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SOME STABILITY RESULTS FOR MARKOVIAN ECONOMIC SEMIGROUPS

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SOME STABILITY RESULTS FOR MARKOVIAN ECONOMIC SEMIGROUPS

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Abstract. The paper studies existence, uniqueness and stability of stationary equilibrium distributions in a class of stochastic dynamic models common to economic analysis. The stability conditions provided are suitable for treating multi-sector models and nonlinear time series models with unbounded state.

1. Introduction

Stability and instability of random dynamic systems are among the most fundamental themes of economic modeling. In the theory of long-run growth, stability is the key criterion behind convergence (or divergence) of cross-country income series. Stability analysis also has applications to business cycle fluctuations, demand for credit and real cash balances, sustainable exploitation of renewable resources, and calculation of ruin probabilities given cash flows from insurance premiums and claims. For models of economic learning stability determines the degree of convergence to long-run rational expectations equilibria. In econometrics many Monte Carlo calculations rely on the stability of Markov chains which have as their limit the distribution from which one wishes to sample. In operations research the stability of queues must be analyzed in order to determine their relative cost and optimal

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control, with applications to flexible manufacturing systems and the design of service facilities.[1]

In this paper we study the large class of dynamic economic models whose evolution can be described by a semigroup of operators $(\mathbf{P}_t)_{t \in \mathbb{T}}$ on $L_1 := L_1(S, \mathscr{B}(S), \lambda)$, where topological space S is the state space for the endogenous variables of the economic system, $\mathscr{B}(S)$ is the Borel sets on S, and λ is some σ-finite measure. The idea is that for many Markovian economic models one can construct the semigroup $(\mathbf{P}_t)$ such that if $\psi \in L_1$ is a density that gives the probability distribution of the initial condition, then its image under $\mathbf{P}_t$ is the density which gives the probability distribution of the state variable at time $t \in \mathbb{T}$, so that the map $t \mapsto \mathbf{P}_t \psi$ describes the orbit or *flow* of probability mass over time. Here $\mathbb{T}$ may be either $[0, \infty)$ or $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Our interest is in whether or not this system is (globally) asymptotically stable, in the sense that there is a unique density ψ^* with the property

$$(1) \qquad \mathbf{P}_t \psi^* = \psi^*, \; \forall \, t \in \mathbb{T}, \;\; \text{and} \; \lim_{t \to \infty} \|\mathbf{P}_t \psi - \psi^*\| = 0, \; \forall \psi \in \mathscr{D}.$$

Here $\| \cdot \|$ is the L_1 norm, and $\mathscr{D} := \{\psi \in L_1 \, : \, \psi \geq 0 \text{ and } \|\psi\| = 1\}$ is the collection of all densities on S. The objective is to develop simple sufficient conditions for (1) that are both applicable and easy to verify

[1]A *very* partial list of references is as follows. For stochastic growth see Mirman (1970) and Brock and Mirman (1972). For business cycles and stability see for example Long and Plosser (1983), or Farmer and Woodford (1997); for money demand see Lucas (1980), or Stokey Lucas and Prescott (1989). Sustainable exploitation is discussed in Mitra and Roy (2003). The literature on stability in queues is vast. A reference in manufacturing systems is Courcoubetis and Weber (1994). Bray (1982) and Evans and Honkapohja (2001) are well-known studies of stability in learning processes.

for common economic and econometric models, as well as to extend existing results on asymptotic stability of Markov semigroups.

After stating our main stability result two applications are given. One is a short proof of asymptotic stability for the threshold autoregression model of Chan and Tong (1986) under suitable conditions on parameters and the shock. The second gives stability conditions for discrete time models evolving on the positive cone of finite dimensional vector space. Such models are typical of economic applications.

Conditions for dynamic stability of stochastic economic models with a Markovian structure has been studied by many authors. Early studies include Mirman (1970), Razin and Yahav (1979) and Futia (1982). A summary of these techniques with new material is given in Stokey, Lucas and Prescott (1989). For more recent work see for example Hopenhayn and Prescott (1992), Bhattacharya and Majumdar (2001, 2003) or Stachurski (2003) and their references.[2]

Many dynamic economies have a Markov structure. Recently, conditions for the existence of recursive transition rules have been found for economies with tax distortions, externalities, heterogenous agents, and so on. See, for example, Le Van, Morhaim and Dimaria (2002), or Mirman, Morand and Reffett (2004).

