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OPTIMAL FOREST TAXATION UNDER
PRIVATE AND SOCIAL AMENITY
VALUATION

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Abstract

This paper analyzes socially optimal forest taxation when the government has a binding tax revenue requirement. In the Faustmann model the optimal design of forest taxation consists of non-distortionary taxes, such as site productivity tax, site value tax or profit tax. A combination of distortionary unit (or yield) tax and timber tax can also be used to collect the tax revenue in a non-distortionary way. In the Hartman model with amenity services as a public good, the optimal design consists of a non-distortionary tax and a Pigouvian tax, which adjusts the private rotation age to the socially optimal one. Now only the site productivity tax is non-distortionary, while unit, yield, timber, site value and profit taxes generally serve as a corrective Pigouvian taxes. In the absence of a non-distortionary tax, a combination of unit (or yield) and timber taxes can often be used to both tax revenue collection and Pigouvian correction.

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1. Introduction

Forest taxation has always been a central issue in forest economics – even to the extent that it gave rise to Martin Faustmann's celebrated contribution 1849. Since the renessance of the rotation analysis in forest economics started by Samuelson's seminal review (1976), the effects of alternative forest taxes on the Faustmann rotation age have been extensively studied (see e.g. Chang 1982, Johansson and Löfgren 1985). Respective comparative static analysis of foret taxation within the Hartman model, which includes amenity services of forests, has been solved rigorously in Koskela and Ollikainen 2000.

In contrast to the frequent discussion of forest taxation issues since the beginning of forest economics, the analysis of the socially optimal design of forest taxation is largely an unexplored area. There are two notable attempts to tackle this issue, namely, Gamponia and Mendelsohn (1988) and Englin and Klan (1990).[1] Gamponia and Mendelsohn (1988) focus on the excess burden of yield and timber taxes in the absence of externalities (amenity services) in forestry.[2] They recognize the difficulties in finding analytical solution for the excess burden of forest taxation and concentrate on numerical simulations. They end up to stress the use of neutral forest taxes, or combinations of distortionary forest taxes, which minimize the deadweight loss of forest taxation.

Englin and Klan (1990) study optimal forest taxation policy in the absence of a binding tax revenue requirement and in the very special case, where amenity services of forest stands are a public good, but forest owners value only harvest revenue, not amenity services. Now private harvesting reduces amenities available and causes a negative externality to recreators, so that neutral forest taxes are no longer desirable. What is needed, are distortionary taxes, which shift the market behavior towards the social optimum. To this end Englin and Klan (1990) solve optimal Pigouvian tax rates, which equate the privately optimal rotation age with the socially optimal rotation.

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[1] A closely related literature is also a discussion about the forest taxation in an economy with an ordinary and an Austrian sector with the focus is on how to design distortionary taxation in an economy with an ordinary and an Austrian sector so as to achieve intrasector and intersectoral efficiency in the absence of government budget constraint (see Kovenock and Rothschild 1983 and Kovenock 1986). This literature this is, however, not relevant for our case, because we focus solely on forestry.

[2] Here the excess burden refers to the magnitude of the rotation age distortions caused by forest taxes.

Both these analyses neglect two factors which seem to be highly relevant for the optimal design of forest taxation. First, they neglect the fact that usually the governmental forest tax policy is not chosen freely, because the government has to collect tax revenue from forestry for financing the national budget. Second, empirical evidence suggests that private landowners do value amenity services (see e.g. Binkley 1981, Kuuluvainen et al. 1996). Therefore, one should ask a couple of questions. How does this behavioral feature affect the optimal forest taxation, i.e., will the need for Pigouvian taxes vanish in favor of neutral forest taxation in the true Hartman framework? Does the tax revenue requirement modify the optimal taxation in the Faustmann and Hartman models, respectively.

These two issues are in the focus of our paper. We assume that the government has to collect an exogenously given tax revenue from forestry and it maximizes welfare from forestry in a partial equilibrium setting with an exogenous timber price. The social welfare function depends on the welfare of private landowners, as well as on those by the citizens, who might have free access to the amenity services of private forest stand. We consider optimal taxation first in the Faustmann framework by assuming that the only thing the society values is the net present value of harvest revenue from timber production. Then we allow for the joint production of timber and amenities, and study optimal forest taxation in the Hartman framework augmented with recreators who enjoy amenity services from private forests.

Given that the chosen forest tax forms vary from country to country, we consider a broad set of forest taxes. Our taxes include the most popular forms of property and harvest taxes, as well as the profit tax. More specifically, the class of (i) *property taxes* levied on land value contains three alternative taxes. *The site productivity tax* is paid annually and is based on the yield potentiality of a given site irrespective of the actual harvests or standing timber. *The site value tax* is a proportional tax on the land value and paid annually. *A* property tax may be also levied on the value of trees, and is often called *timber tax*. Second class of forest taxes consists of (ii) *harvest taxes*. The most common version of harvest taxes is *the yield tax*, which is levied on the harvest revenue. Alternatively, *a unit tax* levied on the timber volume harvested can be used. Third, we also study (iii) *a profit tax* levied on the net timber revenue the landowner gets from the forests.

We show that in the Faustmann framework optimal forest taxation consists of neutral forest taxes. Among the class of forest taxes under our study, site value tax, site productivity tax and profit tax have this property. A combination of harvest and timber tax can also be used to collect the tax revenue in a non-distortionary way. Allowing for amenity services (Hartman model) changes the optimal design of tax policy. In most cases optimal design consists of a combination of a neutral tax and a distortionary tax. While the neutral tax helps to collect the tax revenue, the distortionary one adjusts the amenity production to the socially optimal level. Now only the site productivity tax is always non-distortionary, while the site value and profit taxes are neutral only in a special case. Again, in the absence of a non-distortionary tax a combination of a unit (yield) and timber tax can under certain conditions be used to both tax revenue collection and Pigouvian correction of externality.

