# Analytic Approximations for Spread Options

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ABSTRACT

This paper expresses the price of a spread option as the sum of the prices of two compound options. One compound option is to exchange vanilla call options on the two underlying assets and the other is to exchange the corresponding put options. This way we derive a new analytic approximation for the price of a European spread option, and a corresponding approximation for each of its price, volatilty and correlation hedge ratios. Our approach has many advantages over existing analytic approximations, which have limited validity and an indeterminacy that renders them of little practical use. The compound exchange option approximation for European spread options is then extended to American spread options on assets that pay dividends or incur carry costs. Simulations quantify the accuracy of our approach; we also present an empirical application, to the American crack spread options that are traded on NYMEX. For illustration, we compare our results with those obtained using the approximation attributed to Kirk [1996], which is commonly used by traders.

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# 1. Introduction

A spread option is an option whose pay-off depends on the price spread between two correlated underlying assets. If the asset prices are $S_1$ and $S_2$ the payoff to a spread option of strike $K$ is $[\omega(S_1 - S_2 - K)]^+$ where $\omega = 1$ for a call and $\omega = -1$ for a put. Early work on spread option pricing by Ravindran [1993] and Shimko [1994] assumed each forward price process is a geometric Brownian motion with constant volatility and that these processes have a constant non-zero correlation: we label this the '2GBM' framework for short.

The 2GBM framework is tractable but it captures neither the implied volatility smiles that are derived from market prices of the vanilla options on $S_1$ and $S_2$, nor the implied correlation smile that is evident from market prices of the spread options on $S_1 - S_2$. In fact correlation 'frowns' rather than 'smiles' are a prominent feature in spread option markets. This is because the pay-off to a spread option decreases with correlation. Since traders expectations are usually of leptokurtic rather than normal returns, market prices of out-of-the-money call and put spread options are usually higher than the standard 2GBM model prices, which are based on a constant correlation. Hence, the implied correlations that are backed out from the 2GBM model usually have the appearance of a 'frown'.

Numerical approaches to pricing and hedging spread options that are both realistic and tractable include Carr and Madan [1999] and Dempster and Hong [2000] who advocate models that capture volatility skews on the two assets by introducing stochastic volatility to the price processes. And the addition of price jumps can explain the implied correlation frown, as in the spark spread option pricing model of Carmona and Durrleman [2003a]. However pricing and hedging in this framework necessitates computationally intensive numerical resolution methods such as the fast Fourier transform (see Hurd and Zhou [2009]). Other models provide only upper and lower bounds for spread option prices, as in Durrleman [2001] and Carmona and Durrleman [2005], who determine a price range that can be very narrow for certain parameter values. For a detailed survey of these models and a comparison of their performances, the reader is referred to the excellent survey by Carmona and Durrleman [2003b].

Spread option traders often prefer to use analytic approximations, rather than numerical techniques, for their computational ease and the availability of closed form formulae for hedge ratios. By reducing the dimension of the price uncertainty from two to one, the 2GBM assumption allows several quite simple analytic approximations for the spread option price to be derived (see Eydeland and Wolyniec [2003]). The most well-known of these is the approximation stated in Kirk [1996], the exact origin of which is unknown; it is commonly referred to by traders as Kirk's approximation. Another approximation, due to Deng et al. [2008], is derived by expressing a spread option price as a sum of one dimensional integrals, and Deng et al. [2007] extend this approximation, and Kirk's approximation, to price and hedge multi-asset spread options.

All these approximations are based on the 2GBM assumption, where the underlying prices are assumed to have a bivariate lognormal distribution, which is quite unrealistic for most types of financial assets. However, they may be extended to approximations for spread options under more general assumptions for the joint distribution of the underlying prices. For instance, Alexander and Scourse [2004] assume the underlying prices have a bivariate lognormal mixture distribution, and hence express spread option prices as a weighted sum of four different 2GBM spread option prices, each of which may be obtained using an analytic approximation. The prices so derived display volatility smiles in the marginal distributions, and a correlation frown in the joint distribution.

There is an indeterminacy problem with the 2GBM analytic approximations mentioned above. The problem arises because all these approximations assume we know the implied volatilities of the corresponding single asset options. Since the 2GBM assumption is unrealistic, the individual asset implied volatilities are usually not constant with respect to the single-asset option strike - i.e. there is, typically, an implied volatility skew for each asset. So the strike at which we measure the implied volatility matters. However, this is not determined in the approximation. Thus we have no alternative but to apply some ad hoc rule, which we call the *strike convention*. But the implied correlation is sensitive to the strike convention; that is, different rules for determining the single-asset option implied volatilties give rise to quite different structures for the correlation frown. Indeed, many strike conventions infeasible values for implied correlations when calibrating to market prices.

There are two sources of this problem. Firstly, some approximations (including Kirk's approximation) are only valid for spread options of certain strikes. Secondly, and this is due to the indeterminacy described above, correlation risk is not properly quantified in these approximations. In fact, the sensitivity of the spread option price to correlation is constrained to be directly proprtional to the option vega. Indeed, a problem that is common to all approximations that require an ad hoc choice of strike convention, is that the hedge ratios derived from such approximations may be inconsistent with the vanilla option Greeks, and as such, the errors from delta-gamma-vega hedging could be inappropriately attributed to correlation risk.

In this paper we derive a new analytic approximation for spread option prices and hedge ratios, based on the 2GBM assumption. We express the spread option price as the sum of the prices of two compound exchange options. One compound option is to exchange two vanilla call options, one on each of the two underlying assets, and the other compound option is to exchange the corresponding put options. In this compound exchange option (CEO) approximation, the strikes at which to measure the single-asset implied volatilities are endogenous to the model. Thus the CEO approximation is free of any strike convention, and yields a unique implied correlation for each spread option strike, even when there are implied volatility skews on the individual assets. We shall demonstrate, using simulations of spread otption prices, and using real market spread option prices, that the CEO approximation provides a very much closer fit to the spread option price than does Kirk's approximation. Moreover, the CEO hedge ratios are consistent with those for the single asset options. Furthermore, correlation risk is not simply assumed to be proportional to volatility risk, as it is in other analytic approximations. In fact, correlation sensitivities are quantified independently of the single asset option vegas. We also derive a new, general formula for the early exercise premium of an American spread option on spot underlyings. This is because the majority of traded spread options are American-style options on assets that pay dividends or have carry costs.

The outline is as follows: Section 2 provides a critical review of the existing analytic approximations for spread options, exploring in greater depth the claims made above. Section 3 sets out the compound exchange option representation, and derives a new analytic approximation to the price and hedge ratios of European spread options. In Section 4 we derive the early exercise premium for an American spread option on assets that pay dividends or incur carry costs. Section 5 reports the results of two empirical studies: it begins with a simulation exercise that demonstrates the flexibility and accuracy of the CEO approximation, and explores the practical difficulties that arise on attempting to implement the approximation given by Kirk [1996]. Then we calibrate the CEO approximation to market data for American crack spread options, comparing the fit and the spread option hedge ratios with those obtained using Kirk's approximation. Section 6 concludes.

## 2. BACKGROUND

Let $\left(\Theta, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}\right)$ be a filtered probability space, where $\Theta$ is the set of all possible events $\theta$ such that $S_{1t}, S_{2t} \in (0, \infty)$, $(\mathcal{F}_t)_{t\geq 0}$ is the filtration produced by the sigma algebra of the price pair $(S_{1t}, S_{2t})_{t\geq 0}$ and $\mathbb{Q}$ is a bivariate risk neutral probability measure. Assume that the risk-neutral price dynamics are governed by two correlated geometric Brownian motions with constant volatilities, so the dynamics of the two underlying asset prices are given by:

$$dS_{it} = (r - q_i)S_{it}dt + \sigma_i S_{it} dW_{it}, \qquad i = 1,\ 2 \tag{1}$$

where $W_{1t}$ and $W_{2t}$ are Wiener processes under risk neutral measure $\mathbb{Q}$, $r$ is the (assumed constant) risk-free interest rate and $q_1$ and $q_2$ are the (assumed constant) dividend yields of the two assets. The volatilities $\sigma_1$ and $\sigma_2$ are also assumed to be constant as is the covariance:

$$\langle dW_{1t}, dW_{2t}\rangle = \rho dt.$$

When the strike of a spread option is zero the option is called an exchange option, since the buyer has the option to exchange one underlying asset for the other. The fact that the strike is zero allows one to reduce the pricing problem to a single dimension, using one of the assets as numeraire. If $S_{1t}$ and $S_{2t}$ are the spot prices of two assets at time $t$ then the payoff to an exchange option at the expiry date $T$ is given by $[S_{1T} - S_{2T}]^+$. But this is equivalent to an ordinary call option on $x_t = S_{1t}/S_{2t}$ with unit strike. Hence, using risk-neutral valuation, the price of an exchange option is given by

$$P_t = \mathbb{E}_{\mathbb{Q}}\left\{e^{-r(T-t)}[S_{1T} - S_{2T}]^+\right\} = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left\{S_{2T}\ [x_T - 1]^+\right\}.$$

