Meshfree Approximation for Multi-Asset Options

Emmanuel Hanert
Département des sciences du milieu et de l'aménagement du territoire
Université catholique de Louvain, Belgium

Aanand Venkatramanan
ICMA Centre, University of Reading, UK

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ICMA Centre • The University of Reading
Whiteknights • PO Box 242 • Reading RG6 6BA • UK
Tel: +44 (0)1183 788239 • Fax: +44 (0)1189 314741
Web: www.icmacentre.rdg.ac.uk
Director: Professor John Board, Chair in Finance
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ABSTRACT

We price multi-asset options by solving their price partial differential equations using a meshfree approach with radial basis functions under jump-diffusion and geometric Brownian motion frameworks. In the geometric Brownian motion framework, we propose an effective technique that breaks the multi-dimensional problem to multiple 3D problems. We solve the price PDEs or PIDEs with an implicit meshfree scheme using thin-plate radial basis functions. Meshfree approach is very accurate, has high order of convergence and is easily scalable and adaptable to higher dimensions and different payoff profiles. We also obtain closed form approximations for the option Greeks. We test the model on American crack spread options traded on NYMEX.

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Emmanuel Hanert[1]
Départment des sciences du milieu et de l'aménagement du territoire
Université catholique de Louvain
Place Croix du Sud 2/16, B-1348 Louvain-la-Neuve, Belgium
Email: emmanuel.hanert@uclouvain.be

Aanand Venkatramanan[2]
ICMA Centre, University of Reading,
Reading, RG6 6BA, UK.
Email: a.venkatramanan@icmacentre.ac.uk

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[1]Most of the work was completed while the author was at University of Reading, UK
[2]Author to whom all correspondence must be addressed

1. Introduction

The simplest way to price multi-asset options is to approximate the distribution of the returns of a basket of assets to a univariate distribution and thus derive an approximate pricing formula. But such single factor models have serious drawbacks, as they completely ignore the effects of covariance between the individual assets. On the other hand, increasing the number of risk factors can lead to an exponential rise in the complexity of the problem, and solving them would demand extreme computational power.

Numerical methods such as finite difference methods (FDM) and finite element methods (FEM) have been widely used to solve partial differential equations (PDE) to price various derivative securities. For a given accuracy, they are much faster than simulation and more flexible than tree based techniques. Many variations of FDM and FEM such as Galerkin, adaptive mesh and finite volume methods have been proposed to solve complex problems that occur in finance. For instance, Broadie and Glasserman (2004) use a stochastic mesh approach for high-dimensional pricing problems while Lötstedt et al. (2007) use a space-time adaptive grid. In recent years fast Fourier transforms have gained popularity for solving high dimensional PDEs. Carr and Madan (1999), Dempster and Hong (2000) and Borak et al. (2005) use FFT to price spread options when the joint characteristic function of the underlying assets is analytic. Although these methods are scalable to high dimensions in theory, difficulty in implementing them renders them less useful.

A more recent method called meshfree approach that uses radial basis functions has considerable advantages when compared to finite difference or finite element schemes. The meshfree method allows for high-order, accurate approximations and to easily increase the dimension of the problem. Analytical approximate formulae can be easily derived even for higher order option price sensitivities. The meshless nature of the approach makes it more suitable to adapt to problems with complex payoff structures, such as barrier options. They also eliminate the time spent on building the mesh which could be prohibitive for multi-dimensional problems with complex structure.

Recent work on meshfree approaches in finance include Hon and Mao (1999) who apply a collocation scheme using global radial basis functions for solving option price PDEs. For European options, they show that the degree of accuracy is more sensitive to time integration than to spatial approximation. Fasshauer et al. (2004) employ Gaussian radial basis functions to price multi-asset European options while Pettersson et al. (2007) use multi-quadratic radial basis functions to price multi-asset American options. They both compare their approach with various finite difference schemes and show that their approach is significantly faster. Larsson et al. (2007) use a generalised Fourier transform to improve the efficiency of the radial basis functions approach particularly for higher dimensional problems. By choosing the nodal points isometrically they obtain equivariant dense discretization matrices that can be diagonalised by applying a generalised Fourier transform.[3]

Solving partial integro differential equations (PIDE) is more tedious than solving a PDE, as it involves evaluating a non-local integral equation in addition to a partial differential equation. Many approaches have been studied, starting with Andersen and Andreasen (2000) who combine the fast Fourier transform with an alternate direction implicit finite difference scheme. Matache et al. (2004) apply a θ-scheme in time and a wavelet Galerkin scheme in space. Cont and Voltchkova (2005) and Briani et al. (2007) develop an implicit-explicit finite difference schemes, who handle the

[3] An *equivariant* matrix is one that commutes with a group of permutation matrices.

differential operator implicitly and integral operator explicitly. They also study the convergence and stability of the schemes and give bounds on errors due to localisation of the integral term.

In this paper we price multi-asset options under a multi-factor geometric Brownian motion (GBM) and a jump-diffusion framework. Our contribution is two-fold - firstly, for the GBM framework we propose a new accurate technique based on the work of Alexander and Venkatramanan (2009a) to break the dimension of the problem. For a d-factor GBM model, this approach simplifies the d-dimensional problem to solving multiple 3-D parabolic PDEs that can be solved independently. The number of degrees of freedom reduces from N^d to $2(d-1)N^3$, where N is the number of nodes in one direction under the chosen numerical scheme. Secondly, we describe a fully implicit scheme based on meshfree approach using thin plate spline radial basis functions to solve a general price PIDE. The method is unconditionally stable, second order convergent in time and by choosing high order radial basis functions we can achieve high order convergence in space. To our knowledge, there is no literature available on fully implicit meshfree scheme for solving PIDEs that appear in finance.

The structure of this paper is as follows. In the following section we derive a general multi-dimensional PIDE under a jump-diffusion framework. We also introduce the compound exchange option approach that is necessary to develop our fast GBM based pricing model and derive the associated price PDEs. In section 3 we discuss the meshfree approximation that we use to solve the partial integro-differential equations and also derive results on convergence. The empirical results for two asset crack spread options are presented in section 4. Finally we summarize and conclude.

2. Pricing Multi-Asset Options

The aim of this section is to describe the PDE based approach to price multi-asset options when the price of the underlying assets follow: 1) jump-diffusion processes, and 2) correlated GBM processes. In the former case we assume that the individual asset prices are driven by correlated Wiener processes each but by a central jump process. In the GBM case we adopt the compound exchange option approach introduced by Alexander and Venkatramanan (2009a) to speed up the computation of the basket option price by solving a system of 3D PDEs instead of a multidimensional PDE.

2.1. Jump Diffusion Framework - General PIDE

We derive a multi-dimensional PIDE when the underlying assets follow mean reverting jump diffusion processes. The derivation is quite straightforward and there are numerous references available in the literature (see Cont and Tankov (2004), Cont and Voltchkova (2005) for a detailed discussion). We therefore give only an outline of the derivation.

