NONLINEAR PRODUCTION, ABATEMENT, POLLUTION AND MATERIALS BALANCE RECONSIDERED

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Abstract

In the environmental economics literature the standard approach of modeling nonlinear production and abatement processes is to treat waste emissions "simply as another factor of production" (Cropper and Oates 1992). That approach doesn't map the materials flow involved completely and hides, moreover, the exact links between production, residuals generation and abatement. This paper shows that production functions with emissions treated as inputs can be reconstructed as a subsystem of a comprehensive production-cum-abatement technology that is in line with the materials-balance principle. In a simple economy with full regard of the materials flow it also explores the consequences for allocative efficiency and efficiency-restoring taxation of multiple and interdependent residuals generated in the transformation processes of production, abatement and consumption. Finally, the paper demonstrates that efficiency may require setting the emissions tax rate above or below *conventionally* defined marginal abatement cost if the residual subject to abatement is not the only residual causing pollution.

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Nonlinear production, abatement, pollution and materials balance reconsidered [1]

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1. The problem

As is well-known from the law of mass conservation, the flow of materials taken from the environment for economic uses generates a flow of materials from the economy back into the environment that is of equal weight (after accounting for time delays). The economic activities of production and consumption are merely processes of transforming materials that only change the physical and chemical attributes and the composition of the materials flow. Quite obviously, the composition of the flow of residuals from the economy into the environment is of great significance because different kinds of residuals differ in their detrimental impact on the environment.[2]

As a consequence, a sensible strategy for alleviating the problem of environmental degradation is to control the process of materials transformation by reducing the emission of the most harmful pollutants through residuals abatement activities. Like production and consumption, this activity is a process of transforming materials subject to the materials-balance principle: the weight of all material outputs of that process equals the weight of all material inputs.

In their seminal paper on 'production, consumption and externalities', Ayres and Kneese (1969) made a strong case for the need of a consistent and encompassing application of the materials-balance principle to *all* transformation processes. In their formalized materials-balance approach they employed linear technologies with fixed input-output coefficients but since then the profession has revealed a preference for modeling non-linear rather than linear technologies. In fact, the notion and empirical evidence of strictly increasing (real) marginal abatement costs is at the core of many pollution control studies.

To be sure, it is possible to bring non-linear (abatement) technologies into line with materials-balance requirements, too. This has been demonstrated in various previous studies the most general and ambitious of which probably is Krysiak and Krysiak (2003). Yet fully regarding the materials-balance principle in theoretical analysis comes at the cost of enormous addi-

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[2] A more detailed analysis would need to focus on further determinants such as the medium of discharge and the local environmental medium's assimilative capacity.

tional complexity which tends to prevent the derivation of informative results. To avoid such complexities many environmental economists became reluctant to explicitly and properly regard the materials-balance principle as the correct theoretical foundation of their analyses (Pezzey and Toman 2002, 202; Pethig 2003). Therefore Ayres and Kneese's (1969, 283) verdict still applies to much of the present work that production processes are viewed "… in a manner that is somewhat at variance with the law of conservation of mass".

To be more specific, consider the simple production function $Y : \mathbb{R}^3_+ \to \mathbb{R}_+$ with

$$y = Y(e, \ell, m), \tag{1}$$

where two inputs, *labor* ℓ and *material m*, are employed to produce two outputs as joint products, a wanted *consumer good*, *y*, and an unwanted *production residual*, *e* (with *e* for *emissions*). This type of technology was already applied in the early 1970s, e. g. by Forster (1972) and Klevorick and Cramer (1972). Varying grossly in its degree of generality, it became a widespread and accepted tool of analysis within few years (e. g. in Mäler 1974, Pethig 1975, Baumol and Oates 1975).

In their survey on environmental economics, Cropper and Oates (1992) refer to the production function (1) as the standard approach in the environmental economics literature[3]. They observe that the treatment of waste emissions "simply as another factor of production … seems reasonable since attempts … to cut back on waste discharge will involve the diversion of other inputs to abatement activities - thereby reducing the availability of these other inputs for the production of goods" (Cropper and Oates 1992, p. 678). This citation reveals these authors' awareness of technology (1) as a concept that *implicitly* involves both the generation of a production residual and an abatement activity. Cropper and Oates (1992, p. 678) also find it reasonable "...to assume the usual curvature properties ..." that is, they require function *Y* from (1) to exhibit the

> **Properties (Y):** $Y : \mathbb{R}^3_+ \to \mathbb{R}_+$ *is concave and satisfies* $Y_e > 0, Y_\ell > 0$ *and* $Y_m > 0$.

It is not clear, however, what exactly is the link between the production of a consumer good, residuals generation and abatement which Cropper and Oates conjecture as being hidden in (1). To put it differently, it is not clear how an explicit and comprehensive analysis of interde-

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[3] As compared to our equation (1), the equation (2) in Cropper and Oates (1992, p. 678) is slightly more general in that they allow for an arbitrarily large vector of conventional inputs (which is reduced to the two-dimensional vector (ℓ, m) in (1)) and allow the level of pollution to be a negative productivity-reducing externality. This externality is omitted in the present paper to keep the analysis simple.

pendent production, residuals generation and abatement would relate to production functions of type (1).[4]

The main objectives of the present paper are (i) to undertake a fresh exploration of that issue with the important qualification to keep the analysis strictly (and explicitly!) in line with the materials-balance principle and (ii) to assess the consequences of (i) for allocative efficiency and pollution control. An immediate implication of adopting a rigorous materials-balance perspective is to insist that the treatment of emissions in (1) as 'conventional inputs in production functions' is only acceptable as a convenient though purely formal analytical device not meant to deny the emissions' true nature as unwanted by-products generated in the process of producing wanted goods and then discharged into the environment.

To further clarify this point suppose ℓ and m are constant in (1). Since Y_e, the marginal abatement cost in terms of the wanted output, is positive, one can choose from a menu of good things, y, and bad things, e, but more of the good inadvertently comes along with more of the bad. While this property of (1) serves the needs of model building in environmental economics well, (1) is less appealing, if not outright embarrassing, regarding its materials-balance implications. To see that suppose all units of the outputs y and e as well as all units of input m are defined such that each unit is equal to one unit of weight. Suppose further that the technology (1) is understood as a process of transforming the material m into the outputs y and e. With these qualifications one may wish to know whether (1) can be considered the description of a transformation process that involves no material inputs other than m and no material outputs other than y and e. The answer is an outright and definitive no. (1) would blatantly violate the materials-balance principle, since that principle requires $m = e + y$ if no other inputs and outputs are involved. An obvious implication of $m = e + y$ is $dy/de = -1$ for constant m, which is, of course, inconsistent with properties (Y) that requires $Y_e > 0$.

It is not our intention to join in the chorus of those who declare all pieces of research in environmental economics fundamentally flawed that are found guilty of not properly regarding the materials-balance principle. We rather aim at answering the intriguing question whether, and if so how, the production activity (1) can be reconciled with the materials-balance principle. We will show that (1) can be reconstructed, indeed, as part of a comprehensive production-cum-abatement technology that is in line with the materials-balance principle.

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[4] The only theoretical inquiry into that issue we are aware of is offered in Siebert et al. (1980). Yet these authors fail to fully account for the materials-balance principle which is why their approach is of limited relevance for our subsequent analysis.

