



Analytic Approximations for Multi-Asset Option Pricing

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ABSTRACT

We derive a general analytic approximation for pricing basket options on N assets, which is extended to analytic approximations for pricing general rainbow options, including best-of and worst-of N asset options. The key idea is to express the option's price as a sum of prices of various compound exchange options, each with different pairs of sub-ordinate multi- or single-asset options. For some multi-asset options a strong condition holds, whereby each compound exchange option is equivalent to a standard single-asset option under a modified measure, and in such cases an almost exact analytic price exists for the multi-asset option. The underlying asset prices are assumed to follow log-normal processes, although the strong condition can be extended to certain other price processes for the underlying. More generally, approximate analytic prices for multi-asset options are derived using a weak log-normality condition, where the approximation stems from making constant volatility assumptions on the price processes that drive the prices of the sub-ordinate basket options. The analytic formulae for multi-asset option prices, and their Greeks, are defined in a recursive framework. For instance, the option delta is defined in terms of the delta relative to sub-ordinate multi-asset options, and the deltas of these sub-ordinate options with respect to the underlying assets. An empirical study of a particular 4-asset basket option tests the accuracy of our approximation, given some assumed values for the calibrated parameters. Then we demonstrate how to calibrate the model's parameters to market data so that the prices are consistent with the implied volatility and correlation skews of the assets.

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1. INTRODUCTION

The most commonly traded multi-asset options are basket options, rainbow options, best-of- N -assets and worst-of- N -assets options. These are collectively referred to as ‘linear’ multi-asset options because they have a linear structure in their payoffs. That is, the payoff is $\omega[f(\mathbf{S}_t, K)]^+$ where f is an affine transformation, \mathbf{S}_t is a vector of N asset prices, K is the option strike and ω is +1 for a call and -1 for a put. For example, when $N = 3$ then $\mathbf{S}_t = (S_1, S_2, S_3)$ and $f(\mathbf{S}_t, K) = (\theta_1 S_1 + \theta_2 S_2 + \theta_3 S_3) - K$ for the basket option with weights $(\theta_1, \theta_2, \theta_3)$ and $f(\mathbf{S}_t, K) = (S_1 - K, S_2 - K, S_3 - K)$ for a rainbow option.¹

To our knowledge exact analytic solutions only exist for a best-of-two-assets option, which is commonly called an ‘exchange option’, and this only when the underlying asset prices are assumed to follow correlated geometric Brownian motion (GBM) processes.² This is because a linear combination of log normal processes is not log normal. Even in the case of a spread option, i.e. a basket option on two assets with $\theta_1 = 1, \theta_2 = -1$, there is no exact analytic formula for the price.

There have been a number of attempts to approximate prices for standard European basket options: Levy [1992] approximates the basket price distribution with that of a single log-normal variable, matching the first and second moments; Gentle [1993] derives the price by approximating the arithmetic average by a geometric average; extending the Asian option pricing approach of Rogers and Shi [1995], Beißer [1999] expresses a basket option price as a weighted sum of single-asset Black-Scholes prices with adjusted forward price and adjusted strike for every constituent asset, so the price is expressed as an adjusted Black-Scholes price; Milevsky and Posner [1998a] use the reciprocal gamma distribution and Milevsky and Posner [1998b] use the Johnson [1949] family of distributions to approximate the basket distribution; Ju [2002] approximates the ratio of the characteristic function of the arithmetic average to the approximating variable using higher order Taylor’s expansion. For a comparison of the performance of the above models, please refer to Krekel et al. [2004].

Most of the above methods, based on approximating the basket price distribution or average price options, require the basket value to be positive. Hence, they may fail when the basket has a negative weight on one or more underlying asset prices. Models based on approximating the basket price distribution also ignore the possibility that correlation between individual assets, and their individual volatilities, may affect the option price. As a result their prices and hedge ratios need not be consistent with the implied volatility skews and implied correlation skews of the underlying asset prices.

Research on analytic approximations for pricing rainbow options includes an intuitive inductive formula of Johnson [1987], who extends the two asset rainbow option pricing formula of Stulz [1982] to the general case of N assets.³ Topper [2001] uses a finite element scheme to solve the associated non-linear parabolic price PDEs for options on two asset with different payoff profiles. More recently, Ouwehand and West [2006] have used the multivariate normal density approximation of West [2005] to price rainbow options on up to four assets.

¹A best-of option is a rainbow option with zero strike and worst-of options may be priced as best-of options as explained in the next section

²See Margrabe [1978]

³However, no mathematical justification is given for this formula.

In this paper we present a new method for approximating the price of a linear multi-asset option. We begin by assuming each asset price follows a standard GBM process, but later this assumption is relaxed to allow for more general drift and local volatility components. If, under their respective asset price processes, there exists an analytic solution for every single-asset vanilla option, then we have an approximate analytic formula for the price of a linear multi-asset option. The key idea is to rewrite the payoff as a sum of payoffs to various compound exchange options, each with different pairs of sub-ordinate basket options. As a result, the option price can be expressed exactly, as a sum of prices of compound exchange options on various sub-baskets of assets. Hence the pricing of compound exchange options is central to our work. We derive two general log-normality conditions, a strong condition under which almost exact prices can be obtained and a weaker condition under which prices are more approximate. Then we price the compound exchange options using the formula of Margrabe [1978] for exchange options on assets that follow log-normal processes.

The outline of this paper is as follows: Section 2 introduces our recursive framework for pricing linear multi-asset options and applies the recursion to basket options, rainbow options, and options on the best-of- N assets or worst-of- N assets; Having shown that all these linear multi-asset option prices may be expressed in terms of exchange option and basket option prices, and that the latter depends only on the prices of compound exchange options, Section 3 explains how we approximate the prices of the compound exchange options for the recursion that is central to this approach; Section 4 presents our approximations for basket option prices, which are based on either strong or weak log-normality conditions; Section 5 presents some experimental results on the accuracy of our approximation and an example illustrating the recommended approach to model calibration; Section 6 concludes.

2. PRICING FRAMEWORK

We shall first introduce a recursive approach that relies on expressing the price of a European option on a basket of assets as a sum of prices of two compound exchange options. Each compound exchange option (CEO) is an option to exchange two options. The sub-ordinate options in the CEO are basket options, where the assets in the baskets are a subset of the assets in the original multi-asset option. Then we show that the prices of other multi-asset options with a linear structure in their payoffs may be expressed in terms of basket option and CEO prices. This recursive framework allows one to price linear multi-asset options by pricing various CEOs.

Consider N assets with prices $\mathbf{S}_t = (S_{1t}, S_{2t}, \dots, S_{Nt})'$. Let $\mathbf{b}_N = (\theta_1 S_{1t}, \theta_2 S_{2t}, \dots, \theta_N S_{Nt})'$ be a basket of N assets with weights $\Theta_N = (\theta_1, \theta_2, \dots, \theta_N)$, where θ_i are real constants. Let $B_t = \sum_{i=1}^N \theta_i S_{it}$ be the price of the basket at any time t and V_{Nt} be the price of an option on a basket \mathbf{b}_N with strike price K . The payoff at expiry (time T) is given by:

$$\begin{aligned} V_{NT} &= [\omega (B_T - K)]^+ = [\omega (\Theta \mathbf{S}_T - K)]^+ \\ &= [\omega \Theta (\mathbf{S}_T - \mathbf{K})]^+ \end{aligned} \quad (1)$$

where $\omega = 1$ and -1 for calls and puts respectively and $\mathbf{K} = (K_1, K_2, \dots, K_N)'$ is a column vector of strikes such that $\Theta \mathbf{K} = K$.

Now let $\mathbf{b}_m = (\theta_1 S_{1t}, \theta_2 S_{2t}, \dots, \theta_m S_{mt})'$ and $\mathbf{b}_n = (\theta_{m+1} S_{(m+1)t}, \theta_{m+2} S_{(m+2)t}, \dots, \theta_N S_{Nt})'$ denote sub-

baskets of \mathbf{b} of sizes m and n respectively, with $m + n = N$, and denote the weights vectors of the corresponding sub-baskets by Θ_m and Θ_n . Similarly, let $\mathbf{S}'_t = (\mathbf{S}'_{mt}, \mathbf{S}'_{nt})$ and $\mathbf{K}' = (\mathbf{K}'_m, \mathbf{K}'_n)$. With this notation, equation (1) may be rewritten as

$$\begin{aligned} V_{NT} &= [\omega [\Theta_m (\mathbf{S}_{mT} - \mathbf{K}_m) + \chi \Theta_n (\mathbf{S}_{nT} - \mathbf{K}_n)]]^+ \\ &= \left[\omega \left(\Theta_m [\mathbf{S}_{mT} - \mathbf{K}_m]^+ - \Theta_n [\chi (\mathbf{K}_n - \mathbf{S}_{nT})]^+ \right) \right]^+ \\ &\quad + \left[\omega \left(\Theta_n [\chi (\mathbf{S}_{nT} - \mathbf{K}_n)]^+ - \Theta_m [\mathbf{K}_m - \mathbf{S}_{mT}]^+ \right) \right]^+, \end{aligned} \quad (2)$$

and

$$V_{NT} = \begin{cases} [C_{mT} - P_{nT}]^+ + [C_{nT} - P_{mT}]^+ & \text{if } \chi = 1, \\ [C_{mT} - C_{nT}]^+ + [P_{nT} - P_{mT}]^+ & \text{if } \chi = -1, \end{cases} \quad (3)$$

where C_m, P_m and C_n, P_n are prices of basket call and put options on m and n assets respectively. The role of parameter χ is to ensure that the prices of the sub-baskets are non-negative, in order to be able to define call and put options on them. That is, $\chi = 1$ or -1 such that $\Theta = (\Theta_m, \chi \Theta_n)$ and $\Theta_m \mathbf{S}_{mt}$ and $\Theta_n \mathbf{S}_{nt}$ are non-negative.