Let us assume that the risk-neutral price dynamics of d underlying assets are governed by exponential Lévy processes of the form

$$S_{it} = S_{i0} \exp\left(rt + X_{it}\right),$$

for $1 \leq i \leq d$. X_i is a Lévy process given by

$$X_{it} = X_{i0} + \int_0^t \sigma_{is} dW_{it} + \lambda_i \sum_{j=i}^{N_t} \Delta X_{ij}, \tag{1}$$

where ΔX_i are i.i.d. random variables with a certain distribution, W_i are correlated Wiener processes, N_t is a Poisson process with intensity μ and $\mathbf{\Lambda} = (\lambda_1, \ldots, \lambda_d)'$ is a vector of real constants. Let σ_i be a continuous non-anticipating process with

$$\mathbb{E}\left\{\int_0^T \sigma_{it} dt\right\} < \infty\,, \quad \text{for } 1 \leq i \leq d,$$

ρ_{ij} be the correlation between W_i and W_j, and ν a Lévy measure with

$$\int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty.$$

Under the risk-neutral probability measure $\mathbb{Q}$, X_i has the characteristic triplet $(\sigma_i, \gamma_i, \nu)$ with:

$$\gamma_i(\sigma_i, \nu) = -\frac{\sigma_i^2}{2} - \int \left(e^y - 1 - y1_{|y|\leq 1}\right)\nu(dy).$$

Then the risk neutral dynamics of S_i is given by

$$dS_{it} = rS_{it^-}dt + \sigma_{it}S_{it^-}dW_{it} + \int_{-\infty}^{\infty} (e^x - 1)\, S_{it^-} J_{X_i}(dx, ds), \tag{2}$$

where J_{X_i} is the compensated random measure describing the jumps of X_i.

Let $f = f(\mathbf{S}_t, t)$ be the price of an option on d assets, with $\mathbf{S} = (S_1, S_2, \ldots, S_d)$, whose prices are described by equation (2) and let $\mathcal{L}$ be the infinitesimal generator of the Lévy process. Then, f satisfies the partial integro-differential equation:

$$\begin{aligned}
\frac{\partial f}{\partial t}(\mathbf{S}) &= \mathcal{L}f(\mathbf{S}) \\
&= \sum_{i=1}^{d} rS_i \frac{\partial f}{\partial S_i}(\mathbf{x}) + \frac{1}{2}\sum_{i=1}^{d} \sigma_i x_i \left(\sum_{j=1}^{d} \rho_{ij}\sigma_j x_j \frac{\partial^2 f}{\partial S_i \partial S_j}(\mathbf{S})\right) - rf(\mathbf{S}) \\
&\quad + \int_{-\infty} \infty \nu(dy)\left(f(\mathbf{S}e^{\mathbf{\Lambda} y}) - f(\mathbf{S}) - \sum_{i=1}^{d} S_i(e^{\lambda_i y} - 1)\frac{\partial f}{\partial x_i}(\mathbf{S})\right)
\end{aligned} \tag{3}$$

with its associated boundary conditions.

Let, $F(t, \mathbf{Z}) = e^{r(T-t)} f(\mathbf{S}e^{\mathbf{Z}})$, $Z_{it} = \log \dfrac{S_{it}}{S_{i0}}$ and $\tau = (T - t)$. Then the above PIDE can be rewritten as

$$\begin{aligned}
\frac{\partial F}{\partial \tau}(\mathbf{Z}) &= \sum_{i=1}^{d} \frac{1}{2}\sigma^2 \frac{\partial F}{\partial Z}(\mathbf{Z}) + \frac{1}{2}\sum_{i=1}^{d} \sigma_i \left(\sum_{j=1}^{d} \sigma_j \frac{\partial^2 F}{\partial Z_i \partial Z_j}(\mathbf{Z})\right) \\
&\quad + \int_{-\infty} \infty \left(F(\mathbf{Z} + \mathbf{\Lambda} z) - F(\mathbf{Z}) - \sum_{i=1}^{d} (e^{\lambda_i z} - 1)\frac{\partial F}{\partial Z_i}(\mathbf{Z})\right)\nu(dz), \\
\frac{\partial F}{\partial \tau}(\mathbf{Z}) &= (\mathcal{D} + \mathcal{I})\, F(\mathbf{Z}),
\end{aligned}$$

where $\mathcal{D}$ and $\mathcal{I}$ represent the differential and integral operators respectively.

2.2. GBM Framework - A Fast Dimension Reduction Technique

Here we assume the risk-neutral price dynamics of the underlying assets, given by equation (2), are governed only by correlated GBM processes (there is no jump component. Then equation (3) reduces to:

$$\frac{\partial f}{\partial t}(\mathbf{S}) = \sum_{i=1}^{d} rS_{it}\frac{\partial f}{\partial S_i}(\mathbf{S}) + \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{2}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 f}{\partial S_i \partial S_j}(\mathbf{S}) - rf(\mathbf{S}). \tag{4}$$

We propose a new dimension reduction technique based on the recursive approach of Alexander and Venkatramanan (2009a) to efficiently and accurately price basket options. For very high dimensional problems, our approach leads to significant computational gains. By breaking the multi-dimensional problem to multiple three dimensional independent problems, the total number of nodes required at any point of time can be as low as n^3, where n is the number of nodes in one direction. Moreover, as the problem dimension increases, the computation time only increases linearly and since the problems are independent of each other, they can be solved parallelly.

In this approach, the price of a basket option is computed as a sum of prices of two compound exchange options. These compound exchange options are options to exchange European options of same maturity written on a disjoint subset of the baskets of assets. Then the central idea of this approach is to express these compound exchange options as exchange options written on assets whose prices follow GBM processes. The prices of these compound exchange options are computed by solving their associated price PDEs. The sub-basket options on which the compound exchange options are written, are in turn priced in a similar fashion as a sum of compound exchange options on their sub-baskets.

This approach is particularly effective for problems with dimensions greater than 5 as the total number of nodes required in those cases reduces from N^d to $2(2^{d-1}-1)N^3$. In order to price a d dimensional multi-asset option we solve a total number of $2(2^{d-1}-1)$ 3D PDEs along with d Black-Scholes PDEs. For low dimension problems, although this approach may lead to a marginal increase in number of nodes, it yields volatility skew consistent prices and hedge ratios due to its built-in convention to choose the underlying asset volatilities.[4] In section 4 we implement this approach using a meshfree scheme to price two asset spread options.

Since pricing compound exchange options is central to our model we derive a generic price PDE for a compound exchange option in the following subsection. The CEOs are to exchange two options whose underlying asset prices are lognormally distributed.[5] Later we describe the pricing framework and show how a basket option payoff can be decomposed into compound exchange option payoffs and hence priced in a recursive manner.

[4] See Alexander and Venkatramanan (2009b) for a detailed discussion on implied volatility skew consistent spread option pricing.

[5] Since a stochastic integral or a non-linear transformation of a Lévy process (or a diffusion process) is not a Lévy (or diffusion) process, the price of an option on a lognormal asset does not follow a GBM process naturally. Nevertheless, we can express it as a GBM process by approximating the option volatility process. Then the compound exchange options reduce to simple lognormal exchange options and the price of a basket option can be computed by solving for exchange option prices repeatedly. However, when the asset prices follow jump-diffusion processes, such an approximation is not straightforward and also does not lead to any meaningful representation of the option price processes.

