RUHR
ECONOMIC PAPERS

Manuel Frondel
Colin Vance

On Marginal and Interaction Effects: The Case of Heckit and Two-Part Models

#138

Imprint

Ruhr Economic Papers

Published by

Ruhr-Universität Bochum (RUB), Department of Economics
Universitätsstr. 150, 44801 Bochum, Germany

Technische Universität Dortmund, Department of Economic and Social Sciences
Vogelpothsweg 87, 44227 Dortmund, Germany

Universität Duisburg-Essen, Department of Economics
Universitätsstr. 12, 45117 Essen, Germany

Rheinisch-Westfälisches Institut für Wirtschaftsforschung (RWI)
Hohenzollernstr. 1-3, 45128 Essen, Germany

Editors

Editorial Office

Ruhr Economic Papers #138

Responsible Editor: Christoph M.Schmidt

ISSN 1864-4872 (online) – ISBN 978-3-86788-156-2

Ruhr Economic Papers #138

Manuel Frondel and Colin Vance

On Marginal and Interaction Effects: The Case of Heckit and Two-Part Models

Bibliografische Informationen der Deutschen Nationalbibliothek

Die Deutsche Bibliothek verzeichnet diese Publikation in der deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über: *http//dnb.ddb.de* abrufbar.

ISSN 1864-4872 (online)
ISBN 978-3-86788-156-2

Manuel Frondel and Colin Vance[1]

On Marginal and Interaction Effects: The Case of Heckit and Two-Part Models

Abstract

Interaction effects capture the impact of one explanatory variable x_1 *on the marginal effect of another explanatory variable* x_2*. To explore interaction effects, so-called interaction terms* x_1x_2 *are typically included in estimation specifications. While in linear models the effect of a marginal change in the interaction term is equal to the interaction effect, this equality generally does not hold in non-linear specifications (AI, NORTON, 2003). This paper provides for a general derivation of marginal and interaction effects in both linear and non-linear models and calculates the formulae of the marginal and interaction effects resulting from HECKMAN's sample selection model as well as the Two-Part Model, two commonly employed censored regression models. Drawing on a survey of automobile use from Germany, we argue that while it is important to test for the significance of interaction effects, their size conveys limited substantive content. More meaningful, and also more easy to grasp, are the conditional marginal effects pertaining to two variables that are assumed to interact.*

JEL Classification: C34

Keywords: Censored regression models, interaction terms

September 2009

---

1 Both RWI. – We are very grateful for invaluable comments and suggestions by Christoph M. Schmidt. – All correspondence to Manuel Frondel, RWI, Hohenzollernstr. 1-3, 45128 Essen, Germany, e-mail: frondel@rwi-essen.de.

1 Introduction

To explore whether the effect of an explanatory variable x_1 on the expected value $E[y]$ of the dependent variable y depends on the size of another explanatory variable x_2, it is indispensable to estimate the interaction effect, which is formally given by the second derivative $\frac{\partial^2 E[y]}{\partial x_2 \partial x_1}$. To this end, linear estimation specifications typically include so-called interaction terms, consisting of the product $z := x_1 x_2$ of two explanatory variables. In linear contexts, the marginal effect of the interaction term $\frac{\partial E[y]}{\partial (x_1 x_2)}$ equals the interaction effect $\frac{\partial^2 E[y]}{\partial x_2 \partial x_1}$.

This equality, however, generally does not extend to non-linear specifications, as is demonstrated by AI and NORTON (2003) for the example of probit and logit models. Furthermore, NORTON, WANG, and AI (2004) emphasize that in non-linear models, interaction effects are generally conditional on all explanatory variables, rather than being constant, as in the linear case. For both logit and probit models, they calculate the formulae of the interaction effects if the interacted variables are (i) both continuous, (ii) both dummy variables, and (iii) if one variable of each type is included in the interaction term.

The present paper builds on the work of these authors in two respects. First, we calculate the formulae of both the marginal and interaction effects resulting from HECKMAN's sample selection model as well as the Two-Part model, two commonly employed approaches to accommodate censored data. Second, using an empirical example that applies the Two-Part model to travel survey data collected from a sample of motorists in Germany, we illustrate several subtleties inherent to the substantive interpretation of interaction effects gleaned from non-linear models. Most notably, we argue that while testing the statistical significance of an interaction effect is important, its size conveys little information of economic relevance. To this end, we draw a clear distinction between interaction effects, $\frac{\partial E[y]}{\partial (x_1 x_2)}$, and interaction terms, $x_1 x_2$.

The following section provides for a general derivation of interaction effects for both linear and non-linear models. Section 3 presents a concise comparison of the Two-

Part and HECKMAN model. Sections 4 and 5 derive the specific formulae of the marginal and interaction effects for both types of models, followed by the presentation of an example in Section 6. The last section summarizes and concludes.

2 Interaction Effects

To provide a general derivation of interaction effects in both linear and non-linear models, we closely follow NORTON, WANG, and AI (2004).

2.1 Linear Models

We begin by drawing on the following linear specification of the expected value of dependent variable y:

$$E := E[y|x_1, x_2, \mathbf{w}] = \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \mathbf{w}^T \boldsymbol{\beta}, \tag{1}$$

where the parameters $\beta_1, \beta_2, \beta_{12}$, as well as the vector $\boldsymbol{\beta}$ are unknown and vector $\mathbf{w}$ excludes x_1 and x_2.

Assuming that x_1 and x_2 are continuous variables, the marginal effect of x_1 on the expected value E is dependent on x_2 if $\beta_{12} \neq 0$:

$$\frac{\partial E}{\partial x_1} = \beta_1 + \beta_{12} x_2. \tag{2}$$

The impact of a marginal change in x_2 on the marginal effect of x_1, in other words the interaction effect, is then obtained from taking the derivative of (2) with respect to x_2:

$$\frac{\partial^2 E}{\partial x_2 \partial x_1} = \beta_{12}. \tag{3}$$

In linear specifications, therefore, the interaction effect $\frac{\partial^2 E}{\partial x_2 \partial x_1}$ equals the marginal effect $\frac{\partial E}{\partial (x_1 x_2)}$ of the interaction term $x_1 x_2$. For non-linear models, however, this equality generally does not hold, as is demonstrated in the subsequent section.