The rest of the paper is organized as follows. In section 2 we develop the basic framework, while sections 3 and 4 provide the analysis of optimal taxation in the Faustmann and in the Hartman framework, respectively. Finally, there is a brief concluding section 5.

2. Basic Framework

In designing the socially optimal forest taxation we follow conventional public economics approach by modeling the interaction between the government and the representative landowner as a two-stage game. In the first stage the government, acting as a Stackelberg leader, decides about its forest taxation policy and commits to it. In the second stage private landowners choose their harvesting conditional on the chosen tax policy. The structure of this section follows this idea by applying the backward induction. Hence, we analyze first in section 2.1 how the chosen forms of forest taxes affect the privately optimal rotation age to obtain the landowners' reaction function. In section 2.2 we introduce the social welfare function, and the government budget constraint, which are then used to derive the design of socially optimal forest taxation in the subsequent chapters 3 and 4.

2.1 Behavior of the Representative Landowner

Given that the landowners' objectives may differ, we will focus on two basic rotation frameworks, the Faustmann and the Hartman models. The comparative statics of forest taxes in the Faustmann model are already well-known (Chang 1982, and Johansson and Löfgren 1985), while that of the Hartman model is derived in Koskela and Ollikainen 2000. In what follows we present briefly results needed for the analysis of optimal taxation, and ask the interested reader to consult above references for exact derivations. We analyze the Hartman model, and then give the results for the Faustmann model as the special case.

Assume that the private landowner values both the net harvest revenue and the amenity services from forest stands. Based on Hartman (1976) we postulate the following quasi-linear objective function in the absence of taxes

$$W = V + E. \tag{1}$$

In equation (1) the first term is the Faustmann part, defined in the absence of taxes as $V = \dfrac{pf(T) - ce^{rT}}{e^{rT} - 1}$, where p is stumpage price, $f(T)$ is the growth of the stand as a function of its age T with the conventional convex-concave properties and c denotes the regeneration cost. The second term, $E = \dfrac{e^{rT} \int_0^T F(s)e^{-rs}ds}{e^{rT} - 1}$, describes the present value of amenity services over all rotations, where and $F(s)$ is the flow of amenities for the stand of age s.

The first-order condition for the maximization of (1) is $W_T = V_T + E_T = 0$, and it can be rewritten as

$$W_T = pf'(T) - rpf(T) - rV + F(T) - rE = 0 .^3 \tag{2}$$

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[3] We denote the partial derivatives by primes for functions with one argument and by subscripts for functions with many arguments. Hence, e.g. $f'(T) = \frac{\partial f(T)}{\partial T}$ for $f(T)$, while

According to equation (2), the private landowner equates the marginal benefit of delaying the harvest to age T, defined by $pf'(T) + F(T)$, to the marginal opportunity cost of delaying the harvest, defined by $rpf(T) + r(V + E)$. Equation (2) reveals that the relation of the Hartman rotation age to the Faustmann rotation age depends on the sign of $F(T) - rE$. It can be shown that

$$F(T) - rE \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \ \ as \ \ F'(T) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 .^4 \tag{3}$$

Hence, the Faustmann and Hartman rotations coincide if $F'(T) = 0$. But when the amenity valuation increases with the age of the forest stand, the Hartman rotation period is longer than the Faustmann rotation period and vice versa for the decreasing marginal amenity valuation.

Next we turn to study the effects of forest taxes in the Harman framework, and as a special case the Faustmann framework. All forest taxes affect only the site value, i.e., the Faustmann part V, but depending on the nature of the amenity valuation function they may change the relative profitability of timber vis'a'vis amenity production.

- **Harvest taxes**

If the government levies the yield (τ) or the unit tax (t) on harvesting, the after-tax net revenue from harvesting is defined by equation (4), while the amenity part, E, remains unchanged so that

$$\hat{V} = \frac{\hat{p}f(T) - ce^{rT}}{e^{rT} - 1}, \tag{4}$$

$A_x(x, y) = \frac{\partial A(x,y)}{\partial x}$ for $A(x, y)$, etc.

[4] See Bowes and Krutilla (1985), Johansson and Löfgren (1988), and Koskela and Ollikainen (2000).

where $\hat{p} \equiv p(1-\boldsymbol{t})-t$ is the after-tax stumpage price and $\hat{V}$ the after-tax present value of the soil.

Differentiating equation (1) in the presence of harvest taxes (4) with respect to T yields

$$\hat{W}_T(\boldsymbol{t},t) = \hat{p}(f'(T)-rf(T)) - r\hat{V} + F(T) - rE = 0 . \tag{5}$$

The impact of harvest taxes on the private rotation age can be shown to be

$$T_{\boldsymbol{t}}^H, T_t^H \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \quad as \quad rc(1-e^{-rT})^{-1} + F(T) - rE \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 .^5 \tag{6}$$

In the Faustmann model, where $F(T) - rE = 0$, both harvest taxes lengthen the private rotation age since they affect like a decrease in the net stumpage price, while in the Hartman model the same holds true when the marginal valuation of amenities is increasing or constant in the age of the stand, i.e. when $F'(T) \geq 0$. Under the assumption $F'(T) < 0$, however, it is possible that the rotation age is shortened by the harvest taxes.

Property taxes

As for the property taxes we explore first the site value tax (levied directly on the value of forestland), and the site productivity tax (levied indirectly on administratively set site quality classification values of forestland). For the site value tax, we denote the annual tax payment by b and define its present value as

$$\int_0^\infty be^{-rs} ds = \frac{b}{r} . \tag{7}$$

If the fraction of the value of the forestland delivered in taxes is $\boldsymbol{b}$, we have from equation (7) $b = r\boldsymbol{b}V$, so that the after-tax value of the forestland can be rewritten

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[5] For a proof, see Koskela and Ollikainen (2000). The notation T^F and T^H refers to the rotation age in the Faustmann and in the Hartman models, respectively.