Compound Exchange Option Price PDE

Assume that the price of an asset i follows geometric Brownian motion process with constant volatilities σ_i:

$$\begin{aligned} dS_{it} &= rS_{it}dt + \sigma_i S_{it}dW_{it}, \\ \langle dW_{1t}, dW_{2t}\rangle &= \rho dt \qquad i = 1,\ 2. \end{aligned}$$

Consider a compound exchange option on two vanilla options on assets 1 and 2. The payoff of such an option is given by

$$\tilde{f_T} = \omega \left[U_{1T} - U_{2T}\right]^+,$$

where U_{it} is the price of a European option on asset i with strike K_i. By applying Itô's lemma, the two vanilla option price processes can be described by (see Alexander and Venkatramanan (2009a) for details on derivation):[6]

$$dU_{it} = rU_{it}dt + \xi_{it}U_{it}dW_{it}, \tag{5}$$

where

$$\xi_{it} = \sigma_i \frac{S_{it}}{U_{it}} \frac{\partial U_{it}}{\partial S_{it}}. \tag{6}$$

The associated SDE for ξ_i is given by:

$$d\xi_{it} = a_i(\xi_{it}, t)\left(-\xi_{it}dt + dW_{it}\right),$$

where $a_i(\xi_{it}, t) = \xi_{it}\left(\sigma_i - \xi_{it} + c\right)$, and $c = \sigma_i S_{it}\dfrac{\Gamma_{it}}{\Delta_{U_{it}}}$ is assumed constant.[7]

We now express the price of a compound exchange option as

$$\tilde{f_t} = e^{-r(T-t)}U_{2t}\mathbb{E}_{\mathbb{P}}\left\{\omega\left[X_T - 1\right]^+\right\},$$

where $X_t = \frac{U_{1t}}{U_{2t}}$ and $\mathbb{P}$ is a new martingale measure whose Radon-Nikodym derivative with respect to $\mathbb{Q}$ is given by:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left(-\frac{1}{2}\xi_{2t}^2 t + \xi_{2t}W_{2t}\right).$$

The solution to ξ_i is given by

$$\xi_{it} = \tilde{\sigma}_i\left(1 - \left(1 - \frac{\tilde{\sigma}_i}{\xi_{i0}}\right)\exp\left(\frac{1}{2}\tilde{\sigma}_i^2 t - \tilde{\sigma}_i W_{it}\right)\right)^{-1} \tag{7}$$

[6] An alternative formulation of the problem is to price the compound exchange option as an option on the spread X_t with zero strike. By Itô's lemma, the option price processes are described as:

$$dU_{it} = rU_{it}dt + \Sigma_{it}dW_{it}$$

where $\Sigma_{it} = \sigma_i S_{it}\frac{\partial U_{it}}{\partial S_{it}}$. Then, for $X_t = (U_{1t} - U_{2t})$, we have $\tilde{f_T} = \omega X_T^+$ and

$$dX_t = rX_t dt + \Sigma_{1t}dW_{1t} - \Sigma_{2t}dW_{2t}.$$

The associated SDEs for Σ_i can be derived easily by applying Itô's lemma.

[7] In section 4, we justify this assumption and find that the compound exchange option price is less sensitive to c.

where $\tilde{\sigma}_i = \sigma_i + c$, and it is easy to verify that $\mathbb{E}\left\{\int_0^T |\xi_{it}|^2 dt\right\} < \infty$ which is necessary and sufficient for $\frac{d\mathbb{P}}{d\mathbb{Q}}$ to exist.

Let $f_t = f(X_t, \xi_{1t}, \xi_{2t}, t)$ be the price of a European option whose payoff is given by $[X_T - 1]^+$, so that $\tilde{f_t} = U_{2t} f_t$. Under $\mathbb{P}$, X_t is a martingale and

$$dX_t = \xi X_t dZ_t,$$

where $Z_t = W_t + W_{3t}$, $W_t = W_{2t} - \xi_2 t$, $W_{1t} = \rho W_{2t} + \sqrt{1-\rho^2} W_{3t}$, and $\xi^2 = \xi_1^2 + \xi_2^2 - 2\rho\xi_1\xi_2$. Note that W_t, W_{3t} and Z_t are martingales under this measure.

We now have the transformed set of equations:

$$\begin{aligned} dX_t &= \xi_t X_t dZ_t, \\ d\xi_{1t} &= a_1(\xi_{1t}, t)\left(\left(\rho\xi_{2t} - \xi_{1t}\right) dt + dZ_t\right), \\ d\xi_{2t} &= a_2(\xi_{2t}, t) dW_t. \end{aligned} \tag{8}$$

Then, it can be shown that f satisfies the following PDE:

$$\begin{aligned} &\frac{\partial f}{\partial t} + \frac{1}{2}\xi^2 X^2 f_{XX} + \frac{1}{2}a_1^2 f_{\xi_1\xi_1} + \frac{1}{2}a_2^2 f_{\xi_2\xi_2} + X a_1 \xi f_{X\xi_1} \\ &\qquad + X a_2 \xi \rho f_{X\xi_2} + a_1 a_2 \rho f_{\xi_1\xi_2} + a_1\left(\rho\xi_2 - \xi_1\right) f_{\xi_1} - rf = 0 \end{aligned} \tag{9}$$

along with the associated boundary conditions

$$\begin{aligned} f(X, \xi_1\xi_2, T) &= [X-1]^+, \\ f_X(X, \xi_1, \xi_2, t) &= 1 \; as \; X \to \infty, \\ f(X, \xi_1, \xi_2, t) &= X \; as \; \xi_1 \; or \; \xi_2 \to \infty, \\ f(0, \xi_1, \xi_2, t) &= 0, \\ f_t + \frac{1}{2}\xi_1^2 f_{XX} + \frac{1}{2}a_1^2 f_{\xi_1\xi_1} + a_1 X \xi_1^2 f_{X\xi_1} - a_1\xi_1 f_{\xi_1} - rf &= 0 \; , when \; \xi_2 = 0 \; and \; for \; all \; \xi_1, \; X, \; t, \\ f_t + \frac{1}{2}\xi_2^2 f_{XX} + \frac{1}{2}a_2^2 f_{\xi_2\xi_2} + a_2 X \xi_2^2 \rho f_{X\xi_2} - rf &= 0 \; , when \; \xi_1 = 0 \; and \; for \; all \; \xi_2, \; X, \; t. \end{aligned} \tag{10}$$

Multi-Asset Options as Compound Exchange Options

Let $\mathbf{S}_t = (S_{1t}, S_{2t}, ..., S_{Nt})'$ and $\mathbf{b}_N = (\theta_1 S_{1t}, \; \theta_2 S_{2t}, \; ..., \; \theta_N S_{Nt})'$ be a basket of N assets with weights $\mathbf{\Theta}_N = (\theta_1, \theta_2, \; ..., \theta_N)$, where θ_i are real constants. Let $B_t = \mathbf{\Theta}_N \mathbf{S}_t = \sum_{i=1}^N \theta_i S_{it}$ be the price of the basket at any time t and $V_{NT} = [B_T - K]^+$ be the payoff to a call option on a basket $\mathbf{b}_N$ with strike price K.

Now let $\mathbf{b}_m$ and $\mathbf{b}_n$ be sub-baskets of $\mathbf{b}$ of sizes m and n respectively ($m + n = N$), and $\mathbf{\Theta}_m$ and $\mathbf{\Theta}_n$ be the weights of the corresponding sub-baskets such that $\mathbf{\Theta} = (\mathbf{\Theta}_m, \; \chi \; \mathbf{\Theta}_n)$; $\chi = 1$ or -1. Similarly, let $\mathbf{S}_t = (\mathbf{S}_{mt}, \mathbf{S}_{nt})$ and $\mathbf{K} = (\mathbf{K}_m, \mathbf{K}_n)$.

Alexander and Venkatramanan (2009a) show that the payoff to a basket option on N assets can be expressed in terms of compound exchange options on sub-basket options on m and n assets

respectively. That is,

$$V_{NT} = E_{1T} + E_{2T} = \begin{cases} [C_{mT} - P_{nT}]^+ + [C_{nT} - P_{mT}]^+ & if\ \chi = 1 \\ [C_{mT} - C_{nT}]^+ + [P_{nT} - P_{mT}]^+ & if\ \chi = -1 \end{cases}$$

where C_{kT} and P_{kT} are payoffs to call and put options on k assets.

