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CONVERGENCE, PATH DEPENDENCE AND THE NATURE OF STOCHASTIC EQUILIBRIA: A TERATOLOGY OF GROWTH METHODS

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CONVERGENCE, PATH DEPENDENCE AND THE NATURE OF STOCHASTIC EQUILIBRIA: A TERATOLOGY OF GROWTH MODELS

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ABSTRACT. This paper establishes global stability for a class of stochastic increasing returns accumulation models. The nature of the unique stochastic steady state is investigated. It is found that the models generate highly path dependent time series over long horizons. The findings demonstrate that the standard stability concept used in stochastic growth theory is satisfied by models which contradict our intuitive association of globally stability with unique, history-independent outcomes for each set of economic fundamentals. At the same time, the analysis provides a principled theoretical framework for integrating increasing returns models more closely with the cross-country income data.

1. INTRODUCTION

The long-run behavior of deterministic growth models with decreasing returns technology was first studied by Solow [36], Swan [39], Cass [6] and Koopmans [22]. It was shown that Inada-type conditions on technology and preferences imply the existence of a unique steady state that acts as a global attractor for all initial values of capital per head. The major implication of these findings was clear: each set of economic fundamentals was shown to be associated with a unique long-run outcome. This observation in turn motivated the large empirical literature on conditional convergence.[1]

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[1]See, for example, Barro and Sala-i-Martin [4], Mankiw, Romer and Weil [27], Galor [14] and Durlauf and Quah [12].

These results on uniqueness and global stability of equilibrium were extended to include nondeterministic models by Brock and Mirman [5]. Using identical Inada-type assumptions, they proved that analogous results hold for models where the production function includes a random component. Based on careful inferences from the model primitives, their "stable interval" proof showed clearly that even with the addition of uncertainty, the correspondence from economic fundamentals to long-run outcomes was indeed unique.

Subsequently the major focus of theoretical research on stochastic growth became simplification of the proofs and investigation of alternative sufficient conditions.[2] These conditions extended stability to new classes of models. Relatively little attention was paid to investigation of the nature of the stochastic steady state itself [28].

This paper studies stability in a class of increasing returns growth models originally analysed by Azariadis and Drazen [2]. Despite the existence of multiple equilibria in the deterministic case, the stochastic version is shown to have a unique and globally stable stochastic steady state. At the same time, the models generate highly path dependent time series over long horizons for each given set of economic fundamentals. These finding imply that in general, stability results for stochastic growth models cannot be reduced to a statement of the existence of a unique and globally stable stochastic steady state: the implied association of global stability with unique long-run outcomes is problematic.

[2]See, for example, the contributions of Futia [13], Stokey *et al.* [38] and Hopenhayn and Prescott [17].

In the model, uniqueness and stability of the steady state is induced by the introduction of noise. The notion of noise-induced stability is not new to economics. Kandori, Mailath and Rob [21] study the effects of adding noise to a game-theoretic model with random matching. The introduction of mutations to player strategies in a repeated one-shot game leads to the existence of a unique, globally stable distribution of agents over strategies. This result is observed for games that without mutation exhibit multiple strict Nash equilibria.

Previously, the phenomenon of noise-induced stability was observed in physical systems. Mackey, Longtin and Lasota [26] demonstrated global stability when additive or multiplicative Gaussian white noise is included in the Fokker-Planck differential equation. In the deterministic model, variation of a parameter leads to switching between single and multiple steady states. With Gaussian noise, global stability holds and is invariant across parameter values, although the switching behavior of the original model is paralleled by varying degrees of nonergodic state dynamics.

In development, path dependent behavior is driven by persistent shocks coupled to a nonlinear feedback system of locally increasing returns over phases of the development process. These in turn are generated by self-reinforcing or "autocatalytic" growth mechanisms. Examples of such phenomena include externalities associated with human capital formation [2, 15], investment with barriers to capital mobility [32], complementarities across industries [10], and stages of growth [16], as well as the less formal models of earlier development theorists.[3]

[3]For a survey of growth and development with locally increasing returns see Azariadis [3].

All of the above models share the property that returns to scale are decreasing in the limit, in the sense that there exists a bounded subset of the state space such that returns to accumulation decrease on its complement. This property is referred to here as global decreasing returns, and is critical to proving existence and asymptotic stability of equilibrium in the sense of Brock and Mirman; noise does not induce stability in the "persistent growth" models of Romer [34] and others.

It is arguably the case that local path dependency combined with global stability is a natural feature of many economic, physical and biological systems. The classic example is Darwinian evolution, where shocks are initially amplified and then stabilized as successive mutations trigger progress and change [18]. Similar dynamics can be observed in some nonlinear business cycle models, where self-reinforcing slumps in the level of activity are eventually damped as falling prices alter real money balances. Even the strongest feedback systems—such as hyperinflation—must ultimately be limited by systemic constraints and the adjustment of underlying behavior.[4]

Thus while the analysis developed here runs counter to intuition on the behaviour of globally stable models, it should not be regarded as nonconstructive. Indeed the salience of the model is apparent from the empirical growth literature. The transition probability matrix for income estimated from cross-country data by Quah [31] has a unique and stable limiting distribution, while at the same time exhibiting persistence at the extremes of the distribution. Mankiw, Romer and Weil

[4]See in particular the discussion in Krugman [23, pp. 1–7], where these types of models are referred to as "self-reinforcing" in the short-run and "self-limiting" in the long-run.

[27] find evidence to support conditional convergence at an aggregate level; Durlauf and Johnson [11] find multiple regimes.

Section 2 outlines the techniques used to prove global asymptotic stability of the stochastic Azariadis-Drazen model. Section 3 considers a general growth problem, and gives the major definitions. Section 4 discusses path dependence and its relationship to global stability. A persistence concept called finite-horizon path dependence is introduced. Section 5 formulates a stochastic Azariadis-Drazen model. The model is proved to be both globally stable and finite-horizon path dependent. Section 6 discusses econometric implications of these results.

2. Techniques

This section provides a discussion of methods that can be used to establish existence of unique, globally stable equilibria in increasing returns models. The ideas are based on a framework for studying perturbed nonlinear systems due to Lasota [24].