$\hat{V}(\boldsymbol{b}) = (1 - \boldsymbol{b})V$. We can express the landowner's objective function for the site value tax as

$$\hat{W}(\boldsymbol{b}) = (1 - \boldsymbol{b})V + E , \tag{8}$$

The first-order condition for the maximization of (8) is given by

$$\hat{W}_T(\boldsymbol{b}) = (1 - \boldsymbol{b})(pf'(T) - rpf(T) - rV) + F(T) - rE = 0 . \tag{9}$$

Differentiating (9) with respect to β gives

$$T_{\boldsymbol{b}}^{H} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \ \ as \ \ F'(T) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 . \tag{10}$$

According to equation (10) the site value tax has no effect on the Faustmann rotation age, where $F'(T) = 0$ by definition. In the Hartman model a rise in the site value tax makes amenity production relatively more (less) profitable, so that the landowner lengthens (shortens) the rotation age when $F'(T) > (<) \ 0$.

In the presence of the site productivity tax, denoted by a(i) where i refers to site index i of the land, the after-tax land value is given by $\hat{V}(a(i)) = V - \dfrac{a(i)}{r}$, so that the objective function can be written as

$$\hat{W}(a(i)) = V - \frac{a(i)}{r} + E . \tag{11}$$

where a(i)/r is the present value of the site productivity tax. The first-order condition is $\hat{W}_T(a(i)) = V_T + F(T) - rE = 0$. The site productivity tax is neutral, because it does not distort the relative profitability of timber and amenity production. This holds also for the Faustmann model.

The timber tax, α, is levied annually on the stumpage value of growing timber volume so that the net present value of harvest revenue is given by

$$\hat{V}(\boldsymbol{a}) = \frac{pf(T) - ce^{rT} - \boldsymbol{a}e^{rT}\int_0^T pf(s)e^{-rs}ds}{e^{rT} - 1}. \tag{12}$$

If the present value of annual timber earnings is denoted by $U = \dfrac{e^{rT}\int_0^T pf(s)e^{-rs}ds}{e^{rT} - 1}$, then

the objective function of the landowner can be written as

$$\hat{W}(\boldsymbol{a}) = V - \boldsymbol{a}U + E, \tag{13}$$

The first-order condition for the privately optimal rotation age is

$$\hat{W}_T(\boldsymbol{a}) = pf'(T) - rpf(T) - rV - \boldsymbol{a}(pf(T) - rU) + F(T) - rE = 0, \tag{14}$$

where $pf(T) - rU > 0$ when $f'(T) > 0$ (see Koskela and Ollikainen 2000). It is straightforward to show that timber tax shortens private rotation age irrespective of the sign of $F(T) - rE$, i.e. both $T_{\boldsymbol{a}}^F < 0$ and $T_{\boldsymbol{a}}^H < 0$. Timber tax decreases both the value of standing timber at the harvest time and the opportunity cost of harvesting with the former effect dominating.

- **Profit tax**

In the presence of the profit tax $\boldsymbol{q}$ the net harvest revenue is $\hat{V}(\boldsymbol{q}) = (1 - \boldsymbol{q})V$, and the private landowner maximizes $\hat{W}_T(\boldsymbol{q}) = (1 - \boldsymbol{q})V + E$. Choosing T optimally gives

$$\hat{W}_T(\boldsymbol{q}) = (1 - \boldsymbol{q})V_T + F(T) - rE = 0. \tag{15}$$

The comparative statics of the profit tax is

$$T_q^H \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \ \ as \ \ F'(T) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0. \tag{16}$$

The outcome is qualitatively similar to that of the site value tax for both Faustmann and Hartman models.

Comparative statics of forest taxation is collected in Table 1. Only the site productivity tax and the timber tax have qualitatively similar effects in both models. The site value tax and the profit tax, which are neutral in the Faustmann model, are generally distortionary in the Hartman model with the exception of site-specific amenities. They, as well as the harvest taxes, may have positive or negative effects depending on the nature of marginal amenity valuation.

Table 1. Comparative statics of forest taxation

Forest tax	Faustmann Model	Hartman Model
Harvest tax $(x=t,\tau)$	$T_x^F > 0$	$T_x^H \geq (<)0$ as $rc(1-e^{-rT})^{-1} + F(T) - rE \geq (<)0$
Site prod. tax, a	$T_a^F = 0$	$T_a^H = 0$
Site value tax, β	$T_b^F = 0$	$T_b^H \geq (<)0$ as $F'(T) \geq (<)0$
Timber tax, α	$T_a^F < 0$	$T_a^H < 0$
Profit tax θ	$T_a^F = 0$	$T_q^H \geq (<)0$ as $F'(T) \geq (<)0$

2.2 Social Welfare Function

In designing forest taxation policy the government is assumed to maximize the social welfare function. We consider two cases. First, we assume that only harvest revenue from forestry counts, so that the social welfare function is defined by the indirect net revenue function of the landowner V^* (equation 17a). Second, we study a case where both the landowner and recreators value amenity services from forests. For simplicity the

amenity valuation function is postulated to be the same for the representative landowner and recreators.[6] When citizens have full access to enjoy the amenity services from private forests and there are no congestion effects associated with enjoying amenity services of forests we have the social welfare function (17b).

$$SW^F = V^*(a(i), \boldsymbol{t}, t, \boldsymbol{b}, \boldsymbol{q}, \boldsymbol{a}) \tag{17a}$$

$$SW^H = V^*(a(i), \boldsymbol{t}, t, \boldsymbol{b}, \boldsymbol{q}, \boldsymbol{a}) + E^*(a(i), \boldsymbol{t}, t, \boldsymbol{b}, \boldsymbol{q}, \boldsymbol{a}) + (n-1)E, \tag{17b}$$

where n is the number of citizens and (n-1) is the number of recreators.