The basket option price can then be computed as a sum of prices of two compound exchange options:

$$V_{Nt} = e^{-r(T-t)} \left(\mathbb{E}_{\mathbb{Q}} \{E_{1T}\} + \mathbb{E}_{\mathbb{Q}} \{E_{2T}\} \right).$$

By approximating the prices of the sub-baskets to follow lognormal processes we may be able to express the sub-basket option price and volatility processes as in equations (5) and (7) respectively. Then the prices of the two compound exchange options in the right hand side of above equation can be computed by solving exchange option PDEs as in equation (9).

For the compound exchange option on call options on m and n assets respectively, when $\chi = -1$, we replace $U_{1t} = C_{mt}$ and $U_{2t} = C_{nt}$, S_{it} and σ_i as the prices and volatilities of the two sub-baskets, in equations (5) and (6). The compound exchange option on put options is priced similarly but with $U_{1t} = P_{nt}$ and $U_{2t} = P_{mt}$. In both cases, the correlation ρ is the correlation between the two sub-basket prices with m and n assets.

This procedure is repeated recursively where the call and put sub-basket option prices C_{mt}, C_{nt}, P_{mt} and P_{nt} are in turn computed as a sum of compound exchange options. However, for a given sub-basket $\mathbf{b}_m$ or $\mathbf{b}_n$, we only need to compute either the call or put sub-basket option prices as the other can be found using put-call parity. This leads to a tree like structure where every node represents a basket option that involves solving two 3D PDEs and has two daughter nodes that represent sub-basket options. At the final iteration or terminal nodes, the basket size reduces to one and the option is just a plain vanilla option whose price can be computed by solving the Black-Scholes PDE.

As an example, consider the case of a spread option on two underlying assets. The payoff to a spread option of strike K is given by $[\omega(S_1 - S_2 - K)]^+$ where $\omega = 1$ for a call and $\omega = -1$ for a put. The risk neutral price of a European spread option may be expressed as the sum of risk neutral prices of two compounded exchange options. That is,

$$f_t = e^{-r(T-t)} \left(\mathbb{E}_{\mathbb{Q}} \left\{ [\omega [U_{1T} - U_{2T}]]^+ \middle| \mathcal{F}_t \right\} + \mathbb{E}_{\mathbb{Q}} \left\{ [\omega [V_{2T} - V_{1T}]]^+ \middle| \mathcal{F}_t \right\} \right), \tag{11}$$

where U_{1T}, V_{1T} are pay-offs to European call and put options on asset 1 with strike K_1 and U_{2T}, V_{2T} are pay-offs to European call and put options on asset 2 with strike K_2, respectively. The prices of the two compound exchange options are found by solving two PDEs as in equation (9) with $X_t = \frac{U_{1t}}{U_{2t}}$ and $\xi_{it} = \sigma_i \frac{S_{it}}{U_{it}} \frac{\partial U_{it}}{\partial S_{it}}$ in one PDE, and $X_t = \frac{V_{1t}}{V_{2t}}$ and $\xi_{it} = \sigma_i \frac{S_{it}}{V_{it}} \frac{\partial V_{it}}{\partial S_{it}}$ in the other.

Pricing spread options using this approach resolves the ambiguity in choosing the two underlying asset volatilities and yields prices and hedge ratios that are consistent with the implied volatility skew. Moreover, calibrating the model is simple and requires only a one dimensional solver.

3. Numerical Solution of PDE

In this section we describe an implicit scheme based on meshfree approach to solve PIDEs. The meshfree approach involves approximating the PIDE solution using a linear combination of certain basis functions, known as radial basis functions, over a scattered set of nodes in the problem domain. In the spacial domain, we use a collocation scheme to find the solution and we adopt an implicit Crank-Nicolson scheme to integrate in time.

Due to the absence of an underlying mesh, the meshfree approach can be easily scaled and adapted to changes in the geometry of the domain. This eliminates the time spent on building a mesh which could be substantial. Although the approximate solution is expressed in terms of a finite set of nodal values, the solution and its derivatives are uniquely defined over the entire computational domain. Moreover, the functional form of the solution also lets us derive explicit formulae for option Greeks.

Unlike the FDM and FEM, the meshfree method is a global method. The solution value at a given node depends on the value of the solution over the entire computational domain and not just on its value in the immediate neighbourhood of the node. As a result the accuracy and computational cost per degree of freedom is larger for the meshfree method. This property of meshfree approach makes it more suitable for path-dependent options like American and barrier options and even more attractive for solving PIDEs which involve a non-local integral term.

3.1. Meshfree Approach

We approximate the model solution with a linear combination of basis functions defined in the d-dimensional space $\mathbb{R}^d$. The unknown solution is evaluated over a distribution of nodes in the computational domain leading to a system of linear equations. We then obtain the solution by solving the system of linear equations. However, the problem is well-posed and a solution exists if and only if the coefficient matrix is non-singular. Therefore we restrict ourselves to strictly positive definite basis functions, such as radial basis functions, that are known to yield strictly positive definite system matrices which are always non-singular.

Let us define

$$f(\mathbf{x}, t) \approx f^h(\mathbf{x}, t) = \sum_{i=1}^{N} f_i(t)\varphi(r_i), \tag{12}$$

where $\mathbf{x} = (x_1, x_2, ..., x_d)$ denotes the coordinate of an arbitrary point i in the computational domain $D \subset \mathbb{R}^d$, f_i is the unknown nodal value at node i, $r_i = \|\mathbf{x} - \mathbf{x}_i\|_2$ is the Euclidian distance between $\mathbf{x}$ and $\mathbf{x}_i$, and φ is a radial basis function (RBF). Note that since φ only depends on the Euclidean distance, increasing the dimension of the problem just requires one to redefine the Euclidean distance in that dimension. Also, unlike FDM or FEM, the system matrices in a meshfree approach are dense. But they are much smaller in size as the basis functions are high-order global functions and hence the approximate solution requires less degrees of freedom.

Some commonly used RBFs that are globally supported functions and conditionally strictly positive definite are:

Finitely Smooth RBF		
Thin plate spline	: $r^{2k}\log r$	$k \in \mathbb{N}$
Power	: r^{β}	$\beta > 0,\ \beta \notin 2\mathbb{N}$
Infinitely Smooth RBF		
Gaussian	: $\varphi(r) = e^{-\varepsilon r^2}$	
Multiquadratic	: $\varphi(r) = (1+\varepsilon^2 r^2)^{\beta}$	$\beta > 0,\ \beta \notin \mathbb{N}$

where ε is a smoothing parameter that can be tuned based on the problem under consideration.

In this paper, we use the finitely smooth thin plate spline radial basis function with $k = 2$:

$$\varphi(r) = r^4 \log(r). \tag{13}$$

Note that the solution to equation (3) must be at least twice continuously differentiable and the above choice of RBF satisfies that.

Collocation method to compute the discrete solution

The unknown nodal values f_i are found by using a collocation method, which requires that the discrete solution f^h satisfies the model equation on each node $\mathbf{x}_i$. For the sake of simplicity, let us write the model equation as:

$$\frac{\partial f}{\partial t}(\mathbf{x}, t) = (\mathcal{D} + \mathcal{I})\, f(\mathbf{x}, t), \tag{14}$$

where $\mathcal{D}$ and $\mathcal{I}$ are differential and integral operators representing the right-hand side of the model equation. The collocation method then amounts to imposing that the discrete solution f^h satisfies the model equation (14) at every node i. Such a condition can be expressed by the following set of N equations:

$$\frac{\partial f^h}{\partial t}(\mathbf{x}_i, t) = (\mathcal{D} + \mathcal{I})\, f(\mathbf{x}_i, t) \quad \forall i = 1, \ldots, N \text{ and } \forall t. \tag{15}$$

These N equations will allow us to compute the N unknown coefficients f_i defining the discrete solution.