Mathematically, this work extends techniques developed by Lasota (1994). In that paper, Lasota developed a fundamental new method to prove asymptotic stability of integral Markov semigroups. He shows that their stability is closely connected to L_1 weak precompactness of trajectories. The present paper provide new ways to verify this property, by identifying simple conditions under which uniformly tight flows

[2]For an analysis of economic models that do not necessarily have a Markovian structure see for example Schenk-Hoppé (2002) and references therein.

of densities generated by integral Markov semigroups are also uniformly integrable. Also, we introduce Meyn and Tweedie's (1993) very general notion of norm-like functions to help identify uniformly tight trajectories. The latter method proves to be useful when the state space is a positive cone of finite-dimensional vector space, as often happens in economic theory.

2. Formulation of the Problem

First we give some definitions and examples. A linear operator $\mathbf{P}$ sending L_1 into itself (a self-mapping) is called a Markov operator if $\mathbf{P}\mathscr{D} \subset \mathscr{D}$. From the definition it follows that every Markov operator is both positive and a contraction.[3] By a Markov semigroup is meant a collection $(\mathbf{P}_t)$ of self-mappings on L_1 such that

1. $\mathbf{P}_t$ is a Markov operator for each $t \geq 0$;
2. $\mathbf{P}_0 = I$, the identity map on L_1; and
3. $\mathbf{P}_s \circ \mathbf{P}_t = \mathbf{P}_{s+t}$ for all $s, t \geq 0$ (semigroup under composition).

In practice Markov semigroups appear in several ways, probably the most common being via transition probability functions of Markovian random systems. By a transition probability function we mean a map $p \colon \mathbb{T} \times S \times S \to [0, \infty)$ such that $(x, y) \mapsto p(t, x, y)$ is $\mathscr{B}(S) \otimes \mathscr{B}(S)$-measurable, $\forall t \in \mathbb{T}$; and $p(t, x, \cdot) \in \mathscr{D}$ for each $t \in \mathbb{T}$ and $x \in S$. Heuristically, one thinks of $p(t, x, y)\lambda(dy)$ as the probability of travelling to y from x after t units of time have elapsed.

For example, many financial time series are assumed to follow an Ornstein–Uhlenbeck process

$$dX_t = -\mu X_t dt + \sigma dB_t,$$

[3]That is, $\psi \geq 0$ implies $\mathbf{P}\psi \geq 0$, and $\|\mathbf{P}\psi\| \leq \|\psi\|$, $\forall \psi \in L_1$.

where μ, σ are positive constants and $(B_t)_{t=0}^{\infty}$ is a Brownian motion. In this case it is well-known that $(X_t)_{t=0}^{\infty}$ has transition probability function

$$p(t,x,y) = \frac{1}{\sqrt{2\pi h(t)}} \exp\left(-\frac{(y - xe^{-\mu t})^2}{2h(t)} \right),$$

where $h(t) := (\sigma^2/2\mu)(1 - e^{-2\mu t})$.

It is not difficult to verify that if p is a transition probability function then the collection of operators $(\mathbf{P}_t)$ defined by

$$(\mathbf{P}_t \psi)(y) = \int_S p(t,x,y)\psi(x)\lambda(dx) \tag{2}$$

is a Markov semigroup. The density $\mathbf{P}_t\psi$ is the marginal distribution of the time t state given that p is the law of motion and ψ is the initial distribution of the state. Markov semigroups with the representation (2) for some transition probability function p will be called *integral* Markov semigroups.

Discrete time Markovian systems may also generate integral Markov semigroups. Suppose that $p\colon S \times S \to [0,\infty)$ is jointly measurable and satisfies $p(x,\cdot) \in \mathscr{D}$ for all $x \in S$, where $p(x,y)\lambda(dy)$ is thought of as representing the probability that the state variable transits from x to y in one step. If we define $p(1,x,y) := p(x,y)$, and, for each $t \in \mathbb{N}$,

$$p(t+1,x,y) := \int_S p(t,x,u)p(u,y)\lambda(du), \tag{3}$$

then $p\colon \mathbb{T} \times S \times S \to [0,\infty)$ is a transition probability function for $\mathbb{T} = \mathbb{N}_0$, and $(\mathbf{P}_t)_{t\in\mathbb{T}}$ defined as in (2) is an integral Markov semigroup when $\mathbf{P}_0 := I$. It is easy to check that in this case $\mathbf{P}_t = \mathbf{P}^t$, the t-th iterate of the map $\mathbf{P}\colon L_1 \to L_1$ defined by

$$(\mathbf{P}\psi)(y) = \int_S p(x,y)\psi(x)\lambda(dx). \tag{4}$$

Consider for example the nonlinear autoregression

$$X_{t+1} = T(X_t, \xi_t), \quad T\colon S \times E \to S, \quad t \in \mathbb{N}_0, \tag{5}$$

where (ξ_t) is an i.i.d. sequence of valued random variables on probability space $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in measurable space $(E, \mathscr{E})$, and T is a $(\mathscr{B}(S) \otimes \mathscr{E}, \mathscr{B}(S))$-measurable map. The random sequence (X_t) represents the endogenous variables, and T is some transition rule.