Consider a model of economic growth where the state space is income per head. Discretize the state space into N disjoint regions, or "bins," where N is a positive integer. Very little generality is lost assuming that income evolves according to a first order Markov process, which can be associated with an $N \times N$ matrix $\mathbf{P} = (p_{ij})$, where p_{ij} is the conditional probability that the economy moves from bin i to bin j in one period. Evidently $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$. Matrices with this property are called *stochastic.*

Suppose that the current state of the economy is drawn according to distribution $\pi = (\pi_1, \ldots, \pi_N)$, where π_i is the probability that the realized value of income is in bin i. In this case, the distribution π' for the next period state is

$$\pi' = \pi \mathbf{P} = \left(\sum_i p_{i1} \pi_i, \ldots, \sum_i p_{iN} \pi_i \right). \tag{1}$$

The intuition is that $\sum_i p_{ij} \pi_i$ sums the probability p_{ij} of the state moving from i to j across all i, weighted by the probability π_i of i occurring as the current state. Hence the sum gives the (unconditional) probability of entering bin j next period, and $\pi \mathbf{P}$ is the distribution that governs the next-period state.

By identical reasoning, $\pi \mathbf{P} \mathbf{P} = \pi \mathbf{P}^2$ is the distribution two periods hence, and so on. A stochastic equilibrium is a distribution π^* such that $\pi^* = \pi^* \mathbf{P}$. The equilibrium is globally stable if $\pi \mathbf{P}^t \to \pi^*$ in norm as $t \to \infty$ for all possible initial distributions π.

It is assumed that transition from any bin i to any other bin j occurs with positive (possibly very small) probability.[5]

That there exists a unique and globally stable equilibrium whenever $p_{ij} > 0$ for all i, j is a classical result due to A. A. Markov. Alternatively, consider the following operator-theoretic argument. To each $N \times N$ matrix $\mathbf{P}$ there corresponds a unique linear operator $P \colon \mathbb{R}^N \to \mathbb{R}^N$, where for row vector $x \in \mathbb{R}^N$, the image Px of x under P is $x\mathbf{P}$. In the current context a probability distribution is an element of the $N-1$ dimensional simplex $\Delta \subset \mathbb{R}^N$. It is straightforward to verify that when $\mathbf{P}$ is stochastic, P maps Δ into itself.

Distance between points in $\mathbb{R}^N$ can be measured by the norm $\|x\| = \sum_i |x_i|$. We now show that the operator P is strongly contractive on

[5]Based on postwar data, Quah [32] calculates a 5% transition probability from first to last income decile in three generations.

Δ in this norm, in the sense that

$$(2) \qquad \|P\pi - P\pi'\| < \|\pi - \pi'\|, \quad \forall \pi, \pi' \in \Delta, \quad \pi \neq \pi'.$$

Recall that $p_{ij} > 0$ for each pair i, j. Note also that if π and π' are any two distributions, $\pi \neq \pi'$, then $\pi_i - \pi'_i$ is positive for at least one i and negative for at least one i. But then

$$\begin{aligned} \|P\pi - P\pi'\| &= \sum_j \left| \sum_i p_{ij}(\pi_i - \pi'_i) \right| \\ &< \sum_j \sum_i |p_{ij}(\pi_i - \pi'_i)| = \sum_i |\pi_i - \pi'_i| \sum_j p_{ij} \\ &= \sum_i |\pi_i - \pi'_i| = \|\pi - \pi'\|, \end{aligned}$$

which proves (2).

In addition, under $\|\cdot\|$ the set Δ is a closed, bounded subset of $\mathbb{R}^N$. Therefore Δ is compact under the same metric.[6] Moreover, it is known [20, Theorem 4.1.6, Corollary 1] that a strongly contractive operator P mapping a compact space into itself has a unique fixed point π^*, and that all points in the space are norm-convergent to π^* under iteration of P. Clearly π^* is a unique and globally stable equilibrium for the economy in question.[7]

The advantage of the above approach is that it can be extended to growth models with infinite, noncompact state space. For such models, strong contractiveness holds for a large and important class of productivity shocks. The main difficulty to overcome is that the operator P now acts in a space of infinite dimension, corresponding to the set of

[6] All norms are equivalent in finite dimensional space.

[7] An alternative contraction-based argument for discrete space Markov chains is given in Stokey *et al.* [38, Lemma 11.3].

all distributions on an arbitrary (rather than finite) state space. In this case compactness of the distribution space fails. However, in some cases it can be proved that the closure of the set of iterates $\{P^t\pi : t \geq 1\}$ is compact for any initial distribution π. The key criterion for obtaining this property is global decreasing returns to capital accumulation.

When combined with strong contractiveness, compactness of the closure of the set of iterates is sufficient to obtain identical stability results.

3. Formulation of the Problem

This section sets out the main concepts discussed in the paper using the framework of a generic growth problem. In contrast to the previous section, the state space is now treated as continuous. In particular, the model evolves on state space X, where X is a Borel subset of the real line $\mathbb{R}$. The collection of all Borel subsets of X is denoted $\mathcal{B}$, and μ is Lebesgue measure on $(X, \mathcal{B})$. The symbol $L_1(\mu)$ denotes the space of μ-integrable real functions on X.[8]

3.1. **Stochastic growth models.** For the purposes of this paper, a growth model E on X is a pair (S, Ψ), where S is a period-to-period transition rule and Ψ is a "random number generator" from which uncorrelated and identically distributed shocks are drawn. Given current state $x_t \in X$, an X-valued shock ξ_t is drawn by Ψ and the next period

[8] As usual, $L_1(\mu)$ is a Banach lattice of equivalence classes; functions equal off a μ-null set are identified and "almost everywhere" notation is suppressed throughout. Real functions introduced in the paper are assumed to be Borel functions, and subsets of X are assumed to be Borel sets. Integration is always with respect to μ, and is denoted by dx, dy, etc.

state $x_{t+1} \in X$ is determined as

$$x_{t+1} = S(x_t, \xi_t). \tag{3}$$

The state variable x can be thought of as representing per capita income or some proxy thereof. The transition rule $S\colon X \times X \to X$ encodes the implications of the model primitives and the restrictions imposed by optimizing behavior.