Collocation method is a special case of weighted residual methods such as minimum variance (MV) and Galerkin methods, with Dirac function as the weight function. This implies that the residual are minimised only at the nodal points while in the case of a MV or Galerkin method it is minimised over the entire domain. Larsson and Fornberg (2003) compare various schemes and show that RBF collocation is far superior in accuracy to standard second-order finite differences or even a standard Fourier-Chebyshev pseudospectral method. Also, the difficulty in implementing MV and Galerkin schemes in higher dimensions makes collocation schemes a preferred alternative.

The collocation method is quite susceptible to the distribution of the nodes in the domain and the solution might vary with the choice of nodes. Therefore, in order to reduce the total residual of the approximate solution the nodes should be chosen carefully. An easy alternative to minimise the variance of the solution is to randomly scatter the nodal points according to certain distributions (see Fasshauer (2007)). In this paper we use uniform distribution but other commonly used distributions include Halton, Sobol, Chebyshev, latin hypercube and normal distributions.

The PIDE given by equation (3) involves a non-local integral term over $\mathbb{R}$. But, this can be evaluated only within the localised computational domain of the grid as the solution is defined only within this region. Cont and Voltchkova (2005) give estimates for the truncation error due to this and show that it decays exponentially as the bounds increase. When compared to FDM and FEM, the meshfree method is intrinsically a global method and hence is more suitable and efficient for problems with non-local terms for two main reasons. Firstly, since the FDM or FEM solution is defined only at discrete points, the numerical integration scheme is forced to evaluate the integrand only at the discrete points defined on the grid. Therefore, if the spacial step size is not small enough, the computation error could be very high. On the other hand, making the grid finer will drastically increase the overall computation time. However, in the meshfree approach, the approximate solution is continuously defined over the computation domain and the integral can be evaluated with greater accuracy. Secondly, employing an implicit time stepping scheme for the integral term, in a FDM or FEM, will result in a large, dense mass matrix instead of a sparse matrix. Therefore, one is forced to handle the integral term explicitly in which case the time step has to be sufficiently small to ensure stability and accuracy. In the meshfree method, this does not add to affect the system matrix and the integral term can be handled implicitly.

Eq. (15) requires to evaluate the partial derivatives and integrals of the option price. The former have the following expression:

$$\begin{aligned}
\frac{\partial f^h}{\partial t}(\mathbf{x}_i,t) &= \sum_{j=1}^{N} \frac{\mathrm{d}f_j}{\mathrm{d}t}(t)\varphi(r_{ij}),\\
\frac{\partial f^h}{\partial x_{ik}}(\mathbf{x}_i,t) &= \sum_{j=1}^{N} f_j(t)\frac{\mathrm{d}\varphi}{\mathrm{d}r}(r_{ij})\frac{\partial r_j}{\partial x_{ik}}(\mathbf{x}_i)\\
&= \sum_{j=1}^{N} f_j(t) r_{ij}^2(4\log r_{ij}+1)(x_{ik}-x_{jk}), \qquad (16)\\
\frac{\partial^2 f^h}{\partial x_{ik}^2}(\mathbf{x}_i,t) &= \sum_{j=1}^{N} f_j(t)\left[\frac{(x_{ik}-x_{jk})^2}{r_{ij}^2}\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2}(r_{ij}) + \frac{\mathrm{d}\varphi}{\mathrm{d}r}(r_{ij})\frac{\partial^2 r_j}{\partial x_{ik}^2}(\mathbf{x}_i)\right]\\
&= \sum_{j=1}^{N} f_j(t)\left[(x_{ik}-x_{jk})^2(12\log r_{ij}+7) + \left(r_{ij}^2-(x_{ik}-x_{jk})^2\right)(4\log r_{ij}+1)\right],
\end{aligned}$$

where $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|_2$ and $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{id})^T$.

Let $\mathcal{M} \subset \mathbb{R}^d$ be the localised computational domain and $M^l, M^u \in \mathbb{R}$ such that $\mathcal{M} = [\lambda_k M^l, \lambda_k M^u]^d$, for $1 \le k \le d$. Let $B_i^l = \min\limits_{1\le k\le d}\left\{M^l - \frac{1}{\lambda_k}x_{ik}\right\}$ and $B_i^u = \min\limits_{1\le k\le d}\left\{M^u - \frac{1}{\lambda_k}x_{ik}\right\}$. Then the terms appearing the

integral part are given by:

$$\begin{aligned}
\mathcal{I}f^h(\mathbf{x}_i) &= \int_{\mathcal{M}} \left(f^h(\mathbf{x}_i + \mathbf{\Lambda} z) - f^h(\mathbf{x}_i) - \sum_{k=1}^{d} (e^{\lambda_k z} - 1) \frac{\partial f^h}{\partial x_{ik}}(\mathbf{x}_i) \right) \nu(dz) \\
&= \sum_{j=1}^{N} f_j \left(\int_{B_i^l}^{B_j^u} \varphi(\tilde{r}_{ij}) \nu(dz) - \int_{M^l}^{M^u} \varphi(r_{ij}) \nu(dz) - \int_{M^l}^{M^u} \sum_{k=1}^{d} (x_{ik} - x_{jk})(e^{\lambda_k z} - 1) r_{ij}^2 (4 \log r_{ij} + 1) \nu(dz) \right) \\
&= \sum_{j=1}^{N} f_j \left(r_{ij}^2 (4 \log r_{ij} + 1) \sum_{k=1}^{d} (x_{ik} - x_{jk}) - \varphi(r_{ij}) \right) \int_{M^l}^{M^u} \nu(dz) \\
&\qquad - \sum_{j=1}^{N} f_j r_{ij}^2 (4 \log r_{ij} + 1) \sum_{k=1}^{d} \left((x_{ik} - x_{jk}) \int_{M^l}^{M^u} e^{\lambda_k z} \nu(dz) \right) + \sum_{j=1}^{N} f_j \int_{B_i^l}^{B_j^u} \varphi(\tilde{r}_{ij}) \nu(dz)
\end{aligned}$$

where $\tilde{r}_{ij}$ represents the distance between $\mathbf{x}_i + \bar{\alpha} z$ and $\mathbf{x}_j$. This also shows that the integral does not depend on the distribution of the nodes (step size) as it did in the case of FDM.

The set of equations can then be expressed in matrix form as:

$$\mathbf{M} \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{f} = (\mathbf{D} + \mathbf{I})\, \mathbf{f} \tag{17}$$

where $\mathbf{M}_{ij} = \varphi(\|\mathbf{x}_i - \mathbf{x}_j\|)$ and $\mathbf{f} = (f_1, f_2, \ldots, f_N)'$.

$\mathbf{D}$ and $\mathbf{I}$ are the discrete operators corresponding to $\mathcal{D}$ and $\mathcal{I}$, respectively. Since φ depends only on the Euclidean distance between a pair of nodes, it is radially symmetric about a point and therefore $\mathbf{M}$ is symmetric.