There is of course a large number of dynamic macroeconomic models which have the discrete recursive structure used in (5). See for example Stokey, Lucas and Prescott (1989) or Hamilton (1994) and references therein.

Loosely speaking, we can say that the conditional distribution of the next period state given that the current state $X_t = x$ is $\mathbf{M}(x, dy)$ when

$$\mathbf{M}(x, B) := \int_\Omega \mathbb{1}_B[T(x, \xi_t(\omega))]\mathbb{P}(d\omega). \tag{6}$$

Very often in economic applications it turns out that $\mathbf{M}(x, dy)$ is absolutely continuous with respect to the underlying measure λ (write $\mathbf{M}(x, dy) \precsim \lambda$), which is typically the Lebesgue measure. In this case, the transition probabilities have a density representation p, where $p(x, y)\lambda(dy) := \mathbf{M}(x, dy)$, so that p is a function on $S \times S$ with $p(x, \cdot) \in \mathscr{D}$ for all $x \in S$. If all goes well, p is jointly measurable, so we can construct the transition probability function as in (3), and therefore a semigroup $(\mathbf{P}_t)$.

All of this would be meaningless if $\mathbb{T} \ni t \mapsto \mathbf{P}_t\psi \in \mathscr{D}$ does not describe the flow of density functions for the state variables of the economy when $\psi \in \mathscr{D}$ is the distribution of x_0. Note that X_t can be viewed as a $\sigma\{\xi_0, \dots, \xi_{t-1}\}$-measurable random variable on $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mu_t :=$

$\mathbb{P} \circ X_t^{-1}$ be the marginal distribution of X_t, a measure on $(S, \mathscr{B}(S))$. Given the independence of X_t and ξ_t one has

$$\begin{aligned}\mu_{t+1}(B) &= \mathbb{P}\{T(X_t, \xi_t) \in B\} \\ &= \int_S \int_\Omega \mathbb{1}_B[T(x, \xi_t(\omega))]\mathbb{P}(d\omega)\mu_t(dx) = \int_S \mathbf{M}(x, B)\mu_t(dx).\end{aligned}$$

If $\mu_t \precsim \lambda$, then it is clear that $\mu_{t+1} \precsim \lambda$, and in fact if $\psi_t := d\mu_t/d\lambda$ for each t, then some rearranging of the above expression gives $\psi_{t+1}(y) = \int_S p(x, y)\psi_t(x)\lambda(dx)$. Comparing this with (4) we see that $\psi_{t+1} = \mathbf{P}\psi_t$, so if the initial distribution ψ_0 is in $\mathscr{D}$, then $\psi_t = \mathbf{P}^t\psi_0 = \mathbf{P}_t\psi_0$ as required.

3. Results

In this section the main stability result is proved. First we need some assumptions on the state space and the underlying measure.

Assumption 3.1. The space S is metrizable, locally compact and also σ-compact, in the sense that every open subset of S can be expressed as a countable union of compacts sets, and the measure λ is locally finite.[4]

Definition 3.1. A nonnegative, continuous function $V : S \to \mathbb{R}$ is called norm-like if there exists a sequence of compact sets (K_j) in S with $K_j \uparrow S$ and $\inf_{x \notin K_j} V(x) \to \infty$ as $j \to \infty$.[5]

For example, let S be Euclidean space, let B be the closed unit ball in S and let $V(x) = \|x\|$. Then $K_j := j \cdot B \uparrow S$ and $\inf_{x \notin K_j} V(x) = j \to \infty$.

[4]A measure λ on $(S, \mathscr{B}(S))$ is called locally finite if $\lambda(K) < \infty$ for every compact subset K of S.

[5]As usual, $K_j \uparrow S$ means that $K_j \subset K_{j+1}$, all j, and $\cup_{j=1}^{\infty} K_j = S$.

Norm-like functions were introduced in relation to Markov chains by Meyn and Tweedie (1993).