Once a current state x_t and the pair $E = (S, \Psi)$ are specified, it is possible to calculate a conditional distribution for the next period state x_{t+1} from (3). For fixed growth model E, the probability that x_{t+1} is in $B \subset X$ given that $x_t = x$ is denoted $Q(x, B)$. A function Q which associates a distribution for the next period to each current period value x is called a *transition probability function.*

The sequence of state variables generated by (3) is a stochastic process on X. Given the transition probability function Q and an initial state value x_0, probabilities of events (sample paths) for the process can be calculated. An event can be represented by a subset of the sample space for the stochastic process, which is the infinite Cartesian product $\times_{t=0}^{\infty} X$. Thus for given x_0, the event $x_t \in B_t \subset X$ for t in a subset A of $\mathbb{N}$ is written as $\times_{t \in A} B_t$, and its probability is denoted $\Pi_{x_0}(\times_{t \in A} B_t)$.[9]

[9]More formally, the distribution Π_{x_0} is a probability on measurable space $(\times_{t=0}^{\infty} X, \otimes_{t=0}^{\infty} \mathcal{B})$, where $\times_{t=0}^{\infty} X$ is the space of all X-valued sequences and $\otimes_{t=0}^{\infty} \mathcal{B}$ is the σ-algebra on $\times_{t=0}^{\infty} X$ generated by the Cartesian product $\times_{t=0}^{\infty} \mathcal{B}$. It is known [35, II.9, Theorem 2] that such a distribution Π_{x_0} always exists and can be constructed uniquely from an initial condition x_0 and the transition probability Q.

3.2. The L_1 approach. In this paper stochastic processes evolve in the space of density functions.[10] Here a density function is a nonnegative element of $L_1(\mu)$ that integrates to unity.

In order to work in the space of densities, we require that the common distribution Ψ of the shocks ξ_t can be represented by a density function ψ. In addition, we assume that

Assumption 3.1. For each fixed $x \in X$, the map $z \mapsto S(x, z)$ is nonsingular.[11]

In this case, the transition probability function Q can always be represented by a density function $p \colon X \times X \to \mathbb{R}_+$. Here $p(x, \cdot)$ is a density in its second argument for each fixed $x \in X$. The number $p(x, y)$ can be thought of as analogous to the conditional transition probability p_{ij} in Section 2. The relationship between p and Q is

$$Q(x, B) = \int_B p(x, y)dy,$$

and p is called the *stochastic kernel* for growth model E.

Given stochastic kernel p corresponding to E, define an operator P from the function space $L_1(\mu)$ into itself by

$$(Pf)(y) = \int p(x, y)f(x)dx. \tag{4}$$

It can be verified using Fubini's theorem that if f is a density function on X, then its image Pf is again a density function. The operator P is called the *Markov operator* corresponding to E, and has the following interpretation: If the current state of the economy E is selected

[10] For an introduction to the literature on density techniques, see the monograph of Lasota and Mackey [25]. For further discussion of stochastic growth by L_1 methods see Stachurski [37].

[11] A map is nonsingular when the preimage of every μ-null set is μ-null.

according to density f, then Pf is the density function for the next period state. The intuition is analogous to that given for the discrete version (1) of (4) in Section 2.

Iteration of the operator P is equivalent to moving forward in time. If P^t is defined by $P^t = P \circ P^{t-1}$ and $P^1 = P$, and if f is the distribution that currently describes the probabilistic laws that govern E, then $P^t f$ gives the current distribution for E t periods hence.[12]

The standard Brock-Mirman definition of stochastic equilibrium [5, p. 492] is given below. The definition of stability used here requires convergence in $L_1(\mu)$ norm, which is stronger than stability in the weak topology used by Brock and Mirman.

Definition 3.1. Let E be a growth model on X, and let P be the corresponding Markov operator. An *equilibrium* or *steady state* for E is a density f^* on X such that $Pf^* = f^*$. An equilibrium f^* is called *globally stable* if $P^t f \to f^*$ in $L_1(\mu)$ norm as $t \to \infty$ for every density function f. The economy E is called globally stable if it has a unique, globally stable equilibrium.

Thus an equilibrium or stochastic steady state in the sense of Brock and Mirman is a probability distribution over the state space that is invariant from the current period to the next, given the optimal behavior of agents and the laws of motion that determine the evolution of the system. It is an immediate consequence of the definition of stability that in the infinite limit initial conditions do not matter.

[12] For a stylized diagrammatic representation of the evolution of densities over income space see, for example, Quah [32, Figure 4], [33, Figure 1]. For nonparametric estimation of a sequence of actual cross-country densities see Quah [31, Figure 6], or Jones [19, Figure 1].

4. Path Dependence

This section provides a definition of path dependence, as well as an auxillary persistence concept called finite-horizon path dependence. Path dependent dynamic systems are common to many sciences. In economics, much of the early work on increasing returns and history-dependent selection in explicitly stochastic systems is due to Arthur [1]. In the growth literature, path dependence has recently come to be associated with concepts such as stratification, polarization and convergence clubs.[13]

4.1. **Infinite horizon results.** Arthur defined a stochastic process to be path dependent whenever it is not ergodic (i.e., globally stable) [1, p. 13]. David [7, p. 14] defined a path dependent stochastic process to be one "whose asymptotic distribution evolves as a consequence (function) of the processes' own history." He cited "multiplicity of absorbing states" as a source of this outcome.

The definition of Arthur is mathematically precise but may not in all cases coincide with our intuitive notion of history dependent selection (how to regard a Markov process which is sweeping to infinity?), while David's definition cannot be applied for processes which do not have a limiting distribution for every initial state. Here multiplicity of absorbing states is used as the primitive to define path dependence.[14]

[13]See, for example, Quah [32, 33], Galor [14] and Durlauf and Quah [12].

[14]See also the related definition of Mitra and Nishimura, who call (deterministic) systems on the real line path dependent whenever there exist at least two sets of positive measure such that trajectories originating in these regions have different limit sets [29, p. 261].

Definition 4.1. Let E be a growth model, and let Q be the corresponding transition probability function. The model E is called *path dependent* if there exist at least two disjoint open intervals Λ_1, Λ_2 in X such that

$$Q(x, \Lambda_i^c) = 0, \quad \forall x \in \Lambda_i, \quad i = 1, 2, \tag{5}$$

where Λ_i^c is the complement of Λ_i on X.

A set Λ satisfying (5) has the property that, once entered, the probability of exit is zero. Such sets are called *absorbing* or *ergodic* for the process defined by Q. Thus E is called path dependent if there exist multiple disjoint open intervals which are absorbing for the stochastic process generated by E.

Intuitively, if open interval Λ is absorbing for E, and if the initial state x_0 is drawn according to a density f that vanishes off Λ, then the state is in Λ in every future period with probability one. That is, $\int_\Lambda P^t f = 1$ for every $t \in \mathbb{N}$. The statement holds for $t = 1$, because

$$\begin{aligned}
\int_\Lambda Pf &= \int_\Lambda \int p(x,y) f(x) dx\, dy \\
&= \int \int_\Lambda p(x,y) dy\, f(x) dx \\
&= \int Q(x, \Lambda) f(x) dx \\
&= \int_\Lambda Q(x, \Lambda) f(x) dx + \int_{\Lambda^c} Q(x, \Lambda) f(x) dx = 1.
\end{aligned}$$

A similar argument confirms the result for arbitrary $t \in \mathbb{N}$ under an induction hypothesis for $t - 1$.