Maintaining the standard technological assumptions of concavity, non-linearity,[5] and smooth factor substitution, the proper regard of materials-balance requirements will make it necessary to also account for residuals other than the production residuals represented by the variable e in (1). We will also demonstrate that these additional interdependent residuals do not render incorrect the conventional analysis of pollution control based on (1) if and only if their emission doesn't contribute to environmental degradation. Insofar we provide a rigorous rationale and justification for conventional model building. However, if the emitted production residuals, e, are not the only pollutants, the conditions determining allocative efficiency will be shown to differ markedly from those derived in conventional analysis. In that case, the *conventional* marginal cost of abating production residuals, Y_e, will turn out to deviate from the *social* marginal cost of abating these residuals because we deal with a pollution problem involving multiple *and* interdependent pollutants. This finding will be shown to have non-trivial implications for efficiency-restoring tax schemes.

Section 2 introduces a comprehensive technology of production and residuals abatement based on the materials-balance principle, and we will rigorously derive the production function (1) as a proper though incomplete technological subsystem of the comprehensive production-cum-abatement technology. Moreover, the entire comprehensive production-cum-abatement technology will be shown to be completely represented by (1) and two further production functions mapping the domain of (1) into the abatement residuals. In Section 3 we will incorporate the comprehensive production-cum-abatement technology developed in Section 2 into a simple economy subject to pollution, and we will derive the pertaining rules for an efficient allocation. If residuals other than (unabated) production residuals also cause pollution, the optimality rules become complex, since all these pollutants are generated in strict technological interdependence. Section 4 explores the consequences of that interdependence for the design of efficiency-restoring tax schemes. Taking the conventional Pigouvian tax rule as a benchmark we show that if residuals other than production residuals contribute to pollution in addition to the latter, it is not efficient, in general, to set the tax on production residuals equal to the *conventional* marginal abatement cost, Y_e.

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[5] Ayres and Kneese (1969) developed their 'materials balance approach' in a model with strictly linear production processes. That makes it quite easy to keep track of material balance but fails to account for realistic substitution and transformation possibilities. See also Pethig (2003).

2. Abatement and production in line with materials balance

Suppose during the process of producing a consumer good a production residual is generated with each unit equal to one unit of weight. This residual is assumed to be an unwanted product, useless for consumptive and productive purposes, and harmful if discharged into the environment. It is therefore a candidate for abatement. Although it is quite common to equate abatement with disappearance into the void, abatement is clearly an activity of combining scarce inputs, both material inputs and services, for the purpose of transforming the residuals under consideration into outputs with different physical and/or chemical attributes. Abatement does "not destroy the residuals but only alters their form" (Ayres and Kneese 1969, 283).

To formalize such an abatement activity in a very simple way, let a be the amount of the *production residuals to be abated*. Technically speaking, a is an input in the transformation process to be described and needs to be combined with other inputs to make the transformation work. Suppose two other inputs are necessary: a service, called *labor*, ℓ_a, and a physical input, called *material*, m_a. With these 'ingredients', the aim of abatement is to use labor for transforming both material inputs, a and m_a, into two distinct kinds of material residuals denoted *abatement residuals*, which are assumed to differ from production residuals, a, and material, m_a, in some significant way. But it is also obvious that with given positive amounts ℓ_a and m_a one cannot transform arbitrarily large amounts of production residuals into abatement residuals. In fact, we assume that there is a technology $A : (\ell_a, m_a) \to a$ such that the amount $A(\ell_a, m_a)$ of the production residual can be abated when labor input is ℓ_a and material input is m_a. We assume the function A to have the

> **Properties (A):** $A : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is concave and satisfies* $A(0, m_a) = A(\ell_a, 0) = 0$,
> $A_\ell > 0$, $A_m > 0$, $A_{\ell\ell} < 0$, $A_{mm} < 0$ *and* $A_{\ell m} > 0$.

$A_\ell > 0$ and $A_m > 0$ implies that ℓ_a and m_a can be substituted against each other when a given amount of production residuals is to be abated. To sum up, (a, ℓ_a, m_a) is considered a feasible abatement activity, if $A(\ell_a, m_a) = a$.

The next step is to specify the generation of abatement residuals. How exactly and in which amounts abatement residuals are generated is an engineering issue that will not be pursued here. For expository purposes we will consider here the very simple case where all 'abated' production residuals are turned into one kind of abatement residuals, called abatement residu-

als of type 1, and where all material input used in the process of abatement is turned into another kind of abatement residuals, called abatement residuals of type 2. Denoting the amounts of abatement residuals by r_{a_1} and r_{a_2}, respectively, our simple technological assignment is $a = r_{a_1}$ and $m_a = r_{a_2}$.

The technology of producing consumer goods is introduced in form of the production function

$$y = F\left(\ell_y, m_y\right),$$

where ℓ_y denotes labor input, m_y denotes material input and y is the amount of consumer good produced.

Properties (F): $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is concave and satisfies* $F\left(0, m_y\right) = F\left(\ell_y, 0\right) = 0$,
$$F_\ell > 0, \ F_m \in \left]0,1\right[, \ F_{\ell\ell} < 0 \ F_{mm} < 0 \ and \ F_{\ell m} > 0.^6$$

The constraint $F_m \in \left]0,1\right[$ is absent from conventional production functions and therefore demands an explanation. Our simple production model assumes that there is one and only one material input whose quantity m_y is transformed into at least two different outputs, the consumer good (quantity y) and some production residuals (specified below), since the entropy law prevents the full transformation of material into the desired output. Since we conceive of y as a material output whose units are of constant weight, it follows immediately that

$$F_m \Big|_{\left(\ell_y, m_y\right)} < 1 \quad \text{for } m_y = 0 \ \text{ and } \ell_y \geq 0.$$

When this property of F is combined with the (conventional) assumption $F_{mm} < 0$, we conclude that $F_m < 1$ holds on the entire domain of the function F.[7]

From the preceding discussion it is obvious that $y = F\left(\ell_y, m_y\right)$ is *not* a complete description of the production technology. After all, F allows us to maintain some level (and weight) of output y while varying the amount of material input. A minimum requirement for satisfying

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[6] Note that the materials-balance principle also implies $\int_0^{\ell_y} F_\ell\left(\lambda_y, m_y\right) d\lambda_y \leq m_y$ for all $m_y \geq 0$, since the output is made up of material m_y only and the weight per unit of output y is constant. This constraint is compatible with the assumptions $Y_\ell > 0$ and $Y_{\ell\ell} < 0$ if and only if $\lim_{\ell_y \to \infty} F_{\ell\ell} = 0$.

[7] The upper bound $F_m < 1$ is usually absent in textbook treatments of production functions, and it is violated, in particular, by the popular Cobb-Douglas function.

the materials-balance principle is the existence of at least one more output, say a *production residual* r_y, to fill the materials-balance gap:

$$r_y + F\left(m_y, \ell_y\right) = m_y \qquad \text{(and } r_y \geq 0 \text{).}$$

In other words, bringing the conventional non-linear production function $y = F\left(\ell_y, m_y\right)$ in line with the materials-balance principle requires to look at production as transforming the material input, m_y, into two distinct outputs: the consumer good, y, whose generation is the purpose of the activity, and some production residuals, considered unwanted and environmentally harmful. From $r_y = m_y - F\left(m_y, \ell_y\right)$ follows $\left(dr_y / dm_y\right) = 1 - F_m$ which is positive due to the assumption $F_m \in \left]0,1\right[$.

At this point, production and abatement need to be linked up. Without any abatement, the total amount of production residuals generated, $r_y = m_y - F\left(m_y, \ell_y\right)$, would be discharged into the environment. But one can also hold back part of the production residuals from discharging, say the amount a, for abatement such that only the amount $e = r_y - a \geq 0$.