Condition 3.1. For some $s \in \mathbb{T}$, the transition probability is everywhere positive. That is $\forall x, y \in S$, $p(s, x, y) > 0$.

This condition is a "mixing" or "communication" assumption. Over the time interval $0 \leq t \leq s$, the state variable travels to any open set with positive probability.

Condition 3.2. For some $s \in \mathbb{T}$, there exists a continuous function $h \colon S \to \mathbb{R}$ such that $\sup_{x \in S} p(s, x, y) \leq h(y)$ for all $y \in S$.

Condition 3.3. For some $s \in \mathbb{T}$, there exists a norm-like function V and constants $\alpha, \beta \in [0, \infty)$, $\alpha < 1$, such that

$$\int p(s, x, y) V(y) \lambda(dy) \leq \alpha V(x) + \beta, \quad \forall x \in S.$$

Condition 2 is largely technical. Condition 3 is a drift condition, which ensures that the state variable tends to return to the "center" of the state space over time. Of course in a metric space there is no center, but we can generate the space using the expaning sequence of compact sets discussed in Condition 3.

The main theorem can now be stated. The proof is given in Section 5.

Theorem 3.1. *Let $(\mathbf{P}_t)_{t \in \mathbb{T}}$ be an integral Markov semigroup with transition probability function p. If Conditions 3.1–3.3 hold for common $s \in \mathbb{T}$, then $(\mathbf{P}_t)_{t \in \mathbb{T}}$ is asymptotically stable.*

4. Applications

In this section we give two applications of Theorem 3.1. Both are in discrete time. The first is a very simple proof of stability in L_1 norm of the Threshold Autoregression (TAR) model of Chan and Tong (1986) under suitable hypotheses on the coefficients. (See Chan and Tong, 1986, for an earlier proof.) TAR models have recently found many applications in economics (c.f., e.g., Hansen 2001).

The second application provides a condition for stability of systems evolving on the positive cone of finite dimensional Euclidean space. Many economic models have this property, given that prices and quantities are typically nonnegative.

4.1. **Threshold Autoregression.** The basic model has the form

$$X_{t+1} = \sum_{k=1}^{K} (A_k X_t + b_k) \mathbb{1}_{\{X_t \in B_k\}} + \xi_t, \tag{7}$$

where X_t takes values in $\mathbb{R}^N$, $(B_k)_{k=1}^K$ is a (measurable) partition of $\mathbb{R}^N$, and $(A_k)_{k=1}^K$ and $(b_k)_{k=1}^K$ are $N \times N$-dimensional matrices and $N \times 1$-dimensional vectors respectively. The idea is that when X_t is in the region of the state space B_k, the state variable follows the law of motion $A_k X_t + b_k$. The shock ξ is assumed to be an uncorrelated and identically distributed $\mathbb{R}^N$-valued process with density g.

For this model $S = \mathbb{R}^N$, and λ is the Lebesgue measure. We write dx, dy instead of $\lambda(dx), \lambda(dy)$ etc., and $\int$ for $\int_S$. When the current state is equal to $x \in \mathbb{R}^N$, a simple change of variable argument shows that the conditional density $p(x, \cdot)$ for the next period state is

$$p(x, y) = g\left[y - \sum_{k=1}^{K} (A_k x + b_k) \mathbb{1}_{B_k}(x) \right]. \tag{8}$$

From (8) one can contruct the transition probability functions and Markov semigroup $(\mathbf{P}_t)_{t \in \mathbb{T}}$ corresponding to (7) as described in Section 2.

An application of Theorem 3.1 gives the following stability result.

Proposition 4.1. *Let* $(\mathbf{P}_t)_{t\in\mathbb{T}}$ *be the Markov semigroup generated by the dynamical system* (7)*. Suppose that* g *is strictly positive on* $\mathbb{R}^N$*, that* $g \leq M$ *for some* $M < \infty$*, and that* $\mathbb{E}\|\xi\| := \int \|z\| g(z)dz < \infty$*. If, in addition,* $\alpha := \max_k \alpha_k < 1$*, where* α_k *is the spectral radius of* A_k*, then* $(\mathbf{P}_t)$ *is asymptotically stable.*

For example, if ξ is multivariate normal then g satisfies all of the hypotheses of Proposition 4.1.