It is implicit in the stochastic growth literature that global stability rules out path dependence, in the same sense that global stability in deterministic models rules out the presence of multiple local attractors. The following proposition formalizes this idea.

Proposition 4.1. *If economy E is globally stable, then it is not path dependent.*

Proof. Suppose otherwise. In particular, let E be globally stable with equilibrium f^*, and let disjoint open intervals Λ_1 and Λ_2 be absorbing for E. If, in addition, f_1, f_2 are two densities that vanish off Λ_1 and Λ_2 respectively, then

$$1 \geq \int_{\Lambda_1} f^* + \int_{\Lambda_2} f^* = \lim_{t\to\infty} \int_{\Lambda_1} P^t f_1 + \lim_{t\to\infty} \int_{\Lambda_2} P^t f_2 = 2.$$

Contradiction. □

4.2. **Finite-horizon path dependence.** The following definition captures the idea that, while a model may not be path dependent in the sense of Definition 4.1, it may still have disjoint open subsets of the state space with the property that, given any finite time horizon, it is possible to adjust the parameters of the model such that the conditional probabilities of exiting either region prior to the end of the given time horizon is arbitrarily small.

Let $\mathcal{E}$ denote the class of growth models $E_\theta = (S, \psi_\theta)$, $\theta > 0$. Here θ parameterizes the variance of the productivity shock ξ, in the sense that ξ converges in probability to a constant as $\theta \downarrow 0$.

Definition 4.2. The class $\mathcal{E}$ is defined to be *finite-horizon path dependent* if there exist at least two disjoint open intervals $\Lambda_1, \Lambda_2 \subset X$ with the property that, for any finite time horizon T and any $\varepsilon > 0$, there exists an $E_\theta \in \mathcal{E}$ such that for E_θ the conditional probabilities of

(1) the state x_t leaving Λ_1 while $t \leq T$ given $x_0 \in \Lambda_1$, and

(2) the state x_t leaving Λ_2 while $t \leq T$ given $x_0 \in \Lambda_2$

are both less than ε. That is, $\forall \varepsilon > 0$, $\forall T \in \mathbb{N}$, $\exists\, \theta > 0$ such that

$$\sum_{t=0}^{T-1} \Pi^\theta_{x_0}((\times^t_{\tau=0}\Lambda_i) \times \Lambda_i^c) < \varepsilon, \quad \forall x_0 \in \Lambda_i, \quad i = 1, 2.$$

Here $\Pi^\theta_{x_0}$ is the distribution of the stochastic process generated by E_θ.

5. The Model

Based on the above notions of global stability and path dependence, this section analyses a stochastic version of the "threshold externalities" model due to Azariadis and Drazen [2]. Global stability of the stochastic version is established. Properties of the stochastic equilibirium are then investigated. We use the model to show that Proposition 4.1 fails when path dependence is replaced by finite-horizon path dependence.

5.1. Stability in a model with externalities.

The framework is an overlapping generations model. The state space X is the positive real numbers $(0, \infty)$. Agents live for two periods, working in the first and living off savings in the second. Savings in the first period forms capital stock, which in the following period is combined with the labor of a new generation of young agents for production under the technology

$$y_t = A(k_t) k_t^\alpha \ell_t^{1-\alpha} \xi_t^\theta,$$

where y is output, k is capital and ℓ is labor input. The function $k \mapsto A(k)$ signifies the existence of increasing social returns resulting from sensitivity of "technology" to economy-wide capital aggregates. In particular, it is assumed that $A(k) = A_1$ when $k < k_b$ and A_2 when $k \geq k_b$, $0 < A_1 < A_2$, where the bifurcation point k_b is a "critical mass" level of capital stock. This dependence is external to individual agents, and A is treated as constant with respect to private investment.[15]

For convenience, labor supply is normalized to unity. The productivity shocks ξ_t are uncorrelated and identically distributed by density ψ. The exponent θ on ξ is a positive number which parameterizes the variance.

Let c (c') denote consumption while young (old). Agents maximize utility

$$U(c_t, c'_{t+1}) = \ln c_t + \beta E(\ln c'_{t+1})$$

subject to the budget constraint $c'_{t+1} = (w_t - c_t)(1 + r_{t+1})$, where $\beta \in (0,1)$ is a discount factor, and w_t and r_t are the wage and interest rates at t respectively. In this case optimization implies a savings rate from wage income of $\beta/(1-\beta)$, and hence $k_{t+1} = (\beta/(1-\beta))w_t$.

Assuming that labor is paid its marginal factor product yields the law of motion

$$k_{t+1} = DA(k_t)k_t^{\alpha}\xi_t^{\theta}, \tag{6}$$

where $D = (\beta/(1-\beta))(1-\alpha)$. Define $k_i^* = (DA_i)^{1/(1-\alpha)}$ for $i = 1, 2$. We assume that $k_1^* < k_b < k_2^*$. This situation is illustrated in the plot of $k_t \mapsto DA(k_t)k_t^{\alpha}$ given in Figure 1.

Let $\mathcal{F} = \{F_\theta : \theta > 0\}$ be the class of all such economies, parameterized by the exponent θ on the productivity shock. The implied stochastic kernel for F_θ is

$$p_\theta(k, k') = \psi\left[\left(\frac{k'}{DA(k)k^{\alpha}}\right)^{\frac{1}{\theta}}\right]\left(\frac{k'}{DA(k)k^{\alpha}}\right)^{\frac{1}{\theta}}\frac{1}{\theta k'}. \tag{7}$$

[15]For motivation see Azariadis and Drazen [2].