The initial conditions required to integrate Eq. (19) in time are also found by using a collocation principle. If $f^h(\mathbf{x}, 0) = g(\mathbf{x})$, then by collocating that equation we obtain:

$$f^h(\mathbf{x}_i, 0) = g(\mathbf{x}_i) \quad \forall i = 1, \ldots, N$$

which is equivalent to

$$\mathbf{M}_{ij} \mathbf{f}_j(0) = g(\mathbf{x}_i).$$

Time integration

We implement an implicit Crank-Nicolson scheme to integrate equation (17) in time. This is given by

$$\mathbf{M} \frac{1}{\Delta t} \left(\mathbf{f}^{n+1} - \mathbf{f}^n \right) = (\mathbf{D} + \mathbf{I})\, \frac{1}{2} \left(\mathbf{f}^{n+1} + \mathbf{f}^n \right) \tag{18}$$

This can be rewritten as

$$\mathbf{A} \mathbf{f}^{n+1} = \mathbf{B} \mathbf{f}^n,$$

where the final system and conversion matrices are given by

$$\begin{aligned}
\mathbf{A} &= \mathbf{M} - \frac{1}{2} \Delta t\, (\mathbf{D} + \mathbf{I}) \\
\mathbf{B} &= \mathbf{M} + \frac{1}{2} \Delta t\, (\mathbf{D} + \mathbf{I})
\end{aligned}$$

Such a scheme is second order accurate and unconditionally stable. Note that since all the matrices obtained with the meshfree method are dense, using an implicit time integration scheme does not add to any computational overhead.

3.2 Stability and Convergence

In order to ensure stability of the scheme, we could choose a RBF such that the system matrix is properly conditioned. In the case of Gaussian or multiquadratic RBFs, the condition number can be improved against the rate of convergence by choosing an optimal value for the parameter ε but this is not possible for thin plate spline RBFs. Nevertheless, for a given RBF, we can check the stability of the scheme by deriving bounds for the condition number of the system matrix. For positive definite matrices, the condition number can be computed as a ratio of the largest and the smallest eigenvalues. Since a system matrix is always positive definite, we can find the upper bound for the condition number by evaluating the bounds for the eigenvalues.[8]

We prove the convergence of the meshfree method applied on PIDEs such as (3), by showing that it converges for both, the differential and integral parts. Following Schaback and Wendland (1999) and Iske (2003), we can derive a bound for the error between the discrete and exact solution under a meshfree scheme to solve a general second order linear PDE. They show that

$$|f(\mathbf{x},t) - f^h(\mathbf{x},t)| \leq Ch^k \|f(\mathbf{x},t)\|, \tag{19}$$

where $h = \sup_{x\in\mathcal{M}} \min_{1\leq j\leq N} \|\mathbf{x} - \mathbf{x}_j\|$ is the so-called local fill distance at position $\mathbf{x}$, which is a measure of the nodes density around $\mathbf{x}$ and $\|.\|$ is a suitable norm.

For RBF methods, we have a more general result on the bound for the error between the differentials of approximate and exact solutions:

$$|D^m f(\mathbf{x}) - D^m f^h(\mathbf{x})| \leq P_{X,\varphi}(\mathbf{x})^m \|f\|, \quad m \in \mathbb{N}, \tag{20}$$

where $D^m f$ represents the m^{th} order derivative of f and $P_{X,\varphi}(\mathbf{x})$ is the so called power function. The power function can be bounded above by:

$$P_{X,\varphi}(\mathbf{x})^m \|f\| \leq CF(h)$$

for some constant C and function F that depends on the chosen RBF. For instance, in the case of thin plate spline, $F(h) = h^{k-m}$ (for other RBFs see Iske (2003) and references therein).

Now consider, the integral term in equation (3):

$$\mathcal{I} f^h(\mathbf{x}_i) = \int_{\mathcal{M}} \left(f^h(\mathbf{x}_i + \mathbf{\Lambda} z) - f^h(\mathbf{x}_i) - \sum_{k=1}^{d} (e^{\lambda_k z} - 1) \frac{\partial f^h}{\partial x_{ik}}(\mathbf{x}_i) \right) \nu(dz). \tag{21}$$

We can prove that the meshfree scheme is consistent for solving the integral equation if $|\mathcal{I} f(\mathbf{x}) -$

[8] See Ball et al. (1992) and Narcowich et al. (1994) for bounds on condition numbers that is based on the results of Ball (1992) on eigenvalues of distance matrices.

$\mathcal{I}f^h(\mathbf{x})| \to 0$ as $h \to 0$. Since $\nu(dz) < \infty$, equations (19) and (20) imply

$$\begin{aligned}
|\mathcal{I}f(\mathbf{x}) - \mathcal{I}f^h(\mathbf{x})| &\leq \int_{B_i^l}^{B_j^u} \left|f(\mathbf{x}_i + \mathbf{\Lambda}z) - f^h(\mathbf{x}_i + \mathbf{\Lambda}z)\right| \nu(dz) \\
&\quad - \left|f(\mathbf{x}_i)) - f^h(\mathbf{x}_i)\right| \int_{M^l}^{M^u} \nu(dz) - \left|\frac{\partial f}{\partial x_{ik}}(\mathbf{x}_i) - \frac{\partial f^h}{\partial x_{ik}}(\mathbf{x}_i)\right| \sum_{k=1}^{d} \int_{M^l}^{M^u} (e^{\lambda_k z} - 1)\nu(dz) \\
&\leq C_1 h^k \|f\| - C_2 h^k \|f\| - C_3 h^{k-1} \|f\| \\
&\leq C h^{k-1} \|f\|
\end{aligned}$$

for some constants C, C_1, C_2 and C_3. This shows that the presence of the integral term in a PIDE reduces the overall spacial order of convergence from k to $k-1$. Nevertheless, we can improve the order of convergence by choosing higher order radial basis functions. This is another advantage of using a meshfree scheme which is not possible in the case of FDM or FEM.

4. Empirical Results

We implement the dimension reduction technique for the GBM framework using a meshfree approach in order to price Crude oil - Gasoline crack spread options. The crack spread options were traded on NYMEX in March 2006. The spread options were of one month maturity and were on the prompt futures on crude oil and Gasoline.

In the CEO approach the price of a spread option given by equation (11) is computed as a sum of prices of two compound exchange options - one CEO to exchange a call on Gasoline for a call on crude oil and another to exchange a put on crude oil for a put on Gasoline. Hence, in order to price a spread option we need to solve two 3D price PDEs as in (9), one for each CEO price. We use the thin-plate spline RBF with $k = 2$ to solve these price PDEs. In all the figures, we use the following notation: $X = U_1/U_2$, U_{it}, V_{it} and ξ_{it}, η_{it} are prices and volatilities of single asset call and put options respectively; the correlation was set to 0.8.

Since, U_i, ξ_i, V_i and η_i are uniquely determined by their option strikes K_i respectively, and $K_1 - K_2 = K$, we choose K_1 as the only free parameter. Therefore, calibration just involves finding the strike K_1 for which the final solutions of the two PDEs evaluated at $\left(\frac{U_1}{U_2}, \xi_1, \xi_2\right)$ and $\left(\frac{V_1}{V_2}, \eta_1, \eta_2\right)$, yield option price that matches the market price of the spread options.