The European basket option price can then be computed as a sum of risk-neutral expectations of the two replicating CEO payoffs, E_{1T}, E_{2T} , which appear on the right hand side of (3):

$$V_{Nt} = e^{-r(T-t)} \left(\mathbb{E}_{\mathbb{Q}} \left\{ E_{1T} \middle| \mathcal{F}_t \right\} + \mathbb{E}_{\mathbb{Q}} \left\{ E_{2T} \middle| \mathcal{F}_t \right\} \right). \quad (4)$$

Hence, if $\chi = 1$, E_{1T} and E_{2T} are payoffs to exchange options on a basket call and a basket put; and if $\chi = -1$, they are payoffs to exchange options on two basket calls and two basket puts with a different number of assets in each basket. The prices of the call and put basket options C_m, C_n, P_m and P_n are in turn computed as CEOs on their sub-baskets as above.

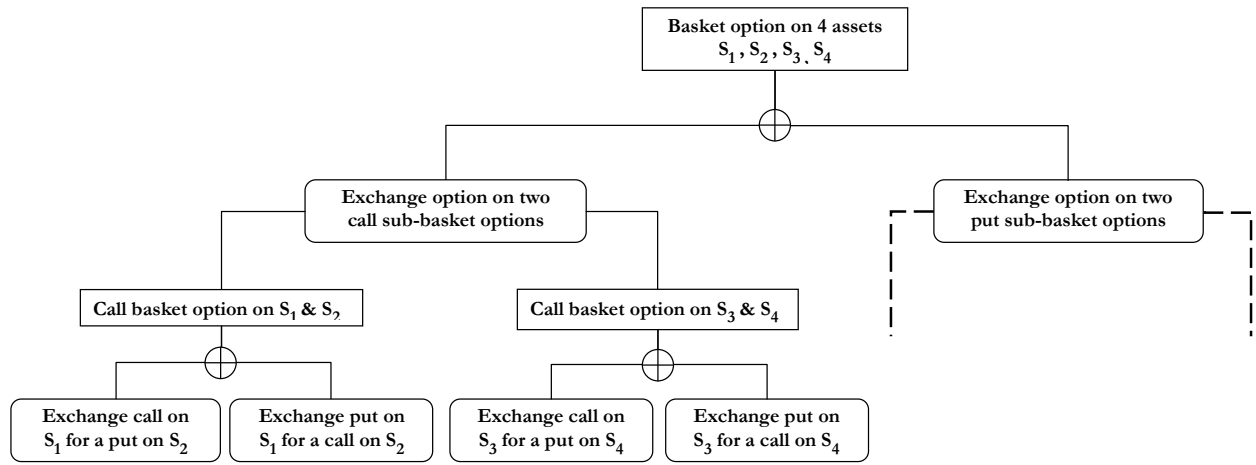
Figure 1 illustrates the pricing of a 4-asset basket option in this framework. It depicts a tree where a basket option price is recursively priced as a sum of exchange option prices. Due to lack of space, we have only shown one leg of the tree as the other leg can be priced in a similar manner. The basket option price is computed as a sum of prices of two CEOs, one on two sub-basket call options and the other on two sub-basket put options. In order to compute the price of the CEOs using Margrabe's formula, we have to first compute the prices of the two pairs of call and put sub-basket options. Extending this argument to the whole tree, we have to compute the prices of one pair of CEOs on call and put two-asset sub-basket options, four pairs of CEOs on call and put vanilla options, and four pairs of standard call and put vanilla options.

Note that we may use put-call parity, to obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left\{ [B_T - K]^+ \right\} - \mathbb{E}_{\mathbb{Q}} \left\{ [K - B_T]^+ \right\} &= \mathbb{E}_{\mathbb{Q}} \{ B_T - K \}, \\ C_{Nt} - P_{Nt} &= \sum_{i=1}^N \theta_i S_{it} e^{-q_i(T-t)} - K e^{-r(T-t)}, \end{aligned} \quad (5)$$

where q_i is the dividend yield of asset i . Therefore, we only need to compute the sub-basket and vanilla option prices for calls, because we can deduce the corresponding put prices using (5). Alternatively, we can derive the put prices, and derive the call prices using (5). So, in the general

FIGURE 1: Option on 4 Asset Basket as a CEO



\oplus denotes sum of prices of the two daughter nodes.

case of N assets, our approach requires $2(N - 1)$ evaluations of Margrabe's formula to compute the CEO prices and N evaluations of the Black-Scholes formula to compute the individual vanilla option prices.

We now demonstrate how the prices of other European linear multi-asset options may be expressed in terms of basket option and exchange option prices. An option on best-of- N assets, whose payoff is given by $\max \{S_{1T}, S_{2T}, \dots, S_{NT}\}$ is just a special case of a rainbow option with $K = 0$. Worst-of- N assets options can be priced as best-of options by noting that

$$\min \{S_{1T}, S_{2T}, \dots, S_{NT}\} = - \max \{-S_{1T}, -S_{2T}, \dots, -S_{NT}\}.$$

Hence we only need to consider a rainbow option on N assets.

The payoff to such an option may be written as sum of two payoffs, one to a best-of option on a sub-basket and the other to a compound option, as follows: Let (n_1, n_2, \dots, n_N) be a permutation of $(1, 2, \dots, N)$ and choose k to be some integer between 1 and N . By splitting the basket of N assets into two sub-baskets, we have

$$\begin{aligned} \max \{S_{1T} - K, S_{2T} - K, \dots, S_{NT} - K\} &= \max \{S_{n_{k+1}T}, S_{n_{k+2}T}, \dots, S_{n_{NT}}\} - K \\ &+ [\max \{S_{n_1T}, S_{n_2T}, \dots, S_{n_kT}\} - \max \{S_{n_{k+1}T}, S_{n_{k+2}T}, \dots, S_{n_{NT}}\}]^+. \end{aligned}$$

In the above, k determines the size of the sub-baskets and the permutation (n_1, n_2, \dots, n_N) determines the assets in these sub-baskets.

For every permutation (n_1, n_2, \dots, n_N) and index k we obtain a different payoff decomposition for the the rainbow option. An illustration of two alternative decompositions, for the case that $N = 4$, are given below. Obviously, the value of the payoff will be the same in each case, and the model should be calibrated in such a way that the option price is invariant to the choice of (n_1, n_2, \dots, n_N) and k .⁴

⁴This is discussed in section 5.

The best-of option payoff terms on the right hand side above may themselves be represented as the sum of two such payoffs, until all sub-baskets are on one or two assets. When the sub-basket size reduces to two, we have

$$\max \{S_{iT}, S_{jT}\} = S_{jT} + [S_{iT} - S_{jT}]^+.$$

For illustration consider a rainbow option on 4 assets. It is convenient to use the notation $X_{n_i t}$ for the price of an option to exchange asset n_i for asset n_{i+1} . Choosing $(n_1, n_2, n_3, n_4) = (1, 2, 3, 4)$ and $k = 2$, the rainbow option's payoff P_{4T} may be written:

$$\begin{aligned} P_{4T} &= \max \{S_{1T} - K, S_{2T} - K, S_{3T} - K, S_{4T} - K\} \\ &= \max \{S_{3T}, S_{4T}\} - K + [\max \{S_{1T}, S_{2T}\} - \max \{S_{3T}, S_{4T}\}]^+ \\ &= S_{4T} + [S_{3T} - S_{4T}]^+ + [S_{2T} + [S_{1T} - S_{2T}]^+ - S_{4T} - [S_{3T} - S_{4T}]^+]^+ - K \\ &= S_{4T} + X_{3T} + [S_{2T} - X_{1T} - S_{4T} + X_{3T}]^+ - K \\ &= S_{4T} + X_{3T} + [B_T]^+ - K, \end{aligned} \tag{6}$$

so that the price of the rainbow option is

$$P_{4t} = S_{4t} + X_{3t} + V_t - Ke^{-r(T-t)},$$

where $V_t = e^{-r(T-t)}\mathbb{E}_Q \{[B_T]^+\}$ is the price of a zero-strike basket option with four assets whose prices are $\{X_{1t}, S_{2t}, X_{3t}, S_{4t}\}$ and with weights $\{-1, 1, 1, -1\}$. Recall that, under the correlated GBM assumption, an analytic solution exists for X_{it} . Hence P_{4t} may be evaluated if we know the price V_t of the basket option; and this may be expressed in terms of CEO prices, as in (4).

The choice $k = 2$ leads to the simplest form of payoff decomposition for a four-asset rainbow option. In this case, given an arbitrary permutation (n_1, n_2, n_3, n_4) , a similar argument to that above may be applied to show that the payoff decomposition is:

$$P_{4T} = S_{n_4T} + X_{n_3T} + [S_{n_2T} - X_{n_1T} - S_{n_4T} + X_{n_3T}]^+ - K,$$

so that a general expression for the price of a 4-asset rainbow option is

$$P_{4t} = S_{n_4t} + X_{n_3t} + V_{4t} - Ke^{-r(T-t)},$$

where $V_{4t} = e^{-r(T-t)}\mathbb{E}_Q \{[B_T]^+\}$ denotes the price of a zero-strike basket option with four assets whose prices are $\{X_{n_1t}, S_{n_2t}, X_{n_3t}, S_{n_4t}\}$ and with weights $\{-1, 1, 1, -1\}$.

Finally, we provide an example that illustrates how to extend this argument to rainbow options on more than 4 assets. Following the lines of the 4-asset rainbow option, the payoff to a rainbow option on 3 assets with prices S_5, S_6 and S_7 can be written as:

$$\begin{aligned} P_{3T} &= \max \{S_{5T}, S_{6T}, S_{7T}\} \\ &= S_{7T} + [S_{6T} - S_{7T}]^+ + [S_{5T} - S_{7T} - [S_{6T} - S_{7T}]^+]^+ \\ &= S_{7T} + X_{6T} + [S_{5T} - S_{7T} + X_{6T}]^+. \end{aligned}$$

Hence, the price of a rainbow option on 7 assets is given by

$$P_{7t} = P_{3t} + \mathbb{E}_Q \{[P_{4T} - P_{3T}]^+\}. \tag{7}$$

3. COMPOUND EXCHANGE OPTIONS

The basket options that were used to price linear multi-asset options in the previous section are defined on a basket which may contain assets, options written on these assets or exchange options. For instance, the basket option on the right hand side of equation (6) is a compound four-dimensional basket option defined on two assets and two exchange options. In the case of a simple basket option, equation (4) expresses its price as a sum of prices of CEOs. Since multi-asset option prices are based on multiple CEOs, pricing CEOs is central to our framework. In this section we derive an analytic approximation for the price of such an option, first under the correlated GBM assumption and then under more general assumptions for the asset price processes.