Margrabe [1978] shows that under these assumptions the price $P_t$ of an exchange option is given by

$$P_t = S_{1t}e^{-q_1(T-t)}\Phi(d_1) - S_{2t}e^{-q_2(T-t)}\Phi(d_2) \tag{2}$$

where $\Phi$ denotes the standard normal distribution function and

$$d_1 = \frac{ln\left(\frac{S_{1,t}}{S_{2,t}}\right) + \left(q_2 - q_1 + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}; \quad d_2 = d_1 - \sigma\sqrt{T-t};$$
$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

A well-known approximation for pricing European spread options on futures or forwards, which is valid for small, non-zero strikes, appears to have been stated first by Kirk [1996]. When $K \ll S_{2t}$ the displaced diffusion process $S_{2t} + K$ can be assumed to be approximately log-normal. Then, the ratio between $S_{1t}$ and $(S_{2t} + Ke^{-r(T-t)})$ is also approximately log-normal and can be expressed as a geometric Brownian motion process. Rewrite the pay-off to the European spread option as:

$$[\omega(S_{1T} - S_{2T} - K)]^+ = (K + S_{2T})[\omega(Z_T - 1)]^+$$

where $\omega = 1$ for a call and $\omega = -1$ for a put, $Z_t = \frac{S_{1t}}{Y_t}$ and $Y_t = S_{2t} + Ke^{-r(T-t)}$. Let $W$ be a Brownian motion under a new probability measure $\mathbb{P}$ whose Radon-Nikodym derivative with respect to $\mathbb{Q}$ is given by:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left(-\frac{1}{2}\bar{\sigma}_2^2 T + \bar{\sigma}_2 W_{2t}\right).$$

Then $Z$ follows a process described by

$$\frac{dZ_t}{Z_t} = (r - \tilde{r} - (q_1 - \tilde{q}_2))\, dt + \sigma_t dW_t$$

with

$$\sigma = \sqrt{\sigma_1^2 + \tilde{\sigma}_2^2 - 2\rho\sigma_1\tilde{\sigma}_2},$$

where $\tilde{\sigma}_2 = \sigma_2 \frac{S_{2t}}{Y_t}$, $\tilde{r} = r\frac{S_{2t}}{Y_t}$, $\tilde{q}_2 = q_2\frac{S_{2t}}{Y_t}$. Therefore, the price $f_t$ at time $t$ for a spread option on $S_1$ and $S_2$ with strike $K$, maturity $T$ and payoff $[\omega(S_1 - S_2 - K)]^+$ is given by:

$$\begin{aligned} f_t &= \mathbb{E}_{\mathbb{Q}}\left\{Y_t e^{-r(T-t)}\left[\omega(Z_T - 1)\right]^+\right\} \\ &= \omega\left[S_{1,t}e^{-q_1(T-t)}\Phi\left(\omega d_{1Z}\right) - \left(Ke^{-r(T-t)} + S_{2t}\right)e^{-(r-(\tilde{r}-\tilde{q}_2))(T-t)}\Phi\left(\omega d_{2Z}\right)\right], \end{aligned} \quad (3)$$

where

$$\begin{aligned} d_{1Z} &= \frac{ln\left(Z_t\right) + \left(r - q_1 - (\tilde{r} - \tilde{q}_2) + \frac{1}{2}\sigma_t^2\right)(T-t)}{\sigma_t\sqrt{T-t}}; \\ d_{2Z} &= d_{1Z} - \sigma_t\sqrt{T-t}. \end{aligned}$$

Under Kirk's approximation, the spread option's deltas and gammas are given by

$$\begin{aligned} \Delta^f_{S_1} &= \omega e^{-q_1(T-t)}\Phi(\omega d_{1Z}), \\ \Delta^f_{S_2} &= -\omega e^{-q_1(T-t)}\Phi(\omega d_{2Z}), \\ \Gamma^f_{S_1S_1} &= e^{-q_1(T-t)}\frac{\phi(d_{2Z})}{S_{1t}\sigma_t\sqrt{T-t}}, \\ \Gamma^f_{S_2S_2} &= e^{-(r-\tilde{r}+\tilde{q}_2)(T-t)}\frac{\phi(d_{2Z})}{(Ke^{-r(T-t)} + S_{2t})\sigma_t\sqrt{T-t}}. \end{aligned} \quad (4)$$

The cross gamma, i.e., the second order derivative of price with respect to both the underlying assets is given by

$$\Gamma^f_{S_1S_2} = -e^{-q_1(T-t)}\frac{\phi(d_{1Z})}{(Ke^{-r(T-t)} + S_{2t})\sigma_t\sqrt{T-t}} = -e^{-(r-\tilde{r}+\tilde{q}_2)(T-t)}\frac{\phi(d_{2Z})}{S_{1t}\sigma_t\sqrt{T-t}}, \quad (5)$$

where $\Delta^z_x$ denotes the delta of $y$ with respect to $x$ and $\Gamma^z_{xy}$ denotes the gamma of $z$ with respect to $x$ and $y$. The Kirk-approximation vegas are similar to Black-Scholes vegas and are easy to derive using chain rule.

Under the 2GBM assumption, other price approximations exist that also reduce the dimension of the uncertainty from two to one.[1] For instance let $S_t = S_{1t}e^{-q_1(T-t)} - S_{2t}e^{-q_2(T-t)}$ and choose an

[1] See Eydeland and Wolyniec [2003]. The approximation derived in Deng et al. [2007, 2008] is not based on dimension reduction. Nevertheless, it has the problem that the prices still depend on a subjective choice for the strikes of the single asset implied volatilities.

arbitrary $M >> max\{S_t, \sigma_t\}$. Then another analytic spread option price, this time based on the approximation that $M + S_t$ has a lognormal distribution, is:

$$f_t = (M + S_t)\Phi(d_{1M}) - (M + K)e^{-r(T-t)}\Phi(d_{2M})$$

where

$$\begin{aligned} d_{1M} &= \frac{ln\left(\frac{M+S_t}{M+K}\right) + \left(r - q + \frac{1}{2}\sigma_t^2\right)(T-t)}{\sigma_t\sqrt{T-t}}; \\ d_{2M} &= d_{1M} - \sigma_t\sqrt{T-t}; \\ \sigma_t &= \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}/(M + S_t). \end{aligned}$$

We remark that, in most of the analytic approximations which are derived by reducing the price dimension from two to one, the spread option volatility will take the form:

$$\sigma = \sqrt{\omega_1\sigma_1^2 + \omega_2\sigma_2^2 - 2\omega_3\rho\sigma_1\sigma_2}\,, \tag{6}$$

where the terms on the right hand side are assumed to be constant.

To avoid arbitrage, a spread option must be priced consistently with the prices of vanilla options on $S_1$ and $S_2$. This implies setting $\sigma_i$ in (6) equal to the implied volatility of $S_i$, for $i = 1, 2$. Then the implied correlation is calibrated by equating the model and market prices of the spread option. But although the 2GBM model assumes constant volatility, the market implied volatilities are not constant with respect to strike. So the strikes $K_1$ and $K_2$ at which the implied volatilities $\sigma_1$ and $\sigma_2$ are calculated can have a significant influence on the results. Each of the spread option price approximations reviewed above requires the single asset implied volatilities to be determined by some convention for choosing $(K_1, K_2)$ such that $K_1 - K_2 = K$. There are infinitely many possible choices for $K_1$ and $K_2$ and, likewise, infinitely many combinations of market implied $\sigma_1$, $\sigma_2$ and $\rho$ that yield the same $\sigma$ in equation (3). Hence the implied volatility and implied correlation parameters are ill-defined. Moreover, the sensitivity of volatility to correlation is constant, and this implies that the price sensitivity to correlation is directly proportional to the option vega (i.e. the partial derivative of the price w.r.t. $\sigma$).

Spread options may be delta-gamma hedged by taking positions in the underlying assets and options on these. But hedging volatility and correlation may be much more difficult. Vega hedging is complicated by the fact that the hedge ratios depend on the strike convention. For a given strike $K$, there could be very many pairs $(K_1, K_2)$ with $K_1 - K_2 = K$ which provide an accurate price. However, for each such pair $(K_1, K_2)$ the spread option Greeks will be different, and if the strikes are chosen without regard for vega risk, the hedging errors accruing from incorrect vega hedging, along with every other unhedged risk, will be collectively attributed to correlation risk.