2.2 Non-Linear Models

Instead of expectation (1), we now depart from

$$E := E[y|x_1, x_2, \mathbf{w}] = F(\beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \mathbf{w}^T \boldsymbol{\beta}) = F(u), \tag{4}$$

where $F(u)$ is a non-linear function of its argument $u := \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \mathbf{w}^T \boldsymbol{\beta}$. In the Probit model, for example, $F(u)$ equals the cumulative normal distribution $\Phi(u)$. We now derive general formulae for the interaction effects resulting from non-linear models if (1) x_1 and x_2 are both continuous variables, (2) both are dummy variables, and (3) x_1 is continuous, while x_2 is a dummy variable.

(1) If $F(u)$ is a twice differentiable function, with the first and second derivatives being denoted by $F'(u)$ and $F''(u)$, respectively, the marginal effect with respect to x_1 reads:

$$\frac{\partial E}{\partial x_1} = \frac{\partial F(u)}{\partial x_1} = F'(u)\frac{\partial u}{\partial x_1} = F'(u)(\beta_1 + \beta_{12} x_2), \tag{5}$$

while the interaction effect of two continuous variables x_1 and x_2 is symmetric[1] and given by

$$\frac{\partial^2 E}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}(\frac{\partial E}{\partial x_1}) = \frac{\partial}{\partial x_2}(\frac{\partial F(u)}{\partial x_1}) = F'(u)\beta_{12} + (\beta_1 + \beta_{12} x_2)(\beta_2 + \beta_{12} x_1) F''(u). \tag{6}$$

As, in general, $(\beta_1 + \beta_{12} x_2)(\beta_2 + \beta_{12} x_1) F''(u) \neq 0$, the interaction effect $\frac{\partial^2 E}{\partial x_2 \partial x_1}$ generally differs from the marginal effect $\frac{\partial E}{\partial (x_1 x_2)}$ of the interaction term $z = x_1 x_2$:

$$\frac{\partial E}{\partial (x_1 x_2)} = \frac{\partial E}{\partial z} = F'(u)\frac{\partial u}{\partial z} = F'(u)\beta_{12}. \tag{7}$$

(2) If x_1 and x_2 are dummy variables, the discrete interaction effect, which in analogy to $\frac{\partial^2 E}{\partial x_2 \partial x_1}$ shall be designated by $\frac{\Delta^2 E}{\Delta x_2 \Delta x_1}$, is given by the discrete change in E due to a unitary change in both x_1 and x_2, $\Delta x_1 = 1, \Delta x_2 = 1$:

$$\frac{\Delta^2 E}{\Delta x_2 \Delta x_1} := \frac{\Delta}{\Delta x_2}(\frac{\Delta E}{\Delta x_1}) = \frac{\Delta}{\Delta x_2}(E[y|x_1 = 1, x_2, \mathbf{w}] - E[y|x_1 = 0, x_2, \mathbf{w}])$$

[1] It becomes obvious from the symmetry of u and second derivative (6) with respect to x_1 and x_2 that the interaction effects are symmetric: $\frac{\partial^2 E}{\partial x_1 \partial x_2} = \frac{\partial^2 E}{\partial x_2 \partial x_1}$. This is a special case of the more general mathematical result called YOUNG's theorem that the second derivatives of a twice differentiable function F are symmetric if they are continuous: $\frac{\partial^2 F}{\partial x_1 \partial x_2} = \frac{\partial^2 F}{\partial x_2 \partial x_1}$.

$$= \{E[y|x_1 = 1, x_2 = 1, \mathbf{w}] - E[y|x_1 = 0, x_2 = 1, \mathbf{w}]\} \qquad (8)$$
$$-\{E[y|x_1 = 1, x_2 = 0, \mathbf{w}] - E[y|x_1 = 0, x_2 = 0, \mathbf{w}]\}.$$

Note that the discrete interaction effects are symmetric: $\frac{\Delta^2 E}{\Delta x_2 \Delta x_1} = \frac{\Delta^2 E}{\Delta x_1 \Delta x_2}$, as can be seen from (8) by rearranging the terms in the middle of the double difference. Using the non-linear representation of expected value (4), the general expression (8) translates into:

$$\frac{\Delta^2 E}{\Delta x_2 \Delta x_1} = \{F(\beta_1 + \beta_2 + \beta_{12} + \mathbf{w}^T \boldsymbol{\beta})) - F(\beta_2 + \mathbf{w}^T \boldsymbol{\beta})\} - \{F(\beta_1 + \mathbf{w}^T \boldsymbol{\beta}) - F(\mathbf{w}^T \boldsymbol{\beta})\}. \quad (9)$$

(3) If x_1 is a continuous variable and x_2 is a dummy variable, the mixed interaction effect $\frac{\Delta}{\Delta x_2}(\frac{\partial E}{\partial x_1})$ can be computed as follows:

$$\frac{\Delta}{\Delta x_2}(\frac{\partial E}{\partial x_1}) := \frac{\Delta}{\Delta x_2}(\frac{\partial F(u)}{\partial x_1}) = \frac{\Delta}{\Delta x_2}(F'(u)(\beta_1 + \beta_{12} x_2)) \qquad (10)$$
$$= F'(\beta_1 x_1 + \beta_2 + \beta_{12} x_1 + \mathbf{w}^T \boldsymbol{\beta})(\beta_1 + \beta_{12}) - F'(\beta_1 x_1 + \mathbf{w}^T \boldsymbol{\beta})\beta_1.$$

The symmetry observed for the cases when both variables are either continuous or dummies also holds true for the mixed interaction effects: $\frac{\partial E}{\partial x_1}(\frac{\Delta}{\Delta x_2}) = \frac{\Delta}{\Delta x_2}(\frac{\partial E}{\partial x_1})$.[2]

All in all, it bears noting that for linear functions such as $F(u) = u$, for which $F'(u) = 1$, all three kinds of interaction effects collapse to β_{12}. Furthermore, we shall re-emphasize the point raised by AI and NORTON (2003:124) that, in contrast to linear specifications, the interaction effect gleaned from non-linear models is generally non-vanishing even if $\beta_{12} = 0$, that is, even if no interaction term is included.

