{\frac{1}{\alpha}}, & \alpha \neq 0 \\ \prod_{j \in K} \left[\xi(\mathbf{y}_j) \right]^{1/k}, & \alpha = 0 \end{cases} \qquad [1]$$

Morever, those properties are independent.

Proof.-

Let $Y \in \Omega$ and $\alpha \neq 0$ be given. By separability of degree α we can write:

$$\left[I(Y^{*}_{-1}, \mathbf{y}_1) \right]^{\alpha} = \left[I(Y^{*}) \right]^{\alpha} + \left[s_1\left(\mathbf{1}_n(1), \left(\mathbf{y}_1 - \mathbf{1}_n(1) \right) \right) \right]^{\alpha}$$

Define now: $t_j(\mathbf{y}_j) := s_1\left(\mathbf{1}_n(j), \left(\mathbf{y}_j - \mathbf{1}_n(j) \right) \right)$. Then,

$$\left[I(Y^{*}_{-1}, \mathbf{y}_1) \right]^{\alpha} = \left[I(Y^{*}) \right]^{\alpha} + \left[t_1(\mathbf{y}_1) \right]^{\alpha}$$

$$\left[I(\mathbf{y}_1, \mathbf{y}_2, \mathbf{1}_n(3), ..., \mathbf{1}_n(k)) \right]^{\alpha} = \left[t_2(\mathbf{y}_2) \right]^{\alpha} + \left[I(Y^{*}) \right]^{\alpha} + \left[t_1(\mathbf{y}_1) \right]^{\alpha}$$

....

$$\left[I(Y) \right]^{\alpha} = \left[I(Y^{*}) \right]^{\alpha} + \sum_{j \in K} \left[t_j(\mathbf{y}_j) \right]^{\alpha}$$

By neutrality, $t_j(.) = t(.)$ for all *j*. That is, separability of degree α and neutrality imply: $I(Y) = \left(\left[I(Y^{*}) \right]^{\alpha} + \sum_{j \in K} \left[t(\mathbf{y}_j) \right]^{\alpha} \right)^{1/\alpha}$.

For the special case in which $\mathbf{y}_j = \mathbf{1}_n q$ for all *j*, by normalization we get:

$$\left[I(qY^{*}) \right]^{\alpha} = 1 + k \left[t(q\mathbf{1}_n) \right]^{\alpha} = q^{\alpha}$$

which implies $\left[t(q\mathbf{1}_n) \right]^{\alpha} = \dfrac{q^{\alpha} - 1}{k}$

Now observe that, for each $\mathbf{y}_j \in [c,1]^n$, there exists a scalar $\xi(\mathbf{y}_j)$ such that $t(\mathbf{y}_j) = t(\mathbf{1}_n \xi(\mathbf{y}_j))$ (by normalization this is true in particular for $\xi(\mathbf{y}_j) = t(\mathbf{y}_j)$). Therefore, we conclude,

$$I(Y) = \left[1 + \sum_{j \in K} \left(\frac{\left[\xi(\mathbf{y}_j)\right]^{\alpha} - 1}{k} \right)\right]^{\frac{1}{\alpha}} = \left[\frac{1}{k} \sum_{j \in K} \left[\xi(\mathbf{y}_j)\right]^{\alpha}\right]^{\frac{1}{\alpha}}$$

For the case $\alpha = 0$ we substitute the value of the function, which is not defined, by the value of its limit as $\alpha \to 0$. The standard procedure (L'Hôpital rule) gives us the desired result.

Consider now the following indices:

(1) $I(Y) = \left[\prod_{i \in N} \prod_{j \in K} y_{ij}\right]^{1/nk}$.

(2) $I(Y) = \left[\sum_{j \in K} \left[\xi(\mathbf{y}_j)\right]^{\alpha}\right]^{\frac{1}{\alpha}}$.

(3) $I(Y) = \left[\frac{1}{k} \sum_{j \in K} \left[\beta_j \xi(\mathbf{y}_j)\right]^{\alpha}\right]^{\frac{1}{\alpha}}$ with $\sum_{j \in K} \beta_j = 1$ and $\beta_j \neq 1/k$, for some *j*.