For this reason we should impose a further condition in the strike convention, i.e. that the spread option Greeks are consistent with the vanilla option Greeks. If they are not, there could be substantial hedging error from gamma and/or vega hedging the spread option with vanilla options. We define a *compatible* strike pair $(K_1, K_2)$ with $K_1 - K_2 = K$, to be such that the price and the hedge ratios of the spread option with strike $K$ are consistent with the prices and the hedge ratios of the two vanilla options at strikes $K_1$ and $K_2$. A compatible pair $(K_1, K_2)$ can be found by equating four ratios: two of the spread option deltas relative to the single asset option deltas, and two of the spread option vegas relative to the single asset option vegas. There is no condition for gamma because it is proportional to vega in the GBM framework.

Now, conditional on the two prices for the underlying assets at expiry, we can choose a unique pair of vanilla options on the respective assets that replicates the spread option's payoff.[2] This unique pair of single asset options best reflects the market expectations of the prices of their respective underlying assets at the option's expiry. Since the price of a spread option is a linear combination of the conditional expectation of the underlying asset prices, these volatilities are expected to give the most accurate spread option prices, assuming the market expectations are correct.

This hedging argument suggests that the strike convention should be selecting a compatible strike pair; this is one of the strike conventions that we have followed in this paper. In addition, to explore whether more realistic implied correlations are obtained using non-compatible strike pairs, we have employed several other strike conventions: using the single asset's at-the-money (ATM) forward volatility to calibrate spread options of all strikes; several conventions for which each $K_i$ is a linear function of $K$ and $S_i$, for $i = 1, 2$; and calibrating $K_1$ as a model parameter, then setting $K_2 = K_1 - K$. However, as we shall see in Section 5, in no case did we obtain reasonable results for spread options of all strikes, when applied to either simulated or market data.

In the next section we present a new analytic approximation where a compatible pair of single asset option strikes is endogenous. It is determined by calibrating the model to the vanilla option implied volatility skews and to the implied correlation frown of the spread options of different strikes.

# 3. Compound Exchange Option Approach

In this section we express the price of a spread option as a sum of prices of two compound exchange options, one on vanilla call options and the other on vanilla put options. The spread option pricing problem thus reduces to finding the right call option pair (and the right put option pair) and then calibrating the implied correlation between the two vanilla options. By establishing a conditional relationship between the strikes of vanilla options and the implied correlation, the spread option pricing problem reduces to a one dimensional problem. We also derive a correlation sensitivity for the spread option price that is independent of the volatility hedge ratios.

**Theorem 1.** *The risk neutral price of a European spread option may be expressed as the sum of risk neutral prices of two compounded exchange options. That is,*

$$f_t = e^{-r(T-t)} \left( \mathbb{E}_{\mathbb{Q}} \left\{ \left[ \omega \left[ U_{1T} - U_{2T} \right] \right]^+ \middle| \mathcal{F}_t \right\} + \mathbb{E}_{\mathbb{Q}} \left\{ \left[ \omega \left[ V_{2T} - V_{1T} \right] \right]^+ \middle| \mathcal{F}_t \right\} \right) \quad (7)$$

*where $U_{1T}$, $V_{1T}$ are pay-offs to European call and put options on asset 1 and $U_{2T}$, $V_{2T}$ are pay-offs to European call and put options on asset 2, respectively.*

[2] To see why, consider a call spread option with zero strike, for example. In the above construction, when $S_{1,T} \geq S_{2,T}$ the pay off will be equal to $K_1 - K_2$.

*Proof.* Let $K_1$ and $K_2$ be positive real numbers such that $K_1 - K_2 = K$ and

$$\begin{aligned} \mathcal{L} &= \{\theta \in \Theta : \omega\ (S_{1T} - S_{2T} - K) \geq 0\} \\ \mathcal{A} &= \{\theta \in \Theta : S_{1T} - K_1 \geq 0\} \\ \mathcal{B} &= \{\theta \in \Theta : S_{2T} - K_2 \geq 0\} \end{aligned}$$

Since a European option price at time $t$ depends only on the terminal price densities, we have

$$\begin{aligned} f_t &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\{\omega \mathbf{1}_{\mathcal{L}}\,[S_{1T} - S_{2T} - K]\} \\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\Big\{\omega \mathbf{1}_{\mathcal{L}}\Big(\,\mathbf{1}_{\mathcal{A}}[S_{1T} - K_1] - \mathbf{1}_{\mathcal{B}}[S_{2T} - K_2] \\ &\qquad + (1 - \mathbf{1}_{\mathcal{A}})[S_{1T} - K_1] - (1 - \mathbf{1}_{\mathcal{A}})[S_{2T} - K_2]\Big)\Big\} \\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\Big\{\omega\Big(\mathbf{1}_{\mathcal{L}\cap\mathcal{A}}[S_{1T} - K_1] - \mathbf{1}_{\mathcal{L}\cap\mathcal{B}}[S_{2T} - K_2] \\ &\qquad + \mathbf{1}_{\mathcal{L}}(1 - \mathbf{1}_{\mathcal{B}})\,[K_2 - S_{2T}] - \mathbf{1}_{\mathcal{L}}(1 - \mathbf{1}_{\mathcal{A}})\,[K_1 - S_{1T}]\,\Big)\Big\} \\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\Big\{\,\big[\omega\left([S_{1T} - K_1]^+ - [S_{2T} - K_2]^+\right)\big]^+\Big\} \\ &\qquad + e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\Big\{\big[\omega\left([K_2 - S_{2T}]^+ - [K_1 - S_{1T}]^+\right)\big]^+\Big\} \\ &= e^{-r(T-t)}\left(\mathbb{E}_{\mathbb{Q}}\left\{[\omega\,[U_{1T} - U_{2T}]]^+\right\} + \mathbb{E}_{\mathbb{Q}}\left\{[\omega\,[V_{2T} - V_{1T}]]^+\right\}\right). \end{aligned} \tag{8}$$

where $U_{1T}$, $V_{1T}$ are pay-offs to European call and put options on asset 1 with strike $K_1$ and $U_{2T}$, $V_{2T}$ are pay-offs to European call and put options on asset 2 with strike $K_2$ respectively. □

The CEO representation of a spread option is a special case of the general framework for multi-asset option pricing introduced by Alexander and Venkatramanan [2009]. Let $U_{it}$ and $V_{it}$ be the Black-Scholes option prices of the calls and puts in equation (7), and set $K_1 = mK$ to be the strike of $U_1$ and $V_1$ and $K_2 = (m-1)K$ to be the strike of $U_2$ and $V_2$, for some real number $m \geq 1$. Choosing $m$ so that the single asset call options are deep in-the-money (ITM),[3] the risk neutral price at time $t$ of a European spread option on 2GBM processes may be expressed as:

$$f_t = e^{-r(T-t)}\omega\ \left[U_{1t}\Phi(\omega\ d_{1U}) - U_{2t}\Phi(\omega\ d_{2U}) - (V_{1,t}\Phi(-\omega\ d_{1V}) - V_{2t}\Phi(-\omega\ d_{2V}))\right] \tag{9}$$

where

$$\begin{aligned} d_{1A} &= \frac{ln\left(\frac{A_{1t}}{A_{2t}}\right) + \left(q_2 - q_1 + \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}}; \\ d_{2A} &= d_{1A} - \sigma_A\sqrt{T-t}; \end{aligned} \tag{10}$$

[3] In the framework of Alexander and Venkatramanan [2009], the exchange options and the vanilla calls and puts in the exchange options need not be traded. Hence we are free to choose the strikes of the vanilla options as we please. However, it should be noted that if their strikes are very far outside the normal range for traded options, the spread option price will be subject to model risk arising from the method used to extrapolate the volatility smile.

and

$$\begin{aligned}
\sigma_U &= \sqrt{\xi_1^2 + \xi_2^2 - 2\rho\xi_1\xi_2}, \\
\sigma_V &= \sqrt{\eta_1^2 + \eta_2^2 - 2\rho\eta_1\eta_2}, \\
\xi_i &= \sigma_i \frac{S_{it}}{U_{it}} \frac{\partial U_{it}}{\partial S_{it}}, \\
\eta_i &= \sigma_i \frac{S_{it}}{V_{it}} \left| \frac{\partial V_{it}}{\partial S_{it}} \right|.
\end{aligned} \tag{11}$$

The correlation $\rho$ used to compute the exchange option volatility $\sigma_U$ in equation (11) is the implied correlation between the two vanilla calls (of strikes $K_1$ and $K_2$), which is the same as the implied correlation between the two vanilla puts, because the puts have the same strikes as the calls. And, since each vanilla option is driven by the same Wiener process as its underlying price, the implied correlation between the vanilla options is the implied correlation of the spread option with strike $K = K_1 - K_2$. As the deltas of the two vanilla options vary with their strikes, the implied correlation does too. For instance, if we fix $K_1$ then, as the spread option strike increases, $K_2$ decreases and the difference between the two deltas increases. Hence, the implied correlation will decrease as $K$ increases, and increase as $K$ decreases. In other words the correlation skew or frown becomes endogenous to the model.