Finally, for the special case of the Probit model, the interaction effects are given by (6), (9), and (10) if $F(u)$ is replaced by the cumulative normal distribution $\Phi(u)$, $F'(u)$ is replaced by the density function of the standard normal distribution, $\phi(u) := \exp\{-u^2/2\}/\sqrt{2\pi}$, and $F''(u)$ is replaced by $\phi'(u) = -u\phi(u)$. Similarly, formulae (6), (9), and (10) can be applied to the Logit model if $F(u)$ is replaced by $\Lambda(u) := 1/(1 + \exp\{-u\})$, $F'(u)$ is replaced by $\Lambda'(u) = \Lambda(u)(1 - \Lambda(u))$, and $F''(u)$ is substituted by $\Lambda''(u) = (\Lambda(u)(1 - \Lambda(u)))' = \Lambda(u)(1 - \Lambda(u))(1 - 2\Lambda(u))$.

[2]Yet, note that $\frac{\Delta}{\Delta x_2}(\frac{\partial E}{\partial x_1}) \neq \frac{\partial}{\partial x_2}(\frac{\Delta E}{\Delta x_1})$.

3 Two-Part and Heckit

To accommodate the feature of zero values in observed data, two-stage estimation procedures such as the Two-Part or HECKMAN's (1979) sample selection model are frequently employed. Both types of models order observations of y into two regimes, where the first stage defines a dichotomous variable R, indicating the regime into which the observation falls:

$$R = 1, \text{if} \quad R^* = \mathbf{x_1}^T \boldsymbol{\tau} + \epsilon_1 > 0 \quad \text{and} \quad R = 0, \text{if} \quad R^* \leq 0. \tag{11}$$

R^* is a latent variable, vector $\mathbf{x_1}$ includes its determinants, $\boldsymbol{\tau}$ is a vector of associated parameter estimates, and ϵ_1 is an error term assumed to have a standard normal distribution. $R = 1$ indicates that $y > 0$, whereas $R = 0$ is equivalent to $y = 0$.

After estimating $\boldsymbol{\tau}$ using Probit estimation methods, the second stage of both models involves estimating the parameters $\boldsymbol{\beta}$ via an OLS regression conditional on $R = 1$, i. e. $y > 0$:

$$E[y|R = 1, \mathbf{x_2}] = E[y|y > 0, \mathbf{x_2}] = \mathbf{x_2}^T \boldsymbol{\beta} + E(\epsilon_2|y > 0, \mathbf{x_2}), \tag{12}$$

where $\mathbf{x_2}$ includes the determinants of the dependent variable y, and ϵ_2 is the error term.

The prediction of the dependent variable consists of two parts, with the first part resulting from the first stage (11), $P(y > 0) = \Phi(\mathbf{x_1}^T \boldsymbol{\tau})$, and the second part being the conditional expectation $E[y|y > 0]$ from the second stage (12):

$$E[y] = P(y > 0) \cdot E[y|y > 0] + P(y = 0) \cdot E[y|y = 0] = P(y > 0) \cdot E[y|y > 0].$$

In the Two-Part Model, where it is assumed that $E(\epsilon_2|y > 0, \mathbf{x_2}) = \mathbf{0}$ and, hence, $E[y|y > 0] = \mathbf{x_2}^T \boldsymbol{\beta}$, the unconditional expectation $E[y]$ is given by

$$E[y] = \Phi(\mathbf{x_1}^T \boldsymbol{\tau}) \cdot \mathbf{x_2}^T \boldsymbol{\beta}. \tag{13}$$

By contrast, the second stage OLS regression of the so-called Heckit model includes the inverse MILL's ratio as an additional regressor to control for sample selectivity:

$$E[y|y > 0] = \mathbf{x_2}^T \boldsymbol{\beta} + \beta_\lambda \cdot \frac{\phi(\mathbf{x_1}^T \boldsymbol{\tau})}{\Phi(\mathbf{x_1}^T \boldsymbol{\tau})}, \tag{14}$$

where β_λ is called the sample-selection parameter and the inverse MILL's ratio $\frac{\phi(\mathbf{x_1}^T\boldsymbol{\tau})}{\Phi(\mathbf{x_1}^T\boldsymbol{\tau})}$ is proportional to $E(\epsilon_2|y > 0, \mathbf{x_2}) \neq 0$ when ϵ_2 is assumed to be normally distributed with constant variance: $\text{Var}(\epsilon_2) = \sigma^2$.

As DOW and NORTON (2003) have noted, the distinction between the conditional and unconditional expectation underlying the Heckit and Two-Part Model gives rise to different interpretations of the regression output in terms of potential and actual outcomes. The actual outcome refers to that which is observed, whereby zeros in the data are treated as true zeros rather than as missing values. The potential outcome, by contrast, refers to a partially observed latent variable that treats zeros in the data as missing, thereby necessitating the inclusion of the inverse Mills ratio in the second stage regression.

4 Marginal and Interaction Effects in Two-Part Models

Using a slightly more detailed version of prediction (13),

$$E := E[y|x_1, x_2, \mathbf{w_1}, \mathbf{w_2}] = \Phi(u_1)u_2,$$

where $u_1 := \tau_1 x_1 + \tau_2 x_2 + \tau_{12} x_1 x_2 + \mathbf{w_1}^T\boldsymbol{\tau}$, $u_2 := \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \mathbf{w_2}^T\boldsymbol{\beta}$, and $\mathbf{w_1}$ and $\mathbf{w_2}$ neither include x_1 nor x_2, we now derive formulae for the interaction effects if (1) x_1 and x_2 are both continuous variables, (2) x_1 is continuous, while x_2 is a dummy variable, and (3) both are dummy variables.