A compound exchange option is an option to exchange one option for another. In general, the two underlying options may be on different assets, have different maturities or may themselves be compound options. Here we describe an analytical approach to price CEOs that are written on options on two different underlying assets with the same maturity as the CEO, as this will always be the case for the framework described in the previous section. The key result is that the price of such a CEO has an equivalent representation as a single-asset option price whose solution can be easily derived.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ be a filtered probability space, where Ω is the set of all possible events such that $S_{1t}, S_{2t} \in (0, \infty)$, $(\mathcal{F}_t)_{t \geq 0}$ is the filtration produced by the sigma algebra of the price pair $(S_{1t}, S_{2t})_{t \geq 0}$ and \mathbb{Q} is a bivariate risk neutral probability measure. Assume that the risk-neutral price dynamics are governed by :

$$\begin{aligned} dS_{it} &= rS_{it}dt + \sigma_i S_{it} dW_{it} \\ \langle dW_{1t}, dW_{2t} \rangle &= \rho dt \quad i = 1, 2, \end{aligned}$$

where, W_1 and W_2 are Wiener processes under risk neutral measure \mathbb{Q} , σ_i is the volatility of asset i (assumed constant) and ρ is the correlation between them (assumed constant).

Let U_{1T} and V_{1T} denote the payoffs to European call and put options respectively, on asset 1 with strike K_1 ; similarly, let U_{2T} and V_{2T} denote the payoffs to European call and put options on asset 2 with strike K_2 . We consider compound options to exchange a European call or put option on one asset with another. For instance, if the CEO is on two calls, the payoff is given by $[\omega (U_{1T} - U_{2T})]^+$, for $\omega = 1$ or -1 . The price f_t of such an option can be obtained as a risk-neutral expectation of the terminal payoff:

$$f_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ [\omega (U_{1T} - U_{2T})]^+ \mid \mathcal{F}_t \right\}. \quad (8)$$

We first describe the price processes for the two call and put options U_i and V_i , $i = 1, 2$. Applying Itô's lemma to (8) yields

$$\begin{aligned} dU_{it} &= \frac{\partial U_{it}}{\partial t} dt + \frac{\partial U_{it}}{\partial S_{it}} dS_{it} + \frac{1}{2} \frac{\partial^2 U_{it}}{\partial S_{it}^2} dS_{it}^2 \\ &= \left(\frac{\partial U_{it}}{\partial t} + rS_{it} \frac{\partial U_{it}}{\partial S_{it}} + \frac{1}{2} \sigma_i^2 S_{it}^2 \frac{\partial^2 U_{it}}{\partial S_{it}^2} \right) dt + \sigma_i S_{it} \frac{\partial U_{it}}{\partial S_{it}} dW_{it} \\ &= rU_{it} dt + \zeta_{it} U_{it} dW_{it}, \end{aligned} \quad (9)$$

where, $\zeta_{it} = \sigma_i \frac{S_{it}}{U_{it}} \frac{\partial U_{it}}{\partial S_{it}}$. Similarly, for the put option price:

$$dV_{it} = rV_{it}dt + \eta_{it}V_{it}dW_{it}, \quad (10)$$

where, $\eta_{it} = \sigma_i \frac{S_{it}}{V_{it}} \left| \frac{\partial V_{it}}{\partial S_{it}} \right|^5$.

From now on we restrict our analysis to a CEO on calls, and solve for ζ_i and U_i . We derive a closed form solution to the CEO price by first solving for the prices of the underlying options U_i and their volatilities ζ_i . Finally, our main result provides a general condition under which a CEO can be priced by reducing it to a single-asset option price under an equivalent measure. The solutions for the put option volatilities and prices can be derived along similar lines.

Lemma 1. *Given option price process (9), ζ_i follows a process described by:*

$$d\zeta_{it} = \zeta_{it} \left(\sigma_i - \zeta_{it} + \sigma_i S_{it} \frac{\Gamma_{it}}{\Delta_{it}} \right) [-\zeta_{it}dt + dW_{it}], \quad (11)$$

where $\Delta_{it} = \frac{\partial U_{it}}{\partial S_{it}}$ and $\Gamma_{it} = \frac{\partial \Delta_{it}}{\partial S_{it}}$ are the delta and gamma of the call option with respect to underlying i .

Proof. Let $\theta_{it} = \frac{\partial U_{it}}{\partial t}$ and $X_{it} = \frac{\Delta_{it}}{U_{it}}$. Dropping the subscripts, we have, by Itô's lemma:

$$\begin{aligned} d\Delta &= \frac{\partial \Delta}{\partial t} dt + \frac{\partial \Delta}{\partial S} dS + \frac{1}{2} \frac{\partial^2 \Delta}{\partial S^2} dS^2 \\ &= \frac{\partial \theta}{\partial S} dt + \frac{\partial \Delta}{\partial S} (rSdt + \sigma SdW) + \frac{1}{2} \frac{\partial^2 \Delta}{\partial S^2} \sigma^2 S^2 dt \\ &= \frac{\partial}{\partial S} \left(\theta + rS\Delta + \frac{1}{2} \sigma^2 S^2 \Gamma \right) dt - (r\Delta + \Gamma \sigma^2 S) dt + \sigma \Gamma dW \\ &= \frac{\partial}{\partial S} (rU) dt - (r\Delta + \Gamma \sigma^2 S) dt + \sigma \Gamma dW \\ &= -\Gamma \sigma^2 S dt + \sigma \Gamma dW, \end{aligned}$$

and

$$\begin{aligned} dX &= \frac{1}{U} d\Delta - \frac{\Delta}{U^2} dU + \frac{\Delta}{U^3} dU^2 - \frac{1}{U^2} d\Delta dU \\ &= \frac{1}{U} (-\Gamma \sigma^2 S dt + \sigma \Gamma dW - \Delta (rdt + \zeta dW) + \Delta \zeta^2 dt - \sigma \Gamma \zeta dt) \\ &= \frac{1}{U} (\Delta \zeta^2 - \sigma \Gamma \zeta - r\Delta - \Gamma \sigma^2 S) dt + \frac{1}{U} (\sigma \Gamma - \zeta \Delta) dW. \end{aligned}$$

Thus,

$$\begin{aligned} d\zeta &= \sigma (XdS + SdX + dSdX) \\ &= \left(\sigma \zeta - \zeta^2 + \frac{\sigma^2 S^2}{U} \Gamma \right) [-\zeta dt + dW], \end{aligned}$$

which can be rewritten as (11). □

⁵Since the delta of a put option is negative, we take the absolute value here. However, due to the symmetry of Wiener process - i.e., upward and downward movements have equal probabilities - the sign of volatility does not affect the distribution of the diffusion term $\eta_i dW_{it}$. The effect of using the absolute value is to change the sign of the correlation between the Wiener process driving the put option, and any other process.

Now we define $\tilde{\sigma}_i = \sigma_i (1 + S_{it}\Gamma_{it}/\Delta_{it})$ and treat $\sigma_i S_{it}\Gamma_{it}/\Delta_{it}$ as constant in equation (11). This is the only approximation we make for pricing CEOs.⁶ Then, the (approximate) option volatility processes are described by

$$d\tilde{\zeta}_{it} = \tilde{\zeta}_{it}(\tilde{\zeta}_{it} - \tilde{\sigma}_i) (\tilde{\zeta}_{it}dt - dW_{it}). \quad (12)$$

Lemma 2. *The solution to equation (12) is given by*

$$\tilde{\zeta}_{it} = \tilde{\sigma} \left(1 - \left(1 - \frac{\tilde{\sigma}}{\tilde{\zeta}_{i0}} \right) \exp \left(\frac{1}{2} \tilde{\sigma}^2 t - \tilde{\sigma} W_{it} \right) \right)^{-1}. \quad (13)$$

Proof. Dropping subscript i for convenience and letting $y_t = \frac{1}{\tilde{\sigma}} \ln \left(\frac{\tilde{\zeta}_t - \tilde{\sigma}}{\tilde{\zeta}_t} \right)$, we have

$$\begin{aligned} dy &= \frac{1}{\tilde{\sigma}} \left(\frac{1}{\tilde{\zeta}_t - \tilde{\sigma}} - \frac{1}{\tilde{\zeta}_t} \right) dx + \frac{1}{2\tilde{\sigma}} \left(\frac{-1}{(\tilde{\zeta}_t - \tilde{\sigma})^2} + \frac{1}{\tilde{\zeta}_t^2} \right) dx^2 \\ &= \frac{dx}{\tilde{\zeta}_t(\tilde{\zeta}_t - \tilde{\sigma})} + \frac{1}{2}(\tilde{\sigma} - 2\tilde{\zeta}_t) \left(\frac{dx}{\tilde{\zeta}_t(\tilde{\zeta}_t - \tilde{\sigma})} \right)^2 \\ &= \tilde{\zeta}_t dt - dW + \frac{1}{2}(\tilde{\sigma} - 2\tilde{\zeta}_t) dt \\ &= \frac{1}{2}\tilde{\sigma} dt - dW. \end{aligned}$$

Thus

$$y_t = y_0 + \frac{1}{2}\tilde{\sigma}t - W_t.$$

Substituting y_t into the above equation we obtain

$$\begin{aligned} \frac{1}{\tilde{\sigma}} \ln \left(\frac{\tilde{\zeta}_t - \tilde{\sigma}}{\tilde{\zeta}_t} \right) &= \frac{1}{\tilde{\sigma}} \ln \left(\frac{\tilde{\zeta}_0 - \tilde{\sigma}}{\tilde{\zeta}_0} \right) + \frac{1}{2}\tilde{\sigma}t - W_t \\ 1 - \frac{\tilde{\sigma}}{\tilde{\zeta}_t} &= ke^{\frac{1}{2}\tilde{\sigma}^2 t - \tilde{\sigma}W_t} \\ \Rightarrow \tilde{\zeta}_t &= \tilde{\sigma} (1 - ke^{\tilde{\sigma}Z_t})^{-1}, \end{aligned}$$

where $k = 1 - \frac{\tilde{\sigma}}{\tilde{\zeta}_0}$ and $Z_t = \frac{1}{2}\tilde{\sigma}t - W_t$. □