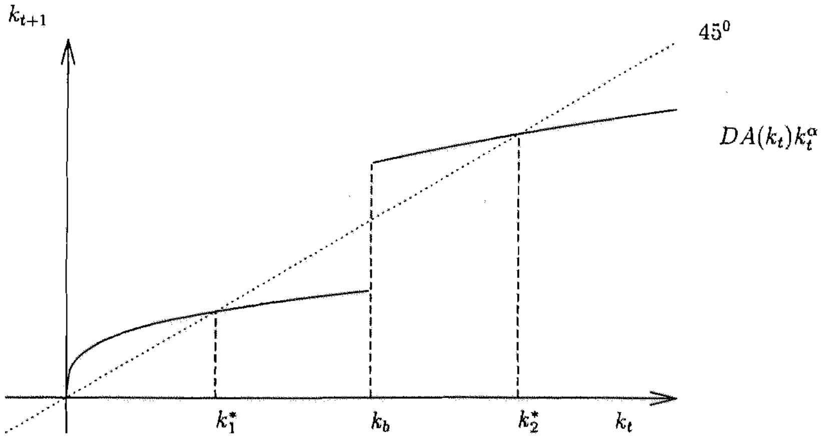

FIGURE 1. The map $k_t \mapsto DA(k_t)k_t^\alpha$.

The kernel can be obtained from (6) using a standard change of variable argument.

In addition to the formulation of Azariadis and Drazen, the following three assumptions are required.

Assumption 5.1. The density ψ satisfies $\psi(z) > 0$, $\forall z \in X$.

Assumption 5.2. The shock ξ satisfies $E|\ln \xi| < \infty$.

Assumption 5.3. The density ψ satisfies $\psi(z)z \leq M$, $\forall z \in X$, M a given constant.

Assumption 5.1 provides a "mixing" condition, which is crucial to the proof of global stability. In the current context, the implication of the assumption is to admit the possibility of "growth miracles" and "growth disasters." Provided that they are nonzero, the possibility of these occurrences may be made arbitrarily small.[16] Most common distributions on the positive reals satisfy this assumption (e.g., the log-normal, gamma, exponential, χ-squared, Weibull and F distributions).

[16]For a summary of the data on such phenomena, see Parente and Precott [30].

Assumptions 5.2 and 5.3 enforce small left- and right-hand tails on the density of the shock: very small and very large shocks are rare. Small-tail assumptions are used to prove existence of equilibrium; the economy does not collapse or grow without bounds.

We are now ready to state the main technical result of the paper.

Proposition 5.1. *The following statements are true.*

1. *If the shock ξ satisfies Assumption 5.1, then no economy in $\mathcal{F}$ has more than one equilibrium.*
2. *If, in addition, Assumptions 5.2 and 5.3 are also satisfied, then the class $\mathcal{F}$ is globally stable. That is, $F_\theta \in \mathcal{F}$ is globally stable for each $\theta > 0$.*

The proof is given the Appendix. The methodology is as was described in Section 2. The Markov operator associated with the stochastic kernel (7) is shown to be strongly contractive and generate precompact trajectories on the space of all density functions. Neither compactness of the state space X nor continuity of the underlying transition rule $k \mapsto DA(k)k^\alpha$ is required. Many of the ideas used in the proof were inspired by results on real dynamical systems with additive perturbations found in the monograph of Lasota and Mackey [25].

Figure 2 presents a sequence of densities generated by iterating the Markov operator implied by (7) on an arbitrary initial distribution f_0. All variables are in logs for convenience. The horizontal axis is the logarithm of capital per head. Here f_0 can be thought of as an initial distribution of a "large" number of Azariadis-Drazen economies. The density f_0 is the left-most distribution, with probability mass shifting rightwards over time, and developing the bimodal structure observed in

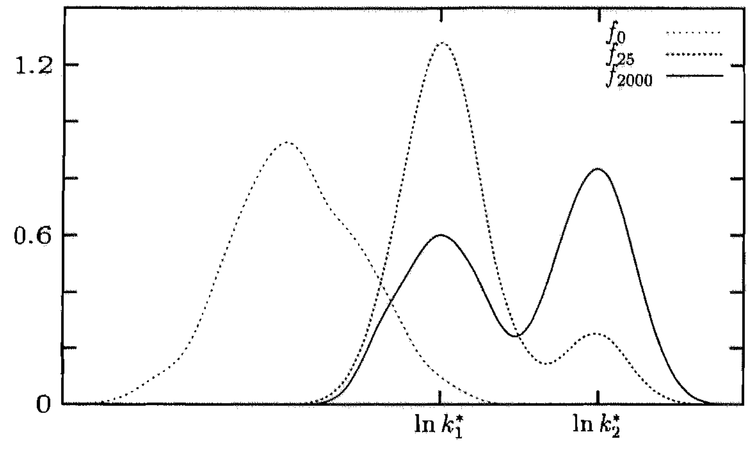

FIGURE 2. Convergence to equilibrium

the actual cross-country growth data by, among others, Quah [32, 33], Jones [19] and Durlauf and Quah [12]. The two modes correspond to (the logs of) the two local attractors in the deterministic case.[17]

Proposition 5.1 implies that the sequence of densities (f_t) converge to a unique limiting density f^*. There is little observable change after $t = 2000$ (the third density plotted in the figure).

5.2. **Persistence in a stable model.** Despite the global stability result obtained in Proposition 5.1, intuition suggests that when the variance of the shock is low (i.e., when θ is close to zero), initial conditions may be very persistent over finite time intervals. In fact the following result holds.

[17]The parameters used in the simulation are $\theta = 1$, $D = 1$, $\alpha = 0.5$, $A_1 = 0.5$, $A_2 = 2$, $k_b = 0.6875$, ξ lognormal, $\ln \xi \sim N(0, 0.5)$. The densities are generated using Monte Carlo simulation and estimated nonparametrically by the Parzen window method with Gaussian kernel and bandwidth 0.38. Such estimates are known to converge to the true density in L_1 norm for large sample size [8]. Here each generation is represented by 200 sample points.

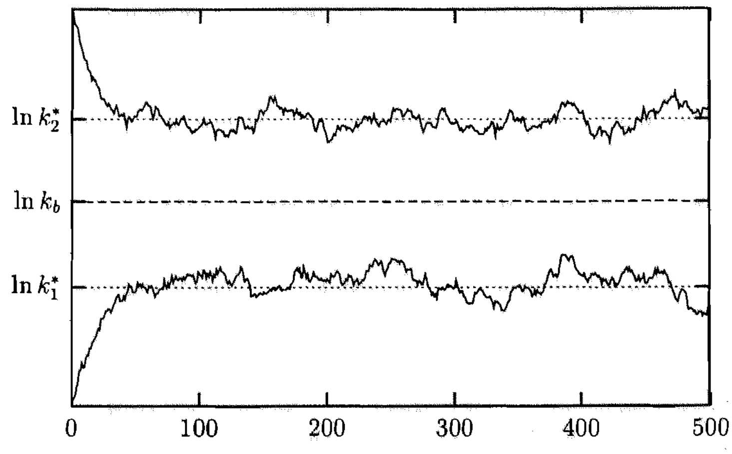

FIGURE 3. Finite-horizon path dependence

Proposition 5.2. *The class $\mathcal{F}$ is finite-horizon path dependent.*

The proof is given in the Appendix. The intervals supporting the two modes of the limiting distribution in Figure 2 become progressively more "absorbing" as the variance of the shock is reduced.