Proof. We check that the conditions of Theorem 3.1 hold for $s = 1$, recalling that $p(1, x, y) := p(x, y)$, where in this case $p(x, y)$ is given by (8). Condition 3.1 follows immediately from positivity of g and (8). Condition 3.2 is immediate from the assumption $g \leq M$. Regarding Condition 3.3, let $V := \|\cdot\|$, the Euclidean norm on $\mathbb{R}^N$. Then for any $x \in \mathbb{R}^N$ we have

$$\begin{aligned}
\int p(x,y)\|y\|dy &= \int \left\| \sum_{k=1}^{K} (A_k x + b_k)\mathbb{1}_{B_k}(x) + z \right\| g(z)dz \\
&\leq \sum_{k=1}^{K} \|A_k x + b_k\| \mathbb{1}_{B_k}(x) + \mathbb{E}\|\xi\| \\
&\leq \sum_{k=1}^{K} \alpha_k \|x\| \mathbb{1}_{B_k}(x) + \sum_{k=1}^{K} \|b_k\| + \mathbb{E}\|\xi\| \\
&\leq \alpha\|x\| + \beta, \qquad \beta := \sum_{k=1}^{K} \|b_k\| + \mathbb{E}\|\xi\|.
\end{aligned}$$

□

4.2. **Models on the Positive Cone.** Consider again the model (5), when $S = \times_{n=1}^{N}(0, \infty)$, the interior of the positive cone in finite-dimensional space, and λ is the Lebesgue measure. The vector of shocks ξ_t takes values in S with density g. As before the sequence (ξ_t) is uncorrelated over time.

Let the map T be described by

$$T(x,z) = \begin{pmatrix} T_1(x)\cdot z_1 \\ \vdots \\ T_N(x)\cdot z_N \end{pmatrix}, \tag{9}$$

where each $T_n \colon S \to (0,\infty)$ is a measurable map. As above, a standard change of variable argument shows that when the current state is equal to x, the next period state has distribution

$$p(x,y) = g\left(\frac{y_1}{T_1(x)}, \dots, \frac{y_N}{T_N(x)}\right) \prod_{n=1}^{N} \frac{1}{T_n(x)}, \tag{10}$$

Consider the following conditions.

Condition 4.1. There is an $r > 0$ and a $k \in \mathbb{N}$ such that for all n between 1 and N, $T_n(x) \geq x_1 \wedge \cdots \wedge x_N$ on A_r and $T_n(x) \geq 1/k$ on $A_r^c := S \setminus A_r$, where $A_r := \cup_{n=1}^{N} \{x \in S : x_n \leq r\}$.

The effect of Condition 4.1 is to push the state variable away from the boundaries of the state space, which prevents it from becoming too "small." The effect of the next condition is to prevent it from becoming too large.

Condition 4.2. There exist constants $C, \gamma \in [0,\infty)$ such that $\gamma < 1$ and

$$\int \|T(x,z)\| g(z) dz \leq C + \gamma \|x\|, \quad \forall x \in S.$$

Condition 4.3. The joint distribution of $\xi = (\xi_1, \dots, \xi_N)$ satisfies

$$\int \sum_{n=1}^{N} \frac{1}{z_n} g(z) dz < 1.$$

Proposition 4.2, which is proved in Section 5, establishes the most difficult part of the following theorem.

Proposition 4.2. *Let* $V : S \to \mathbb{R}$ *be defined by* $V(x) = \sum_{n=1}^{N} \frac{1}{x_n} + \|x\|$. *If Conditions 4.1–4.3 hold, then there exist constants* $\alpha, \beta \in [0, \infty)$ *with* $\alpha < 1$ *and*

$$\int p(x,y)V(y)dy \leq \alpha V(x) + \beta, \quad \forall x \in S.$$

The main result of this section is

Theorem 4.1. *Let* $(\mathbf{P}_t)_{t \in \mathbb{N}_0}$ *be the Markov semigroup generated by the dynamical system* (9). *Let Conditions 4.1–4.3 by satisfied. If, in addition,* $g > 0$ *everywhere on* S *and there is a constant* M *such that* $g(z) \prod_{n=1}^{N} z_n \leq M$ *for all* $z \in S$, *then* $(\mathbf{P}_t)$ *is asymptotically stable.*

For example, if ξ is multivariate lognormal then g satisfies all of the hypotheses of Theorem 4.1.