**Proposition 2.** *The spread option deltas, gammas and vegas of the price - refer to equation (9), are given by:*

$$\begin{aligned}
\Delta^f_{S_i} &= \Delta^f_{U_i}\Delta^{U_i}_{S_i} + \Delta^f_{V_i}\Delta^{V_i}_{S_i} \\
\Gamma^f_{S_iS_i} &= \Gamma^f_{U_i}\left(\Delta^{U_i}_{S_i}\right)^2 + \Gamma^{U_i}_{S_i}\Delta^f_{U_i} + \Gamma^f_{V_i}\left(\Delta^{V_i}_{S_i}\right)^2 + \Gamma^{V_i}_{S_i}\Delta^f_{V_i} \\
\Gamma^f_{S_1S_2} &= \Gamma^f_{S_2S_1} = \Gamma^f_{U_1U_2}\Delta^{U_1}_{S_1}\Delta^{U_2}_{S_2} + \Gamma^f_{V_1V_2}\Delta^{V_1}_{S_1}\Delta^{V_2}_{S_2} \\
\mathcal{V}^f_{\sigma_i} &= \mathcal{V}^f_{\sigma_U}\frac{\partial\sigma_U}{\partial\sigma_i} + \mathcal{V}^f_{\sigma_V}\frac{\partial\sigma_V}{\partial\sigma_i} + \mathcal{V}^{U_i}_{\sigma_i}\Delta^f_{U_i} + \mathcal{V}^{V_i}_{\sigma_i}\Delta^f_{V_i}
\end{aligned} \tag{12}$$

*where, $\Delta^z_x$ and $\mathcal{V}^z_x$ denotes the delta and vega of $z$ with respect to $x$ respectively, and $\Gamma^z_{xy}$ denote the gamma of $z$ with respect to $x$ and $y$.*

*Proof.* Differentiate equation (9) using chain rule. □

Equation (12) shows that the CEO model Greeks are functions of their respective single asset option Greeks. Therefore, it is possible to construct a portfolio with single asset call and put options to replicate the spread option. For instance, to hedge the price and volatility risk of a call spread option due to of asset 1, we can buy $(\Delta^f_{U_1} + \mathcal{V}^{U_1}_{\sigma_1})$ call options on asset 1 with price $U_1$ and $\left(\Delta^f_{V_1} + \mathcal{V}^{V_1}_{\sigma_1}\right)$ put options on asset 1 with price $V_1$. Other risks can be hedged in a similar manner.[4]

A limitation of the analytic approximations reviewed in the previous section is that correlation risk is not properly quantified: the spread option correlation sensitivity must be a constant times

[4] Hedge portfolios of single asset call and put options are constructed by picking the coefficients of the corresponding single asset Greeks on the right hand side of equation (12) and adding them together. This implies that the CEO model hedge ratios are indeed consistent with the endogenous single asset option strikes given in Theorem 1.

the option vega. By contrast, the CEO approximation yields a closed form formula for the sensitivity of the spread option price to correlation. Write the approximate spread option price as $f = f(U_1, U_2, V_1, V_2, \sigma_U, \sigma_V)$. The approximation is structured so that the implied correlation is directly related to $m$, the only independent and therefore central parameter. The exchange option volatilities in equation (11) are therefore also determined by $m$, so we may write

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial m}\frac{\partial m}{\partial \rho},$$

$$\text{where} \quad \frac{\partial f}{\partial m} = \frac{\partial f}{\partial U_1}\frac{\partial U_1}{\partial m} + \frac{\partial f}{\partial U_2}\frac{\partial U_2}{\partial m} + \frac{\partial f}{\partial V_1}\frac{\partial V_1}{\partial m} + \frac{\partial f}{\partial V_2}\frac{\partial V_2}{\partial m} + \frac{\partial f}{\partial \sigma_U}\frac{\partial \sigma_U}{\partial m} + \frac{\partial f}{\partial \sigma_V}\frac{\partial \sigma_V}{\partial m}.$$

Now, unlike other analytic approximations, in the CEO approximation the volatility and correlation hedge ratios could be independent of each other, depending on our choice for $m$. The correlation affects the spread option price only through its effect on $\sigma_U$ and $\sigma_V$, and we may choose $m$ such that $\frac{d\sigma_U}{d\rho} = \frac{d\sigma_V}{d\rho} = 0$. We call the spread option volatility at such a value for $m$ the *pure* spread option volatility. Thus, we may choose $m$ so that

$$\frac{\partial f}{\partial m} = K\left(\frac{\partial f}{\partial U_1}\frac{\partial U_1}{\partial K_1} + \frac{\partial f}{\partial U_2}\frac{\partial U_2}{\partial K_2} + \frac{\partial f}{\partial V_1}\frac{\partial V_1}{\partial K_1} + \frac{\partial f}{\partial V_2}\frac{\partial V_2}{\partial K_2}\right),$$

in which case, at the pure spread option volatility, we have:

$$\frac{\partial f}{\partial \rho} = Kg\left(\xi_1, \xi_2, \rho; m\right)^{-1}\left(\frac{\partial f}{\partial U_1}\frac{\partial U_1}{\partial K_1} + \frac{\partial f}{\partial U_2}\frac{\partial U_2}{\partial K_2} + \frac{\partial f}{\partial V_1}\frac{\partial V_1}{\partial K_1} + \frac{\partial f}{\partial V_2}\frac{\partial V_2}{\partial K_2}.\right) \tag{13}$$

where

$$g\left(x, y, z; m\right) = (xy)^{-1}\left(\left(\frac{\partial x}{\partial m}x + \frac{\partial y}{\partial m}y\right) - z\left(\frac{\partial x}{\partial m}y + \frac{\partial y}{\partial m}x\right)\right).$$

We call (13) the *pure* correlation sensitivity of the spread option price because it is independent of the volatility sensitivities $\frac{\partial f}{\partial \sigma_U}$ and $\frac{\partial f}{\partial \sigma_V}$. That is, the pure correlation sensitivity of the spread option price is the correlation sensitivity at the pure spread option volatility. In the following proposition we establish a precise relationship between the pure correlation sensitivity of the spread option price and our central parameter $m$:

**Proposition 3.** *Let $m = m(\rho, t)$ be such that $m : [-1, 1] \times [0, T] \to [1, \infty)$. Then at the pure spread option volatility we have*

$$g\left(\xi_1, \xi_2, \rho; m\right) = g\left(\eta_1, \eta_2, \rho; m\right). \tag{14}$$

*Proof.* The total derivative of $\sigma_U$ is:

$$d\sigma_U = \frac{\partial \sigma_U}{\partial \xi_1}d\xi_1 + \frac{\partial \sigma_U}{\partial \xi_2}d\xi_2 + \frac{\partial \sigma_U}{\partial \rho}d\rho.$$

Hence

$$\begin{aligned}\frac{d\sigma_U}{d\rho} &= \frac{\partial \sigma_U}{\partial \xi_1}\frac{d\xi_1}{dm}\frac{dm}{d\rho} + \frac{\partial \sigma_U}{\partial \xi_2}\frac{d\xi_2}{dm}\frac{dm}{d\rho} + \frac{\partial \sigma_U}{\partial \rho} \\ &= \mathbb{A}\frac{dm}{d\rho} - \frac{\xi_1\xi_2}{\sigma_U} \end{aligned} \tag{15}$$

where

$$\mathbb{A} = \sigma_U^{-1}\left(\frac{d\xi_1}{dm}\left(\xi_1 - \rho\xi_2\right) + \frac{d\xi_2}{dm}\left(\xi_2 - \rho\xi_1\right)\right).$$

Similarly

$$\frac{d\sigma_V}{d\rho} = \mathbb{B}\frac{dm}{d\rho} - \frac{\eta_1\eta_2}{\sigma_V} \tag{16}$$

$$\text{where} \qquad \mathbb{B} = \sigma_V^{-1}\left(\frac{d\eta_1}{dm}\left(\eta_1 - \rho\eta_2\right) + \frac{d\eta_2}{dm}\left(\eta_2 - \rho\eta_1\right)\right).$$