(1) To calculate the interaction effect $\frac{\partial^2 E}{\partial x_2 \partial x_1}$, we first need to calculate the marginal effect:

$$\frac{\partial E}{\partial x_1} = (\tau_1 + \tau_{12}x_2) \cdot \phi(u_1) \cdot u_2 + \Phi(u_1) \cdot (\beta_1 + \beta_{12}x_2). \tag{15}$$

Apparently, marginal effects resulting from non-linear models generally depend on all other variables. As elaborated in the empirical example below, accurate interpretation necessitates that in calculating the marginal effect $\frac{\partial E}{\partial x_1}$ the derivatives $\tau_{12}x_2$ and $\beta_{12}x_2$ of the interaction terms must be taken into account. Standard software such as STATA, however, erroneously ignores these derivatives when marginal effects are calculated using the *mfx* command.

By now taking the derivative with respect to x_2 and employing $\phi'(u_1) = -u_1\phi(u_1)$, we get the interaction effect:

$$\begin{aligned}\frac{\partial^2 E}{\partial x_2 \partial x_1} &= \tau_{12} \cdot \phi(u_1) \cdot u_2 - (\tau_1 + \tau_{12}x_2) \cdot (\tau_2 + \tau_{12}x_1) \cdot \phi(u_1) \cdot u_1 \cdot u_2 \\ &\quad +(\tau_1 + \tau_{12}x_2) \cdot \phi(u_1) \cdot (\beta_2 + \beta_{12}x_1) \\ &\quad +(\tau_2 + \tau_{12}x_1) \cdot \phi(u_1) \cdot (\beta_1 + \beta_{12}x_2) + \Phi(u_1) \cdot \beta_{12}.\end{aligned} \tag{16}$$

Note that, in general, it would be incorrect to calculate the interaction effect by taking the marginal effect of the interaction term $z = x_1x_2$:

$$\frac{\partial E}{\partial z} = \tau_{12} \cdot \phi(u_1) \cdot u_2 + \Phi(u_1) \cdot \beta_{12}. \tag{17}$$

(2) The mixed interaction effect $\frac{\Delta}{\Delta x_2}(\frac{\partial E}{\partial x_1})$ follows immediately from the marginal effect (15):

$$\begin{aligned}\frac{\Delta}{\Delta x_2}(\frac{\partial E}{\partial x_1}) &= \frac{\partial E}{\partial x_1}|_{x_2=1} - \frac{\partial E}{\partial x_1}|_{x_2=0} \\ &= (\tau_1 + \tau_{12}) \cdot \phi(\tau_1 x_1 + \tau_2 + \tau_{12}x_1 + \mathbf{w_1}^T\boldsymbol{\tau}) \cdot \{\beta_1 x_1 + \beta_2 + \beta_{12}x_1 + \mathbf{w_2}^T\boldsymbol{\beta}\} \\ &\quad +\Phi(\tau_1 x_1 + \tau_2 + \tau_{12}x_1 + \mathbf{w_1}^T\boldsymbol{\tau}) \cdot (\beta_1 + \beta_{12}) \\ &\quad -\tau_1 \cdot \phi(\tau_1 x_1 + \mathbf{w_1}^T\boldsymbol{\tau}) \cdot \{\beta_1 x_1 + \mathbf{w_2}^T\boldsymbol{\beta}\} - \Phi(\tau_1 x_1 + \mathbf{w_1}^T\boldsymbol{\tau}) \cdot \beta_1.\end{aligned} \tag{18}$$

(3) Applying formula (8) to $E[y|x_1, x_2, \mathbf{w_1}, \mathbf{w_2}]$, the discrete interaction effect $\frac{\Delta^2 E}{\Delta x_2 \Delta x_1}$ is obtained as follows:

$$\begin{aligned}\frac{\Delta^2 E}{\Delta x_2 \Delta x_1} &= \{[E[y|x_1 = 1, x_2 = 1, \mathbf{w_1}, \mathbf{w_2}] - E[y|x_1 = 0, x_2 = 1, \mathbf{w_1}, \mathbf{w_2}]\} \\ &\quad -\{[E[y|x_1 = 1, x_2 = 0, \mathbf{w_1}, \mathbf{w_2}] - E[y|x_1 = 0, x_2 = 0, \mathbf{w_1}, \mathbf{w_2}]\} \\ &= \Phi(\tau_1 + \tau_2 + \tau_{12} + \mathbf{w_1}^T\tau) \cdot \{\beta_1 + \beta_2 + \beta_{12} + \mathbf{w_2}^T\beta\} \\ &\quad -\Phi(\tau_2 + \mathbf{w_1}^T\tau) \cdot \{\beta_2 + \mathbf{w_2}^T\beta\} - \Phi(\tau_1 + \mathbf{w_1}^T\tau) \cdot \{\beta_1 + \mathbf{w_2}^T\beta\} \\ &\quad +\Phi(\mathbf{w_1}^T\tau) \cdot \{\mathbf{w_2}^T\beta\}.\end{aligned} \tag{19}$$

5 Marginal and Interaction Effects in Heckit Models

The second stage of the Heckit model relies upon the conditional expectation

$$E = E[y|x_1, x_2, \mathbf{w_1}, \mathbf{w_2}, y > 0] = u_2 + \beta_\lambda \cdot \lambda(u_1),$$

where $u_1 := \tau_1 x_1 + \tau_2 x_2 + \tau_{12} x_1 x_2 + \mathbf{w_1}^T \boldsymbol{\tau}$, $u_2 := \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \mathbf{w_2}^T \boldsymbol{\beta}$, and $\lambda(u_1) := \frac{\phi(u_1)}{\Phi(u_1)}$ denotes the inverse MILL's ratio, β_λ is the respective coefficient, and $\mathbf{w_1}$ and $\mathbf{w_2}$ exclude x_1 and x_2.