⁶For in the money options, $\sigma_i S_{it}\Gamma_{it}/\Delta_{it} \ll (\tilde{\zeta}_{it} - \sigma_i)$. For example, in the Black-Scholes model, when $S = 100$, $T = 30$ days, $\sigma = 0.2$, $r = 5\%$, then the values are (4.13, 2.5), (1.66, 0.2), (0.78, 4E-04), (0.46, 2.9E-09), (0.29, 3.3E-18) for call options with strikes 100, 90, 80, 70 and 60 respectively. For out of the money options, $\sigma_i S_{it}\Gamma_{it}/\Delta_{it} \approx (\tilde{\zeta}_{it} - \sigma_i)$, which implies $(\tilde{\zeta}_{it} - \sigma_i + \sigma_i S_{it}\Gamma_{it}/\Delta_{it}) \approx 2(\tilde{\zeta}_{it} - \sigma_i)$. Therefore, by treating $\sigma_i (1 + S_{it}\Gamma_{it}/\Delta_{it})$ as constant we may derive an approximate solution to $\tilde{\zeta}_i$ irrespective of the moneyness of the option. Moreover since the pricing framework described in section 2 only requires the weighted sum of strikes of the single-asset options on all assets to be equal to K , we can choose the strikes such that the approximation error is a minimum. However, the approximation error will be high when the moneyness of the option swings between in-the-money and out of the money (when the asset price crosses the strike frequently). This can be easily avoided by choosing either low or high strike values.

Lemma 3. *The call option price at time t is given by*

$$U_{it} = U_{i0}e^{rt} \left(\exp \left(-\frac{1}{2}\tilde{\sigma}^2 t + \tilde{\sigma} W_{it} \right) - \left(1 - \frac{\tilde{\sigma}}{\xi_{i0}} \right) \right). \quad (14)$$

Proof. Dropping subscript i and substituting ξ_t in equation (9) gives

$$\begin{aligned} U_t &= U_0 \exp \left(rt - \frac{1}{2} \int_0^t \xi_t^2 dt + \int_0^t \xi_t dW_t \right) \\ &= U_0 \exp \left(rt - \frac{1}{2} \int_0^t (\xi_t^2 - \tilde{\sigma} \xi_t) dt - \int_0^t \xi_t dZ_t \right) \\ &= U_0 \exp \left(rt - \frac{1}{2} \tilde{\sigma}^2 k \int_0^t e^{\tilde{\sigma} Z_t} (1 - ke^{\tilde{\sigma} Z_t})^{-2} dt - \tilde{\sigma} \int_0^t (1 - ke^{\tilde{\sigma} Z_t})^{-1} dZ_t \right). \end{aligned} \quad (15)$$

Also

$$\begin{aligned} \frac{1}{\tilde{\sigma}} d \left(\ln \left(\frac{e^{\tilde{\sigma} Z_t}}{1 - ke^{\tilde{\sigma} Z_t}} \right) \right) &= \frac{1}{1 - ke^{\tilde{\sigma} Z_t}} dZ_t + \frac{1}{2} \tilde{\sigma} k \frac{e^{\tilde{\sigma} Z_t}}{(1 - ke^{\tilde{\sigma} Z_t})^2} dt \\ \Rightarrow \ln \left(\frac{e^{\tilde{\sigma} Z_t}}{1 - ke^{\tilde{\sigma} Z_t}} \right) &= \int_0^t \frac{\tilde{\sigma}}{1 - ke^{\tilde{\sigma} Z_t}} dZ_t + \int_0^t \frac{1}{2} \tilde{\sigma}^2 k \frac{e^{\tilde{\sigma} Z_t}}{(1 - ke^{\tilde{\sigma} Z_t})^2} dt. \end{aligned}$$

Substituting this in equation (15) gives

$$\begin{aligned} U_t &= U_0 \exp \left(rt - \ln \left(\frac{e^{\tilde{\sigma} Z_t}}{1 - ke^{\tilde{\sigma} Z_t}} \right) \right) \\ &= U_0 e^{rt} \left(\exp \left(-\frac{1}{2} \tilde{\sigma}^2 t + \tilde{\sigma} W_t \right) - \left(1 - \frac{\tilde{\sigma}}{\xi_0} \right) \right) \\ &= U_0 e^{rt} (e^{-\tilde{\sigma} Z_t} - k). \end{aligned}$$

□

From equation (14), we can see that when the initial option volatility $\xi_{i0} = \tilde{\sigma}_i$, then U follows a log-normal process. By definition, we have

$$\xi_{it} = \sigma_i \frac{S_{it}}{U_{it}} \frac{\partial U_{it}}{\partial S_{it}}. \quad (16)$$

This implies that $\xi_{it} \rightarrow \sigma_i$ when $\frac{\partial U_{it}}{\partial S_{it}}$ and $\frac{S_{it}}{U_{it}}$ both tend to one. This is possible when the strikes of the call options are chosen such that they are deep-in-the-money. Then, $\sigma_i \approx \tilde{\sigma}_i$ and therefore $\xi_{i0} \approx \tilde{\sigma}_i$. Moreover, from equation (13), $\xi_{it} \approx \tilde{\sigma}_i$ for all $t \in [0, T]$.⁷ We call this the *weak form* of log-normality condition, under which the option price process follows approximate log-normal process. It is important to note that the weak condition does not depend on the choice of drift of the stochastic process of the underlying asset price. However, it is necessary for σ_i to be a constant. This is particularly useful for pricing basket options, as discussed in section 4.

In the following theorem we provide a stronger condition under which the relative option price follows a log-normal process. We call this the *strong form* of log-normality condition, and under

⁷An intuitive explanation of this is that the price of a deep ITM exchange option is a linear function of the relative price of the two underlying assets and, under the GBM assumption, the relative price distribution is log-normal. The price of a deep OTM exchange option is approximately zero.

this condition a CEO price may be expressed as a single-asset option price. This leads to an almost exact solution to the price of a CEO.

Theorem 4. *If the following condition holds*

$$U_{1t} \left(1 - \frac{\tilde{\sigma}_1}{\xi_{1,0}} \right) - U_{2t} \left(1 - \frac{\tilde{\sigma}_2}{\xi_{2,0}} \right) = 0, \quad (17)$$

then the CEO on calls has the same price as a standard single-asset option under a modified yet equivalent measure.

Proof. The price of a CEO on two call options is given by:

$$\begin{aligned} f_t &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ [\omega (U_{1T} - U_{2T})]^+ \right\} \\ &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ [U_{1t} e^{r(T-t)} (e^{-\tilde{\sigma}_1 Z_1(T-t)} - k_1) - U_{2t} e^{r(T-t)} (e^{-\tilde{\sigma}_2 Z_2(T-t)} - k_2)]^+ \right\} \\ &= \mathbb{E}_{\mathbb{Q}} \left\{ \left[U_{1t} e^{-\tilde{\sigma}_1 Z_1(T-t)} - U_{2t} e^{-\tilde{\sigma}_2 Z_2(T-t)} - (U_{1t} k_1 - U_{2t} k_2) \right]^+ \right\} \\ &= \mathbb{E}_{\mathbb{Q}} \left\{ \left[U_{1t} \exp \left(-\frac{1}{2} \int_t^T \tilde{\sigma}_1^2 ds + \int_t^T \tilde{\sigma}_1 dW_{1s} \right) - U_{2t} \exp \left(-\frac{1}{2} \int_t^T \tilde{\sigma}_2^2 ds + \int_t^T \tilde{\sigma}_2 dW_{2s} \right) \right]^+ \right\}. \end{aligned}$$

Let $dW_{1t} = \rho_1 dW_{2t} + \sqrt{1 - \rho_1^2} dW_{3t}$, where W_{2t} and W_{3t} are independent Wiener processes and \mathbb{P} be a probability measure whose Radon-Nikodym derivative with respect to \mathbb{Q} is given by:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp \left(-\frac{1}{2} \tilde{\sigma}_2^2 T + \tilde{\sigma}_2 W_{2t} \right).$$

Let $Y_t = U_{1t}/U_{2t}$ and W be a Brownian motion under \mathbb{P} . Then the dynamics of Y can be described by

$$dY_t = rY_t dt + \tilde{\sigma}_t Y_t dW_t,$$

where $\tilde{\sigma}_t^2 = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\rho\tilde{\sigma}_1\tilde{\sigma}_2$, and the price of the CEO can be written as the price of a single-asset option written on Y , as

$$\begin{aligned} f_t &= U_{2t} e^{-r(T-t)} \mathbb{E}_{\mathbb{P}} \left\{ \left[Y_t \exp \left(\int_t^T r dt - \frac{1}{2} \int_t^T \tilde{\sigma}_s^2 ds + \int_t^T \tilde{\sigma}_s dW_s \right) - 1 \right]^+ \right\} \\ &= U_{2t} e^{-r(T-t)} \mathbb{E}_{\mathbb{P}} \left\{ [Y_T - 1]^+ \right\}. \end{aligned}$$

□

The above proof shows that, under (17), the relative prices of the vanilla options behave like log-normal processes, so that a CEO can be reduced to a simple log-normal exchange option. The price of such an option can be easily found by change of numeraire, as in Margrabe [1978]. The strikes K_1 and $K_2 = (K_1 - K)$ for which the log-normality condition (17) holds can be found by using a simple one-dimensional solver.