Index (1) satisfies neutrality and normalization but not α-consistency. Index (2) satisfies neutrality and α-consistency but not normalization. Index (3) satisfies α-consistency and normalization but not neutrality.

Q.e.d.

Remark 1.- *The restriction $y_{ij} > 0$ is introduced in order to avoid the indeterminacy of the formula for negative values of the parameter α.*

3. Discussion

Theorem 1 above identifies a family of indices that permits measuring development in a multidimensional context, in terms of the generalized mean of order α of the corresponding egalitarian equivalent values. The properties of the generalized means are well known: they are homogeneous of degree one, monotone in ξ, increasing in α, concave for $\alpha \leq 1$ and convex for $\alpha \geq 1$.

There are some specific values of the parameter α worth considering. For $\alpha = 1$ we get additive separability and the formula yields the arithmetic mean (the type of the HDI). By taking limits, as $\alpha \to 0$, we obtain the geometric mean. For $\alpha = -1$ we get the harmonic mean (used for the analysis of gender differences in the Human Development Reports). Finally, taking limits when $\alpha \to -\infty$ we get the leximin criterion: $I(Y) = \min_{j \in K} \{\xi_j\}$.

The parameter α may thus be regarded as controlling the degree of substitutability between the different dimensions (equation 1 is actually a symmetric CES function). Or, put differently, the parameter α measures our concern for equality across dimensions, with lower values of α corresponding to higher concern and viceversa. Additive separability implies constant rates of substitution (linear indifference curves) whereas separability of degree zero implies decreasing rates of substitution (a standard Cobb-Douglas symmetric function). In the limit, separability of degree $\alpha \to -\infty$ implies full complementarity (Leontief indifference curves).

Indeed, we can singularize the leximin criterion as the only member of this family that satisfies "minimal lower boundedness" (Bossert and Peters, (2000), Herrero, Martínez & Villar (2009)). That property says that there is no trade-off between dimensions when all members of the society are at their worst level in some of them. Formally: for all $Y \in \Omega$, all $j \in K$, $I(Y_{-j}, c\mathbf{1}_n(j)) \geq I(Y)$.

The following result is obtained:

Corollary: *A social evaluation index satisfies neutrality, normalization α-consistency, and minimal lower boundedness if an only if it corresponds to the leximin criterion. That is:*

$$I(Y) = \min_{j \in K} \left\{ \xi_j \right\}$$

(The proof is trivial and thus omitted)

Following the theory of income inequality measurement [see for instance Cowell (1995), Sen & Foster (1997), Goerlich & Villar (2009)] we can identify the egalitarian equivalent value of a characteristic, $\xi(\mathbf{y}_j)$, with its mean value deflated by some inequality measure, that is:

$$\xi(\mathbf{y}_j) = \mu(\mathbf{y}_j)\left[1 - f(\mathbf{y}_j)\right] \qquad [2]$$

where $f(\mathbf{y}_j)$ is an inequality index (e.g. the Gini index or a member of the entropy or the Atkinson family). For that we need to assume that the inequality index satisfies anonymity (a permutation of the rows does not change the value of the index) and quasi-concavity (redistribution improves the value of the index).[4]

Note that a generalized mean of order α corresponds to Atkinson's egalitarian equivalent value for $\alpha = 1 - \varepsilon$. Therefore, one could choose precisely this notion of egalitarian equivalent and obtain a formula that gives us the geometric mean of order α of the geometric mean of order α of each dimension. This is precisely the proposal in Foster, López-Calva & Székely (2005). Yet one may consider that inequality among people with respect to some dimension (e.g. income) may be treated differently that inequality across dimensions. In that respect our formulation is flexible enough to allow for a different treatment of people and dimensions. Be as it may, observe that the neutrality property

[4] If we also impose monotonicity (larger values of the variables imply larger values of the function), then the inequality index will have to vary into the interval [0,1].

prevents us from using different definitions of the egalitarian equivalent values for different characteristics.

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