Before deriving the formulae for the marginal and interaction effects, it should be recognized that

$$\delta(u_1) := \lambda'(u_1) = \frac{-u_1 \phi(u_1) \Phi(u_1) - \phi^2(u_1)}{\Phi^2(u_1)} = -[\lambda(u_1)]^2 - u_1 \cdot \lambda(u_1)$$

and

$$\delta'(u_1) := -2\lambda(u_1) \cdot \lambda'(u_1) - \lambda(u_1) - u_1 \cdot \lambda'(u_1) = -[2\lambda(u_1) + u_1] \cdot \delta(u_1) - \lambda(u_1).$$

(1) To calculate the interaction effect of two continuous variables, we once again need the marginal effect:

$$\frac{\partial E}{\partial x_1} = (\beta_1 + \beta_{12} x_2) + \beta_\lambda \cdot \delta(u_1) \cdot (\tau_1 + \tau_{12} x_2). \tag{20}$$

By taking the derivative with respect to x_2 and employing the expressions $\delta(u_1)$ and $\delta'(u_1)$ from above, we get the interaction effect:

$$\frac{\partial^2 E}{\partial x_2 \partial x_1} = \beta_{12} + \beta_\lambda \cdot \{\delta'(u_1) \cdot (\tau_2 + \tau_{12} x_1) \cdot (\tau_1 + \tau_{12} x_2) + \delta(u_1) \cdot \tau_{12}\}. \tag{21}$$

(2) On the basis of the marginal effect (20), the mixed interaction effect $\frac{\Delta}{\Delta x_2}(\frac{\partial E}{\partial x_1})$ is now given by:

$$\begin{aligned} \frac{\Delta}{\Delta x_2}\left(\frac{\partial E}{\partial x_1}\right) &= \frac{\partial E}{\partial x_1}\Big|_{x_2=1} - \frac{\partial E}{\partial x_1}\Big|_{x_2=0} \qquad (22) \\ &= \beta_{12} + \beta_\lambda \cdot \{\delta(\tau_1 x_1 + \tau_2 + \tau_{12} x_1 + \mathbf{w_1}^T \boldsymbol{\tau}) \cdot (\tau_1 + \tau_{12}) - \delta(\tau_1 x_1 + \mathbf{w_1}^T \boldsymbol{\tau}) \cdot \tau_1\}. \end{aligned}$$

(3) The discrete interaction effect reads as follows:

$$\begin{aligned}\frac{\Delta^2 E}{\Delta x_2 \Delta x_1} &= \{[E[y|x_1=1, x_2=1, \mathbf{w_1}, \mathbf{w_2}] - E[y|x_1=0, x_2=1, \mathbf{w_1}, \mathbf{w_2}]\} \\ &\quad -\{[E[y|x_1=1, x_2=0, \mathbf{w_1}, \mathbf{w_2}] - E[y|x_1=0, x_2=0, \mathbf{w_1}, \mathbf{w_2}]\} \qquad (23) \\ &= \beta_{12} + \beta_\lambda \{\lambda(\tau_1+\tau_2+\tau_{12}+\mathbf{w_1}^T\tau) - \lambda(\tau_2+\mathbf{w_1}^T\tau) - \lambda(\tau_1+\mathbf{w_1}^T\tau) + \lambda(\mathbf{w_1}^T\tau)\}.\end{aligned}$$

Note that in all three cases the interaction effect collapses to the coefficient β_{12} of the interaction term if $\beta_\lambda = 0$, that is, when the inverse MILL's ratio is neglected and the Heckit model degenerates to the classical linear regression model.

6 Empirical Example

To illustrate the estimation of marginal and interaction effects gleaned from a Two-Part model, we employ household data drawn from the German Mobility Panel (MOP 2007) using the following specification:

$$E[s] = \Phi(\mathbf{x}^T \boldsymbol{\tau}) \cdot \{\mathbf{x}^T \boldsymbol{\beta}\}, \qquad (24)$$

where the dependent variable s is the daily distance driven for non-work travel and the set of explanatory variables **x** includes the individual and household attributes that are hypothesized to influence the extent of this travel. Variable definitions and descriptive statistics are presented in Table A1 in Appendix A. A detailed data description can be found in FRONDEL, PETERS, VANCE (2008), and FRONDEL, VANCE (2009a, 2009b).

The key attributes of interest in the following example are the individual's *age*, the number (#) of *children*, and the dummy variable *enoughcars* indicating whether the individual lives in a household in which the number of cars is at least equal to the number of licensed drivers. Each of these variables is interacted with a *female* dummy variable, which is intended to capture the role played by household responsibilities, social status, and competition among household members in dictating access to the car. In addition, we interact the variable measuring the number of children with the age of the individual. The specification thus yields all combinations of interactions: between

(1) two continuous variables, (2) two dummies, and (3) a dummy and continuous variable.

Table 1 reports the results of two model specifications, one in which several interaction terms are included and another in which these are omitted entirely. Appendix B presents the results from a Heckit model for comparison. To focus on the salient results, we refrain here from reporting the estimation results of the first-stage Probit models, which can be found in Appendix C, and instead present both the coefficient estimates of the (second-stage) OLS regression, as well as the marginal and interaction effects of the explanatory variables on distance driven resulting from the Two-part model. Given that the marginal and interaction effects are comprised of multiple parameters that makes analytical computation of the variance impossible, the standard errors are calculated by applying the Delta method, which uses a first-order Taylor expansion to create a linear approximation of a non-linear function.

Turning first to the model that includes the interaction terms, the OLS estimates and associated marginal effects are seen to differ markedly, both with respect to their magnitude and statistical significance. For some of the variables, such as *commute distance* and *city region*, significant coefficient estimates correspond to insignificant estimates of the marginal effects, while for others, e.g. # *children* and *enoughcars*, the converse is true. In this regard, it appears to be particularly important to distinguish between interaction terms and interaction effects[3]: For example, while the OLS estimates of the coefficients of the interaction terms *female* $\times$ *age* and *female* $\times$ # *children* do not statistically differ from zero, the associated interaction effects are significantly negative and positive, respectively.