To sum up, the combined technologies of production and abatement are given by

$$y \leq F\left(\ell_y, m_y\right) \qquad \text{(2a)} \qquad\qquad \ell_a + \ell_y = \ell \qquad \text{(2e)}$$

$$r_y = m_y - y \qquad \text{(2b)} \qquad\qquad m_a + m_y = m \qquad \text{(2f)}$$

$$a \leq A\left(\ell_a, m_a\right) \qquad \text{(2c)} \qquad\qquad r_{a_1} = a \qquad \text{(2g)}$$

$$e = r_y - a \qquad \text{(2d)} \qquad\qquad r_{a_2} = m_a \qquad \text{(2h)}$$

$$\textit{The properties (A) and (F) are satisfied} \qquad \text{(2i)}$$

As discussed in the introduction, environmental economists have always been serious about joint production of wanted and unwanted outputs, about environmental damage caused by the emission of the latter and about residuals abatement to reduce emissions. But rather than focusing on comprehensive production-cum-abatement technologies such as (2), many of them used to employ production functions of type (1), i. e. the functional form $y = Y\left(e, \ell, m\right)$, where y, e, ℓ and m are defined in the same way as in (2).

The comparison between (1) and (2) readily confirms that if the production function of type (1) is at all compatible with the production-cum-abatement technology (2) it is an incomplete

description of that technology (2). The intriguing question is, therefore, what the precise relationship is between (1) and (2).[8] Can a function of type (1) with property (Y) be shown to be implied by (2) or is (1) incompatible with (2)? In the latter case, one would need to discard the technology (1) since the materials-balance principle cannot be dispensed with.

To see what the link between (1) and (2) is like we now scrutinize the comprehensive technology (2) to elicit its major properties in several steps.

Proposition 1:

(i)   *Define* $\Upsilon := \left\{ (y, e, \ell, m) \in \mathbb{R}^4 \mid z = z(v) \text{ for } z = y, e, \ell, m \text{ and } v \in \Omega \right\}$, *where* $z(v)$ *is the component z of vector* $v = \left(a, e, \ell, \ell_a, \ell_y, m, m_a, m_y, r_{a_1}, r_{a_2}, r_y, y \right) \in \mathbb{R}_+^{12}$ *and where* $\Omega := \left\{ v \mid v \text{ satisfies } (2a) - (2i) \right\}$. *The set* Υ *is convex.*[9]

(ii)   *If (2a) and (2b) hold as equalities there is a mapping* $G : (e, \ell, m) \rightarrow y$ *implied by the production system (2) whose image is set valued.*

To establish proposition 1(i), it suffices to prove convexity of the set Ω, since if Ω is convex, its projection Υ into the subspace of all vectors (y, e, ℓ, m) is convex, too. Consider $\alpha \in [0,1]$, $v^\alpha := \alpha v^1 + (1-\alpha)v^2$ and $v^i := \left(a^i, e^i, \ell^i, \ell_a^i, \ell_y^i, m^i, m_a^i, m_y^i, r_{a_1}^i, r_{a_2}^i, r_y^i, y^i \right)$ for $i = 1, 2$. By definition, Ω is convex if for any pair $v^1, v^2 \in \Omega$ it is true that $v^\alpha \in \Omega$ for all $\alpha \in [0,1]$. Hence we need to show that (2a) - (2i) is satisfied for v^α. Consider $r_y^\alpha, m_y^\alpha, y^\alpha$ and confirm that these variables satisfy (2b) by calculating

$$
\begin{aligned}
m_y^\alpha - y^\alpha &= \alpha m_y^1 + (1-\alpha)m_y^2 - \alpha y^1 - (1-\alpha)y^2 = \alpha\left(m_y^1 - y^1\right) + (1-\alpha)\left(m_y^2 - y^2\right) = \\
&= \alpha r_y^1 + (1-\alpha)r_y^2 = r_y^\alpha
\end{aligned}
$$

Quite obviously, the same procedure can be applied to all linear equations in (2). It therefore only remains to show that (2a) and (2c) are satisfied for v^α. This is easily established by observing that $y^\alpha = \alpha y^1 + (1-\alpha)y^2 \leq \alpha F\left(\ell_y^1, m_y^1\right) + (1-\alpha)F\left(\ell_y^2, m_y^2\right) \leq F\left(\ell_y^\alpha, m_y^\alpha\right)$, where the last inequality is due to the assumption that *F* is a concave function. Since the function *A* is also assumed concave, convexity of Ω is proved.

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[8] An early discussion on the relationship between technological concepts similar to (1) and (2) can be found in Siebert et al. (1980), where the technology of type (1) is referred to as net-emissions approach and that of type (2) as gross-emissions approach. Yet the technological concepts employed in Siebert et al. (1980) are not in line with the materials-balance principle. See also Pethig (2003).

[9] The sets Ω and Υ are also closed since (2) doesn't contain inequalities excluding the equality sign.

We now turn to proving proposition 1(ii). Let (2a) and (2c) hold as equalities and consider (2a), (2c) and (2d) - (2f) to turn (2b) into $m_y = e + A(\ell - \ell_y, m - m_y) + F(\ell_y, m_y)$. Total differentiation of this equation yields, after some rearrangement of terms,

$$dm_y = \frac{1}{1 + A_m - F_m} de + \frac{A_\ell}{1 + A_m - F_m} d\ell - \frac{A_\ell - F_\ell}{1 + A_m - F_m} d\ell_y + \frac{A_m}{1 + A_m - F_m} dm. \qquad (3)$$

Next we set $de = d\ell = dm = 0$ in (4), totally differentiate (2a) and (2c) to obtain, under consideration of (2e) and (2f),

$$dy = F_\ell d\ell_y + F_m dm_y = \frac{F_\ell(1 + A_m) - F_m A_\ell}{1 + A_m - F_m} d\ell_y \gtreqless 0, \qquad (4a)$$

$$da = -A_\ell d\ell_y - A_m dm_y = -\frac{A_m F_\ell + A_\ell(1 - F_m)}{1 + A_m - F_m} d\ell_y < 0 \quad \text{and hence} \qquad (4b)$$

$$\frac{dy}{da} = \frac{F_m A_\ell - F_\ell(1 + A_m)}{A_m F_\ell + A_\ell(1 - F_m)} \gtreqless 0. \qquad (4c)$$

Since $F_m \in \,]0,1[$ and $A_m > 0$ by assumption we have $1 + A_m - F_m > 0$. However, the numerator on the right side of (4a) may attain either sign so that y may increase or decrease by shifting inputs from production, (ℓ_y, m_y), to abatement, (ℓ_a, m_a), while leaving net emissions e unchanged. Equation (4c) demonstrates that for given (e, ℓ, m) shifting the inputs labor and material between production and abatement affects both the amount of production residuals abated and the amount of consumer goods produced. The sign of *dy/da* in (4c) is unclear.

The message of proposition 1(i) is that if there is a function of type (1) implied by (2) it will be concave. Unfortunately, proposition 1(ii) informs us that there is a correspondence rather than a function. Yet this lack of uniqueness can be overcome in a natural way since we are interested, of course, in the level of abatement which, for any given (e, ℓ, m), yields the maximum possible amount of the *wanted* output. That particular level of abatement will be called *efficient*. To characterize the efficient abatement we maximize with respect to a, ℓ_a, ℓ_y, m_a, m_y and r_y the wanted output $F(\ell_y, m_y)$ subject to (2b) – (2f). The associated Lagrangean reads

$$L = y + \lambda_y \left[F(\ell_y, m_y) - y \right] + \lambda_a \left[A(\ell_a, m_a) - a \right] + \lambda_e \left[e + a + y - m_y \right]$$
$$+ \lambda_\ell \left[\ell - \ell_a - \ell_y \right] + \lambda_m \left[m - m_a - m_y \right],$$

where λ_y, λ_a, λ_e, λ_ℓ and λ_m are Lagrangean multipliers. Since the objective function is linear and all terms in the cornered brackets are concave functions the Kuhn-Tucker conditions are necessary and sufficient for a maximum. An interior solution implies

$$\frac{F_\ell}{F_m} \cdot (1 + A_m) = A_\ell \quad \text{as well as} \quad A_\ell > F_\ell . \tag{5}$$

Note that (5) also follows from setting $dy/da = 0$ in (5c). $A_\ell > F_\ell$ follows from rearranging the equation in (5): $F_\ell (1 + A_m - F_m) = F_m (A_\ell - F_\ell) > 0$.