Proof. We check that the conditions of Theorem 3.1 hold for $s = 1$. Condition 3.1 follows immediately from positivity of g and the expression (10). Condition 3.2 follows from the assumptions on g, because

$$p(x,y) = g\left(\frac{y_1}{T_1(x)}, \ldots, \frac{y_N}{T_N(x)}\right) \prod_{n=1}^{N} \frac{y_n}{T_n(x)} \cdot \prod_{n=1}^{N} \frac{1}{y_n} \leq M \prod_{n=1}^{N} \frac{1}{y_n}.$$

Finally, Condition 3.3 follows from Proposition 4.2, as V is clearly norm-like. □

5. Proofs

For the remainder of the paper, let us agree to call Markov operator $\mathbf{P}$ asymptotically stable if the semigroup $(\mathbf{P})_{t \in \mathbb{N}_0}$ defined by $\mathbf{P}_0 = I$, $\mathbf{P}_t = \mathbf{P}^t$ is asymptotically stable. The following result simplifies the proof of Theorem 3.1 by showing that in the case of Markov semigroups it is sufficient to verify stability for the discrete semigroup formed by iteration of some fixed member.

Lemma 5.1. *Let* $(\mathbf{P}_t)_{t\in\mathbb{T}}$ *be a Markov semigroup. If* $\mathbf{P}_s$ *is asymptotically stable for some* $s \in \mathbb{T}$*, then so is* $(\mathbf{P}_t)_{t\in\mathbb{T}}$*.*

We provide a proof for completeness, although the ideas are available in the literature—see for example the discussion in Lasota and Mackey (1994, pp. 201–2 and Remark 7.4.2).

Proof. Write $\mathbf{P}$ for $\mathbf{P}_s$. Let $\mathbf{P}$ be asymptotically stable with fixed point $\psi^* \in \mathscr{D}$. Pick any $\varepsilon > 0$ and any $t \in \mathbb{T}$. Choose $N \in \mathbb{N}$ so that $\|\mathbf{P}^N(\mathbf{P}_t\psi^*) - \psi^*\| < \varepsilon$. Then

$$\|\mathbf{P}_t\psi^* - \psi^*\| = \|\mathbf{P}_t(\mathbf{P}^N\psi^*) - \psi^*\| = \|\mathbf{P}^N(\mathbf{P}_t\psi^*) - \psi^*\| < \varepsilon.$$

$$\therefore \quad \|\mathbf{P}_t\psi^* - \psi^*\| = 0.$$

Regarding asymptotic stability, for $\psi \in \mathscr{D}$ choose $N \in \mathbb{N}$ so that $\|\mathbf{P}^N\psi - \psi^*\| < \varepsilon$. Then $t \geq N$ implies

$$\|\mathbf{P}_t\psi - \psi^*\| = \|\mathbf{P}_{t-N}(\mathbf{P}^N\psi^*) - \mathbf{P}_{t-N}\psi^*\| \leq \|\mathbf{P}^N\psi^* - \psi^*\| < \varepsilon,$$

where we have used the fact that every Markov operator is an L_1 contraction (Lasota and Mackey, Proposition 3.1.1). □

We need the following two auxiliary notions.

Definition 5.1. Markov operator $\mathbf{P}$ on L_1 is said to overlap supports if, $\forall\, \psi, \psi' \in \mathscr{D}$, $\lambda(\operatorname{supp} \mathbf{P}\psi \cap \operatorname{supp} \mathbf{P}\psi') > 0$. Also, $\mathbf{P}$ is called Lagrange stable on $\mathscr{D}$ if the collection of points $\{\mathbf{P}^t\psi_0\} \subset \mathscr{D}$ is precompact for every $\psi_0 \in \mathscr{D}$.[6]

The following result is due to Lasota (1994, Theorem 3.3).[7].

[6]Precompact sets are those with compact closure. Here and below, unless otherwise stated, all topological concepts are with respect to the norm topology.

[7]See also Stachurski (2002, 2003) for a proof of a slightly weaker result

Theorem 5.1. *Markov operator* $\mathbf{P}$ *on* L_1 *is asymptotically stable if and only if it is Lagrange stable on* $\mathscr{D}$ *and overlaps supports.*

To establish Lagrange stability is in general difficult, as the criteria for norm-compact subsets of L_1 are quite restrictive. However, Lasota (1994, Theorem 4.1) has pointed out that in the case of *integral* Markov operators, Lagrange stability holds if and only if every trajectory $\{\mathbf{P}^t\psi\}$ is *weakly* precompact in L_1.[8]

Theorem 5.2. *Let* $\mathbf{P}$ *be an integral Markov operator on* L_1*. The operator is Lagrange stable if and only if there exists a set* $D_0 \subset \mathscr{D}$ *such that* D_0 *in norm dense in* $\mathscr{D}$ *and* $\{\mathbf{P}^t\psi\}$ *is weakly precompact for every* $\psi \in D_0$*.*

Weakly precompact sets in L_1 are relatively easy to identify. For example, order intervals are weakly compact. Also, there is the following characterization.