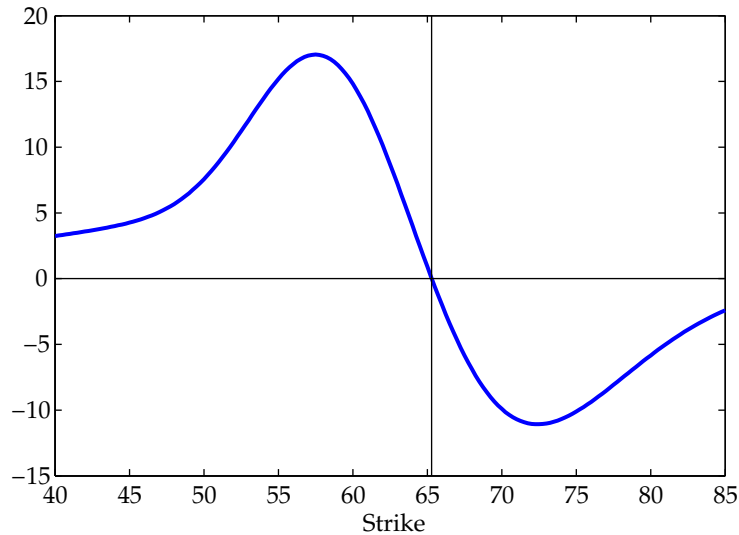
Theorem 4 allows one to price CEOs almost exactly when the underlying option prices satisfy condition (17) almost exactly.⁸ In the next section we shall see that the theorem plays a vital role in

⁸The only approximation we make is to assume $\sigma_i S_{it} \Gamma_{it} / \Delta_{it}$ is constant, in order to obtain an approximate option volatility process described by equation (12). The derivation of the CEO price formula is otherwise exact. As discussed earlier, the approximation error can be extremely small for certain strikes of the vanilla options. Since we are free to choose these strikes, such an approximation can be justified.

our main objective of pricing specific basket options. In its application to basket options note that we are free to choose the strikes K_i of the underlying vanilla options of the CEOs that replicate the basket option, as long as $\sum \theta_i K_i = K$. Therefore we may be able to find particular strikes K_i for which the relative price of vanilla options is log-normal. In order to demonstrate this, in figure 2 we plot the behaviour of condition (17) for two sample vanilla options. We can see that the condition holds when the strikes of the options are equal to 65.3.

FIGURE 2: Plot of condition (17) against strike

($S_1 = 75, S_2 = 65, \sigma_1 = 0.25, \sigma_2 = 0.25, r = 4\%, T = 6$ months)



So far we have only discussed the pricing of CEOs when the underlying asset prices followed GBM processes. However, theorem 4 may be extended to cases where the underlying asset prices follow certain non-GBM processes. In particular, assume that the risk-neutral price dynamics are governed by a more general three-factor model:

$$\begin{aligned} dS_{it} &= \mu_i(S_{it}, t)dt + \sigma_i(S_{it}, t)dW_{it} \\ \langle dW_{1t}, dW_{2t} \rangle &= \rho dt \quad i = 1, 2, \end{aligned}$$

where ρ is assumed constant. It is easy to show that when $\mu_i(S_{it}, t)$ and $\sigma_i(S_{it}, t)$ satisfy

$$\left(\frac{\partial \sigma_{it}}{\partial t} + \mu_{it} \frac{\partial \sigma_{it}}{\partial S_{it}} + \frac{1}{2} \sigma_{it}^2 \frac{\partial^2 \sigma_{it}}{\partial S_{it}^2} \right) = \sigma_{it} \frac{\partial \mu_{it}}{\partial S_{it}}, \quad (18)$$

the option price processes will still be given by (14). For example, this holds for affine functions $\mu_i = a_i + b_i S_i$ and $\sigma_i = \alpha_i + \beta_i S_i$ with $a_i/\alpha_i = b_i/\beta_i$; where a_i, b_i, α_i and β_i are real constants.

4. BASKET OPTION PRICE FORMULA

In this section we derive analytic approximations to the price of a basket option based on the recursive approach described in section 2. In order to price the CEOs, we apply theorem 4 and this leads to a recursive application of the log-normal exchange option pricing formula of Margrabe [1978]. For specific examples of basket options with sizes equal to two or three, we show that our

approach leads to an almost exact solution to the basket option price. However, when the number of assets in the option is four or more the log-normality condition in theorem 4 is too strong. Thus we introduce a more general, but weaker condition under which we derive approximate analytic prices for linear multi-asset options on N assets.

4.1. Pricing under Strong Form of Log-Normality Condition

Recall that when the single-asset option price processes are described by equations (9) or (10), then a CEO on them can be priced by dimension reduction under (17). Now consider a CEO written on two log-normal exchange options, both having a common asset. By Margrabe [1978], the exchange options can be equivalently priced as vanilla options on a single asset by choosing the price of the common asset as numeraire. Then, by using the single-asset argument of section 3, the exchange option price processes may be described by equations (9) or (10). Thus, the CEO can be priced by applying theorem 4 under the strong log-normality condition.⁹

To illustrate this we consider 2-asset basket options with non-zero strike and 3-asset basket options with zero strike.

(a) The payoff to a 2-asset basket option can be written as a sum of payoffs of two CEOs on single-asset call and put options, as in section 3. Since the CEOs are written on vanilla options it is straightforward to compute their prices using theorem 4 under the strong log-normality condition.

(b) Consider a 3-asset basket option with zero strike, when the signs of the asset weights Θ are a permutation of $(1, 1, -1)$ or $(-1, -1, 1)$. This is just an extension of the 2-asset case, where we have an additional asset instead of the strike. The 3-asset basket option can be priced as a CEO either to exchange a 2-asset exchange option for the third asset or to exchange two 2-asset exchange options with a common asset. For example, consider a 3 : 2 : 1 spread option that is commonly traded in energy markets:

$$\begin{aligned} P_T &= [3S_{1T} - 2S_{2T} - S_{3T}]^+ \\ &= [3[S_{1T} - S_{2T}]^+ - [S_{3T} - S_{2T}]^+]^+ + [[S_{2T} - S_{3T}]^+ - 3[S_{2T} - S_{1T}]^+]^+ . \end{aligned}$$

Here the two CEOs can be priced by dimension reduction by choosing S_2 as the numeraire.

In the general case of basket options on N underlying assets, except for the ones discussed above, the two replicating CEOs are no longer written on plain vanilla or log-normal exchange options, but on sub-basket options. Since the prices of these sub-basket options are computed as a sum of prices of CEOs, it is not straightforward to express their processes in a form that would lead to an exact solution to the basket option price. Thus exact pricing under the strong log-normality condition is only possible in special cases, and in general we can only find an approximate price under the weak log-normality condition described in the next sub-section.

⁹Carr [1988] introduced an alternative change of numeraire approach to price sequential exchange options where an exchange may lead to further exchanges. His approach is based on Geske [1977] and Margrabe [1978] who discuss the pricing of compound vanilla options and log-normal exchange options respectively. However, he only discusses the pricing of CEOs on a log-normal exchange option and a log-normal asset that is the same as the asset delivered in the exchange option. His approach is difficult to extend to higher dimensions and also does not apply to our case where both the underlying assets of the CEO are options.

For instance, consider a 4-asset basket option with zero strike. Here we can write the payoff as:

$$\begin{aligned} P_T &= [S_{1T} - S_{2T} - S_{3T} + S_{4T}]^+ \\ &= [[S_{1T} - S_{2T}]^+ - [S_{3T} - S_{4T}]^+]^+ + [[S_{4T} - S_{3T}]^+ - [S_{2T} - S_{1T}]^+]^+ . \end{aligned}$$

Since the two replicating CEOs are written on log-normal exchange options with no common asset, the CEO prices cannot be priced using theorem 4. Nevertheless, by adjusting the volatilities of the CEOs under the weak log-normality condition, the sub-basket option price processes can be approximately described by GBM processes. Then the relative sub-basket option prices follow approximate log-normal processes and the two replicating CEO prices can be computed by using the formula of Margrabe [1978].

4.2. Pricing under Weak Form of Log-Normality Condition

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ be a filtered probability space, where Ω is the set of all possible events such that $(S_{1t}, S_{2t}, \dots, S_{Nt}) \in (0, \infty)^N$, $(\mathcal{F}_t)_{t \geq 0}$ is the filtration produced by the sigma algebra of the N -tuple $(S_{1t}, S_{2t}, \dots, S_{Nt})_{t \geq 0}$ of asset prices and \mathbb{Q} is a multi-variate risk neutral probability measure. We assume that the underlying asset prices processes S_i are described by:

$$\begin{aligned} dS_{it} &= \mu_i(S_{it}, t)S_{it}dt + \sigma_i S_{it}dW_i \\ \langle dW_{it}, dW_{jt} \rangle &= \rho_{ij}dt, \quad 1 \leq i, j \leq N, \end{aligned} \tag{19}$$

where W_i are Wiener processes under the risk neutral measure \mathbb{Q} , σ_i is the volatility of i^{th} asset (assumed constant), $\mu_i(\cdot)$ is a well defined function of S_{it} and t , and ρ_{ij} is the correlation between i^{th} and j^{th} assets (assumed constant).

To price a European CEO on basket options we need to know the distribution of their payoffs at time T . We have already discussed the pricing of CEOs when the underlying assets followed GBM processes, in section 3. Hence we start by describing the price process V_N of the basket option on N assets when $\chi = -1$. As before, we assume that the prices of the call and put sub-basket options on m and n assets follow log-normal processes. We then show that, when the basket option volatility is approximated as a constant, the basket option price process V_N can be expressed as a GBM process. Since C_m, C_n, P_m and P_n are prices of basket options themselves, we may also express their processes as GBM process if their sub-basket option prices follow log-normal processes. In the end, an assumption that sub-basket call and put option prices follow log-normal processes will lead to an approximate log-normal process for the price of a basket option on N assets.