Although no interaction terms are included in the specification presented on the right-hand panel, the corresponding interaction effects, which are calculated using the

[3]Note that the coefficient estimate of 0.009 of the interaction effect pertaining to age and the number of children, for example, which appears on the left-hand panel of Table 2, is calculated on the basis of (10) rather than (17) and, hence, is not simply the marginal effect of the interaction term *age* $\times$ # *children*. This example shows that interaction effects are not to be confused with marginal effects, but are differences between marginal effects, as is demonstrated below.

Table 1: Estimation Results of the Two-Part model (2PM) on Distance driven.

	Interaction Terms Included: $\tau_{12} \neq 0, \beta_{12} \neq 0$				No Interaction Terms: $\tau_{12} = \beta_{12} = 0$			
	OLS		2PM		OLS		2PM	
	Coeff.s	Errors	Effects	Errors	Coeff.s	Errors	Effects	Errors
female	-1.624	(1.292)	-0.285	(0.305)	**-1.491	(0.299)	** -0.581	(0.164)
employed	**-1.181	(0.427)	** -1.087	(0.237)	**-1.212	(0.413)	** -1.302	(0.229)
commute distance	** 0.030	(0.007)	0.006	(0.003)	** 0.029	(0.007)	0.003	(0.003)
age	-0.028	(0.019)	0.008	(0.013)	* -0.033	(0.013)	0.005	(0.007)
female × *age*	-0.020	(0.023)	** -0.058	(0.012)	–	–	** -0.003	(0.001)
age × *# children*	0.012	(0.020)	0.009	(0.011)	–	–	-0.002	(0.001)
high-school diploma	** 0.947	(0.320)	* 0.346	(0.173)	** 1.024	(0.316)	** 0.502	(0.173)
# children	-0.996	(0.873)	** 0.711	(0.211)	-0.277	(0.157)	** 0.722	(0.094)
female × *# children*	0.426	(0.331)	** 1.113	(0.222)	–	–	** -0.110	(0.024)
# employed	-0.033	(0.240)	-0.080	(0.129)	-0.022	(0.240)	0.003	(0.128)
enoughcars	0.741	(0.422)	** 1.854	(0.223)	** 1.368	(0.292)	** 1.966	(0.165)
female × *enoughcars*	* 1.308	(0.552)	** 1.855	(0.298)	–	–	** -0.180	(0.046)
city region	* -0.714	(0.288)	-0.090	(0.162)	* -0.712	(0.289)	-0.079	(0.161)
# observations used for estimation: 17,798								

Note: * denotes significance at the 5 %-level and ** at the 1 %-level, respectively. In the 2PM, interaction terms, such as *female* × *enoughcars*, stand for the interaction effect, here $\frac{\Delta^2 E}{\Delta x_2 \Delta x_1}$.

formulae (16), (18), and (19) and setting $\tau_{12} = \beta_{12} = 0$, still significantly differ from zero in three of four cases. This serves to highlight the fact that the marginal effect of a variable x_1 depends on variable x_2, even when no interaction term $x_1 x_2$ is included in the model. While the interaction effects are more pronounced on the left-hand panel of Table 1 and display different signs in two cases, the qualitative result regarding their statistical significance remains the same.

Moreover, we now illustrate that while the size of interaction effects eludes a straightforward interpretation, their statistical significance warrant testing. For example, with *enoughcars*=1 designating that there are at least as many cars as licensed drivers in a household, the interaction effect of 1.855 of the dummy variables *female* and *enoughcars* indicates a statistically significant difference of the conditional marginal effects of a sufficient versus an insufficient number of cars among male and female persons, and hence signals gender competition for cars. Beyond this qualitative aspect, however, the size of the interaction effect is difficult to interpret.

A major reason is that the interaction effect may be split up in either of two ways with equal justification. Given that we are dealing with a double-difference in this instance, the first way involves calculating the impact of sufficient cars among females and males. For females, this is given by:

$$\begin{aligned}\frac{\Delta E}{\Delta enoughcars}|_{female=1} &= E[y|enoughcars=1, female=1, \mathbf{w_1}, \mathbf{w_2}] \\ &\quad -E[y|enoughcars=0, female=1, \mathbf{w_1}, \mathbf{w_2}] = 2.761^{**},\end{aligned}$$

and for males by:

$$\begin{aligned}\frac{\Delta E}{\Delta enoughcars}|_{female=0} &= E[y|enoughcars=1, female=0, \mathbf{w_1}, \mathbf{w_2}] \\ &\quad -E[y|enoughcars=0, female=0, \mathbf{w_1}, \mathbf{w_2}] = 0.906^{**}.\end{aligned}$$

The difference of this pair of conditional marginal effects, which equals the interaction effect of 1.855 reported for the variables *female* and *enoughcars* on the left-hand panel of Table 1, differs from zero, as the respective interaction effect $\frac{\Delta^2 E}{\Delta enough \Delta female}$ is non-vanishing and statistically different from zero according to Table 1.

The same interaction effect of 1.855 also results from the difference of the following two marginal effects: first, the statistically insignificant marginal effect

$$\begin{aligned}\frac{\Delta E}{\Delta female}|_{enoughcars=1} &= [E[y|female=1, enoughcars=1, \mathbf{w_1}, \mathbf{w_2}] \\ &\quad -E[y|female=0, enoughcars=1, \mathbf{w_1}, \mathbf{w_2}] = 0.410,\end{aligned}$$

which indicates that among households with a sufficient number of cars, there are no significant differences between female and male car use for non-work purposes. By contrast, in households with less cars than licensed drivers, females drive 1.445 non-work kilometers less per day than males, confirming a large body of literature on gender differences in mobility behavior (e.g. WHITE, 1986; LEE, MCDONALD, 2003;

MCDONALD, 2005):

$$\frac{\Delta E}{\Delta female}|_{enoughcars=0} = [E[y|female = 1, enoughcars = 0, \mathbf{w_1}, \mathbf{w_2}] - E[y|female = 0, enoughcars = 0, \mathbf{w_1}, \mathbf{w_2}] = -1.445^{**}.$$

In short, while interaction effects are only of qualitative relevance, we have exemplified that useful quantitative interpretations can be gleaned from breaking the interaction effect into its constituent parts and testing the statistical significance of each conditional marginal effect, as is further demonstrated for the case of the Probit model in Appendix C.