With the concept of efficient abatement we now continue our inquiry into the relationship between the production system (2) and the production function (1).

Proposition 2: *If abatement is efficient, the production-cum-abatement technology (2) implies a production function $Y : D \to \mathbb{R}_+$ that exhibits the properties (Y*) defined as*

(a)   Y exhibits the properties (Y),

(b)   Y satisfies $Y_m \in]0, F_m[$ [where $F_m \in]0,1[$ due to properties (F)],

(c)   The domain of Y is $D := \{(e, \ell, m) \mid e \le m - F(\ell, m)\} \subset \mathbb{R}_+^3$;

Proposition 2 will now be proved in six steps.

Step 1: *If abatement is efficient, the set of equations (2a) – (2f) implies a function $M : (e, \ell, m) \to m_y$ such that*[10]

$$m_y = M(e, \ell, m) . \tag{6}$$
$$\text{+ + +}$$

Rewrite (5a) as $F_\ell (1 + A_m) = A_\ell F_m$, totally differentiate this equation and combine the result with $d\ell = d\ell_a + d\ell_y$ and $dm = dm_a + dm_y$ from (2e) and (2f). After some rearrangements, these operations result in

$$d\ell_y = \frac{\gamma_{m_y}}{\gamma_{\ell_y}} dm_y + \frac{\gamma_\ell}{\gamma_{\ell_y}} d\ell - \frac{\gamma_m}{\gamma_{\ell_y}} dm , \tag{7}$$

where $\gamma_{\ell_y} := A_\ell F_{m\ell} + F_\ell A_{m\ell} - (1 + A_m) F_{\ell\ell} - F_m A_{\ell\ell} > 0$, $\gamma_\ell := F_\ell A_{m\ell} - F_m A_{\ell\ell} > 0$, $\gamma_{m_y} := F_m A_{\ell m} + (1 + A_m) F_{\ell m} - F_\ell A_{mm} - A_\ell F_{mm} > 0$ and $\gamma_m := F_m A_{\ell m} - F_\ell A_{mm} > 0$.

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[10] A plus or minus sign underneath an argument of a function indicates the (assumed) sign of the corresponding first partial derivative.

Next we insert $d\ell_y$ from (7) into (3):

$$dm_y\left(1+A_m-F_m\right)=de+A_\ell d\ell+A_m dm-\frac{\left(A_\ell-F_\ell\right)\gamma_{m_y}}{\gamma_{\ell_y}}dm_y-\frac{\left(A_\ell-F_\ell\right)\gamma_\ell}{\gamma_{\ell_y}}d\ell+\frac{\left(A_\ell-F_\ell\right)\gamma_m}{\gamma_{\ell_y}}dm.$$

Rearranging this equation yields

$$dm_y=\frac{1}{\pi_{m1}}de+\frac{\pi_{m\ell}}{\pi_{m1}}d\ell+\frac{\pi_{mm}}{\pi_{m1}}dm,\tag{8}$$

where $\quad\pi_{m1}=\left(1+A_m-F_m\right)\gamma_{\ell_y}+\left(A_\ell-F_\ell\right)\gamma_{m_y}>0$, $\quad\pi_{mm}=A_m\gamma_{\ell_y}+\left(A_\ell-F_\ell\right)\gamma_m>0$

and $\quad\pi_{m\ell}=A_\ell\gamma_{\ell_y}-\left(A_\ell-F_\ell\right)\gamma_\ell=\left(A_\ell+F_\ell\right)\gamma_\ell+\left[A_\ell F_{m\ell}-\left(1+A_m\right)F_{\ell\ell}\right]A_\ell>0$.

Obviously, equation (8) determines the first derivatives of a function $M:(e,\ell,m)\to m_y$. Hence (6) is established.

Step 2: *If abatement is efficient, the set of equations (2a) – (2f) implies a function $L:(e,\ell,m)\to\ell_y$ such that*

$$\ell_y=L(e,\ell,m).\tag{9}$$
$$+\ \ +\ \ ?$$

dm_y from (3) is now plugged into (7) yielding after some calculations

$$d\ell_y=\frac{\gamma_{m_y}}{\pi_{m1}}de+\frac{\rho_\ell}{\pi_{m1}}d\ell+\frac{\rho_m}{\pi_{m1}}dm,\tag{10}$$

where $\quad\rho_\ell:=\left(1+A_m-F_m\right)\gamma_\ell+A_\ell\gamma_{m_y}>0\quad$ and $\quad\rho_m:=\left(1-F_m\right)\gamma_m-\left[\left(1+A_m\right)F_{\ell m}-A_\ell F_{mm}\right]A_m$.

In view of (10) there is a function $L:(e,\ell,m)\to\ell_y$, satisfying $\ell_y=L(e,\ell,m)$.
$$+\ \ +\ \ ?$$

Step 3: *The preceding steps 1 and 2 imply a production function $Y:(e,\ell,m)\to y$ satisfying $Y_e>0$, $Y_\ell>0$ and $Y_m\in\,]0,F_m[$.*

Invoking (6) and (9) the production function (2a) is turned into

$$y=F\left[L(e,\ell,m),M(e,\ell,m)\right]=:Y(e,\ell,m).\tag{11}$$
$$+\ \ +\ \ ?+\ \ +\ \ +$$

The function Y defined above obviously exhibits $Y_e>0$ and $Y_\ell>0$ but the sign of

$$Y_m=F_\ell L_m+F_m M_m\tag{12}$$

is unclear. To prove $Y_m \in \,]0, F_m[$, let (2a) and (2c) hold as equalities and combine the equations (2a) – (2f), (6) and (9) to rewrite (2b) as

$$e + A\big[\ell - L(e, \ell, m), \; m - M(e, \ell, m)\big] + F\big[L(e, \ell, m), \; M(e, \ell, m)\big] = M(e, \ell, m).$$

We differentiate this equation with respect to m (keeping e and ℓ constant):

$$-A_\ell L_m + A_m\big(1 - M_m\big) + F_\ell L_m + F_m M_m = M_m$$

which yields, after rearrangement of terms,

$$L_m = \frac{A_m - \big(1 + A_m - F_m\big)M_m}{A_\ell - F_\ell}.$$

Next we substitute this term for L_m in (11) and get

$$Y_m = F_\ell L_m + F_m M_m = \frac{A_m F_\ell}{A_\ell - F_\ell} - \left[\frac{F_\ell\big(1 + A_m - F_m\big)}{A_\ell - F_\ell} - F_m\right] M_m = \frac{A_m F_\ell}{A_\ell - F_\ell},$$

since owing to (5) the term in the cornered brackets is zero. Moreover, A_ℓ, A_m and F_ℓ are positive due to the properties (A) and (F), and $A_\ell - F_\ell$ is positive due to (5). Therefore

$$Y_m = \frac{A_m F_\ell}{A_\ell - F_\ell} > 0. \text{ By invoking (5) again we find } Y_m = \frac{A_m F_\ell}{A_\ell - F_\ell} = F_m - \frac{F_\ell\big(1 - F_m\big)}{A_\ell - F_\ell} < F_m.$$

Step 4: *The production function Y from (11) is concave.*

In view of the complex terms constituting its first derivatives there is no way to further specify its curvature by determining the sign of its second-order derivatives. Yet concavity of Y is straightforward from the convexity of the set Υ that was established in proposition 1(i).