When $\chi = -1$, the basket option price is computed as a sum of two CEOs - one on two call sub-basket options and the other on two put sub-basket options. Now, consider the CEO on calls. For $i = m$ or N , we have

$$dC_{it} = rC_{it}dt + \sigma_{C_i}C_{it}d\tilde{W}_{it}, \tag{20}$$

where $\sigma_{C_m}, \sigma_{C_n}$ are the volatilities and \tilde{W}_m, \tilde{W}_n are the Wiener processes driving the two call basket

options. Applying Itô's lemma:

$$\begin{aligned}
 dE_{1t} &= \frac{\partial E_{1t}}{\partial t} dt + \sum_{i=m,n} \left(\frac{\partial E_{1t}}{\partial C_{it}} dC_{it} + \sum_{j=m,n} \frac{1}{2} \frac{\partial^2 E_{1t}}{\partial C_{it} \partial C_{jt}} dC_{it} dC_{jt} \right) \\
 &= \left(\frac{\partial E_{1t}}{\partial t} + \sum_{i=m,n} \left(r C_{it} \frac{\partial E_{1t}}{\partial C_{it}} + \sum_{j=m,n} \gamma_{ij} \sigma_{C_i} \sigma_{C_j} C_{it} C_{jt} \frac{\partial^2 E_{1t}}{\partial C_{it} \partial C_{jt}} \right) \right) dt + \sum_{i=m,n} \sigma_{C_i} C_{it} \frac{\partial E_{1t}}{\partial C_{it}} d\tilde{W}_{it} \\
 &= rE_{1t} dt + \sum_{i=m,n} \sigma_{C_i} C_{it} \Delta_{C_{it}} d\tilde{W}_{it}. \tag{21}
 \end{aligned}$$

The price of the CEO on puts E_{2t} will follow a similar process to the one described by equation (21) with C_i replaced by P_i .¹⁰

Then, by equation (4), $V_{Nt} = E_{1t} + E_{2t}$ and so we have

$$\begin{aligned}
 dV_{Nt} &= dE_{1t} + dE_{2t} \\
 &= r(E_{1t} + E_{2t}) dt + \sum_{i=m,n} \sigma_{C_i} C_{it} \Delta_{C_{it}} d\tilde{W}_{it} - \sum_{i=m,n} \sigma_{P_i} P_{it} \Delta_{P_{it}} d\tilde{W}_{it} \\
 &= rV_{Nt} dt + V_{Nt} \left(\left(\sigma_{C_m} \frac{C_{mt}}{V_{Nt}} \frac{\partial E_{1t}}{\partial C_{mt}} - \sigma_{P_m} \frac{P_{mt}}{V_{Nt}} \frac{\partial E_{2t}}{\partial P_{mt}} \right) d\tilde{W}_{mt} - \left(\sigma_{C_n} \frac{C_{nt}}{V_{Nt}} \frac{\partial E_{1t}}{\partial C_{nt}} - \sigma_{P_n} \frac{P_{nt}}{V_{Nt}} \frac{\partial E_{2t}}{\partial P_{nt}} \right) d\tilde{W}_{nt} \right) \\
 &= rV_{Nt} dt + V_{Nt} \left((\xi_{mt} - \eta_{mt}) d\tilde{W}_{mt} - (\xi_{nt} - \eta_{nt}) d\tilde{W}_{nt} \right).
 \end{aligned}$$

Let \tilde{W}_m and \tilde{W} be independent Wiener processes with $d\tilde{W}_{nt} = \gamma_{mn} d\tilde{W}_{mt} + \sqrt{1 - \gamma_{mn}^2} d\tilde{W}_t$ and γ_{mn} is the correlation between the basket options written on \mathbf{b}_m and \mathbf{b}_n . Define, $\tilde{\sigma}_t = (\xi_{mt} - \eta_{mt})$ and $\tilde{\sigma}_{nt} = (\xi_{nt} - \eta_{nt})$, with

$$\xi_{it} = \sigma_{C_i} \frac{C_{it}}{V_{Nt}} \frac{\partial E_{1t}}{\partial C_{it}} \quad \text{and} \quad \eta_{it} = \sigma_{P_i} \frac{P_{it}}{V_{Nt}} \frac{\partial E_{2t}}{\partial P_{it}}. \tag{22}$$

Then V_N can be described as

$$dV_{Nt} = rV_{Nt} dt + \tilde{\sigma}_t V_{Nt} d\tilde{W}_t, \tag{23}$$

where the basket option volatility, $\tilde{\sigma}_t$ is given by

$$\tilde{\sigma}_t = \sqrt{\tilde{\sigma}_{mt}^2 + \tilde{\sigma}_{nt}^2 - 2\gamma_{mn}\tilde{\sigma}_{mt}\tilde{\sigma}_{nt}}, \tag{24}$$

and the covariance between the sub-basket options written on baskets \mathbf{b}_m and \mathbf{b}_n is given by

$$\begin{aligned}
 \gamma_{mn} dt &= \mathcal{C}(\mathbf{b}_m, \mathbf{b}_n) \\
 &= \mathcal{C}(\mathbf{b}_{m1}, \mathbf{b}_{n1}) - \mathcal{C}(\mathbf{b}_{m1}, \mathbf{b}_{n2}) - \mathcal{C}(\mathbf{b}_{m2}, \mathbf{b}_{n1}) + \mathcal{C}(\mathbf{b}_{m2}, \mathbf{b}_{n2}), \tag{25}
 \end{aligned}$$

for $i = m, n$ and $j = 1, 2$. $\mathcal{C}(a, b)$ is the covariance between the Wiener processes driving the assets in baskets a and b , and \mathbf{b}_{ij} s are the sub-baskets of \mathbf{b}_i , that is, $\mathbf{b}_i = \{\mathbf{b}_{i1}, \mathbf{b}_{i2}\}$.

¹⁰When $\chi = 1$, the two CEOs are written on call and put sub-basket options. Their price processes will be similar to equation (21) but now each has a call and put option component. For instance,

$$dE_{1t} = rE_{1t} dt + \sigma_{C_m} C_{mt} \Delta_{C_{mt}} d\tilde{W}_{mt} + \sigma_{P_n} P_{nt} \Delta_{P_{nt}} d\tilde{W}_{nt}.$$

The weak form of the log-normality condition is most likely to hold when we choose the two CEOs to be deep in-the-money (ITM). That is, we shall choose the strikes of the sub-basket call and put options such that the call option on m assets and the put option on n assets are deep in the money. The reason for such a choice is that both $\frac{\partial E_{1t}}{\partial C_{mt}}$ and $\frac{\partial E_{2t}}{\partial P_{nt}}$ tend to one, while $\frac{\partial E_{1t}}{\partial C_{nt}}$ and $\frac{\partial E_{2t}}{\partial P_{mt}}$ tend to zero. Thus, by choosing the strikes of the sub-basket options so that the basket option price is obtained from deep ITM call and put options, we may approximate ξ_{it} and η_{it} as constants. Thus, $\xi_m \approx \sigma_{C_m}$, $\eta_n \approx \sigma_{P_n}$ and $\xi_n, \eta_m \approx 0$ and we may therefore approximate the basket option volatility to be constant; $\tilde{\sigma}_t = \tilde{\sigma}$. The basket option price will then follow an approximate log-normal process described by

$$dV_{Nt} = rV_{Nt}dt + \tilde{\sigma}V_{Nt}d\tilde{W}_t.$$

Equation (20), describing the two call sub-basket option prices, can be derived in a similar fashion, starting with their sub-basket option price processes. Ultimately, when the sub-basket size reduces to one, we will have a plain vanilla option whose processes will be described by equations (9) and (10).

Theorem 5. *Let E_{1t} and E_{2t} be the prices of the two CEOs above. Then the price of the basket option on \mathbf{B} at time t is given by the recursive formula:*

$$\begin{aligned} V_{Nt}(\Theta, \mathbf{S}_t, \mathbf{K}, T, \omega) &= \mathbb{E}_{\mathbb{Q}} \left\{ V_{NT} \mid \mathcal{F}_t \right\} \\ &= E_{1t}(\Theta, \mathbf{S}_t, \mathbf{K}, T, \omega) + E_{2t}(\Theta, \mathbf{S}_t, \mathbf{K}, T, \omega), \end{aligned} \quad (26)$$

where

$$\begin{aligned} E_{1t}(\Theta, \mathbf{S}_t, \mathbf{K}, T, \omega) &= \omega (V_{mt}(\Theta_m, \mathbf{S}_{mT}, \mathbf{K}_m, +1)\Phi(\omega d_{11}) - V_{nt}(\Theta_n, \mathbf{S}_{nT}, \mathbf{K}_n, -\chi)\Phi(\omega d_{12})) \\ &= \omega (V_{mt}^1\Phi(\omega d_{11}) - V_{nt}^1\Phi(\omega d_{12})), \text{ say} \\ E_{2t}(\Theta, \mathbf{S}_t, \mathbf{K}, T, \omega) &= \omega (V_{nt}(\Theta_n, \mathbf{S}_{nT}, \mathbf{K}_n, \chi)\Phi(\omega d_{21}) - V_{mt}(\Theta_m, \mathbf{S}_{mT}, \mathbf{K}_m, -1)\Phi(\omega d_{22})) \\ &= \omega (V_{nt}^2\Phi(\omega e_1) - V_{mt}^2\Phi(\omega e_2)), \text{ say} \end{aligned} \quad (27)$$

and

$$d_{i1} = \frac{\ln(V_{mt}^i/V_{nt}^i) + \frac{1}{2}\sigma_{Ei}^2(T-t)}{\sigma_{Ei}\sqrt{T-t}}; \quad d_{i2} = d_{i1} - \sigma_{Ei}\sqrt{T-t};$$

where σ_{E1} and σ_{E2} are the volatilities of the two CEO prices E_1 and E_2 respectively.

Proof. Recall that C_m, P_m and C_n, P_n are themselves prices of options on baskets of sizes m and n respectively. Therefore these prices can be computed by applying equation (26) recursively. Their volatilities $\sigma_{C_m}, \sigma_{C_n}, \sigma_{P_m}$ and σ_{P_n} will be given by equation (24). This procedure is followed until the size of a sub-basket reaches one.

When the size of the basket reduces to one, the basket option price is merely the price of a vanilla option under the chosen model. Then for $\mathbf{S}_t = (S_{it}), \mathbf{K}_1 = (K_i)$ and $\Theta_1 = \theta_i$, for some $1 \leq i \leq N$, the single-asset option price is given by

$$\begin{aligned} E_1(\Theta, \mathbf{S}_t, \mathbf{K}, T, \omega) &= \omega e^{-r(T-t)}\theta_i (F_{itT}\Phi(\omega d_1) - K_i\Phi(\omega d_2)), \\ E_2(\Theta, \mathbf{S}_t, \mathbf{K}, T, \omega) &= 0, \end{aligned} \quad (28)$$

where F_{itT} is the i^{th} asset futures price and

$$d_1 = \frac{\ln\left(\frac{F_{itT}}{K_i}\right) + \left(r + \frac{1}{2}\Sigma_i^2\right)(T-t)}{\Sigma_i\sqrt{T-t}}; \quad d_2 = d_1 - \Sigma_i\sqrt{T-t}$$

For instance, when $\mu_i = (r - q_i)$ in equation (19), $F_{itT} = S_{it}e^{(r-q_i)(T-t)}$ and $\Sigma_i = \sigma_i$. Or more generally, when $\mu_i = \kappa(\theta(t) - \ln S_{it})$:¹¹

$$F_{itT} = \exp\left(e^{-\kappa(T-t)} \ln S_{it} + \int_t^T e^{-\kappa(T-s)} \theta(s) ds + \frac{\sigma_i^2}{2\kappa} (1 - e^{-2\kappa(T-t)})\right)$$

$$\Sigma_i = \sigma_i \sqrt{\frac{1 - e^{-2\kappa(T-t)}}{2\kappa}}$$

□

One of the main advantages of our approximation is that we can derive analytic formulae for the multi-asset option Greeks which, unlike most other approximations, capture the effects that individual asset's volatilities and correlations have on the hedge ratios.