In the subsequent analysis we assume that there is an exogenous tax revenue target, denoted by $\bar{R}$. There are alternative ways to formulate the tax revenue requirement in the case of forestry. Here we assume that the short run government debt or surplus is not regarded as an important factor, so that all what counts is the discounted sum the tax revenue collected from forestry given by

$$R = \left[\frac{(p\boldsymbol{t} + t)f(T) + \boldsymbol{a}\int_0^T pf(s)e^{-rs}ds}{e^{rT} - 1} + \frac{a}{r}\boldsymbol{y} \right], \tag{18}$$

where $\boldsymbol{y} = \boldsymbol{q}, \boldsymbol{b}$.

When studying optimal tax policy in next sections we will not assume that all taxes are present at the same time. We indicate in each case, which taxes are assumed to be operative.

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[6] When the recreators value forest amenities, but the private landowner does not, the social welfare function would read as $SW^F = V^*(a(i), \boldsymbol{t}, t, \boldsymbol{b}, \boldsymbol{q}, \boldsymbol{a}) + (n-1)E$. This case yields, however, qualitatively similar results as the case of (17b) and, therefore, we do not consider it explicitly. This rather special case has been partly analyzed in Englin and Klan (1990), but in the absence of the government tax revenue requirement. Moreover, their analysis did not include the site productivity tax, which is always neutral in the Hartman framework, and they used the site value tax and the profit tax as neutral taxes. Hence, their results are very special in the sense that i) they do not allow for tax revenue requirement and ii) they assume that $F'(T) = 0$ for landowners.

3. Socially Optimal Forest Taxation in the Faustmann Framework

A neutral tax is optimal, when the society sees no need to distort the private decisions. As the comparative statics of rotation age in the Faustmann model revealed, site productivity tax, site value tax and profit tax are neutral taxes, and have this desirable property. But in the Hartman model only the site productivity tax is neutral, if the amenity valuation is not site specific, i.e., $F'(T) \neq 0$. Therefore, we will use the site productivity tax as our benchmark tax when studying whether we need other forest taxes for the socially optimal design of forest taxation.

3.1 Optimal forest taxation in the presence of a neutral tax

The government's problem is to maximize the social welfare function (17a) subject to the tax revenue requirement (18). Differentiating the Lagrangian, $\Omega = V^* - \boldsymbol{l}(\overline{R} - R)$, with respect to the site productivity tax a(i) yields $\Omega_a = -\dfrac{1}{r} + \boldsymbol{l}\dfrac{1}{r} = 0$, so that $\boldsymbol{l} = 1$. The shadow price λ represents the marginal cost of public funds which is equal to one. This means that the government can raise tax revenue without imposing any cost (over collected tax revenue) on the society.[7] Hence, it is an ideal tax instrument to collect the required revenue without distorting landowner's privately optimal behavior. This outcome is due to the fact that site productivity tax is neutral, i.e., $T_a^H = 0$.

When the site productivity tax has been set at the optimal level, $a = a^*$, the optimal yield tax $\boldsymbol{t}$, in the absence of other taxes, can be obtained by differentiating the Lagrangian Ω with respect to $\boldsymbol{t}$ under the condition $\boldsymbol{l} = 1$. This gives

$$\Omega_{t|a=a^*} = \boldsymbol{t}\left\{\frac{(e^{rT}-1)pf'(T) - re^{rT}pf(T)}{(e^{rT}-1)^2}\right\}T_t^F = 0, \qquad \text{(19a)}$$

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where $T_t^F > 0$ and $\dfrac{(e^{rT}-1)pf''(T) - re^{rT}pf(T)}{(e^{rT}-1)^2} < 0$, due to the first-order condition (2).

Hence, $\Omega_{t\,|a=a^*} = 0$ necessitates that the optimal yield tax, $t = t^*$, is zero. This result holds also for the unit tax, because of the definition of these taxes.[8]

Analogously, when the site productivity tax is set at the optimal level, the first-order condition for the timber tax in the absence of other taxes is given by

$$\Omega_{a\,|a=a^*} = \frac{a}{(e^{rT}-1)}\left\{\frac{1}{(e^{rT}-1)}(pf'(T)-rU)\right\}T_a^F = 0, \qquad \textbf{(19b)}$$

where $T_a^F < 0$, and that $(pf(T)-rU) > 0$, when $f'(T) > 0$. Hence, the optimal timber tax rate, $a = a^*$, is zero.

To summarize, we have obtained

Result 1. *If the society values only harvest revenue from forests and wishes to collect a given forest tax revenue, then it should use only a neutral site productivity tax (or site value or profit tax) and set all distortionary taxes equal to zero.*

This result makes sense. In the absence of externalities a neutral tax is optimal, because it minimizes the deadweight loss of taxation. This result confirms what e.g. Gamponia and Mendelsohn (1988) pointed out forestry context, though without the exact proof.

3.2 Optimal forest taxation in the absence of neutral taxes

What about if the government does not have site productivity tax or other neutral, lump-sum tax available? Could we then find a tax mix to collect the required forest tax revenue without distorting the landowners' behavior? Our answer is positive.

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[8] To obtain the optimality condition for the unit tax, just multiply equation (19a) by the factor tT_t^F / ptT_t^F.

Differentiating the Lagrangian Ω with respect to the unit and timber taxes in the absence of other taxes we get after some manipulations (see Appendix 1)

$$\Omega_t = (\mathbf{1} - 1)\frac{f(T)}{(e^{rT} - 1)} + \frac{\mathbf{1}}{(e^{rT} - 1)^2}\{tA + \mathbf{a}B\}T_t^F = 0 \qquad (20a)$$

$$\Omega_a = (\mathbf{1} - 1)U + \frac{\mathbf{1}}{(e^{rT} - 1)^2}\{tA + \mathbf{a}B\}T_a^F = 0 \qquad (20b)$$

where $A = (e^{rT} - 1)f'(T) - rf(T)e^{rT} < 0$ due to first-order condition (3), $B = (pf(T) - rU) > 0$, $T_t^F > 0$ and $T_a^F < 0$.