Step 5: *The domain of the production function Y*

We now determine the domain D of function Y by observing that (3) yields

$$\frac{de}{dm_y} = 1 + A_m - F_m > 0 \quad \text{and} \quad \frac{de}{d\ell_y} = A_\ell - F_\ell > 0$$

for given ℓ and m. That implies, in turn,

$$\max_{\ell_y, m_y} \Big[m_y - A\big(\ell - \ell_y, m - m_y\big) - F\big(\ell_y, m_y\big)\Big] = m - A(0,0) - F(\ell, m) = m - F(\ell, m).$$

Consequently, for given inputs ℓ and m the amount of production residuals emitted is largest when no abatement takes place at all. As an implication, the domain of Y from (11) is D as

defined above. The upper bound which is placed on e in D is due to the fact that e is an output and the law of mass conservation doesn't allow for an arbitrary expansion of a material output in a production process with a limited (finite) amount of material input. This completes the proof of proposition 2.

Proposition 2 constitutes an important step toward reconciling the use of the production function (1) with material-balance requirements. It shows (i) that if efficient abatement is presupposed, the output of the consumer good is uniquely determined by (e, ℓ, m) and (ii) that the implied production function satisfies the properties (Y) since it satisfies the properties (Y*). Thus we confirmed that the properties (Y) are necessary for any production function to be compatible with the technology (2). We also showed, however, that (2) imposes further constraints on the function Y from (11) concerning its domain and the derivative Y_m. Both these additional restrictions can be easily violated if production functions Y of type (1) are employed that exhibit the properties (Y) only. In particular, $Y_m \in \,]0, F_m[$ implies that popular parametric production functions such as Cobb-Douglas functions don't qualify for representing technologies of type (2).

In proposition 2 we didn't account for the residuals resulting from the abatement process. Now we make up for this omission in

Proposition 3:

(i)   *Provided that the abatement activity is always kept at an efficient level, the production-cum-abatement technology (2) is equivalent to three functions Y, R^1, and R^2 that map $(e, \ell, m) \in D$ into $Y(e, \ell, m), R^1(e, \ell, m)$ and $R^2(e, \ell, m)$, where D and Y are specified as in proposition 2, and where*

$$r_{a_1} = R^1(e, \ell, m) := A\left[\ell - L(e, \ell, m), m - M(e, \ell, m)\right] \geq 0, \tag{13}$$

$$r_{a_2} = R^2(e, \ell, m) := T\left[R^1(e, \ell, m), \ell - L(e, \ell, m)\right] \geq 0, \text{ and} \tag{14}$$

$T : (a, \ell_a) \rightarrow m_a$ *is a function such that* $m_a = T(a, \ell_a)$, *if and only if* $a = A(\ell_a, m_a)$.

(ii)   *The functions R^1 and R^2 are concave. Its derivatives are indeterminate in sign except for $R_e^1 < 0$ and $R_e^2 < 0$.*

First we specify the functions R^1 and R^2 from (13) and (14), respectively. As for the abatement residuals of type 1, we combine (2c), (2e) - (2g), (8) and (10) to obtain (13). Since $r_{a_2} = m_a$ due to (2h), we obtain $r_a = T(a, \ell_a)$ (with T as defined in proposition 3(i)). When combined with (9), (13), (2e) and (2g), this equation yields (14). Concavity of the functions R^1 and R^2 follows immediately from the convexity of the set Ω established in the proof of proposition 1(i).

To see how the generation of abatement residuals reacts on the emission of production residuals we take the partial derivatives

$$R_e^1 = -\left(A_\ell L_e + A_m M_e\right) \quad \text{and} \quad R_e^2 = -\left[T_a A_m M_e + \left(T_a A_\ell + T_\ell\right) L_e\right] = -M_e.$$

While $R_e^1 < 0$ is obvious, $R_e^2 = -M_e < 0$ is explained by observing that total differentiation of $a = A(\ell_a, m_a)$ yields $dm_y = \dfrac{1}{A_m} da - \dfrac{A_\ell}{A_m} d\ell_a$ and that in view of the definition of function T above we clearly have $T_a = (1/A_m)$ and $T_\ell = -(A_\ell/A_m)$. The signs of the partial derivatives R_e^1 and R_e^2 are as expected: If the scale of abatement is stepped up (such that the emissions e are reduced) more abatement residuals are generated and vice versa. Checking the remaining partial derivatives of (13) and (14) reveals that their sign is ambiguous. This indeterminacy demonstrates that the impact of ℓ and m on r_{a_1} and r_{a_2} is quite complex in spite of the simple hypotheses (2g) and (2h).

In view of the propositions 2 and 3 the technology (2) is completely and compactly described by

$$y = Y(e, \ell, m), \quad r_{a_1} = R^1(e, \ell, m) \quad \text{and} \quad r_{a_2} = R^2(e, \ell, m). \tag{15}$$

where Y, R^1 and R^2 are as defined in (11), (13) and (14), respectively. In proposition 2 the function Y from (11) has been shown to satisfy properties (Y*) while the curvature of R^1 and R^2 is less well known (proposition 3(ii)). With proposition (2) our main goal is clearly achieved, namely to show that the properties (Y) are necessary conditions for functions (1) to represent a subsystem of the comprehensive technology (2). In proposition (2) we even proved that the properties (Y) are not sufficient since the properties (Y*) are more restrictive than the properties (Y).

Yet the characterization of the relationship between (1) and (2) is still incomplete from a theoretical point of view. One may want to know which the comprehensive set of conditions is that must be satisfied by the functions Y, R^1 and R^2 from (15) to secure that these functions are equivalent to some technology (2). Unfortunately there is little hope to make progress toward this end because the properties of Y, R^1 and R^2 from (15) are made up of a complex mix of first and higher-order derivatives of the functions A and F from (2). The basic production functions A and F determine the curvature of Y, R^1 and R^2 in a very complex and interdependent way involving a constrained maximization procedure (efficient abatement).

It is true that without a complete set of conditions we cannot decide whether a given conventional production function (1) is compatible with the comprehensive technology (2) or not. Yet the value of such a result is not so clear. The information provided in propositions 2 and 3 appears to be sufficient for most *theoretical* modeling exercises, since in those studies first and higher-order derivatives are usually not quantitatively (let alone numerically) specified. For applied research, it is hardly appropriate to start out with some 'arbitrary' function Y satisfying the properties (Y*). One would rather have to start with the empirically valid specification of the *true* technology (2) anyway, since (11), (13) and (14) are and always will be derived from the true empirically observable technology (2).