Below we present the deltas, gammas and vegas of a basket option; the corresponding formulae for a rainbow option (or a best-of or worst-of option) would then follow from it's basket-option representation, which we have shown how to derive, using the principles of Section 2. Differentiating the basket option price given in theorem 5, using the chain rule, yields the following:

Proposition 6. *The basket option deltas, gammas and vegas of our basket option price f are given by:*

$$\begin{aligned} \Delta_{S_i}^f &= \Delta_{C_j}^f \Delta_{S_i}^{C_j} + \Delta_{P_j}^f \Delta_{S_i}^{P_j} \\ \Gamma_{S_i}^f &= \Gamma_{C_j}^f \left(\Delta_{S_i}^{C_j}\right)^2 + \Gamma_{S_i}^{C_j} \Delta_{C_j}^f + \Gamma_{P_j}^f \left(\Delta_{S_i}^{P_j}\right)^2 + \Gamma_{S_i}^{P_j} \Delta_{P_j}^f \\ \mathcal{V}_{\sigma_i}^f &= \mathcal{V}_{\sigma_{E_1}}^f \frac{\partial \sigma_{E_1}}{\partial \sigma_i} + \mathcal{V}_{\sigma_{E_2}}^f \frac{\partial \sigma_{E_2}}{\partial \sigma_i} + \mathcal{V}_{\sigma_i}^{C_j} \Delta_{C_j}^f + \mathcal{V}_{\sigma_i}^{P_j} \Delta_{P_j}^f \end{aligned} \quad (29)$$

where j is equal to m when $1 \leq i \leq m$ and equal to n when $m+1 \leq i \leq N$; Δ_x^z , Γ_x^z and \mathcal{V}_x^z denote the delta, gamma and vega of z with respect to x respectively.

5. EMPIRICAL RESULTS

This section describes how to implement our approximation to price a basket option. For illustration we assume the option is written on four assets, and has payoff $[S_1 - S_2 - S_3 + S_4]^+$. The asset prices follow correlated GBM processes with constant volatilities and correlations. The interest rate was 4% and dividend yields were assumed to be zero.

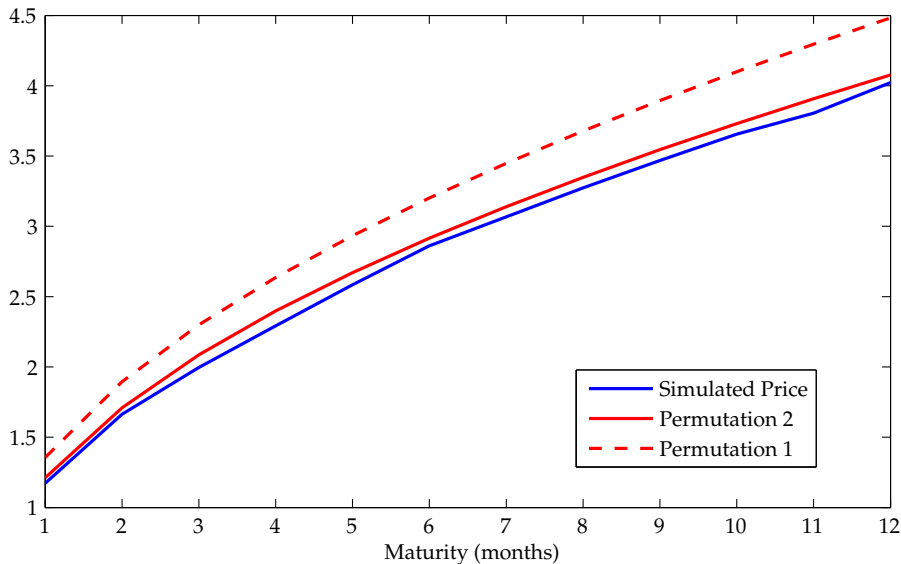
We begin with an experiment which takes the model parameters as given, and compares the prices obtained using our approximation with the prices that are obtained using simulation of the multivariate log-normal asset price processes. The initial underlying asset prices, basket weights,

¹¹see Pilipovic [2007]

volatilitites and correlations were chosen to be:

$$\mathbf{S} = \begin{pmatrix} 100 \\ 90 \\ 85 \\ 75 \end{pmatrix}, \quad \mathbf{\Theta} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} 0.10 \\ 0.15 \\ 0.18 \\ 0.20 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 0.8 & 0.6 & 0.2 \\ 0.8 & 1 & 0.55 & 0.65 \\ 0.6 & 0.55 & 1 & 0.57 \\ 0.2 & 0.65 & 0.57 & 1 \end{pmatrix}. \quad (30)$$

FIGURE 3: Comparison of our model prices with simulated prices across various basket option maturities



Following the payoff decomposition discussed in section 2, we can decompose the payoff into a CEO written on two simple exchange options, each on a pair of assets. By choosing the size of the sub-baskets to be equal to 2 (i.e. setting $k = 2$), this can be done in four different ways.¹² That is, the two pairs of assets can be permuted in four different ways. However, due to symmetry, a call option to exchange asset 1 for 2 is equivalent to a put option to exchange asset 2 for 1. Hence the permutations (1, 2, 3, 4) and (1, 3, 2, 4) are equivalent to (4, 3, 2, 1) and (4, 2, 3, 1) respectively and yield the same prices. We therefore evaluate the performance of our approximation for the former two permutations only, and our experiment will compare our results with the simulated prices.¹³ We call these permutations p_1 and p_2 .

Figure 3 shows the simulated basket option price and model price under permutations p_1 and p_2 , with the same set of model parameters, but for different maturities, from 1 to 12 months. The model price under p_2 seems to match the simulated prices closely for all maturities, but this is not the case when we price the CEO using p_1 . The price difference between the two permutations mainly arises due to the difference in the way the individual volatilities and correlation

¹²While there are $4!$ permutations possible, most of them lead to redundant representations while some others, for instance, (1, 4, 2, 3), lead to a CEO on two 2-asset options written on the sum of asset prices. This is equivalent to an exchange option with negative strike.

¹³We also tried to compute the price of the basket option using the Black-Scholes model. Here we expressed the basket option as a simple exchange option to exchange a 2-asset basket on assets 1 and 4 for another on assets 2 and 3. By assuming that the 2-asset basket prices followed GBM processes, price was found using Margrabe [1978]. However we do not use these prices because they were highly inaccurate, and ranged between 0.5 and 1.5 across all maturities.

affect the basket option price. In permutation p_1 , the correlations ρ_{12} and ρ_{34} affect basket option price through the exchange option prices and volatilities, while in p_2 they affect the price only through the correlation γ between the sub-baskets. A similar argument applies to ρ_{13} and ρ_{24} in permutations p_2 and p_1 . The approximation error creeps in while approximating the basket option volatility, as in equation (24), whose sensitivity to different correlations depends on the chosen permutation.

The results in figure 3 are for a particular choice of volatilities and correlations for the underlying assets. In order to check the performance of our approximation with other correlation values, we calculated our model prices, and the simulated prices, for a random sample of uniformly distributed correlation matrices. The maturity of the basket option was 6 months. We calculated the range of our model prices, and the range of simulated prices, as the correlation matrices were changed randomly. The minimum and maximum simulated prices were 2.12 and 10.61 respectively, while the minimum model prices under permutations p_1 and p_2 were 0.2 and 1.13, and the maximum model prices were 10.48 and 10.83 respectively. Hence, the simulated prices form a subset of the possible model prices under permutation p_2 .

We now move to an illustration of how to price a basket option in practice. Instead, we would want to calibrate the model parameters, including the volatilities and correlations to match the benchmark price. For an N -asset basket option, there are N asset price volatility and $N(N - 1)/2$ correlation parameters and there are infinitely many possible combinations of values for which the model price of the basket option would be equal to its market price, if indeed a sufficiently liquid market price exists. But not all values will yield implied volatility and implied correlation skew consistent prices and hedge ratios.

Our approximation provides a natural convention for choosing the underlying asset volatilities; they should be set equal to the implied volatilities of the vanilla options on individual assets with the strikes that appear in the terminal step of the recursive procedure. For instance, in the present example, for the exchange option on assets 1 and 2, σ_1 will be equal to the implied volatility of an option on S_1 with strike S_2 , and σ_2 will be equal to the implied volatility of an option on S_2 with strike S_1 . The volatilities σ_3 and σ_4 are chosen in a similar manner. This yields prices and hedge ratios that are volatility skew consistent.