Finally, it bears noting that the marginal effects of variables that are interacted with others are distinct to those when no interaction terms are employed in a specification. For example, the marginal effect of the gender dummy *female* is statistically significant and negative in the specification without interaction terms, but insignificant in the more general model specification including interaction terms. To sum up, while interaction effects are commonly hard to interpret and generally differ from zero even when no interaction terms are included, this section has demonstrated that the pair of conditional marginal effects of two variables that are assumed to interact are both more easy to grasp and of more importance than the magnitude of the corresponding interaction effect.

7 Summary and Conclusion

By providing a general derivation of marginal and interaction effects in both linear and non-linear models and the specific formulae of marginal and interaction effects gleaned from Heckit and Two-Part Models, this paper has analyzed the significance of these effects. Drawing on a survey of automobile use from Germany, we have illustrated that a non-vanishing interaction effect of two variables indicates differing marginal effects of one variable conditional on alternative values of the other variable, as one would expect for two interacting variables. The size of an interaction effect, however, hardly conveys any information. More meaningful, and also more easy to grasp, are the conditional marginal effects pertaining to two variables that are assumed to interact.

In linear specifications, so-called interaction terms, consisting of the product x_1x_2 of two explanatory variables, are typically included to capture the interaction effect, that is, the impact of an explanatory variable x_1 on the marginal effect of another explanatory variable x_2. In non-linear models, however, the marginal effect $\frac{\partial E}{\partial(x_1x_2)}$ of the interaction term generally differs from the interaction effect, whose formal description is given by the second derivative $\frac{\partial^2 E}{\partial x_2 \partial x_1}$. This difference, along with the fact that interaction effects are generally non-vanishing even when no interaction terms are included in any non-linear specification, raise the question as to whether interaction terms are irrelevant in non-linear contexts.

It might be argued that it is not necessary to include any interaction term in non-linear specifications, such as the Two-Part model, as in this case the marginal effect of an explanatory variable x_1 generally depends on all other variables. This line of reasoning would be incorrect, however, since this dependence always prevails, irrespective of whether a particular effect of another variable x_2 is taken into account by including the interaction term x_1x_2.

This can be seen from general expression (5), describing the marginal effect of variable x_1:

$$\frac{\partial E}{\partial x_1} = F'(u)(\beta_1 + \beta_{12}x_2).$$

The derivative $F'(u)$ captures the impact of a marginal change in $u = \beta_1x_1 + \beta_2x_2 + \beta_{12}x_1x_2 + \mathbf{w}^T\boldsymbol{\beta}$ induced by the variation of any of the included variables, whereas a special effect of varying x_2 is only to be observed if an interaction term x_1x_2 is included and the respective coefficient β_{12} is non-vanishing. In sum, the inclusion of interaction terms such as *female* $\times$ *age* is indispensable if one wants to meaningfully test the hypothesis of whether, for example, there are gender-specific differences in the impact of age on distance driven.

Appendix A: Data

Table A1: Variable Definitions and Descriptive Statistics

Variable Definition	Variable Name	Mean	Std. Dev.
Daily Kilometers driven for non-work purposes	*s*	4.505	10.801
Kilometers from home to work	*commute distance*	14.097	25.659
Dummy: 1 if person is female	*female*	0.480	0.500
Dummy: 1 if person is employed in a full-time or part-time job	*employed*	0.573	0.495
Age of the person	*age*	47.531	15.175
Dummy: 1 if person has a high school diploma	*high-school diploma*	0.340	0.474
Number of children younger than 18	*# children*	0.553	0.894
Number of employed household members	*# employed*	1.165	0.884
Dummy: 1 if number of cars $\geq$ number of licensed drivers	*enoughcars*	0.565	0.496
Dummy: 1 if household resides in a city	*city region*	0.323	0.468
Dummy: 1 if household has a private parking space	*private parking*	0.858	0.349
Walking time to the nearest public transportation stop	*minutes*	5.580	4.685
Dummy: 1 if the nearest public transportation stop is serviced by rail transit	*rail transit*	0.109	0.312

Appendix B: Comparison of Heckit and Two-Part

The marginal effects of the Heckit model, which are derived from the conditional expectation and incorporate the inverse Mills ratio, are interpreted in terms of potential outcomes (see DOW and NORTON, 2003). In several cases, e. g. for the variables *commute distance*, *high-school diploma*, and #*children*, this leads to qualitatively different conclusions than those pertaining to the actual outcomes of the Two Part Model.

Table B1: Estimation Results of Heckit and Two-Part Models

	Heckit Model				Two-Part Model			
	OLS		2. Stage		OLS		Both Stages	
	Coeff.s	Errors	Effects	Errors	Coeff.s	Errors	Effects	Errors
female	* -2.739	(1.336)	-0.294	(0.366)	-1.624	(1.292)	-0.285	(0.305)
employed	-0.475	(0.521)	** -1.061	(0.280)	** -1.181	(0.427)	** -1.087	(0.237)
commute distance	** 0.039	(0.008)	** 0.030	(0.007)	** 0.030	(0.007)	0.006	(0.003)
age	** -0.079	(0.028)	-0.035	(0.026)	-0.028	(0.019)	0.008	(0.013)
female × *age*	0.038	(0.031)	-0.018	(0.025)	-0.020	(0.023)	** -0.058	(0.012)
age × # *children*	0.006	(0.020)	0.012	(0.021)	0.012	(0.020)	0.009	(0.011)
high-school diploma	** 1.080	(0.330)	0.373	(0.205)	** 0.947	(0.320)	* 0.346	(0.173)
# *children*	-1.137	(0.875)	-0.208	(0.398)	-0.996	(0.873)	** 0.711	(0.211)
female × # *children*	-0.670	(0.575)	0.555	(0.372)	0.426	(0.331)	** 1.113	(0.222)
# *employed*	-0.013	(0.241)	-0.088	(0.257)	-0.033	(0.240)	-0.080	(0.129)
enoughcars	0.022	(0.511)	** 1.826	(0.268)	0.741	(0.422)	** 1.854	(0.223)
female × *enoughcars*	-0.574	(0.927)	* 1.366	(0.617)	* 1.308	(0.552)	** 1.855	(0.298)
city region	** -0.922	(0.298)	-0.072	(0.194)	* -0.714	(0.288)	-0.090	(0.162)
Inverse Mill's ratio	* -7.370	(3.047)	–	–	–	–	–	–
# observations used for estimation: 17,798								