3. Allocative efficiency and materials balance in an economy with production, abatement and pollution

We now envisage a simple economy where an (aggregate) firm applies technology (2). The consumption of the only wanted output y is modeled as a process of material transformation like production and abatement: it consists in turning the amount y of the consumer good into the amount r_c of a post-consumption residual. Both are equal in weight,

$$y = r_c. \tag{16}$$

Thus we now deal with four different types of residuals: e, r_{a1}, r_{a2} and r_c, each of which has the potential to degrade the environment after having been discharged. Denote by x an index of the ambient concentration of pollutants, called *pollution* for short. We define

$$x = X\left(\underset{+}{e}, \underset{\geq 0}{r_{a_1}}, \underset{\geq 0}{r_{a_2}}, \underset{\geq 0}{r_c}\right) \quad \text{with } X(0,0,0,0) = 0 \tag{17}$$

as the pollution that results from releasing the residuals e, r_{a1}, r_{a2} and r_c into the environment. X is assumed to be non-decreasing and convex. If $X_v = 0$ for $v = r_{a1}, r_{a2}$, or r_c the level of the residual under consideration is small enough to be fully neutralized by nature's assimilative capacity. Environmental degradation as described by x negatively affects the consumers' utility

$$u_i = U^i \left(\underset{-}{x}, \underset{+}{y_i} \right) \quad i = 1, \ldots, n. \tag{18}$$

The general equilibrium model is completed by introducing the standard resource constraints

$$y \geq \sum_i y_i, \quad \overline{\ell} \geq \ell \quad \text{and} \quad \overline{m} \geq m, \tag{19}$$

where $\overline{\ell}$ and $\overline{m}$ are the economy's fixed factor endowments. To characterize a Pareto efficient allocation of the economy (15) - (19) we solve the Lagrangean

$$\sum_i \alpha_i U^i \left(x, y_i \right) + \lambda_f \left[Y(e, \ell, m) - y \right] + \lambda_x \left[x - X \left(e, r_{a1}, r_{a2}, r_c \right) \right] + \lambda_1 \left[r_{a1} - R^1 (e, \ell, m) \right] +$$
$$+ \lambda_2 \left[r_{a2} - R^2 (e, \ell, m) \right] + \lambda_y \left(y - \sum_i y_i \right) + \lambda_c \left(r_c - y \right) + \lambda_\ell \left(\overline{\ell} - \ell \right) + \lambda_m \left(\overline{m} - m \right), \tag{20}$$

where α_i are arbitrary positive numbers for $i = 1, \ldots, n$. Under the assumption that an interior[11] solution exists the first-order conditions are

$$\lambda_y = \alpha_i U_y^i \quad i = 1, \ldots, n \tag{21a}$$
$$\lambda_x = -\sum_i \alpha_i U_x^i \tag{21b}$$
$$\lambda_f Y_e = \lambda_x X_e + \sum_j \lambda_j R_e^j \tag{21c}$$
$$\lambda_f Y_\ell = \lambda_\ell + \sum_j \lambda_j R_\ell^j \tag{21d}$$
$$\lambda_f Y_m = \lambda_m + \sum_j \lambda_j R_m^j \tag{21e}$$

$$\lambda_y = \lambda_f + \lambda_c \tag{21f}$$
$$\lambda_c = \lambda_x X_{r_c} \tag{21g}$$
$$\lambda_1 = \lambda_x X_{r_{a1}} \tag{21h}$$
$$\lambda_2 = \lambda_x X_{r_{a2}} \tag{21k}$$

Proposition 4: *The efficient allocation of the economy (15) - (19) is characterized by*

$$MD_x = \frac{Y_e}{X_e} \cdot Q \quad \text{with} \quad MD_x := -\sum_i \frac{U_x^i}{U_y^i} \quad \text{and} \quad Q := \frac{\left(1 + X_{r_c} \sum_i \frac{U_x^i}{U_y^i} \right) X_e}{X_e + \sum_j X_{r_{aj}} R_e^j}. \tag{22}$$

As a consequence,

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[11] In the present context, an interior solution implies that (e, ℓ, m) is in the interior of the domain D of function Y. This doesn't only require all variables to be bounded away from zero but also that the inequality $e \leq m - F(\ell, m)$ is not binding. For the sake of completeness one would have had to consider this inequality as an additional constraint in (20). But with the strict inequality sign in the solution the associated Lagrange multiplier would be zero and therefore need not be introduced in the first place.

$$MD_x \cdot X_e \left\{ \begin{matrix} \geqq \\ < \end{matrix} \right\} Y_e \quad if \quad \begin{cases} X_{r_{a1}} > 0 \ and / or \ X_{r_{a2}} > 0 \ and \ X_{r_c} = 0, \\ X_{r_{a1}} = X_{r_{a2}} = X_{r_c} = 0, \\ X_{r_{a1}} = X_{r_{a2}} = 0 \ and \ X_{r_c} > 0. \end{cases} \tag{23}$$

To derive (22) from (21), we first consider (21f) and (21g) to obtain $\dfrac{\lambda_y}{\lambda_f} = \dfrac{\lambda_f + \lambda_c}{\lambda_f} =$

$= 1 + \dfrac{\lambda_x}{\lambda_f} X_{r_c}$. Next we combine this equation with (21a) to turn (21b) into

$$\frac{\lambda_x}{\lambda_f} = -\frac{\lambda_y}{\lambda_f} \sum_i \frac{U_x^i}{U_y^i} = -\left(1 + \frac{\lambda_x}{\lambda_f} X_{r_c}\right) \sum_i \frac{U_x^i}{U_y^i} = -\frac{\sum_i \dfrac{U_x^i}{U_y^i}}{1 + X_{r_c} \sum_i \dfrac{U_x^i}{U_y^i}} . \tag{24}$$

In (24), $\left(1 + X_{r_c} \sum_i \dfrac{U_x^i}{U_y^i}\right) \in \,]0,1[$, since $\lambda_x / \lambda_f > 0$ in an interior solution. We proceed by infer-

ring $\dfrac{\lambda_x}{\lambda_f} = \dfrac{Y_e}{X_e + \sum_j X_{r_{aj}} R_e^j}$ from (21c), (21h) and (21k). (22) is then obtained by inserting

this equation into (24).

Clearly, (22) characterizes the efficient level of pollution generated jointly via the emission of *all* residuals. MD_x is the damage from a marginal *increase in pollution* evaluated by the consumers' aggregate marginal willingness-to-pay in terms of the consumer good for avoiding that increase. The right side of (22) represents the marginal benefit of pollution that comes in form of an increase in the consumer good made possible by a small increase in pollution through stepping up emissions. Equivalently, one can interpret MD_x as the marginal benefit and $Y_e Q / X_e$ as the marginal cost of *reducing pollution*. To further interpret the term $Y_e Q / X_e$ observe that Y_e is the amount of consumer goods that cannot be produced anymore when the emission of production residuals is reduced by a small amount. It is known as *marginal abatement costs of production residuals* (in terms of the consumer good) in models where no residuals other than production residuals are considered. For convenience, we will refer to Y_e as *conventional* marginal abatement costs of production residuals.

If we multiply both sides of (22) by X_e we obtain $MD_x \cdot X_e = Y_e \cdot Q$. The left side of this equation represents the marginal benefit of reducing the emission of production residuals and

the right side consists of the marginal abatement costs of production residuals. Those costs differ from Y_e unless $Q = 1$. In fact, inspection of the definition of Q in (22) reveals that given $X_e > 0$ (which we presuppose throughout) Q may be greater or smaller than unity depending on which other residuals contribute to pollution. More specifically, the optimality condition $MD_x \cdot X_e = Y_e \cdot Q$ and the definition of Q from (22) imply (23).

The striking result is that Y_e, the *conventional* marginal abatement cost of production residuals, is an incorrect measure of the social marginal abatement costs of production residuals, in general, when in addition to production residuals other residuals contribute to pollution, too.[12] As shown in (23), if abatement residuals are pollutants in addition to production residuals but post-consumption residuals are not, Y_e underestimates the true marginal abatement costs of production residuals. This is so because the abatement of production residuals generates a negative pollution externality through the increase of abatement residuals which it unavoidably brings about but which is not accounted for in Y_e. Conversely, if post-consumption residuals pollute in addition to production residuals but abatement residuals don't, we infer from (23) that Y_e overestimates the (true) marginal abatement costs of production residuals since this time abatement induces a positive externality in form of a small reduction in pollution caused by post-consumption residuals which is not captured by Y_e.