On the basis of the first-order conditions (20a) and (20b) one can see that, under the condition $tA + \mathbf{a}B = 0$, the privately optimal rotation age of the forest stand is determined by the following first-order condition

$$V_T = 0 \iff p\left[(e^{rT} - 1)f'(T)re^{rT}f(T)\right] + cre^{rT} = 0,$$

which is exactly the same condition that holds without forest taxes. Hence, this is the condition for the optimal combination of unit and timber taxes. Solving $tA + \mathbf{a}B = 0$ for the ratio of taxes and applying the definitions of A and B yields

$$\frac{t^*}{\mathbf{a}^*} = -\frac{(e^{rT} - 1)(pf(t) - rU)}{(e^{rT} - 1)f'(T) - rf(T)e^{rT}} > 0 \qquad (21)$$

This tax mix gives a non-distortionary private rotation age. Due to the definition of unit and yield taxes, this result holds also for a combination of the yield tax and timber tax. Summing up, we have

Result 2. *If the society values only harvest revenue from forests and wishes to collect a given forest tax revenue, then in the absence of neutral taxes a combination of unit tax and timber tax (or a combination of yield tax and timber tax) collects the required tax revenue in a non-distortionary way.*

The economic intuition behind this result lies in the use of two distortionary taxes which affect rotation in an opposing way so that an appropriate combination cancel out each other's distortionary effect.[9]

4. Socially Optimal Forest Taxation in the Hartman Framework

Next we turn to consider the case, which has not been analyzed in the literature, where the amenity services from forests are valued by the private landowner, as well as by recreators. The relevant social welfare function is (17b). We follow the same strategy as in the previous section and start by assuming that the government has available a non-distortionary tax.

4.1 Optimal forest taxation in the presence of a neutral tax

We consider first the use of the site productivity tax, the site value tax, and the profit tax. These are all neutral in the Faustmann model, but depending on the nature of amenity valuation the last two may be distortionary in the Hartman framework. Does the neutral site productivity tax suffice for an optimal tax policy, or do we have to complement it with another tax?

Recalling the social welfare function (17b) and the tax revenue requirement (in the absence of other taxes), we can write down the Lagrangian as $\Omega^H = V^* + E^* + (n-1)E - \mu(\bar{R} - R)$, where subscript H refers to Hartman case and μ is the marginal cost of public funds. Choosing $a(i)$ so as to maximize the Lagrangian yields a first-order condition $\Omega_a = -\dfrac{1}{r} + \mu\dfrac{1}{r} = 0 \iff \mu = 1$. The optimal site productivity tax is non-distortionary which results from the fact that $T_a^H = 0$.

Would it be socially optimal to use also the site value tax? Differentiating the Lagrangian with respect to β and assuming that the site productivity tax is set at the optimal level

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results in the following first-order condition for the socially optimal site value tax b (see Appendix 2 for the details):

$$\Omega_b^H\big|_{a=a^*} = T_b^H\left\{(n-1)E_T + \frac{bV_T}{(1-b)}\right\} = 0 \tag{22}$$

To evaluate the optimal site value tax we utilize two facts. First, from the private first-order condition we have that $E_T \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \Leftrightarrow V_T \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0$ Second, we recall that

$T_b^H \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \; as \; F'(T) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$, so that the effect of the site value tax depends on how the

marginal amenity valuation evolves with the age of the forest stand.

There are two possibilities depending on the nature of the amenity valuation function. First, if amenity valuation is site-specific ($F'(T) = 0$) the optimal site value tax is zero. Second, note that $\hat{V}_T^H(b) = E_T + V_T = 0$ implies $E_T = -V_T$. Using this in (22) allows us to express the term in braces as $(n-1)E_T + \frac{bV_T}{(1-b)} = (n-1)E_T - \frac{b(n-1)E_T}{(1-b)}$, so that solving for b yields:

$$b^* = \frac{(n-1)}{n} \tag{23}$$

Equation (23) is a classical Pigouvian tax: the size of the tax reflects the size of the externality, which in the case of identical preferences for amenity services is given by the share of recreators among citizens.[10]

According to equation (23) the optimal site value tax is the same independently of whether the privately optimal rotation age is shorter or longer than the socially optimal one. What accounts for this interesting result? The answer lies in the sensitivity of the

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[10] The analogous result has been derived in Koskela and Ollikainen (1997) in the case of two

comparative statics of the site value tax on the nature of the marginal amenity valuation. When the marginal valuation of amenity services increases with the age of the stand ($F'(T) > 0$) the site value tax will have a positive effect on private rotation, while when it decreases with the age ($F'(T) < 0$) the site value tax will have a negative effect on it. For $F'(T) > 0$ the privately optimal rotation age is too short from the viewpoint of social optimum, and too long for $F'(T) < 0$. In both cases society should use site value tax: in the former case to lengthen and in the latter case to shorten the privately optimal rotation age.

Since the effect of the profit tax θ is identical to that of the site value tax (see equations 10 and 16), we have

Result 3. *If the society values both harvest revenue and amenities from forests and wishes to collect a given forest tax revenue, then in addition to the optimal site productivity tax, the design of forest tax policy consists of*
  a) no other taxes when amenity valuation is site specific, i.e., independent of the age of forest stand
  b) a combination of the neutral site productivity tax and a distortionary site value (profit) tax, which reflects the size of the externality, when the amenity valuation increases or decreases with the age of the stand.

These findings make sense. In the presence of externality and the optimal site productivity tax, no other taxes are needed when amenity valuation is independent of the age of the forest stand, because all that counts are the amenities associated with the bare land. But if the amenity valuation depends on the age of the forest stand, the site productivity tax should be complemented with the site value tax or the profit tax by the amount, which depends on the size of externality associated with amenity services.

If the site value and profit taxes are not feasible and then one can ask whether the harvest and timber taxes have any role in the optimal design of forest taxation in the presence of the site productivity tax. Since the comparative statics of the yield tax and the unit tax are

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period model.

qualitatively similar the Hartman model, we concentrate on the yield tax (and assume other taxes to be absent in equation 18).