Table 1: *Computation time and condition numbers*

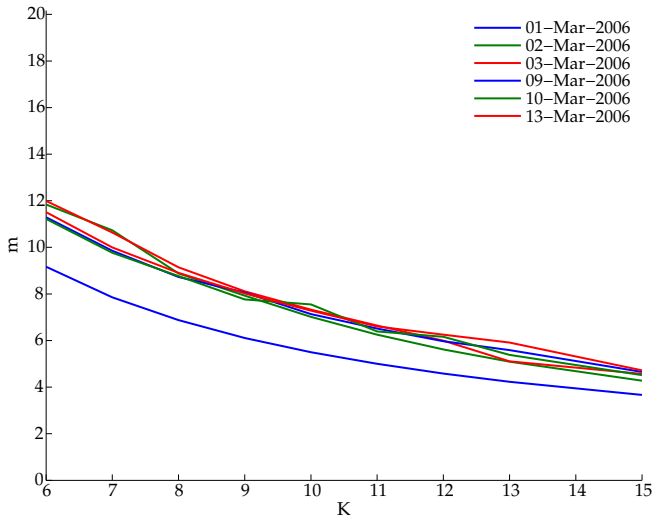
No. of nodes	3	6	9	12
Generating nodes distribution	0.0245	0.0080	0.0235	0.0274
Initialisation	0.458	1.409	1.634	3.078
Building the system matrix	0.0268	0.1297	1.748	9.606
Integration in time	1.96	2.88	12.78	81.27
Condition number of system matrix	4058	1.51E+06	2.92E+07	2.27E+08

Since the true solution is uniquely defined over the entire computational domain the Greeks may be evaluated continuously over the same. For instance, the delta of a spread option with respect to asset k in the CEO approach is given by:

$$\frac{\partial f}{\partial S_k} \approx \frac{\partial f^h}{\partial S_k} = \sum_{i=1}^{N} f_i r_i^2 (4 \log r_i + 1) X \frac{\partial X}{\partial U_1} \frac{\partial U_1}{\partial S_k}, \; k = 1, 2.$$

Analytic expressions for other model Greeks can be easily derived from equations (16) by applying chain rule to differentiate.

FIGURE 1: *Calibrated parameter* $m = \frac{K_1}{K}$ *against spread option strike*



In section 3.2 we showed that, for solving PIDEs, the scheme is unconditionally stable, second order convergent in time and have high order of convergence in space. Although the presence of an integral term may alter the condition numbers of the system matrix, we can ensure that the convergence and stability are not compromised by choosing a suitable high-order RBF. Therefore, we suppose that the results on convergence and accuracy for solving PIDEs will be similar to the ones presented in this section for solving PDEs.

Table 1 shows the computation times of different tasks and condition numbers of matrices for a number of nodes. The most expensive step is that of solving a dense system of linear equations at every time step. This seems to increase exponentially as the number of nodes increases.[9]

Figure 1 plots the calibrated strike parameter $m = K_1/K$ of the compound exchange option approach against the spread option strike over various trading dates. In order to justify approximating $c = \sigma_i S_{it} \Gamma_{it} / \Delta_{U_{it}}$ as constant in equation (7), we checked the sensitivity of the solution to c and found it to be insignificant. A possible reason for this could be that the calibrated strikes of the single asset options, in figure 1 are such that the call options are ITM and the put options are

[9]The calculations were run on a standard Dell Inspiron laptop with a 1.8 GHz processor and 2GB RAM. The authors would be happy to provide other results upon request.

OTM. Then $c << (\xi_i - \sigma_i)$ and its effect on the solution will be insignificant as observed. The strike parameter m also shows reasonable stability over consecutive trading days allowing one to choose suitable starting values to speed up calibration.

FIGURE 2: *Convergence of solution*

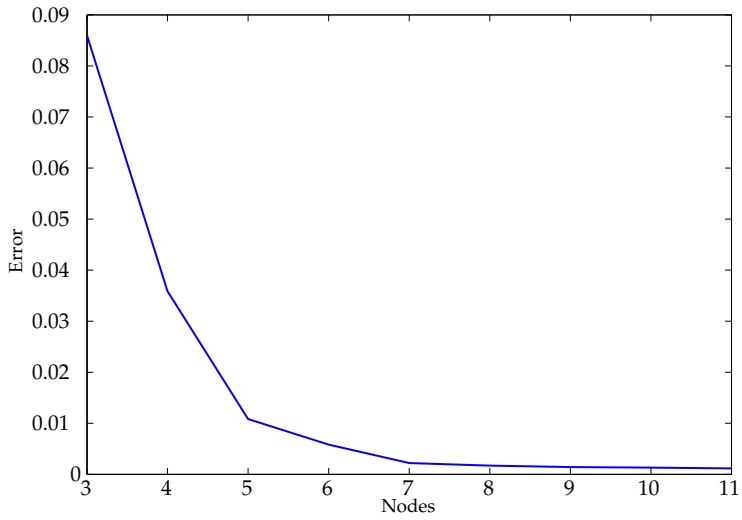


Figure 2 shows the rate of convergence of the meshfree solution as the number of nodes increases. The convergence result was obtained by using the calibrated strikes values of Gasoline and crude oil options from $N = 12$ to compute the model price for different values of N. It is easy to observe that the rate of convergence is at least quadratic as expected. Since accurate solutions can be computed even with a few number of nodes, the scheme can be scaled to higher dimensions by keeping the number of nodes to a minimum. This allows for solving high dimensional problems even on a PC with limited memory and computation resources.

FIGURE 3: *Solution error with respect to Gasoline strike* K_1

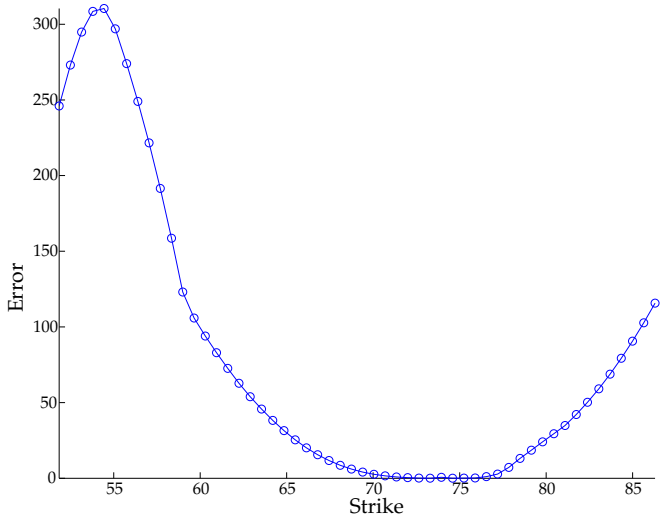


Figure 3 plots the squared model error as a function of strike parameter m for a spread option with