When $\frac{d\sigma_U}{d\rho} = 0$, equation (15) implies that

$$\begin{aligned}
\frac{dm}{d\rho} &= \mathbb{A}^{-1}\frac{\xi_1\xi_2}{\sigma_U} \\
&= \xi_1\xi_2\left(\frac{d\xi_1}{dm}\left(\xi_1 - \rho\xi_2\right) + \frac{d\xi_2}{dm}\left(\xi_2 - \rho\xi_1\right)\right)^{-1} \\
&= \xi_1\xi_2\left(\left(\frac{d\xi_1}{dm}\xi_1 + \frac{d\xi_2}{dm}\xi_2\right) - \rho\left(\frac{d\xi_1}{dm}\xi_2 + \frac{d\xi_2}{dm}\xi_1\right)\right)^{-1} \\
&= g\left(\xi_1, \xi_2, \rho; m\right)^{-1}.
\end{aligned}$$

Similarly, when $\frac{d\sigma_V}{d\rho} = 0$,

$$\begin{aligned}
\frac{dm}{d\rho} &= \eta_1\eta_2\left(\left(\frac{d\eta_1}{dm}\eta_1 + \frac{d\eta_2}{dm}\eta_2\right) - \rho\left(\frac{d\eta_1}{dm}\eta_2 + \frac{d\eta_2}{dm}\eta_1\right)\right)^{-1} \\
&= g\left(\eta_1, \eta_2, \rho; m\right)^{-1}.
\end{aligned}$$

Therefore $g\left(\xi_1, \xi_2, \rho; m\right) = g\left(\eta_1, \eta_2, \rho; m\right)$. Finally, note that $\frac{dm}{d\rho}$ is well-defined throughout $\rho \in [-1, 1]$ because if

$$\rho = \left(\frac{d\xi_1}{dm}\xi_1 + \frac{d\xi_2}{dm}\xi_2\right)\left(\frac{d\xi_1}{dm}\xi_2 + \frac{d\xi_2}{dm}\xi_1\right)^{-1}$$

then $g\left(\xi_1, \xi_2, \rho; m\right) = 0$ and $g\left(\eta_1, \eta_2, \rho; m\right) = 0$ if $\xi_i = \eta_i$ or $\eta_i = 0$. But $\xi_i$ can never be equal to $\eta_i$, and when $\eta_i = 0$, the spread option is replicated by the CEO on calls (there is no CEO on puts) and we do not need equation (14). Therefore,

$$\rho \neq \left(\frac{d\eta_1}{dm}\eta_1 + \frac{d\eta_2}{dm}\eta_2\right)\left(\frac{d\eta_1}{dm}\eta_2 + \frac{d\eta_2}{dm}\eta_1\right)^{-1}.$$

□

Proposition 2 provides a condition (i.e. $g\left(\xi_1, \xi_2, \rho; m\right) = g\left(\eta_1, \eta_2, \rho; m\right)$) that we shall use to calibrate the CEO approximation at the stationary spread option volatililty. This way we obtain a correlation sensitivity for the spread option price that is not constrained to be directly proportional to its volatility sensitivity. At time $t$ we calibrate a single parameter $m = m(\rho, t)$ for each spread option, by equating the market price of the spread option to its model price (9). Let $f_{Mt}$ be the market price of the spread option and $f_t(m, \rho)$ be the price of a spread option given by equation (9). Then, for this option, we choose $m$ such that $||f_M - f(m,\rho)||$ is minimized, subject to the constraint that $g(\xi_1, \xi_2, \rho; m_j) = g(\eta_1, \eta_2, \rho; m_j)$ at each iteration $j$.[5] Then, a compatible pair of single asset options' strikes is uniquely determined by setting $K_1 = mK$ and $K_2 = (m-1)K$, where $K$ is the strike of the spread option.

The calibration problem can be solved using a one-dimensional gradient method. The first order differential of $f$ with respect to $\rho$ is given by equation (13). The first order derivatives of $\xi_i$ and $\eta_i$ with respect to $m$ can be calculated from their respective implied volatilities $\sigma_1$ and $\sigma_2$ either numerically or by assuming a parametric function, such as a cubic spline, on their strikes. Then $m$ can be calculated either numerically, or by finding the roots of the resulting polynomial equation.[6] Therefore, the model calibration will just involve using a one dimensional solver method, so the computation time will be minimal.

## 4. Pricing American Spread Options

The price of an American-style option on a single underlying asset is mainly determined by the type of the underlying asset, the prevailing discount rate, and the presence of any dividend yield. The option to exercise early suggests that these options are more expensive than their European counterparts but there are many instances when early exercise is not optimal, for instance for calls on non-dividend paying stocks, and calls or puts on forward contracts (see James [2003]). Since no traded options are perpetual, the expiry date forces the price of American options to converge to the price of their European counterparts. Before expiry, the prices of American calls and puts are always greater than or equal to the corresponding European calls and puts. In the moving boundary pricing method, the price of an American option can be expressed as a sum of its European counterpart and an early exercise premium (EEP).[7] Here the optimal stopping problem is transformed to one that involves finding the boundary point at which it is optimal to exercise. The EEP is then the expected value of the net gains from the payoff, conditional on the underlying asset price crossing the optimal boundary.

Pricing American spread options is more complicated than pricing single-asset American options, for two reasons: 1) the optimal boundary value of one asset is now a function of the other asset's price (see Detemple [2005] for more details), and 2) the conditional expectation of the net gains from the payoff upon early exercise is not easy to compute. Even in the case of European spread options, we saw that the option price, which is just the conditional expectation of the payoff at expiry, does not have an analytic solution.

In order to overcome this problem we use the CEO conditional probabilities for the spread option to be ITM, to express the EEP as a sum of three components. In so doing we may separate the early exercise boundary into two boundaries, one for each underlying asset. This allows us to treat the American spread option problem as an extension of the single-asset American option price problem. As a result, we may use a two-dimensional extension of an existing numerical scheme, such as Kim [1990], to compute the optimal boundaries.

[5] Recall that the approximation error will be smallest when we choose the vanilla options in the exchange options to be as deep ITM as possible.

[6] For instance, when implied volatilities can be closely fitted by a cubic function, equation (14) reduces to a cubic equation, whose roots can be found very easily.

[7] See McKean [1965], Carr et al. [1992], Kim [1990] and Jacka [1991] for single-asset American option examples.

Let $\mathcal{S}_{t,T}$ be the set of all stopping times between $t$ and $T$. Then the price of an American option is given by

$$P_t^A(S_1, S_2, K, q_1, q_2, \sigma, T) = \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}_{\mathbb{Q}} \left\{ e^{-r(\tau-t)} Y_\tau \middle| \mathcal{F}_t \right\}, \tag{17}$$

where $Y_t = \omega [S_{1t} - S_{2t} - K]^+$ is the spread option payoff at time $t$. Applying Tanaka-Meyer's formula (see Karatzas and Shreve [1991]) to the pay-off, we may write

$$Y_t = Y_0 + A_t^Y + M_t^Y,$$

where $M^Y$ is a $\mathbb{Q}$-martingale and $A^Y$ is a difference of non-decreasing processes null at 0, adapted to the filtration $(\mathcal{F})_{t\geq 0}$. Now the value of an American spread option at $t \in [0, \tau_0]$ can be expressed as

$$P_t^A(S_1, S_2, K, q_1, q_2, \sigma, T) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ Y_T \right\} + \mathbb{E}_{\mathbb{Q}} \left\{ \int_{\tau_0}^{T} e^{-r(s-\tau_s)} \mathbf{1}_{\tau_s = s} \left( rY_s ds - dA_s^Y \right) \right\} \tag{18}$$

where $\tau_t = \inf \left\{ s \in [t, T] : Y_s = \sup_{\tau \in \mathcal{S}_{s,T}} \mathbb{E}_{\mathbb{Q}} \left\{ e^{-r(\tau-t)} Y_\tau \right\} \right\}$.

Next we note that there is an alternative formulation for the CEO spread option price (9) that may be used to derive the price of an American spread option. This is:

$$f_t = S_{1t} e^{-q_1(T-t)} \mathbb{P}_1 - S_{2t} e^{-q_2(T-t)} \mathbb{P}_2 - K e^{-r(T-t)} \mathbb{P}_3$$

where

$$\begin{aligned}
\mathbb{P}_1 = \mathbb{P}_1(S_{1t}, S_{2t}, K, q_1, q_2, \sigma, t) &= \mathbb{P}\left(S_{1T} \geq S_{2T} + K\right) \\
&= \Phi\left(d_{11}\right) \Phi(\omega\, d_{1U}) - \Phi\left(-d_{11}\right) \Phi(-\omega\, d_{1V}), \\
\mathbb{P}_2 = \mathbb{P}_2(S_{1t}, S_{2t}, K, q_1, q_2, \sigma, t) &= \mathbb{P}\left(S_{2T} \leq S_{1T} - K\right) \\
&= \Phi\left(-d_{12}\right) \Phi(-\omega\, d_{2V}) - \Phi\left(d_{12}\right) \Phi(\omega\, d_{2U}), \\
\mathbb{P}_3 = \mathbb{P}_3(S_{1t}, S_{2t}, K, q_1, q_2, \sigma, t) &= \mathbb{P}\left(K \leq S_{1T} - S_{2T}\right) \\
&= \omega \left[ (m-1) \left( \Phi(d_{22}) \Phi(\omega d_{2U}) + \Phi(-\omega d_{2V}) \Phi(-d_{22}) \right) \right. \\
&\qquad \left. - \, m \left( \Phi(d_{21}) \Phi(\omega d_{1U}) + \Phi(-\omega d_{1V}) \Phi(-d_{21}) \right) \right]
\end{aligned} \tag{19}$$

Note that $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ are just the ITM conditional probabilities when the numeriare is each of the underlying asset prices and the bond price, respectively.