Maximizing the Lagrangian $\Omega^H = V^* + E^* + (n-1)E - \mathbf{l}(\overline{R} - R)$ yields the familiar condition $\mathbf{m} = 1$ for the site productivity tax. Choosing now the yield tax, when the site-productivity tax is at the optimal level, gives the following first-order condition

$$\Omega_t^H\big|_{a=a^*} = \left\{(n-1)(e^{rT}-1)(F(T)-rE) + \mathbf{t}\left[(e^{rT}-1)pf'(T) - re^{rT}pf(T)\right]\right\}T_t^F = 0 \quad (24)$$

Recalling from section 2 that $T_t^H \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \quad as \quad rc(1-e^{-rT})^{-1} + F(T) - rE \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$ we first

note that in the special case where $T_t^H = 0$, the optimal yield tax rate is zero. When $T_t^H \neq 0$ equation (24) can be solved for the optimal yield tax to get

$$\mathbf{t}^*\big|_{a=a^*} = -\frac{(n-1)(e^{rT}-1)(F(T)-rE)}{(e^{rT}-1)pf'(T) - re^{rT}pf(T)} \quad (25)$$

From equation (25) we can derive several properties for the optimal yield tax depending on the nature of the amenity valuation. First, neglecting the public goods property of forest amenities (i.e. setting n = 1) implies that optimal yield tax would be zero. Second, if amenities are site-specific (for which $F'(T) = 0$ and $F(T) - rE = 0$) optimal yield tax would again be zero. Third, when n > 1 and $F'(T) > 0$, we have $F(T) - rE > 0$ but $(e^{rT} - 1)pf'(T) - re^{rT}pf(T) < 0$, so that optimal yield tax is positive. Fourth, we have the case where $F(T) - rE < 0$ and $(e^{rT} - 1)pf'(T) - re^{rT}pf(T) = ?$, where the size and sign of the optimal yield tax/subsidy remains unclear.

We can summarize our findings in

Result 4. *If the society values both harvest revenue and amenities from forests and wishes to collect a given forest tax revenue, then in addition to the optimal site productivity tax, the design of forest tax policy consists of*

> a) *no other taxes if amenity valuation is site specific, i.e., independent of the age of forest stand*
>
> b) *a combination of the neutral site productivity tax and a distortionary yield tax (or a unit tax) when marginal valuation of amenities increases with the age of the stand.*
>
> c) *a combination of the neutral site productivity tax and a distortionary yield tax/subsidy (or a unit tax/subsidy) remains unclear when marginal valuation of amenities increases with the age of the stand.*

An economic interpretation of case a) is familiar. If the amenity valuation is site-specific, the rotation age is not distorted by the amenity valuation. Therefore the neutral site productivity tax is enough. Cases b) and c) reflect the classical Pigouvian policy. If the amenity valuation increases with the age of forest stand, then the private rotation age is too low from the viewpoint of the society. By complementing the neutral site productivity tax with the positive yield (or unit) tax has the effect of increasing the rotation age. Finally, if the amenity valuation decreases with the age of forest stand, then the rotation age is too long from the society's viewpoint. But the optimal policy remains ambiguous because under these circumstances the effect of the harvest tax on the rotation is ambiguous a priori.

Assume next that the site-productivity tax is set at the optimal level and ask whether we need to introduce a timber tax to maximize the social welfare? The first-order condition for the constrained social welfare maximization is given by

$$\Omega_t^H\big|_{a=a^*} = \left\{(n-1)(e^{rT}-1)(F(T)-rE) + a(e^{rT}-1)\left[pf(T)-rU\right]\right\}T_a^H = 0 \quad \textbf{(26)}$$

Solving for the optimal timber tax yields

$$a^*\big|_{a=a^*} = -\frac{(n-1)(F(T)-rE)}{pf(T)-rU}, \qquad\qquad \textbf{(27)}$$

where the denominator of (27) is positive irrespective amenity valuation, while the numerator is positive (negative) for $F'(T) < 0$ ($F'(T) > 0$). The optimal timber tax is zero

if there is no externality involved in forestry (n = 1) or amenities are site-specific ($F'(T) = 0$). For other cases, we can conclude that

Result 5.   *If the society values both harvest revenue and amenities from forests and wishes to collect a given forest tax revenue, then in addition to the optimal site productivity tax, the design of forest tax policy consists of*
a) *a timber tax, if the amenity valuation decreases with the age of the stand*
b) *a timber subsidy if the amenity valuation increases with the age of the stand.*

The interpretation of Result 5 is obvious. When the marginal valuation of amenity services decreases with the age of the stand, privately optimal rotation age is too long from the society's perspective. Therefore, a positive timber tax, which shortens the rotation age, can be used to reduce the externality caused by private harvesting to recreators. But when the marginal valuation of amenity services increases with the age of the stand, privately optimal rotation age is too short from the society's perspective, and timber subsidy can be used to lengthen the privately optimal rotation age and reduce the externality to recreators.

4.2 Optimal forest taxation in the absence of neutral taxes

Let us finally ask can we find, in the absence of the site productivity tax, a combination of a unit tax and a timber tax (or a yield tax and timber tax), which in the same vein as in the case of the Faustmann model, would produce the socially optimal rotation. Note that in the Hartman model the neutrality property is desirable only in the case of site-specific amenities. Otherwise we need which would internalize the externality caused by private harvesting on recreators.