FIGURE 4: *Initial solution : $N = 3$*

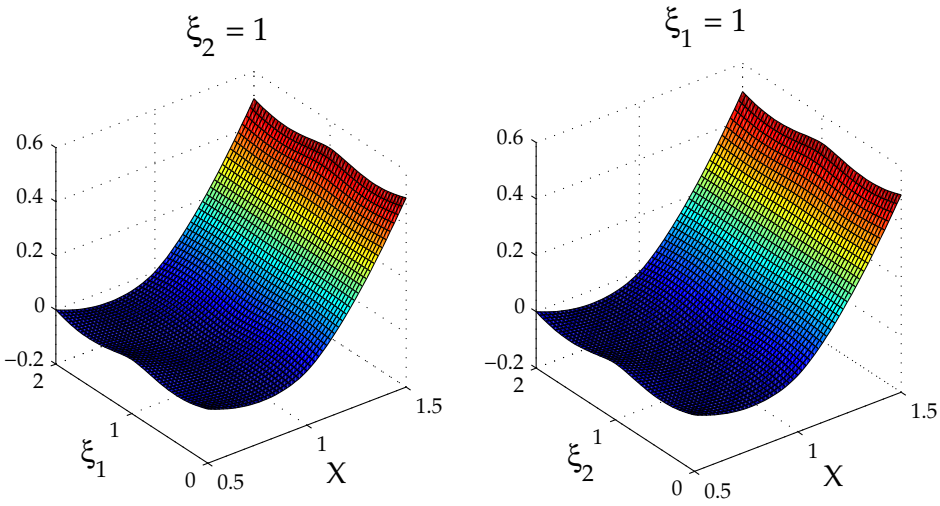


FIGURE 5: *Initial solution : $N = 12$*

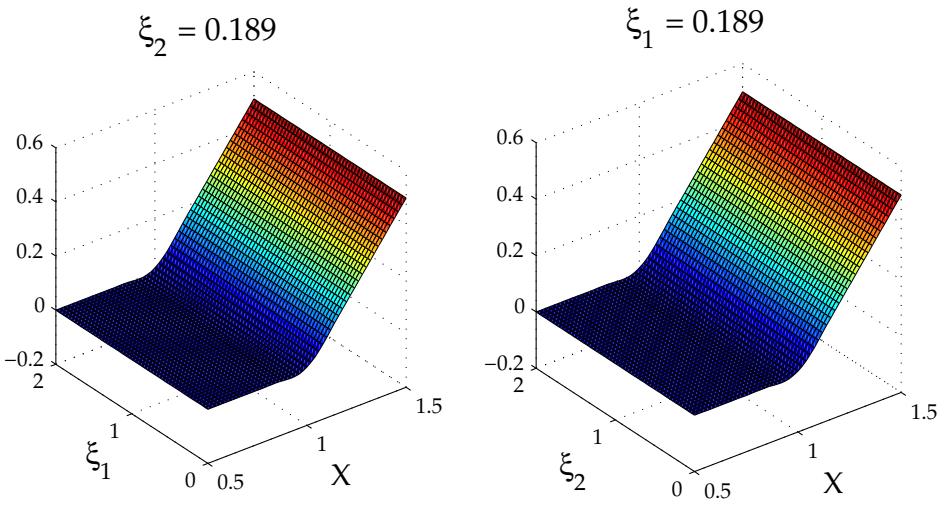
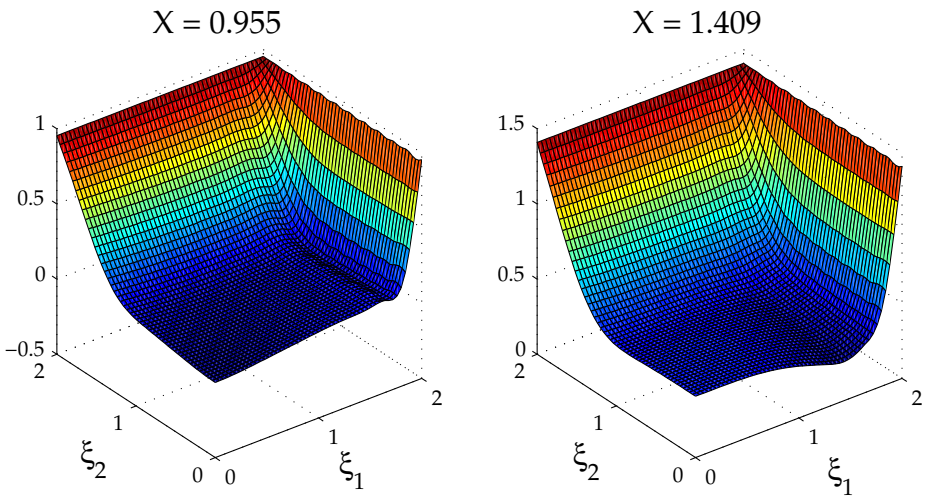


FIGURE 6: *Final solution : $N = 12$*



$K = 13$. Calibration basically involves finding the right value of single asset option strike K_1 (or K_2) for which this error is a minimum. The plot shows that the value of m at which the error is minimum can be uniquely found. By choosing the implied volatilities σ_i of the single asset options corresponding to these strike values, we resolve the ambiguity in choosing the volatilities of the underlying assets. Thus we obtain volatility skew consistent spread option price and hedge ratios making calibration more meaningful.

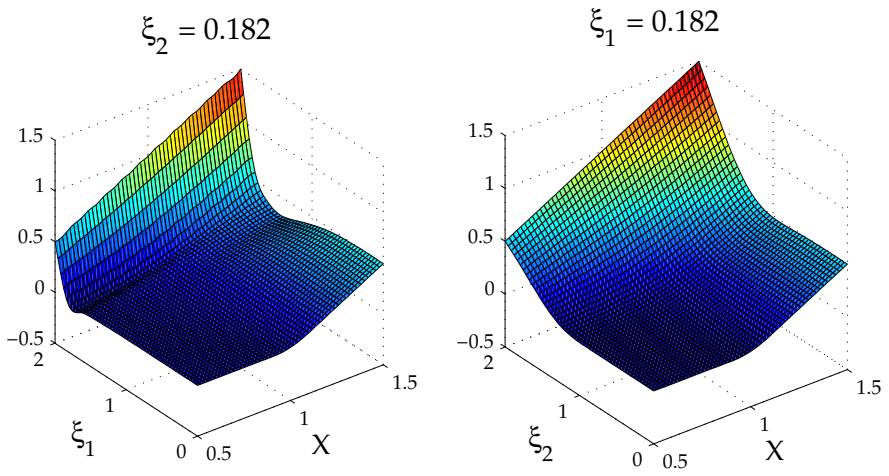
Figures 4 and 5 show the initial solution of the CEO on calls for $N = 3, 12$, across a cross section of ξ_1 and ξ_2. Recall that the meshfree approach involves approximating the solution using radial basis functions that are at least finitely differentiable throughout the domain. Therefore in order to approximate functions with singular points, the nodal points have to be carefully distributed. In figures 4 and 5, the initial solution is not differentiable at $X = 1$. When $N = 3$ we can see that the initial solution is very different to a typical payoff curve. Whereas when the N is increased to 12, the initial solution fits the payoff well even around $X = 1$. Since any interpolation scheme is prone to have oscillations around the nodes, the number of nodes and their distribution have to be suitably chosen. For instance, when $N = 3$ such oscillations are present around the nodes along ξ_1 and ξ_2 axes.

Figures 6 and 7 show the final solution for a CEO on calls across different cross sections. When the option volatilities range between 0.5 and 1.5, the CEO price is more sensitive to the volatility of crude oil than to Gasoline but for higher values of volatilities the price behaves otherwise.

5. SUMMARY AND CONCLUSIONS

We discussed the pricing of multi-asset options when the underlying asset prices followed jump-diffusion or geometric Brownian motion processes. When the underlying assets followed geometric Brownian motion processes we employed the compound exchange option approach to

FIGURE 7: *Final solution : $N = 12$*



decompose a multi dimensional problem into multiple 3D problems. This approach yields accurate prices while substantially reducing the computational time and memory requirements. In the jump-diffusion case, we outline the derivation of the multi-dimensional price PIDE.

The multi-asset option prices are computed by solving the associated partial (integro) differential equation using a fully implicit meshfree method with thin-plate spline radial basis functions. The meshfree method has several advantages compared to the existing schemes such as finite differences and finite elements, particularly for solving PIDEs. It is easily scalable to higher dimensions and flexible to changes in payoff profiles or geometry of the domain. We can also derive explicit analytic formulae to various option Greeks.

In this paper we applied a collocation method in space and Crank-Nicolson method in time to find the solution of the multi-dimensional price PIDE. We showed that our method is unconditionally stable and has high order comvergence in both space and time. We tested the meshfree scheme with compound exchange option approach on crack spread option data obtained from NYMEX. We provided empirical results to suggest that our approach has considerable benefits when compared to existing finite difference or finite element methods.

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