An illustration of path dependence in the Azariadis-Drazen economy over 500 generations is given in the Figure 3. The x-axis is time, and the y-axis is the log of the state variable k. In the figure, two time series are generated by simulation, one with a low initial level of capital and the other with a high initial level.[18] Figure 3 corresponds to the situation in Figure 1, where k_b is half-way between k_1^* and k_2^*. Despite the fact that the economy satisfies global stability, individual time series exhibit strong path dependence.

[18]The parameters are $D = 1$, $\alpha = 0.95$, $A_1 = 1.0$, $A_2 = 1.05$, $\theta = 1$, ξ lognormal, $\ln \xi \sim N(0, 0.05)$.

6. Implications

In this section some implications of the analysis are considered in the light of the evolving cross-country income panel.

Following the contributions of Romer [34], Mankiw, Romer and Weil [27] and others, empirical analysis of growth data has typically been underpinned by explicit theoretical structure. In recent years, however, it has been forcefully argued that standard regression analysis—justified by linearizing a deterministic Solow-Ramsey model in the neighborhood of its long-run steady state—does not provide a sufficiently flexible architecture to capture the major features of the cross-country data [32, 12]. In other words, linear analysis suffers from specification bias.

In contrast, more flexible descriptive and nonparametric methods have proved revealing. Several authors have obtained empirical support for the deterministic multiple equilibria models [2, 31, 32, 33]. Nevertheless, theoretical aspects of the link between these deterministic models and the inherently stochastic growth data have not always been clear. For example, Parente and Prescott [30, p. 13] argue that the evidence does not support "poverty traps," given the observable fact of mobility across the income distribution.

The stochastic increasing returns model developed in this paper offers a theoretical structure suitable for integrating the multiple equilibria development literature into the empirical growth research. In particular, it has been shown that the relationship between the deterministic multiple equilibria models and the globally stable stochastic models is not dichotomous; the degree of path dependence instead depends on the parameters that determine persistence and mixing across

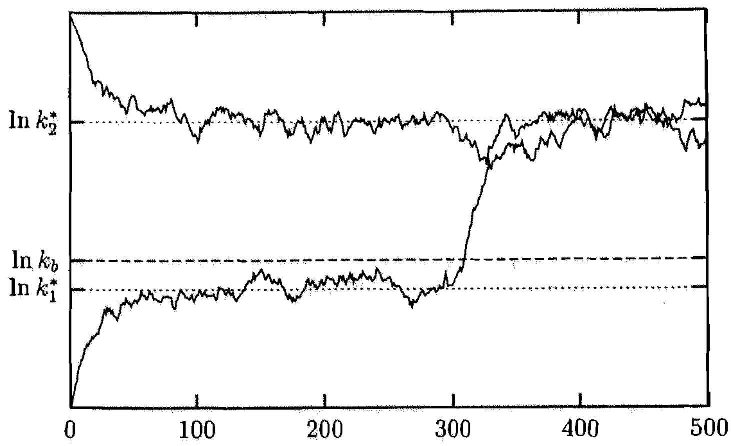

FIGURE 4. Phase transition

the income distribution. This observation highlights the theoretical structure provided for investigation of convergence, take-off, persistence, polarization, growth miracles and other salient aspects of the evolution of the cross-country income distribution.

As one illustration of these ideas, consider the simulated growth miracle in the time series shown in Figure 4. Relative to Figure 3, k_b is taken to be closer to the lower fixed point k_1^*. A growth miracle occurs after some 300 generations, and the economy makes the transition to the mode associated with the higher "attractor" k_2^*. As further research clarifies the nature of the increasing returns mechanism and the necessary model parameters, ensemble Monte Carlo simulations of this time series could, for example, provide estimates of take-off or "first passage" probabilities under different policy regimes (determining the critical value k_b).

7. Appendix

We begin with the proof of Proposition 5.1. The framework used here is due to Lasota [24]. An application of these methods to the classical problem of Brock and Mirman is found in Stachurski [37].

The basic construct for the proof is a *semidynamical system*, which is a pair (U, T), where U is a metric space with distance ϱ and T is a continuous map from U into itself. The system (U, T) is called *strongly contractive* if

$$\varrho(Tx, Tx') < \varrho(x, x'), \quad \forall x, x' \in U, \quad x \neq x'.$$

Every strongly contractive system (U, T) has at most one fixed point in U. To see this, suppose that x and x' are both fixed points. Then $\varrho(Tx, Tx') = \varrho(x, x')$, which is only possible if $x = x'$.

The system (U, T) is called *Lagrange stable* if the trajectory $\{T^n x : n \in \mathbb{N}\}$ is precompact (i.e., has compact closure) for every $x \in X$. If (U, T) is both strongly contractive and Lagrange stable then there exists a unique fixed point x^* of T on U, and $T^n x \to x^*$ as $n \to \infty$ for every $x \in X$. (See Lasota [24, Theorem 3.3]. An alternative proof is available in Stachurski [37, Theorem 5.2].)

Let $D(\mu)$ be the set of density functions on $(X, \mathcal{B}, \mu)$, X the positive reals. The set $D(\mu)$ is a metric space under the $L_1(\mu)$ norm. In addition, let $P_\theta \colon L_1(\mu) \to L_1(\mu)$ be the Markov operator associated with the stochastic kernel (7) by (4). Since P_θ is a positive linear operator on a Banach lattice it must be norm-continuous. It can also be readily verified that $P_\theta \colon D(\mu) \to D(\mu)$. Hence $(D(\mu), P_\theta)$ is a semidynamical system.

Proof of Proposition 5.1. There is no loss of generality in assuming that the exponent $\theta = 1$, because if ψ is the density of ξ and ψ_θ is the density of ξ^θ, then ψ_θ satisfies Assumptions 5.1, 5.3 and 5.2 whenever ψ does, for any given $\theta > 0$.

When Assumption 5.1 holds the system $(D(\mu), P)$ is strongly contracting. That this is the case can be verified by noting that positivity of ψ on X implies positivity of the stochastic kernel (7) for all k, k', and then extending in a straightforward manner the proof of contractiveness on finite state space given in Section 2. This proves the first assertion of Proposition 5.1.