Differentiating first the Lagrangian, $\Omega^H = V^* + E^* + (n-1)E - \boldsymbol{m}(\overline{R} - R)$ in the absence of other taxes than unit and timber taxes with respect to t and α yields (see Appendix 3 for details)

$$\Omega_t^H = (\boldsymbol{m}-1)f(T) + \boldsymbol{m}T_t^H\{(n-1)[F(T)-rE]+tA+\boldsymbol{a}B\}=0 \tag{28a}$$

$$\Omega_a = (m-1)U(e^{rT}-1) + mT_a^F\{(n-1)[F(T)-rE]+tA+aB\} = 0 \tag{28b}$$

where $A = (e^{rT}-1)f'(T) - rf(T)e^{rT}$ and $B = (pf(T)-rU)$

Now if $(n-1)(F(T)-rE)+tA+aB = 0$ then $m=1$ and this tax combination is non-distortionary as in the Faustmann model. That happens if either n = 1, or $F'(T) = 0$.

$$\frac{t^*}{a^*} = \frac{-(e^{rT}-1)(pf(t)-rU)}{(e^{rT}-1)f'(T)-rf(T)e^{rT}} > 0 \tag{28c}$$

For the cases where the marginal valuation of amenities is not constant the following optimality condition characterizes the combination of unit tax and timber tax, which both collects tax revenues and eliminates the externalities in the production of amenity services

$$tA + aB = -(n-1)(F(T)-rE), \tag{28d}$$

where B > 0, while A is negative for $F'(T) \geq 0$ but ambiguous for $F'(T) < 0$.

Interpretation goes as follows. Under $F'(T) > 0$ the privately optimal rotation age is too short from the society's viewpoint. Relative to equation (28c) (reflecting both site-specific amenities and the Faustmann model), the RHS of (28d) has changed from zero to negative so that the LHS of (28d) is smaller. This means that either the unit tax rate (which lengthens the rotation age) has to be raised above the level of the Faustmann case or that timber tax (which shortens the rotation age) rate has to be decreased. If instead $F'(T) < 0$, the privately optimal rotation age is longer than the socially optimal one. The RHS of (28d) has changed from zero to positive when compared with (28c). The term A, however, is now a priori ambiguous and should be evaluated empirically. If A is negative, then the combination of timber and unit tax still works to collect tax revenues and eliminate externalities provided that the timber tax rate is increased so as to make the LHS of equation (28d) positive. If, however, A turns out to be positive, then both taxes affect into same direction, i.e., they tend to lengthen private rotation age. Then the only

possibility is to introduce a timber subsidy together with the unit tax. Now the tax collects the tax revenue, while the subsidy internalizes the externality.

Now we can summarize our findings in

Result 6:   *If the society values both harvest revenue and amenity services from forests and wishes to collect a given forest tax revenue, then in the absence of neutral taxes,*

a) with the site-specific amenities the government should pick a combination of unit tax and timber tax (or a combination of yield tax and timber tax) which collects the required tax revenue in a non-distortionary way

b) with the amenity valuation increasing with the age of the forest stand, the government should pick a combination of unit tax (or a combination of yield tax and timber tax) and timber tax which collects the required tax revenue in a non-distortionary way

c) in the presence of amenity externalities and with the amenity valuation decreasing with the age of the forest stand, an appropriate combination of unit tax/subsidy and timber tax to correct externalities remains ambiguous a priori.

5. Concluding Remarks

We studied the unexplored problem of the socially optimal design of forest taxation, when the government wishes to collect a given tax revenue under two alternative assumptions. First, the society values only net harvest revenue and second, it values also amenity services from forests. Our analytical frameworks were, thus, the Faustmann and Hartman models. In both cases we assumed that the government first announces credibly its tax policy and then private landowners choose their rotation age in the presence of taxes.

We demonstrated that in the Faustmann model only neutral forest taxes are needed to maximize the welfare and collect the tax revenue. The site productive tax and the site value tax as lump-sum property taxes, as well as the profit tax, have this neutrality

property. If neutral taxes are not available, a combination of unit (yield) tax and timber tax, which affect the rotation age in opposing directions and hence can eliminate each other's distortionary effect, can be used to collect tax revenue without distorting privately optimal rotation age.

When forest amenities are a public good with the exception of site-specific amenity valuation the private valuation of amenities does not reflect their social valuation. The optimal design of forest taxation now consists of a neutral tax to collect the required tax revenue and a distortionary tax/subsidy to correct the provision of amenity services to the socially optimal level. While the site productivity tax is the only neutral tax among the taxes we study, there are many possibilities for the choice of the distortionary Pigouvian tax or subsidy. When the private optimal rotation age is too short from the society's viewpoint (amenity valuation increases with the age of the stand), a yield, unit or timber tax could be levied on the landowner so as to lengthen rotation age. Respectively, by a yield, unit or timber subsidy the society can shorten the privately optimal rotation age if it is too long from the society's viewpoint (amenity valuation decreases with the age of the stand). The site value tax has the same properties as a corrective tax. Interestingly, however, a site value or a profit tax related to the size of externality always corrects the externality irrespectively of the nature of the amenity valuation. Finally, we have shown that a combination of unit (yield) and timber taxes can be used to collect the tax revenue and internalize the externality, when the marginal valuation of amenity services is non-decreasing in the age of the stand, but ambiguous for decreasing marginal valuation.

In this paper we have studied the socially optimal forest taxation when the rotation period of a forest stand is independent of that of other adjacent stands, private or public. Forest stands may, however, be interdependent in producing amenity services. It would be an interesting area for research to analyze the impacts of this potential interdependence for the socially optimal forest taxation.