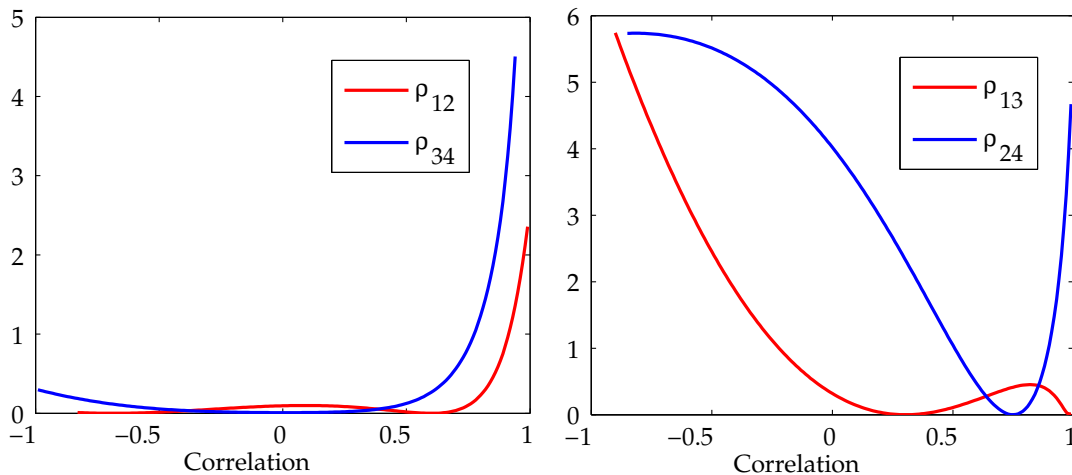
Secondly, certain asset correlations will be set equal to the implied correlations that are backed-out from the market prices of any liquid spread options, or 2-asset basket options, using the volatilities that are calibrated as above. Then we calibrate the remaining correlations so that the basket option model prices under different permutations match each other. This way we eliminate any bias from using a certain permutation.

As in the specific example discussed above, the way that different correlations affect the basket option price depends on the chosen permutation. For a given permutation, correlations between assets that appear in different legs of the tree (figure 1) affect the basket option price only through the correlation γ between the sub-basket options. However, if the assets belong to the same leg of the tree, then the correlations between these assets affect the basket option price through the sub-basket prices and the sub-basket volatilities. For instance, recall that in permutation p_1 the correlations ρ_{12} and ρ_{34} affect the basket option price through the two exchange option prices, whereas in permutation p_2 they affect the basket option price through the CEO price. Therefore, depending on the chosen permutation, these correlation values affect the price in a different man-

ner. A similar argument applies to correlations ρ_{13} and ρ_{24} .

To illustrate this, in figure 4 we plot the squared difference between the prices from permutations p_1 and p_2 against these correlations. We compute the price differences for a correlation, ρ_{ij} , by using the value for all the other parameters given by (30) and we only vary the value of that particular correlation, shown on the horizontal axis. However, these correlation values were constrained, in order to keep the sub-basket correlation, given by equation (25), within -1 and 1 . Figure 4 shows that the prices from permutations p_1 and p_2 can indeed be matched by calibrating the correlations. For instance, the two basket option prices will be equal when ρ_{12} is 0.61, $\rho_{34} = 0.01$, $\rho_{13} = 0.3$ and $\rho_{24} = 0.73$.¹⁴

FIGURE 4: Squared difference between our model prices from permutations (1, 2, 3, 4) and (1, 3, 2, 4)



Thirdly, irrespective of the permutation, some correlations only affect the sub-basket option correlation γ and not the sub-basket option prices or volatilities. Therefore, these would not have been calibrated in the previous step, but they can be calibrated now by matching the model price to the benchmark price of the basket option. In the basket option considered above, for example, the correlations ρ_{23} and ρ_{14} , unlike the rest, affect the basket option price only through the sub-basket correlation γ . While the other correlations were tuned to match the prices obtained under the two possible permutations p_1 and p_2 , ρ_{23} and ρ_{14} can be used to calibrate the model price to the market price of the basket option. For a given value of ρ_{12} , ρ_{34} , ρ_{13} and ρ_{24} , we only need to choose ρ_{23} and ρ_{14} such that the model price is equal to the market price. Although this procedure does not always identify a unique values for each correlation, it identifies the vanilla options and sub-basket options that replicate the basket option.

Figure 5 plots the behaviour of the basket option price with respect to these correlations, when $\rho_{12} = 0.6$ and when 0.9. For example, if the benchmark price for the basket option were 6, as indicated by the dotted horizontal line in the figure, then we would calibrate the following values

¹⁴The convexity of the ρ_{34} curve in the region of 0 is very low, but there is actually only one value for which the two prices are equal.

FIGURE 5: Basket option price variation with respect to correlation between assets 1 and 4, and assets 2 and 3

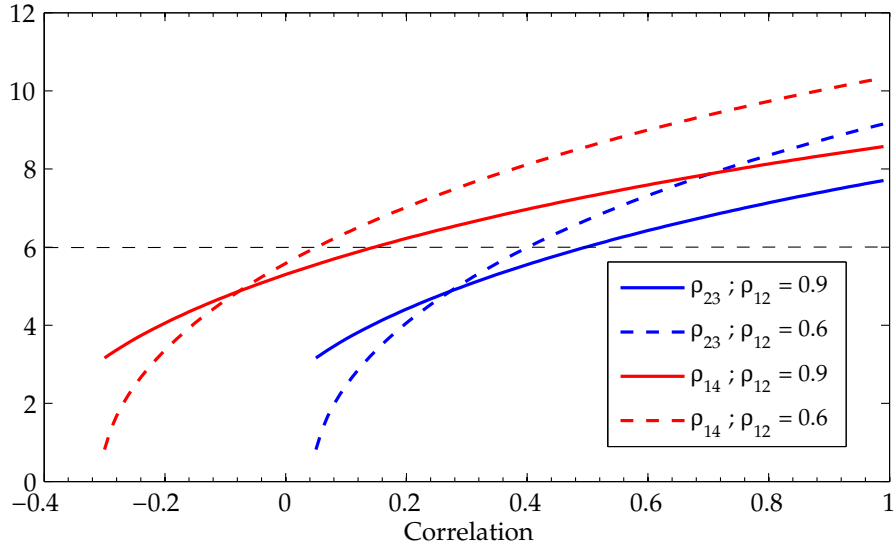
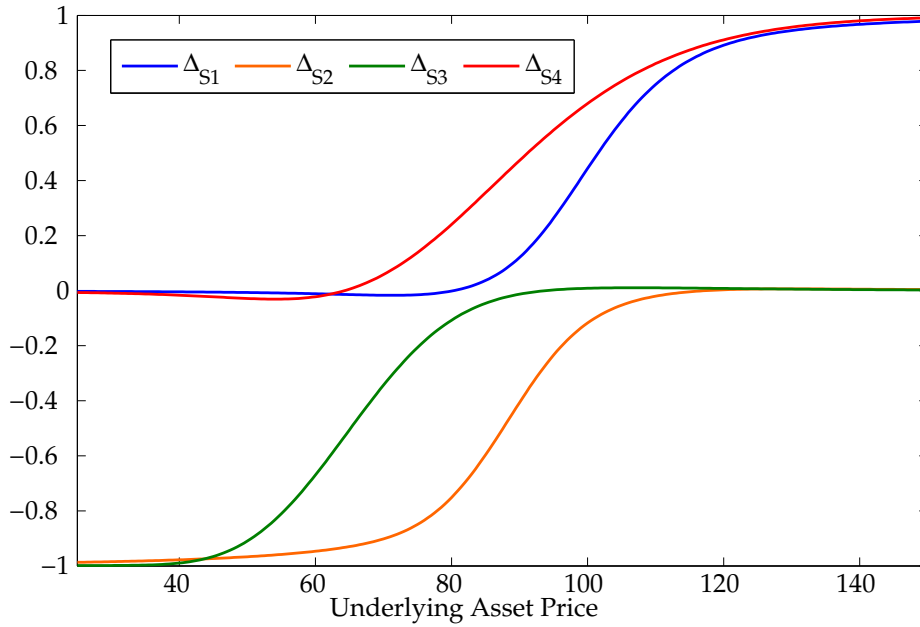


FIGURE 6: Deltas with respect to all 4 underlying asset prices



of ρ_{23} and ρ_{14} . For $\rho_{12} = 0.6$, ρ_{23} and ρ_{14} will be equal to 0.4 and 0.04 respectively; and for $\rho_{12} = 0.9$, ρ_{23} and ρ_{14} will be equal to 0.5 and 0.14 respectively.

Finally figure 6 plots the deltas of the basket option with respect to the four underlying assets. These were computed using equation (29). Due to the positive weights of assets 1 and 4, the deltas with respect to those asset prices resemble the delta of a vanilla call option, whereas the deltas with respect to prices of assets 2 and 3 resemble the delta of a vanilla put option, due to their negative weights. For the parameter values given in (30), the basket option is at the money with respect to every underlying asset. That is, the price of every asset is equal to the weighted sum of prices of the other 3 assets. However, due to differences in their volatilities and correlations, the respective deltas at the given price values are different from each other. For instance, at $S_1 = 100$, $\Delta_1 \approx 0.5$ while at $S_2 = 90$, $\Delta_2 \approx -0.4$. This property is not captured by any other existing approaches to analytic approximations for multi-asset options, because they ignore the effects of asset price volatilities and correlations on the basket option deltas.

6. CONCLUSION

This paper develops a recursive framework for pricing and hedging multi-asset options, such as basket and rainbow options, with a linear payoff structure. Most of the existing approaches to pricing basket options are based on approximating the distribution of the basket price, or they are limited to pricing average price basket options, or they apply only to options on a small number of assets. We derive an approximate pricing formula for a general, N -asset basket option, by expressing the basket option price as a sum of prices of compound exchange options on sub-basket options. For an N -asset basket option, our approach involves computing the prices of $2(N - 1)$ compound exchange options and N vanilla option prices. The approximation error can be minimised by a judicious choice of the strikes of these N vanilla options on the individual assets. We discuss the extension of our approximation to other linear multi-asset options, such as rainbows, best-of and worst-of options, where we express their price in terms of basket option and simple exchange option prices. Also, when the basket contains no more than three assets, our approach yields an almost exact price.

This recursive approach has several advantages over those already developed in the literature. Firstly, the underlying asset prices may follow heterogeneous GBM processes. For instance, some asset prices could follow mean-reverting processes whilst others follow standard GBM processes. Secondly, our framework provides a convention for selecting the implied volatilities of vanilla options on the individual underlying assets that are used to price the basket option. This yields volatility skew consistent prices. Moreover, our prices may also be consistent with implied correlations from any two-asset options used in the calibration set, although there is no convention for setting these, as we have for the implied volatilities. Thirdly, we can derive analytic approximations for multi-asset option Greeks, and unlike other approaches, these Greeks will be influenced by the individual asset price volatilities and correlations. Hence hedge ratios are consistent with the individual asset implied volatility and implied correlation skews. Finally, we have demonstrated how to calibrate the model parameters using a 4-asset rainbow option as an example.

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