The second assertion of the proposition will be established if $(D(\mu), P)$ is shown to be Lagrange stable under the additional Assumptions 5.2 and 5.3. By an important result of Lasota [24, Proposition 3.4], it is sufficient to prove that $\{P^n f\}$ is *weakly* precompact for all $f \in \mathcal{M}$, where $\mathcal{M}$ is a subset of $L_1(\mu)$ with the property that its closure contains the density functions. The remainder of the proof shows that this is in fact the case.

Let $\mathcal{M}$ be the set of all nonnegative functions in $L_1(\mu)$ such that

$$\int_0^\infty |\ln x| f(x) dx \tag{8}$$

is finite. We claim that $\mathcal{M}$ has the desired properties.

Pick any density f. To see that there exists a $(f_k) \subset \mathcal{M}$ with $f_k \to f$, define $f_k = \mathbf{1}_{(\frac{1}{k},k)} f$, where $\mathbf{1}$ is the characteristic function. Clearly $f_k \in L_1(\mu)$. Moreover, $f_k \uparrow f$ pointwise, implying convergence in norm. Finally, $f_k \geq 0$ and

$$\int_0^\infty |\ln x| f_k(x) dx = \int_{\frac{1}{k}}^k |\ln x| f(x) dx \leq \ln k. \tag{9}$$

Hence $f_k \in \mathcal{M}$, $\forall k \in \mathbb{N}$.

It remains to show that if $f \in \mathcal{M}$ then $\{P^n f : n \in \mathbb{N}\}$ is weakly precompact. Note first that $\{P^n f\}$ is norm-bounded, because $\|Pg\| = \|g\|$ for all nonnegative $g \in L_1(\mu)$.[19] By a well-known condition of Dunford and Schwartz [9, IV.13.54], a norm-bounded collection of functions $\{P^n f\}$ in $L_1(\mu)$ is weakly precompact whenever it satisfies

(i) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $A \in \mathcal{B}$ and $\mu(A) < \delta$, then

$$\int_A P^n f < \varepsilon, \quad \forall n; \text{ and}$$

(ii) $\forall \varepsilon > 0$, there exists a bounded set E such that

$$\int_{E^c} P^n f \leq \varepsilon, \quad \forall n.$$

Evidently it is sufficient to prove that these conditions are satisfied for all but a finite $(n < N)$ number of the collection $\{P^n f : n \in \mathbb{N}\}$.

Regarding (i), pick any $\varepsilon > 0$. We require a $\delta > 0$ and an $N \in \mathbb{N}$ such that

$$\int_A P^n f(x) dx < \varepsilon \tag{10}$$

whenever $n \geq N$ and $\mu(A) < \delta$.

[19] This can be verified from the definition using Fubini's theorem.

There is no loss of generality in assuming that the constant D in (6) is equal to 1. Define the map $H\colon X \ni x \mapsto |\ln x| \in \mathbb{R}_+$. We have

$$\begin{aligned} E(H|P^n f) &= \int_0^\infty |\ln k'| P^n f(k') dk' \\ &= \int_0^\infty |\ln k'| \int_0^\infty \psi\left(\frac{k'}{A(k)k^\alpha}\right) \frac{1}{A(k)k^\alpha} P^{n-1} f(k) dk dk' \\ &= \int_0^\infty \int_0^\infty \psi\left(\frac{k'}{A(k)k^\alpha}\right) \frac{1}{A(k)k^\alpha} |\ln k'| dk' P^{n-1} f(k) dk. \end{aligned}$$

But

$$\begin{aligned} \int_0^\infty \psi\left(\frac{k'}{A(k)k^\alpha}\right) \frac{1}{A(k)k^\alpha} |\ln k'| dk' & \\ &= \int_0^\infty |\ln(A(k)k^\alpha z)| \psi(z) dz \\ &\le \alpha |\ln k| + C, \end{aligned}$$

where $C = \max_k |\ln A(k)| + E|\ln \xi|$. Here C is finite by the definition of $A(k)$ and Assumption 5.2. Thus,

$$\begin{aligned} E(H|P^n f) &\le \int_0^\infty (\alpha|\ln k| + C) P^{n-1} f(k) dk \\ &= \alpha E(H|P^{n-1} f) + C\|f\|. \end{aligned}$$

Repeating this argument obtains

$$E(H|P^n f) \le \alpha^n E(H|f) + \frac{C\|f\|}{1-\alpha}. \tag{11}$$

Since $E(H|f)$ is finite by (8), it follows that

$$E(H|P^n f) \le 1 + \frac{C\|f\|}{1-\alpha}, \quad n \ge N \tag{12}$$

for some $N \in \mathbb{N}$.

On the other hand, for arbitrary positive a,

$$\begin{aligned} E(H|P^n f) &= \int_{\{k\,:\,|\ln k| < a\}} |\ln k| P^n f(k) dk + \int_{\{k\,:\,|\ln k| \ge a\}} |\ln k| P^n f(k) dk \\ &\ge \int_{\{k\,:\,|\ln k| \ge a\}} |\ln k| P^n f(k) dk \\ &\ge a \int_{\{k\,:\,|\ln k| \ge a\}} P^n f(k) dk \end{aligned}$$

Combining this result with (12) yields the estimate

$$\int_0^{\exp -a} P^n f(k) dk + \int_{\exp a}^\infty P^n f(k) dk \le \frac{1}{a}\left(1 + \frac{C\|f\|}{1-\alpha}\right) \tag{13}$$

whenever $n \geq N$.

Consider now the decomposition

$$\int_A P^n f = \int_{A\cap(0,\exp -a)} P^n f + \int_{A\cap[\exp -a, \exp a]} P^n f + \int_{A\cap(\exp a, \infty)} P^n f.$$

Substituting in (13) gives

$$\int_A P^n f \leq \int_{A\cap[\exp -a, \exp a]} P^n f + \frac{1}{a}\left(1 + \frac{C\|f\|}{1-\alpha}\right) \tag{14}$$

whenever $n \geq N$.