In the case of a spread option we have

$$Y_t = \omega \left[ S_{1t} - S_{2t} - K \right]^+, \quad \text{and} \quad dA_t^Y = \omega \left( (r - q_1) S_{1t} dt - (r - q_2) S_{2t} dt \right).$$

Therefore, in the exercise region, where the spread option is ITM,

$$\begin{aligned}
rY_s ds - dA_s^Y &= r\omega \left( S_{1s} - S_{2s} - K \right) ds - \omega \left( (r - q_1) S_{1s} - (r - q_2) S_{2s} \right) ds \\
&= \omega (q_1 S_{1s} - q_2 S_{2s} - rK) ds.
\end{aligned}$$

When $\omega = 1$, rewriting equation (18) in terms of moving boundaries $B_i$ for $i = 1, 2$, the price of a call spread option is given by:

$$\begin{aligned}
P_t^A(S_1, S_2, K, q_1, q_2, \sigma, T) &= P_t^E(S_1, S_2, K, q_1, q_2, \sigma, T) \\
&\quad + \mathbb{E}_{\mathbb{Q}} \left\{ \int_t^T e^{-r(s-t)} \mathbf{1}_{\{S_{1s} \geq (B_{1s} + K);\; S_{2s} \leq (B_{2s} - K)\}} \left( q_1 S_{1s} - q_2 S_{2s} - rK \right) ds \right\}.
\end{aligned}$$

Now, using the CEO probabilities (19), the American call spread option price can be expressed as

$$\begin{aligned} P_t^A(S_1,S_2,K,q_1,q_2,\sigma,T) &= P_t^E(S_1,S_2,K,q_1,q_2,\sigma,T) + \int_t^T q_1 S_{1t} e^{-q_1(s-t)} \mathbb{P}_1(S_{1t},B_{1s},K,q_1,q_2,\sigma,s-t)ds \\ &\quad - \int_t^T q_2 S_{2t} e^{-q_2(s-t)} \mathbb{P}_2(B_{2s},S_{2t},K,q_1,q_2,\sigma,s-t)ds \\ &\quad - \int_t^T rKe^{-r(s-t)} \mathbb{P}_3(B_{1s},B_{2s},K,q_1,q_2,\sigma,s-t)ds. \end{aligned} \tag{20}$$

The value match condition is given by

$$\begin{aligned} B_{1t} - B_{2t} - K &= P_t^E(B_{1t},B_{2t},K,q_1,q_2,\sigma,T) + \int_t^T q_1 S_{1t} e^{-q_1(s-t)} \mathbb{P}_1(B_{1t},B_{1s},K,q_1,q_2,\sigma,s-t)ds \\ &\quad - \int_t^T q_2 S_{2t} e^{-q_2(s-t)} \mathbb{P}_2(B_{2s},B_{2t},K,q_1,q_2,\sigma,s-t)ds \\ &\quad - \int_t^T rKe^{-r(s-t)} \mathbb{P}_3(B_{1s},B_{2s},K,q_1,q_2,\sigma,s-t)ds. \end{aligned} \tag{21}$$

For $i = 1,\ 2$, the high contact conditions are given by

$$\begin{aligned} 1 - \frac{\partial P_t^E(B_{1t},B_{2t},K,q_1,q_2,\sigma,T)}{\partial B_{it}} &= \frac{\partial}{\partial B_{it}} \left( \int_t^T q_1 S_{1t} e^{-q_1(s-t)} \mathbb{P}_1(B_{1t},B_{1s},K,q_1,q_2,\sigma,s-t)ds \right. \\ &\quad - \int_t^T q_2 S_{2t} e^{-q_2(s-t)} \mathbb{P}_2(B_{2s},B_{2t},K,q_1,q_2,\sigma,s-t)ds \\ &\quad \left. - \int_t^T rKe^{-r(s-t)} \mathbb{P}_3(B_{1s},B_{2s},K,q_1,q_2,\sigma,s-t)ds \right). \end{aligned} \tag{22}$$

# 5. Empirical Results

We begin by calibrating the CEO approximation to simulated spread option prices and comparing the calibration errors with those derived from Kirk's approximation. For the simulations we have used prices $S_1 = 65$ and $S_2 = 50$, and spread option strikes ranging between 9.5 and 27.5 with a step size of 1.5 and maturity 30 days. To simulate market prices with implied volatility skews and a correlation frown, we used quadratic local volatility and local correlation functions that are assumed to be dependent only on the price levels of the underlying assets and not on time. The dividend yields on both underlying assets are zero, the ATM volatilities were both 30% and the ATM correlation was 0.80.

In Kirk's approximation we set the strike convention and hence fix the single asset implied volatilities. Then we use an iterative method to back-out the implied correlation for each option by setting Kirk's price equal to the simulated price. When we match Kirk's prices to our simulated market prices it is very often impossible to derive a feasible value for the implied volatility, and/or for the implied correlation of the spread option in Kirk's formula. Instead we must constrain both these parameters to lie within their feasible set, and because of this there may be large differences between the Kirk's price and the market price.

Of several strike conventions considered, the one that produced the smallest pricing errors (with both simulated and market data) was

$$K_1 = S_{1,0} - \frac{K}{2}, \quad K_2 = S_{2,0} + \frac{K}{2}.$$

Still, in our simulated data, the root mean square calibration error (RMSE) was very high, at 9%.[8] By contrast, the CEO approximation's pricing errors are extremely small: the RMSE was 0.1% on the simulated data. Moreover, the CEO implied correlation skews showed much greater stability over different simulations than those obtained using the Kirk approximation, with any of the strike conventions.

This simulation exercise illustrates a major problem with the approximations for spread options surveyed in Section 2. That is, we have to apply a convention for fixing the strikes of the implied volatilities $\sigma_1$ and $\sigma_2$, take the implied volatilities from the single asset option prices and then calibrate the implied correlation to the spread option price. Moreover, using Kirk's approximation for illustration, we obtained unrealistic results whatever the strike convention employed, because for high strike spread options the model's lognormality assumption is not valid.

We now test the pricing performance of the CEO approximation using market prices of the 1:1 American put crack spread options that were traded on NYMEX between September 2005 and May 2006. These options are on the gasoline - crude oil spread and are traded on the price differential between the futures contracts of WTI light sweet crude oil and gasoline. Option data for American style contracts on each of these individual futures contracts were also obtained for the same time period, along with the futures prices. The size of all the futures contracts is 1000 bbls.

Figure 1 depicts the implied volatility skews in gasoline and crude oil on several days in March 2006, these being days with particularly high trading volumes. Pronounced negative implied volatility skews are evident in this figure, indicating that a suitable pricing model should be able to capture a skewed implied correlation frown.

Table 1 compares the results of Kirk's approximation with the CEO approximation by reporting the average absolute and percentage pricing errors on spread options with different strikes, where the average is taken over all consecutive trading days between 1st March and 15th March 2006. The models were calibrated to the market prices of both the gasoline - crude oil crack spread options and the individual gasoline and crude oil options.

Using Kirk's approximation led to exactly the same calibration problems as were encountered with our simulated data. For high strikes, Kirk's approximation based on feasible values for the implied correlation (between -1 and +1) gave prices that were far too low, and the opposite was the case with the low strikes. Only for a few strikes in the mid range were feasible values of the implied correlation found without constraining the iteration. Kirk's approximation, with constrained values for the spread option's implied volatility and correlation, gives an error that increases drastically for high strike values, as was also the case in our simulation results. By contrast, the CEO approximation errors were again found to be close to zero for all strikes on all dates.

Figure 2 plots the CEO parameter $m$ as a function of the spread option strike, for the same days as in Figure 1.[9] Notice that, even though the implied volatilities shown in Figure 1 are quite variable from day to day, $m$ is very stable at all strikes. The stability of $m$ allows us to choose

[8]Errors are reported as a percentage of the option price. For comparison, the RMSE was 9.3% when we used the constant ATM volatility to determine $\sigma_1$ and $\sigma_2$. Results for other strike conventions are not reported for reasons of space, but are available on request.