4. Corrective taxes in the competitive market economy

Consider now the economy of Section 3 with perfectly competitive markets for labor, material and the consumer good and denote the associated prices by p_ℓ, p_m and p_y, respectively. Quite obviously, such a market economy fails to be efficient in the absence of environmental policy. We are therefore interested in exploring what kind of tax schemes is capable to restore efficiency. As is well-known, if we would replace in our model (15) by (1) and replace (17) by $x = X(e)$ with $X_e > 0$, the pollution externality would be internalized by the time-honored Pigouvian tax rule according to which the tax rate t_e on emissions e is to be set equal to the *conventional* marginal abatement cost, Y_e. But in our model with the comprehensive production-cum-abatement technology (2) it is also interesting to investigate the deviations

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[12] The added term 'in general' weakens that statement slightly. Yet it is necessary since marginal pollution effects of abatement residuals and post-consumption residuals are opposite in sign. It cannot be excluded, therefore, that both effects compensate each other incidently.

from that *conventional* Pigouvian tax rule required in a model containing (15) and (17) and when X_{ra1}, X_{ra2} and/or X_{rc} are positive.

To address this issue we need to look at the agents' optimization problems as implicitly described by the Lagrangeans[13]

$$U^i\left(x, y_i\right) + \mu_i\left[p_\ell \bar{\ell}_i + p_m \bar{m}_i + \theta_i\left(g + b\right) - p_y y\right] \text{ and} \tag{25}$$

$$\left\{p_y y - \left(p_\ell + t_\ell\right)\ell - \left(p_m + t_m\right)m - t_{a1}r_{a1} - t_{a2}r_{a2} - t_c r_c - t_e e\right\} + \mu_c\left(r_c - y\right) +$$

$$+ \mu_f\left[Y\left(e, \ell, m\right) - y\right] + \mu_1\left[r_{a1} - R^1\left(e, \ell, m\right)\right] + \mu_2\left[r_{a2} - R^2\left(e, \ell, m\right)\right]. \tag{26}$$

With regard to post-consumption residuals, implicit in (25) and (26) is the so-called take-back rule as modeled, e. g., in Eichner and Pethig (2000). Each consumer *i* purchases the amount y_i of consumer goods, and after consumption she returns the post-consumption residuals to the producers. By institutional design, producers are responsible for the (orderly) deposition of these residuals, and they may be therefore charged an emission fee, t_c, if the post-consumption residuals cause pollution after having been emitted. This institutional arrangement explains why post-consumption residuals don't enter the consumers' optimization calculus (25),[14] but rather are part of the (aggregate) firm's profit maximization calculus.

The (aggregate) firm employs the technology (15) and is charged input taxes, t_ℓ and t_m, as well as emission taxes, t_e, t_{a1}, t_{a2} and t_c. Some of these tax rates may be zero, of course. The first-order conditions associated to an (interior) solution to (25) and (26) are conveniently summarized in

$$\mu_i p_y = U_y^i, \tag{27a} \qquad \left(p_y - t_c\right)Y_\ell = p_\ell + t_\ell + \sum_j t_{aj}R_\ell^j, \tag{27c}$$

$$\left(p_y - t_c\right)Y_e = t_e + \sum_j t_{aj}R_e^j, \tag{27b} \qquad \left(p_y - t_c\right)Y_m = p_m + t_m + \sum_j t_{aj}R_m^j. \tag{27d}$$

Proposition 4:

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[13] For $v = \ell, m$, consumer *i's* factor endowment is $\bar{v}_i$ with $\sum_i \bar{v}_i = \bar{v}$. θ_i is her share in the firm's profit, *g*, and the government's budget surplus, *b*. The consumers' profit and surplus shares satisfy $\sum_j \theta_j = 1$.

[14] One can easily account for that alternative by modifying (25) and (26) in the following way: In (25), add the term " $-t_c r_{c_i}$ " to the second expression in brackets and add the Lagrange constraint " $\lambda_{c_i} r_{c_i}\left(r_{c_i} - y_i\right)$ " at the end. In (26), delete the term " $-t_c r_c$ " and the Lagrange constraint " $\mu_c\left(r_{c_i} - y\right)$ ". In the absence of institution-specific costs the conditions for allocative efficiency are the same in both regimes (Eichner and Pethig 2000).

(i) Suppose, that $\lambda_1, \lambda_2, \lambda_c, \lambda_f, \lambda_\ell, \lambda_m, \lambda_x$ and λ_y are the values attained by the Lagrange multipliers in the solution to (20) and that the partial derivatives X_e and R_h^j for j = 1, 2 and $h = \ell, m$ are also evaluated at that solution. Set prices $p_\ell = \lambda_\ell, p_m = \lambda_m, p_y = \lambda_y$ and consider the alternative tax schemes A and B:

- A is defined by: $\quad t_{a1}^A = \lambda_1, \ t_{a2}^A = \lambda_2, \ t_c^A = \lambda_c, \ t_e^A = \lambda_x X_e \ \text{and} \ t_\ell^A = t_m^A = 0;$

- B is defined by: $\quad t_{a1}^B = t_{a2}^B = 0, \ t_c^B = \lambda_c, \ t_e^B = \lambda_x X_e + \sum_j \lambda_j R_e^j, \ t_\ell^B = \sum_j \lambda_j R_e^j \ \text{and}$

$$t_m^B = \sum_j \lambda_j R_m^j.$$

With these prices and either tax scheme A or tax scheme B all markets clear and the equilibrium allocation is efficient.

(ii)   When the tax scheme A is implemented, the efficient tax rates satisfy

$$t_e^A \ \left\{ \begin{matrix} \geqq \\ < \end{matrix} \right\} \ Y_e \quad \text{if} \quad \begin{cases} X_{r_{a1}} > 0 \ \text{and/or} \ X_{r_{a2}} > 0 \ \text{and} \ X_{r_c} = 0, \\ X_{r_{a1}} = X_{r_{a2}} = X_{r_c} = 0, \\ X_{r_{a1}} = X_{r_{a2}} = 0 \ \text{and} \ X_{r_c} > 0. \end{cases} \tag{28}$$

To prove Proposition 4i insert the prices and tax rates as assigned in Proposition 4 into (27) and verify that this substitution makes (27) coincide with (21). The tax scheme *A* in Proposition 4i is a pure emissions tax scheme in the sense that a tax is levied on the emission of *each* polluting residual, while non-emission items like the inputs labor and material are not taxed. We infer from (21) that (in equilibrium) with the tax scheme *A* all tax rates are set equal to the marginal environmental damage of the respective residuals[15]:

$$t_v^A = MD_x X_v := -\sum_i \frac{U_x^i X_v}{U_y^i} \quad \text{for} \ v = e, r_{a_1}, r_{a_2}, c \tag{29}$$

We also know that if abatement takes place, optimal emissions tax rates need to equal marginal abatement costs. In our model production residuals are the only residuals subject to abatement. Therefore we will now focus on those residuals. Invoking (27b) we obtain, after some rearrangement of terms,

$$t_e^A = Y_e - \sum_j t_{aj}^A R_e^j - t_c^A Y_e. \tag{30}$$

Maintaining the assumption $X_e > 0$ (which implies $t_e^A > 0$ owing to (29)) we now wish to juxtapose alternative scenarios where one, two or all of the (efficient!) tax rates t_{a1}^A, t_{a2}^A and t_c^A may be zero. In view of $Y_e > 0$, $R_e^1 < 0$ and $R_e^2 < 0$, it is straightforward that (29) and (30) imply (28). (28) complements (24), in fact, but (28) was derived via (30) rather than via (23) and thus offers insights in the determinants of the efficient emissions tax rate t_e.