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List of symbols:

$f(T)$ growth function of a stand

T rotation age

p real stumpage price

c real regeneration cost

r real interest rate

t yield tax (levied on the stumpage value of timber harvested)

t unit tax (levied on the volume of timber harvested)

b annual lump-sum tax payment (levied on the landowner)

b site value tax (annual lump-sum tax b related to the value of the land)

a(i) site productivity tax for site i (a lump-sum tax levied on the productivity of site i)

a timber tax (levied on the stumpage value)

q profit tax (levied on the net harvest revenue)

$F(T)$ amenity valuation function

T^F Faustmann rotation age

T^H Hartman rotation age

V the net present value of harvest revenue over infinite rotations

E the present value of amenity services over infinite rotations

W the net present value of harvest revenue plus the present value of amenity services over infinite rotations

Appendix 1. Derivation of Equations (22a) and (22b)

The Lagrangian for the choice of optimal unit and timber taxes is $\Omega = V^* - \boldsymbol{l}(\overline{R} - R)$, where $R = \left[f(T) + \boldsymbol{a} \int_0^T pf(s)e^{-rs}ds \right](e^{rT} - 1)^{-1}$. Choosing t and α so as to maximize the Lagrangian yields

$$\Omega_t = (\boldsymbol{l} - 1)\frac{f(T)}{e^{rT} - 1} + \frac{\boldsymbol{l}T_t^F}{(e^{rT} - 1)^2}\left\{t\left[(e^{rT} - 1)f'(T) - rf(T)e^{rT}\right] + (e^{rT} - 1)\boldsymbol{a}[pf(T) - rU]\right\} = 0$$

A1.1

$$\Omega_a = (\boldsymbol{l} - 1)U + \frac{\boldsymbol{l}T_a^F}{(e^{rT} - 1)^2}\left\{t\left[(e^{rT} - 1)f'(T) - rf(T)e^{rT}\right] + (e^{rT} - 1)\boldsymbol{a}[pf(T) - rU]\right\} = 0$$

A1.2

Defining next $A = (e^{rT} - 1)f'(T) - rf(T)e^{rT}$ and $B = (e^{rT} - 1)\left[pf(T) - rU\right]$ and applying them in A1.1 and A1.2 yields the equations given in the text.

$$* \quad * \quad * \quad * \quad *$$

Appendix 2. Derivation of Equation (24)

Differentiating the Lagrangian $\Omega = V^* + E^* + (n-1)E - \boldsymbol{m}(\overline{R} - R)$, where $R = \boldsymbol{b}\left[(pf(T) - ce^{rT})(e^{rT} - 1)^{-1} + a/r\right]$ with respect to a implies that $\mu = 1$. Choosing now β optimally yields

$$\Omega_{\boldsymbol{b}}^H\Big|_{a=a} = T_{\boldsymbol{b}}^H\left\{(n-1)E_T + V_T + E_T + \boldsymbol{m}R_T\right\} = 0 \qquad \textbf{A2.1}$$

Accounting for the fact that $E_T + V_T = 0$ at the landowner's optimum, we can express A2.1 as

$$\Omega_{\boldsymbol{b}}^H\Big|_{a=a^*} = T_{\boldsymbol{b}}^H\left\{(n-1)E_T + R_T\right\} = 0, \qquad \textbf{A2.2}$$

where the derivative of the tax revenue function with respect to T is

$$R_T = \frac{b}{(e^{rT}-1)^2}\left[(e^{rT}-1)(pf'(T)-rce^{rT})-r(pf(T)-ce^{rT})e^{rT}\right]$$

Since $V_T(\boldsymbol{b}) = \frac{1-\boldsymbol{b}}{(e^{rT}-1)^2}\left[(e^{rT}-1)(pf'(T)-rce^{rT})-r(pf(T)-ce^{rT})e^{rT}\right]$, we have

$R_T = \frac{\boldsymbol{b}}{1-\boldsymbol{b}}V_T$, so that A2.2 can be re-expressed as

$$\Omega_b^H\Big|_{a=a^*} = T_b^H\left\{(n-1)E_T + \frac{\boldsymbol{b}}{1-\boldsymbol{b}}V_T\right\} = T_b^H\left\{(n-1)E_T - \frac{\boldsymbol{b}}{1-\boldsymbol{b}}E_T\right\} = 0, \qquad \textbf{A2.3}$$

due to the the first-order condition. This yields

$$\boldsymbol{b}^* = \frac{n-1}{n} \text{, which was given in equation (23) of the text.}$$

<p align="center">*   *   *   *   *</p>

Appendix 3. Derivation of Equations (28a) and (28b)

The Lagrangian function can be written as $\Omega = V^* + E^* + (n-1)E - \boldsymbol{m}(\overline{R}-R)$, where $R = \left[f(T) + \boldsymbol{a}\int_0^T pf(s)e^{-rs}ds\right](e^{rT}-1)^{-1}$. Choosing t and α so as to maximize the Lagrangian yields

$$\Omega_t = (\boldsymbol{m}-1)\frac{f(T)}{e^{rT}-1} +$$
$$\boldsymbol{m}T_t^H\left\{(n-1)E_T + \frac{t}{(e^{rT}-1)^2}\left[(e^{rT}-1)f'(T)-rf(T)e^{rT}\right]+(e^{rT}-1)\boldsymbol{a}[pf(T)-rU]\right\}=0$$
$$\textbf{A3.1}$$

$$\Omega_a = (\boldsymbol{m}-1)U +$$
$$\boldsymbol{m}T_a^F\left\{(n-1)E_T + \frac{t}{(e^{rT}-1)^2}\left[(e^{rT}-1)f'(T)-rf(T)e^{rT}\right]+\frac{\boldsymbol{a}}{(e^{rT}-1)}[pf(T)-rU]\right\}=0$$
$$\textbf{A3.2}$$

Defining $A = (e^{rT} - 1)^{-1} f'(T) - rf(T)e^{rT}$ and $B = [pf(T) - rU] > 0$ and noting that $E_T = F(T) - rE$ allows equations A3.1 and A3.2. to re-express as

$$\Omega_t = (\boldsymbol{m}-1)\frac{f(T)}{e^{rT}-1} + \boldsymbol{m}\boldsymbol{T}_t^H\{(n-1)(F-rE)+tA+\boldsymbol{a}B\} = 0 \qquad \textbf{A3.3}$$

$$\Omega_t = (\boldsymbol{m}-1)U(e^{rT}-1) + \boldsymbol{m}\boldsymbol{T}_a^H\{(n-1)(F-rE)+tA+\boldsymbol{a}B\} = 0, \qquad \textbf{A3.4}$$

which were given in the text.

$$* \quad * \quad * \quad * \quad *$$