[9]The average values of the strike $K_1 = mK$ of the corresponding vanilla call and put options on gasoline are given in paranthesis.

accurate starting values for calibration, and this reduces the calibration time to just a few seconds on a standard PC. Figure 3 shows that the implied correlations that are calibrated from the CEO approximation exhibit a realistic, negatively sloped skew on each day of the sample.

Table 2 compares the two deltas and the two gammas of each model, averaged over the sample period from 1st March and 15th March 2006, as a function of the spread option strike. For low strikes the Kirk's deltas are greater than the CEO deltas, and the opposite is the case at high strikes. On the other hand, the CEO gammas are higher than Kirk's at most strikes. This demonstrates that, in addition to serious mispricing, the use of Kirk's approximation will lead to inaccurate hedging, as was claimed in Section 2.

Figures 4, 5, and 6 depict the two deltas, gammas and vegas of the CEO approximation, as a function of the spread option strike.[10] At the ATM strikes the absolute deltas are close to 0.5 and the gammas are close to their maximum value. The vegas are greatest at ITM strikes, and their values depend on the implied volatility levels of the individual vanilla options used. In our data, the implied volatilities of gasoline were higher than those of crude oil in general, hence the peak of the vega with respect to gasoline is further to the right than the peak of the crude oil vega. Moreover, at every strike, vega varies considerably from day to day, because its value depends on the level of the spread and on the level of the respective underlying asset price. This is in sharp contrast to the hedge ratios derived using Kirk's approximation, which are only affected by the level of the spread and not by the level of the underlying asset prices. Finally, Figure 7 depicts the CEO vega with respect to the pure spread option volatility. Its strike dependence is quite similar to that of a Black-Scholes vanilla option vega, in that it takes its maximum value close to the ATM strike.

## 6. CONCLUSION

This paper begins by highlighting the difficulties encountered when attempting to price and hedge spread options using analytic approximations based on a reduction of the price dimension. Firstly, in the presence of market implied volatility smiles for the two underlyings, an arbitrary strike convention is necessary, and since the approximate prices and hedge ratios depend on this convention, they are not unique. As a result, the implied correlations that are implicit in the approximation may vary considerably, depending on the choice of strike convention. Secondly, the approximation may only be valid for a limited strike range. Thirdly, since the spread option prices are affected only by the relative price levels of the underlying assets and not by their individual levels, the probability that the spread option expires ITM is not tied to the price level. Thus, when equating the price approximation to a market price, we may obtain infeasible values for the implied volatilty and/or correlation of the spread option. Fourthly, depending on the strike convention used, the hedge ratios derived from such approximations may be inconsistent with the single asset option Greeks. And finally, the correlation sensivity of the spread option price is simply assumed to be proportional to the option's vega.

[10]The deltas shown are those for put spread options. The call delta with respect to gasoline increases with strike because the call spread option price increases as gasoline prices increase, and the delta with respect to crude oil decreases with strike because the call spread option price decreases with an increase in crude oil prices. As the options move deeper ITM the absolute value of both CEO deltas approach one, and as the options move deeper OTM, they approach zero.

We have developed a new analytic approximation based on an exact representation of a European spread option price as the sum of the prices of two compound exchange options, and have derived an extension of this approximation to American spread options. Using both market and simulated data we have demonstrated that our approximation provides accurate prices and realistic, unique values for implied correlation at all strikes. Another feature of our approximation that is not shared by other approximations is that the spread option Greeks are consistent with the single asset option Greeks. This should lead to more accurate delta-gamma-vega hedging of spread option positions, using the two underlyings and vanilla calls and puts with strikes that are calibrated in the approximation.

## References

C. Alexander and A. Scourse. Bivariate normal mixture spread option valuation. *Quantitative Finance*, 4(6):637–648, 2004.

C. Alexander and A. Venkatramanan. Analytic approximations for multi-asset options. *ICMA Centre Discussion Papers 2009-05*, 2009.

R. Carmona and V. Durrleman. Pricing and hedging spread options in a log-normal model. Technical report, ORFE, 2003a.

R. Carmona and V. Durrleman. Pricing and hedging spread options. *Siam Review*, 45(4):627–685, 2003b.

R. Carmona and V. Durrleman. Generalizing the Black-Scholes formula to multivariate contingent claims. *Journal of Computational Finance*, 9(2):43, 2005.

P. Carr and D. Madan. Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2(4):61–73, 1999.

P. Carr, R. Jarrow, and R. Myneni. Alternative characterizations of American put options. *Mathematical Finance*, 2(2):87–106, 1992.

M. Dempster and S.S.G. Hong. Spread option valuation and the fast Fourier transform. *Mathematical Finance, Bachelier Congress, Geman, H., Madan, D., Pliska, S.R. and Vorst, T., Eds., Springer Verlag, Berlin*, 1:203–220, 2000.

S. Deng, M. Li, and J. Zhou. Multi-asset spread option pricing and hedging. *SSRN eLibrary*, 2007.

S. Deng, M. Li, and J. Zhou. Closed-form approximations for spread option prices and greeks. *Journal of Derivatives*, 16(4):58–80, Spring 2008.

J. Detemple. American-style derivatives: Valuation and computation. *Chapman & Hall/CRC*, 2005.

V. Durrleman. Implied correlation and spread options. Technical report, ORFE, 2001.

A. Eydeland and K. Wolyniec. Energy and power risk management: new developments in modeling, pricing and hedging. *John Wiley & Sons*, 2003.

T.R. Hurd and Z. Zhou. A Fourier transform method for spread option pricing. *Arxiv preprint arXiv:0902.3643*, 2009.

S.D. Jacka. Optimal stopping and the American put. *Mathematical Finance*, 1(1):14, 1991.

P. James. Option theory. *Wiley New York*, 2003.

I. Karatzas and S.E. Shreve. Brownian motion and stochastic calculus. *Springer*, 1991.

I.J. Kim. The analytic valuation of american options. *Review of Financial Studies*, 3(4):547–572, 1990.

E. Kirk. Correlation in energy markets. *Managing Energy Price Risk, Risk Publications and Enron, London*, Edited by V. Kaminski:71–78, 1996.

W. Margrabe. The value of an option to exchange one asset for another. *The Journal of Finance*, 33 (1):177–186, 1978.

H.P. McKean. Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics. *Industrial Management Review*, 6(2):32–39, 1965.

K. Ravindran. Low-fat spreads. *Risk*, 6(10):56–57, 1993.

D. Shimko. Options on futures spreads: hedging, speculation and valuation. *Journal of Futures Markets*, 14(2):183–213, 1994.

 

FIGURE 1: *Implied volatility of gasoline (left) and crude oil (right)*

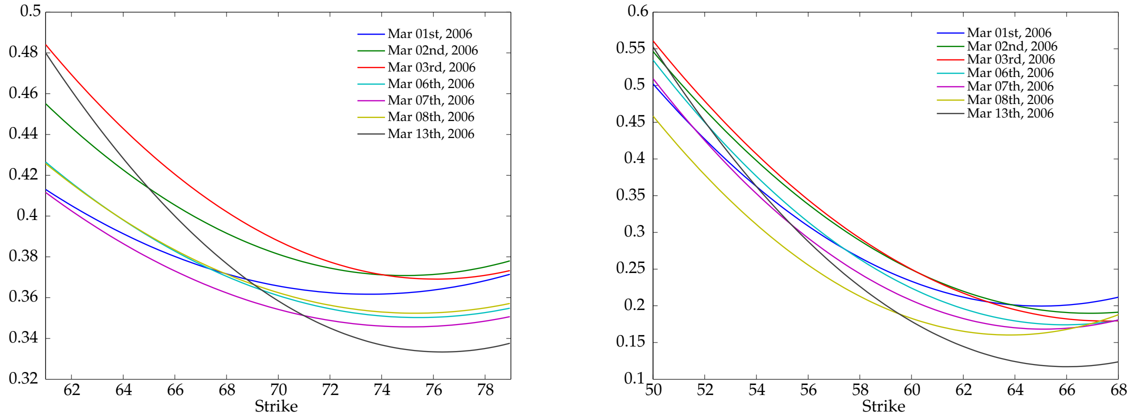


FIGURE 2: *CEO parameter m with average strike of gasoline options* ($K_1 = mK$)