Suppose first, pollution is caused by the emission of production residuals only $\left(X_{ra1} = X_{ra2} = X_{rc} = 0 \right)$. Then $t_e = Y_e$ is optimal due to (28), i. e. the tax rate t_e is required to be set equal to the *conventional* marginal costs of abating production residuals. When combined with (29) $t_e = Y_e$ yields

$$-\sum_i \frac{U_x^i X_e}{U_y^i} = t_e = Y_e \tag{31}$$

which constitutes the *conventional Pigouvian tax rule.* Yet in view of possible deviations from that rule, as described in (29) and (31), (31) is not a trivial result because we specified conditions under which the *conventional* Pigouvian tax rule is valid.

Suppose next that $X_{r_{a1}} > 0$ and/or $X_{r_{a2}} > 0$ but $X_{r_c} = 0$. In this case the social marginal abatement costs of production residuals are $Y_e - \sum_j t_{aj}^A R_e^j > Y_e$. The extra marginal costs, $\sum_j t_{aj} R_e^j$, accrue because an increase in the abatement of production residuals inadvertently raises the generation of environmentally harmful abatement residuals. The result $t_e^A > Y_e$ is worth noting: It is now optimal to abate *less* production residuals than under the *conventional* Pigouvian tax rule. Conversely, if abatement residuals don't degrade the environment but post-consumption residuals do $\left(X_{r_{a1}} = X_{r_{a2}} = 0 \text{ and } X_{r_c} > 0 \right)$, then the social marginal abatement costs of production residuals are $Y_e - t_c^A Y_e < Y_e$. The extra marginal cost savings, $t_c^A Y_e$, occur because an increase in the abatement of production residuals inadvertently reduces the amount of post-consumption residuals and consumer goods. Since post-consumption residuals are (now) pollutants by assumption, curbing r_c reduces that pollution which is, in turn, a beneficial external effect of abating production residuals. With $t_e^A < Y_e$ it is now optimal to abate *more* production residuals than under the *conventional* Pigouvian tax rule.

Despite our main focus on the discussion of (28) one needs to keep in mind that according to (29) the efficient tax scheme *A* requires to levy a tax on *all* polluting residuals. It doesn't follow from setting $t_e^A \neq Y_e$ as prescribed by (28) that no emissions tax other than t_e is needed to achieve efficiency in case that the production residuals are not the only pollutants. For that scenario our analysis provides an important lesson: If all polluting residuals are taxed in an effort to correct for the allocative distortion, it is not efficient, in general, to set the tax rate of the production residuals equal to their marginal abatement cost, *conventionally defined*.

Turning to the interpretation of tax scheme *B* we observe that *B* is also capable of restoring efficiency in the market economy and does so without any tax on abatement residuals. This scheme is a particularly interesting option for efficient pollution control if abatement residuals are difficult and costly to monitor and therefore cannot readily be used as a tax base. However, avoiding taxes on abatement residuals comes at the price of taxing labor and material (in addition to post-consumption and production residuals).[16] Since the signs of the derivatives R_ℓ^j and R_m^j for $j = 1, 2$ are ambiguous, it is not clear, whether t_m and t_ℓ are subsidies or taxes (proper). At any rate, securing efficiency by means of tax scheme *B* requires to drive tax wedges between demand prices and supply prices on all three markets. Taking a closer look at t_e under tax scheme *B* shows that

$$-\sum_i \frac{U_x^i}{U_y^i}\left(X_e + \sum_j X_{r_{aj}} R_e^j\right) = t_e^B = Y_e - t_c^B Y_e. \tag{32}$$

To the left of the first equality sign in (32) we have the sum of the direct (positive) and indirect (negative) marginal benefits of a small reduction in the emission of production residuals. The indirect benefit is, in fact, the marginal environmental damage from the emission of abatement residuals caused by stepping up abatement. Correspondingly, the far right side of (32) represents the social marginal abatement cost which is the same as in tax scheme *A* for the case that $X_{ra1} = X_{ra2} = 0$. Since $t_e^B < Y_e$, it is optimal to abate more production residuals than under the *conventional* Pigouvian tax rule.

5. Concluding remarks

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[16] One may wonder why there isn't a third efficiency restoring tax scheme, that taxes also labor and material (like *B*) but in which t_e captures the impact of *all* kinds of residuals. After all, due to $r_c = y$, post-consumption residuals are generated uno actu with all other outputs. However, when we set $t_{a1} = t_{a2} = t_c = 0$ in (26) there is no way to find values for t_ℓ, t_m and t_e such that market equilibria turn out to be efficient.

It is a standard procedure in environmental economics to model abatement as a non-linear technological process to reduce the discharge into the environment of a residual generated as a by-product of a wanted good. It is also standard in formal analyses of production-cum-abatement to map incompletely the materials flow that is inevitably involved in that process. The present paper shows how the standard way of modeling production-cum-abatement can be brought into line with physical constraints securing materials balance. It demonstrates, in particular, that the production function (1) can be reconstructed from a comprehensive production-cum-abatement technology. Although (1) maps the materials flow incompletely, a production function of type (1) has been shown to be implied by each technology (2) that fully accounts for all materials flows involved. We have also been proved that the implied function Y satisfies the properties (Y*) that are sufficient but not necessary for the properties (Y). It remains an open question, therefore, whether for each function Y satisfying the aforementioned properties (Y*) there exists a technology (2) such that Y is implied by that technology. Therefore we also don't know the comprehensive set of conditions that makes the functions Y, R^1 and R^2 from (15) fully compatible with the technology (2).

Moreover, in a simple general equilibrium model with full regard of the materials flow the paper explores the consequences for allocative efficiency and efficiency-restoring taxation of multiple interdependent pollutants that are inevitably linked to the transformation processes of production, abatement and consumption. Finally, the paper demonstrates that efficiency may require setting the emissions tax rate above or below conventionally defined marginal abatement cost if the residual subject to abatement is not the only residual causing environmental degradation.

Krysiak and Krysiak (2003) address the issue of modeling all processes of materials transformation consistently in an analytical framework that aims at maximum generality. We proceeded, instead, by trading generality for more specific and more informative results about the emergence and consequences of multiple and interdependent residuals and pollutants involved in production-cum-abatement when the materials-balance principle is explicitly and fully regarded.

As for the consequences of multiple residuals, we find, not surprisingly, that if the production residual, r_y, is the only pollutant it suffices to employ the 'truncated' production-cum-abatement technology (1). However, as soon as the emission of at least one additional residual in the pertaining materials flow also contributes to the degradation of the environment, the need for an integrated multi-pollutant control arises (Guruswanny 1991). There is a growing

awareness in both academia and the political arena (U.S. General Accounting Office 1996) that the most efficient control strategy is to consider multiple pollutants simultaneously rather than continue with the prevailing single-pollutant regulations. However, the demand for multi-pollutant control approaches is quite often rationalized by growing concerns about the potential risks to human health and/or to the environment from the interaction of multiple pollutants *after* their emission. These are certainly serious concerns in their own right. Yet the emphasis of the present paper is on allocative problems caused by *multiple pollutants whose generation is interdependent*. In that case, a first-best tax strategy also needs to account for the technical *interdependencies* among the residuals since such linkages impact on the efficient levels of all tax rates. As a consequence, the time-honored rule of equating the emissions tax rate and marginal abatement cost, defined in the conventional way, is shown to be no longer efficient, in general.

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