Note: * denotes significance at the 5 %-level and ** at the 1 %-level, respectively.

Appendix C: Probit Estimation Results

The importance of prudent interpretation in non-linear models with interaction effects is further illustrated by the probit results presented in Table C1. The left-hand column contains the marginal effects reported by Stata's *mfx* command, which applies the formula $\beta_1\phi(\beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \mathbf{w}^T\boldsymbol{\beta})$ to each of the coefficients irrespective of whether they are interacted. The right-hand column contains marginal effects that control for the presence of the interaction terms.

Table C1: Marginal Effects resulting from the first-Stage Probit Estimation

	Using *mfx*		Correct Calculation	
	Effects	Errors	Effects	Errors
female	* 0.067	(0.033)	0.006	(0.016)
employed	** -0.052	(0.011)	** -0.052	(0.011)
commute distance	** -0.001	(0.000)	** -0.001	(0.000)
age	** 0.003	(0.000)	** 0.002	(0.001)
female × *age*	** -0.004	(0.001)	** -0.004	(0.001)
age × *# children*	0.001	(0.001)	0.001	(0.001)
high school diploma	-0.006	(0.009)	-0.006	(0.009)
# children	0.002	(0.026)	** 0.071	(0.012)
female × *# children*	** 0.100	(0.011)	** 0.089	(0.010)
# employed	-0.006	(0.007)	-0.006	(0.007)
enoughcars	** 0.052	(0.011)	** 0.117	(0.011)
female × *enoughcars*	** 0.150	(0.017)	** 0.137	(0.016)
city region	* 0.020	(0.009)	** 0.020	(0.009)
private parking	** 0.041	(0.011)	** 0.041	(0.011)
minutes	** 0.004	(0.001)	** 0.004	(0.001)
rail transit	** -0.048	(0.012)	** -0.048	(0.012)
# observations used for estimation: 44,842				

Note: * denotes significance at the 5 %-level and ** at the 1 %-level, respectively.

As Ai, Dow, and Norton point out, the marginal effects calculated by *mfx* for the interaction terms are incorrect, though in this particular example the differences are seen to be negligible. More notable discrepancies, however, are evident with respect to the marginal effects of the individual variables that comprise the interaction term.

For example, based on the *mfx* estimate, one would falsely conclude that the marginal effect of the number of children is insignificant. By employing the correct calculation given by

$$\begin{aligned}\frac{\partial E}{\partial \#children} \;=\;& (\beta_{\#children} + \beta_{\#children*age}\,\overline{age} + \beta_{\#children*female}\,\overline{female}) \\ & \phi(\beta_{\#children}\,\overline{\#children} + \beta_{\#children*age}\,\overline{\#children} * \overline{age} \\ & + \beta_{\#children*female}\overline{\#children} * \overline{female} + \mathbf{w}^T\boldsymbol{\beta})\end{aligned}$$

the estimate is seen to be significant at the 1% level. In the case of the variable *enough-cars*, the correct marginal effect given by

$$\frac{\Delta E}{\Delta enoughcars} = \Phi(\beta_{enoughcars} + \beta_{enoughcars*female}\overline{female} + \mathbf{w}^T\boldsymbol{\beta}) - \Phi(\mathbf{x}^T\boldsymbol{\beta})$$

is over double the magnitude of that calculated by *mfx*.

References

AI, C. , NORTON, E. C. (2003) Interaction terms in logit and probit models. *Economics Letters* **80**, 123-129.

DOW, W. H. , NORTON, E. C. (2003) Choosing Between and Interpreting the Heckit and Two-Part Models for Corner Solutions. *Health Services & Outcomes Research Methodology* **4**, 5-18.

FRONDEL, M. , VANCE, C. (2009a) Do High Oil Prices Matter? Evidence on the Mobility Behavior of German Households. *Environmental and Resource Economics*, **43 (1)**, 81-94.

FRONDEL, M. , VANCE, C. (2009b) Driving for Fun? Comparing the Effect of Fuel Prices on Weekday and Weekend Fuel Consumption. *Energy Economics*, forthcoming.

FRONDEL, M. , PETERS, J. , VANCE, C. (2008) Identifying the Rebound: Evidence from a German Household Panel. *The Energy Journal*, **29 (4)**, 154-163.

HECKMAN, J. (1979) Sample selection bias as a specification error. *Econometrica* **47**, 153-161.

LEE, B. , MCDONALD, J. (2003) Determinants of commuting time and distance for Seoul residents: The impact of family status on the commuting of women. *Urban Studies* **40 (7)**, 1283-1302.

MCDONALD, N. (2005) Does the Built Environment Affect the Travel 'Gender Gap'? Conference Proceedings 35: Research on Women's Issues in Transportation; Volume 2: Technical Papers. Washington, DC: Transportation Research Board.

NORTON, E. C. , WANG, H. , AI, C. (2004) Computing Interaction Effects and standard errors in Logit and Probit Models. *The Stata Journal* **4 (2)**, 103-116.

MOP (2007) German Mobility Panel. http://www.ifv.uni-karlsruhe.de/MOP.html

WHITE, M. (1986) Sex differences in urban commuting patterns. *American Economic Review* **76(2)**, 368-372.