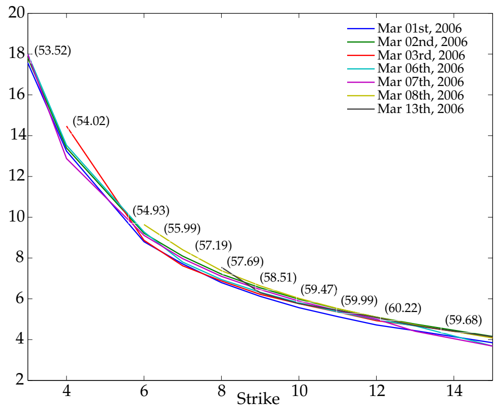


FIGURE 3: *Implied correlation skews of CEO approximation*

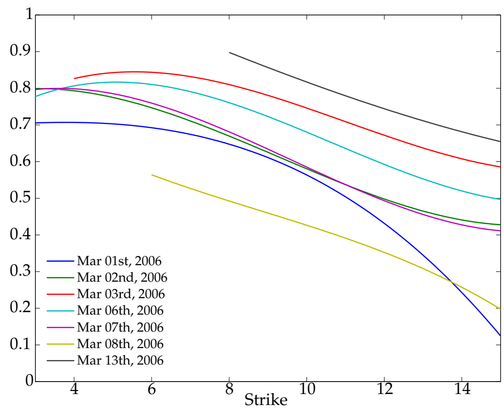

FIGURE 4: *CEO delta with respect to gasoline (left) and crude oil (right)*

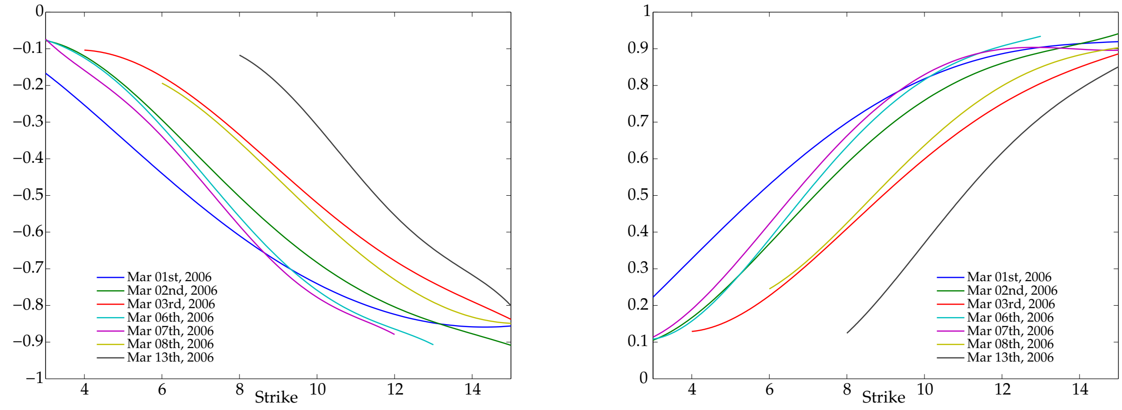


FIGURE 5: *CEO gamma with respect to gasoline (left) and crude oil (right)*

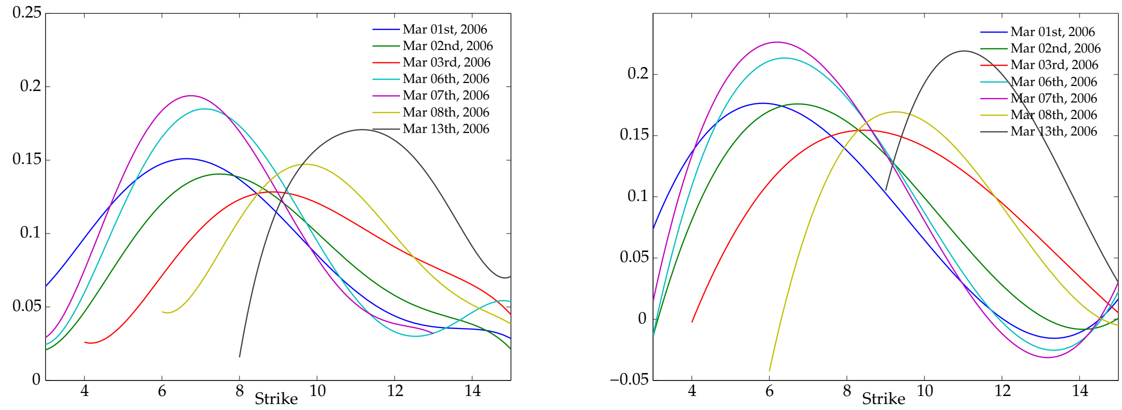


FIGURE 6: *CEO vega with respect to gasoline (left) and crude oil (right)*

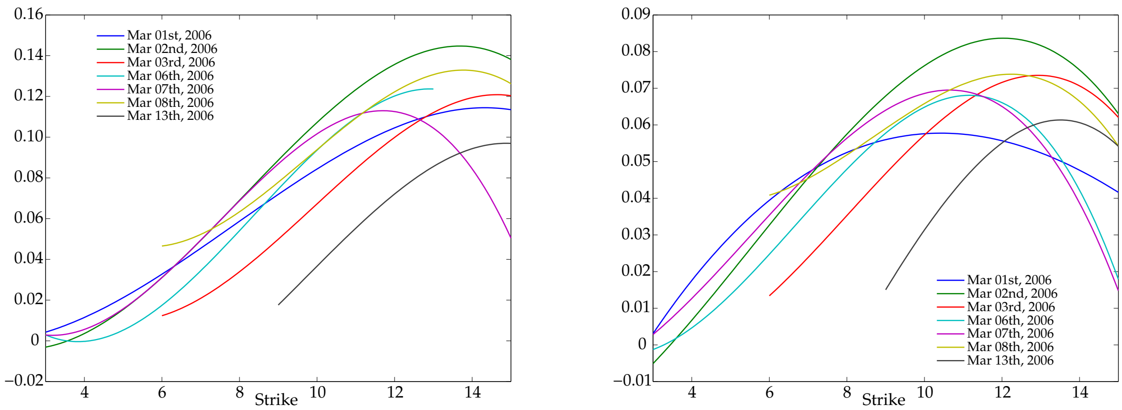


 

FIGURE 7: *CEO vega with respect to spread option volatility*

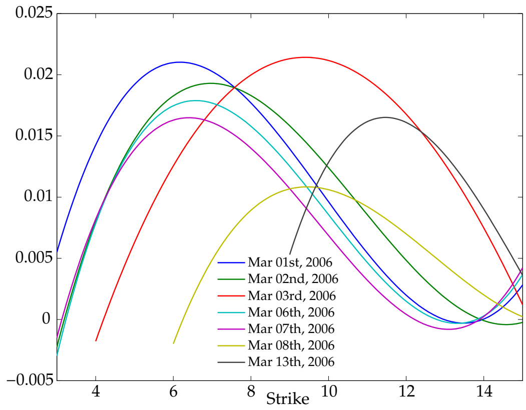


TABLE 1: *Average absolute (percentage) pricing errors*

| **Strike** | 4.0 | 6.0 | 7.0 | 8.0 | 9.0 | 10.0 | 11.0 | 12.0 | 13.0 | 15.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| **CEO** | 0.0017 | 0.0059 | 0.0046 | 0.0048 | 0.0038 | 0.0000 | 0.0001 | 0.0082 | 0.0146 | 0.0003 |
| | (0.64%) | (0.69%) | (0.68%) | (0.47%) | (1.01%) | (0.001%) | (0.002%) | (0.24%) | (0.55%) | (0.005%) |
| **Kirk's** | 0.4969 | 0.6941 | 0.7585 | 0.8422 | 0.8998 | 0.92396 | 0.9158 | 0.8802 | 0.8696 | 0.8236 |
| | (179.1%) | (122.7%) | (176.7%) | (120.1%) | (84.7%) | (60.5%) | (44.1%) | (32.2%) | (26.1%) | (13.2%) |

TABLE 2: *Average difference between Kirk's and CEO deltas and gammas (put spread options)*

| **Strike** | 3.0 | 4.0 | 6.0 | 7.0 | 8.0 | 9.0 | 10.0 | 11.0 | 12.0 | 13.0 | 15.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta_{S_1}$ | 0.0405 | 0.0467 | 0.0462 | 0.0486 | 0.0463 | 0.0112 | -0.0186 | -0.0455 | -0.0652 | -0.0669 | -0.0603 |
| $\Delta_{S_2}$ | 0.0331 | 0.0338 | 0.0206 | 0.0180 | 0.0160 | -0.0248 | -0.0549 | -0.0786 | -0.0928 | -0.1006 | -0.0828 |
| $\Gamma_{S_1}$ | 0.0091 | -0.0010 | -0.0364 | -0.0492 | -0.0478 | -0.0519 | -0.0522 | -0.0469 | -0.0375 | -0.0234 | -0.0080 |
| $\Gamma_{S_2}$ | 0.0136 | -0.0002 | -0.0547 | -0.0623 | -0.0510 | -0.0509 | -0.0576 | -0.0541 | -0.0389 | -0.0111 | 0.0162 |

 