Definition 5.2. Let M be a subset of $\mathscr{D}$. The collection of densities M is called tight if

$$\forall \varepsilon > 0,\ \exists K \subset\subset S \text{ s.t. } \left\{ \int_{K^c} \psi(x)\lambda(dx) < \varepsilon,\ \ \forall \psi \in M \right\}.$$

The notation $K \subset\subset S$ means that K is a compact subset of S, and $K^c := S \setminus K$. The collection M is called uniformly integrable if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } \lambda(A) < \delta \implies \left\{ \int_{A} \psi(x)\lambda(dx) < \varepsilon,\ \ \forall \psi \in M \right\}.$$

Applying a famous theorem of Dunford and Pettis, any subset of $\mathscr{D}$ is weakly precompact whenever it is both tight and uniformly integrable,

[8]As usual, the adjective weakly refers to the topology induced on L_1 by its norm dual L_∞.

provided that the measure λ is locally finite. Thus, in view of Theorem 5.2, to show Lagrange stability one need only check tightness and uniform integrability of all trajectories under $\mathbf{P}$ with initial condition in some dense subset of $\mathscr{D}$. Establishing uniform integrability, however, can itself be quite challenging. In this connection,

Proposition 5.1. *Let $(\mathbf{P}_t)_{t\in\mathbb{T}}$ be an integral Markov operator on L_1 with transition probability function p. Fix $\psi \in \mathscr{D}$ and $s \in \mathbb{T}$. If the set of densities $\{\mathbf{P}_s^t\psi\}_{t\in\mathbb{N}_0}$ is tight, and, in addition, there exists a continuous function $h\colon S \to \mathbb{R}$ such that $p(s,x,y) \leq h(y)$ for all $x, y \in S$, then $\{\mathbf{P}_s^t\psi\}_{t\in\mathbb{N}_0}$ is also uniformly integrable.*

Proof. Fix $\varepsilon > 0$. Write $\mathbf{P}$ for $\mathbf{P}_s$ and $p(x,y)$ for $p(s,x,y)$. Since $\{\mathbf{P}^t\psi\}$ is tight, there exists a compact set K such that

$$\int_{K^c} \mathbf{P}^t\psi \, d\lambda < \frac{\varepsilon}{2}, \quad \forall t \in \mathbb{N}_0. \tag{11}$$

For arbitrary Borel set $A \subset S$, the decomposition

$$\int_A \mathbf{P}^t\psi \, d\lambda = \int_{A\cap K} \mathbf{P}^t\psi \, d\lambda + \int_{A\cap K^c} \mathbf{P}^t\psi \, d\lambda \tag{12}$$

holds. Consider the first term in the sum. We have

$$\begin{aligned}\int_{A\cap K} \mathbf{P}^t\psi(x)\lambda(dx) &= \int_{A\cap K} \left[\int p(x,y)\mathbf{P}^{t-1}\psi(x)\lambda(dx)\right]\lambda(dy)\\ &= \int \left[\int_{A\cap K} p(x,y)\lambda(dy)\right]\mathbf{P}^{t-1}\psi(x)\lambda(dx).\end{aligned}$$

But by the hypothesis and the fact that the image of a continuous real-valued function h on a compact set K is bounded by some constant $N < \infty$,

$$\int_{A\cap K} p(x,y)\lambda(dy) \leq \int_{A\cap K} h(y)\lambda(dy) \leq N \cdot \lambda(A).$$

Therefore,

$$\int_{A\cap K} \mathbf{P}^t \psi \, d\lambda = \int \left[\int_{A\cap K} p(x,y)\lambda(dy) \right] \mathbf{P}^{t-1}\psi \, d\lambda \leq N\lambda(A). \tag{13}$$

Combining (11), (12) and (13), we obtain the bound

$$\int_A \mathbf{P}^t \psi(x)\lambda(dx) \leq N \cdot \lambda(A) + \frac{\varepsilon}{2}$$

for any t and any $A \in \mathscr{B}$. Setting $\delta := \varepsilon/(2N)$ now gives the desired result. □

Regarding tightness, we need the following lemma (Meyn and Tweedie, 1993 Lemma D.5.3—the proof is straightforward).

Lemma 5.2. *A collection of densities $M \subset \mathscr{D}$ is tight whenever there exists a norm-like function V with $\sup_{\psi \in M} \int V\psi \, d\lambda < \infty$.*

The following kind of argument is quite standard (see, for example, the Lasota and Mackey, 1994, §§10.5).