Since $\psi(z)z \leq M$ for all positive z by Assumption 5.3,

$$\begin{aligned} P^n f(k') &= \int_0^\infty \psi\left(\frac{k'}{A(k)k^\alpha}\right) \frac{1}{A(k)k^\alpha} P^{n-1} f(k) dk \\ &= \int_0^\infty \psi\left(\frac{k'}{A(k)k^\alpha}\right) \frac{k'}{A(k)k^\alpha} \frac{1}{k'} P^{n-1} f(k) dk \\ &\leq \frac{\|f\|M}{k'}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{A\cap[\exp -a, \exp a]} P^n f &\leq \int_{A\cap[\exp -a, \exp a]} \frac{\|f\|M}{x} dx \\ &\leq M\|f\| \int_A \exp(a) dx \\ &\leq M\|f\| \exp(a)\mu(A). \end{aligned}$$

We conclude that

$$\int_A P^n f \leq M\|f\|\exp(a)\mu(A) + \frac{1}{a}\left(\frac{1 + C\|f\|}{1-\alpha}\right)$$

when $n \geq N$.

Choose a so small that

$$\frac{1}{a}\left(\frac{1 + C\|f\|}{1-\alpha}\right) \leq \frac{\varepsilon}{2}. \tag{15}$$

Now pick any positive δ satisfying

$$\delta \leq (M\|f\|\exp a)^{-1}\frac{\varepsilon}{2}. \tag{16}$$

Then $n \geq N$ and $\mu(A) < \delta$ implies

$$\int_A P^n f < \varepsilon. \tag{17}$$

This proves condition (i).

It remains to establish that the weak precompactness condition (ii) also holds for the same collection $\{P^n f\}$. Fix again an arbitrary $\varepsilon > 0$. By (13),

$$\int_{\exp a}^{\infty} P^n f \leq \frac{1}{a}\left(1 + \frac{C\|f\|}{1-\alpha}\right) \tag{18}$$

for all positive a, when $n \geq N$. Choose a such that

$$\frac{1}{a}\left(1 + \frac{C\|f\|}{1-\alpha}\right) < \varepsilon. \tag{19}$$

Then the integral of $P^n f$ off the bounded set $(0, \exp a)$ is less than ε for all $n \geq N$, and condition (ii) is also satisfied.

It now follows that $(D(L_1(\mu)), P)$ is Lagrange stable. This completes the proof of the proposition. □

The following lemma is useful in establishing finite-horizon path dependence. We begin with a definition. A subset Λ of the state space X to be *ε-absorbing* for the class $\mathcal{E}$ if it satisfies

$$\lim_{\theta \downarrow 0}\left[\sup_{x \in \Lambda} Q_\theta(x, \Lambda_i^c)\right] = 0, \tag{20}$$

where Q_θ is the transition probability function corresponding to E_θ. Compare this to the definition (5) for an absorbing set. A set is absorbing for an economy E if the probability of exit in one step is zero. A set is ε-absorbing for a class of economies $\mathcal{E} = \{E_\theta\}$ if the maximum probability of exit can be made smaller than any positive ε by reducing the degree of noise, as parameterized by θ.

The definition of finite-horizon path dependence can now be simplified as follows.

Lemma 7.1. *A class $\mathcal{E}$ is finite-horizon path dependent whenever there exist two or more disjoint open intervals that are ε-absorbing for $\mathcal{E}$.*

Proof. Fix $T \in \mathbb{N}$ and $\varepsilon > 0$. By hypothesis, there exists a $\theta > 0$ and two disjoint open intervals Λ_1, Λ_2 such that

$$Q_\theta(x, \Lambda_i^c) < \varepsilon/T, \quad \forall x \in \Lambda_i.$$

Take any t in $0, \dots, T-1$ and any $x_0 \in \Lambda_i$. We have

$$\begin{aligned}
&\Pi^\theta_{x_0}((\times^t_{\tau=0}\Lambda_i) \times \Lambda^c_i) \\
&\quad = \int_{\Lambda_i} \cdots \int_{\Lambda_i} \int_{\Lambda^c_i} Q_\theta(x_{T-1}, dx_T) Q_\theta(x_{T-2}, dx_{T-1}) \cdots Q_\theta(x_0, dx_1) \\
&\quad = \int_{\Lambda_i} \cdots \int_{\Lambda_i} Q_\theta(x_{T-1}, \Lambda^c_i) Q_\theta(x_{T-2}, dx_{T-1}) \cdots Q_\theta(x_0, dx_1) \\
&\quad < \int_{\Lambda_i} \cdots \int_{\Lambda_i} (\varepsilon/T) Q_\theta(x_{T-2}, dx_{T-1}) \cdots Q_\theta(x_0, dx_1) \leq \varepsilon/T.
\end{aligned}$$

But then

$$\sum_{t=0}^{T-1} \Pi^\theta_{x_0}((\times^t_{\tau=0}\Lambda_i) \times \Lambda^c_i) < T(\varepsilon/T) = \varepsilon,$$

as was to be shown. □

We are now able to give the proof of finite-horizon path dependence.

Proof of Proposition 5.2. Let Q_θ be the transition probability function corresponding to E_θ. Let $\Lambda_1 = (0, \lambda_1)$ and $\Lambda_2 = (\lambda_2, \infty)$, where $\lambda_1 \in (k^*_1, k_b)$ and $\lambda_2 \in (k_b, k^*_2)$. Consider first the probability of leaving Λ_1 in one step, given that the current state is $k \in \Lambda_1$:

$$\begin{aligned}
Q_\theta(k, \Lambda^c_1) &= \int_{\lambda_1}^\infty p_\theta(k, k')dk' \\
&= \int_{\lambda_1}^\infty \psi\left[\left(\frac{k'}{DA_1 k^\alpha}\right)^{\frac{1}{\theta}}\right]\left(\frac{k'}{DA_1 k^\alpha}\right)^{\frac{1}{\theta}} \frac{1}{\theta k'} dk' \\
&= \int_{\left(\frac{\lambda_1}{DA_1 k^\alpha}\right)^{\frac{1}{\theta}}}^\infty \psi(z)dz.
\end{aligned}$$

But then

$$Q_\theta(k, \Lambda^c_1) \leq \int_{\left(\frac{\lambda_1^{1-\alpha}}{DA_1}\right)^{\frac{1}{\theta}}}^\infty \psi(z)dz, \quad \forall k \in \Lambda_1.$$

Note that $\lambda_1^{1-\alpha}/(DA_1) > 1$ by construction. It now follows that $Q_\theta(k, \Lambda^c_1) \downarrow 0$ uniformly in $k \in \Lambda_1$ as $\theta \downarrow 0$. A similar argument proves that $Q_\theta(k, \Lambda^c_2) \downarrow 0$ uniformly in $k \in \Lambda_2$ as $\theta \downarrow 0$. Evidently the conditions of Lemma 7.1 are satisfied. □

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