Lemma 5.3. *Let $(\mathbf{P}_t)_{t\in\mathbb{T}}$, p $s \in \mathbb{T}$ and V be as in Theorem 3.1. If $\psi \in \mathscr{D}$ and $\int V\psi \, d\lambda < \infty$, then the trajectory $\{\mathbf{P}_s^t \psi\} \subset \mathscr{D}$ is tight.*

Proof. Let $\mathbf{P} := \mathbf{P}_s$. By Lemma 5.2, it suffices to show that

$$\sup_{t\in\mathbb{N}_0} \int V\mathbf{P}^t\psi \, d\lambda < \infty.$$

By the definition of $\mathbf{P}$,

$$\begin{aligned}
\int V(y)\mathbf{P}^t\psi(y)\lambda(dy) &= \int V(y) \int p(x,y)\mathbf{P}^{t-1}\psi(x)\lambda(dx)\lambda(dy) \\
&= \int \int V(y)p(x,y)\lambda(dy)\mathbf{P}^{t-1}\psi(x)\lambda(dx) \\
&\leq \int [\alpha V(x) + \beta]\mathbf{P}^{t-1}\psi(x)\lambda(dx) \\
&= \alpha \int V(x)\mathbf{P}^{t-1}\psi(x)\lambda(dx) + \beta
\end{aligned}$$

By induction, then,

$$\int V(y)\mathbf{P}^t\psi(y)\lambda(dy) \le \alpha^t \int V\psi\, d\lambda + \frac{\beta}{1-\alpha},$$

which is sufficient for the proof, since $\alpha < 1$. □

Proof of Theorem 3.1. By Lemma 5.1 it suffices to prove asymptotic stability of the Markov operator $\mathbf{P}_s$ when Conditions 1–3 of the theorem hold. From Condition 1 it is easy to see that $\mathbf{P}_s$ overlaps supports. By Condition 3 and Lemma 5.3, $(\mathbf{P}_s^t\psi)_{t\in\mathbb{N}_0}$ is tight whenever $\int V\psi d\lambda < \infty$. By Condition 2 and Proposition 5.1 $(\mathbf{P}_s^t\psi)$ is also uniformly integrable and therefore weakly precompact. Combining Theorems 5.1 and 5.2, the asymptotic stability of $\mathbf{P}_s$ will be established if $D_0 := \{\psi \in \mathscr{D} : \int V\psi d\lambda < \infty\}$ is norm-dense in $L_1(S, \mathscr{B}(S), \lambda)$. This is the case because V is continuous, and, since S is locally compact Hausdorff and λ is Borel regular, the functions with compact support are norm-dense. □

It just remains to prove Proposition 4.2.

Proof of Proposition 4.2. By using a change of variable with the expression (10) we get

$$\begin{aligned}\int p(x,y)V(y)dy &= \int V(T(x,z))g(z)dz \\ &= \int \sum_{n=1}^{N} \frac{1}{T_n(x)z_n} g(z)dz + \int \|T(x,z)\| g(z)dz.\end{aligned}$$

For $x \in A_r$,

$$\begin{aligned}\int \sum_{n=1}^{N} \frac{1}{T_n(x)z_n} g(z)dz &\le \int \sum_{n=1}^{N} \frac{1}{z_n} g(z)dz \frac{1}{x_1 \wedge \cdots \wedge x_N} \\ &\le \int \sum_{n=1}^{N} \frac{1}{z_n} g(z)dz \sum_{n=1}^{N} \frac{1}{x_n}.\end{aligned}$$

Setting $\theta := \int \sum_{n=1}^{N} \frac{1}{z_n} g(z)dz < 1$, then,

$$\int \sum_{n=1}^{N} \frac{1}{T_n(x)z_n} g(z)dz \leq \theta \sum_{n=1}^{N} \frac{1}{x_n} + k, \quad \forall x \in S, \tag{14}$$

where we have used the previous bound on A_r with the bound

$$\int \sum_{n=1}^{N} \frac{1}{T_n(x)z_n} g(z)dz \leq k \text{ on } A_r^c.$$

Combining (14) with Condition 4.2, then,

$$\int p(x,y)V(y)dy \leq \theta \sum_{n=1}^{N} \frac{1}{x_n} + k + C + \gamma \|x\|$$

for all $x \in S$. Setting $\alpha := \theta \vee \gamma < 1$ and $\beta := k + C$ gives the desired result. □

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