AN AXIOMATIC MODEL OF SOCIAL ADAPTIVE BEHAVIOUR

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1 INTRODUCTION

Two important stylized facts about human choice behavior are that this behavior is probabilistic and is affected by social interaction. Probabilistic choice models are routinely used to explain behavior in psychological experiments, see, for example, Bush and Mosteller (1955). In economics the probabilistic choice models were introduced by Luce (1959).

In this paper I consider the general form of a probabilistic adaptive behavior, based on social information. First, I consider a broad class of stochastic behavioral rules. I prove that any stochastic behavioral rule in a social environment is decomposable into four components: deterministic adjustment, exogenous experimentation, imitation, and experimentation based on social information, which I call imitation of scope. Then I define a class of stochastic behavioral rules called local improvement rules. I prove that for these rules the deterministic part induces a generalized gradient dynamics, that is an individual adjusts her behavior in the direction of an increase of some function.

2 A MODEL OF SOCIAL ADAPTATION

Suppose an individual has to make repeated choices over time from a set of alternatives $\Omega \subset R^n$, which is assumed to be compact, and Σ is a sigma-algebra on Ω. At time t the individual observes the current choice of a randomly selected member of a population, $y(t)$. Observations at different moments of time are assumed to be independent. For any $\Gamma \in \Sigma$, define $P(\{x(t)\}, \{y(t)\}, \Gamma, t, \tau)$ to be the transition probability, that is the probability that the individual who at time t has a history of choices $\{x(t)\}$ and a history of observations $\{y(t)\}$, will make a choice $w \in \Gamma$ at time $t + \tau$.

Axiom 1 $P(\{x(t)\}, \{y(t)\}, \Gamma, t, \tau) = P(x, y, \Gamma, t, \tau)$, *where* $x = x(t)$, *and* $y = y(t)$.

Axiom 1 says that the transition probabilities are only determined by the current choice and the current observation. In other words, if the observation made at time t is randomly selected from a distribution with a density function $f_Y(y, t)$, then the process with the transition probability

$$Q(x, \Gamma, t, \tau) = \int_\Omega P(x, y, \Gamma, t, \tau) f_Y(y, t) dy \tag{1}$$

will be a Markov process. Axiom 1 is rather weak. Indeed, assume the agent keeps track about her choices and observations at discrete moments of time, and remembers only finitely many choices and observations. Then one can always redefine the choice space in such a way that Axiom 1 will hold. Hence, Axiom 1 is essentially a finite memory assumption.

Axiom 2 *There exists a function $p(x, y, z, t, \tau) > 0$ measurable in z and twice continuously differentiable in τ such that for any $\Gamma \in \Sigma$ the transition probability is given by*

$$\begin{aligned} P(x, y, \Gamma, t, \tau) &= \int_\Gamma p(x, y, z, t, \tau) dz \\ P(x, y, \Omega, t, \tau) &= 1. \end{aligned}$$

Define a set

$$V_\delta(x) = \{w \in \Omega : \|w - x\| < \delta\},$$

where $\|\cdot\|$ denotes the Euclidean norm. Here, and throughout this paper, X^c denotes the complement of the set X.

Axiom 3 *For any $\delta > 0$ and any $x \in \Omega$, the transition probability satisfies*

$P(x, y, V_\delta^c(x), t, \tau) = o(\tau)$ and the following limits exist:

$$\lim_{\tau \to 0} \frac{1}{\tau} E(x(t+\tau) - x(t)|\ x(t),\ y(t))$$
$$\lim_{\tau \to 0} \frac{1}{\tau} Var(x(t+\tau) - x(t)|\ x(t),\ y(t)).$$

Moreover, for any $x \in \Omega$ and any neighborhood $V(x)$ the transition probability $P(x, y, V^c(x), t, \tau)$ considered as a function of y attains a minimum at $y = x$.

Axiom 2 says that the position at time $t + \tau$, conditional on the position x at time t and observation y at time t, has a strictly positive density function. To understand the meaning of this assumption, let us introduce the following concept.

Definition 1 *Let λ denote Lebesque measure. A distribution F is called singular if there exists a Borel set B_0 such that:*

1. $\lambda(B_0) = 0$;
2. $F(B_0) = 1$;
3. $F(\{x\}) = 0$ *for every* $x \in \Omega$.

If a distribution is not singular it is called regular. Axiom 2 ensures that the distribution will remain regular at all times if it is at time $t = 0$. The fact that the density is strictly positive ensures existence of steady states of the corresponding Markov process in the case when the transition probability does not depend on t. The first part of Axiom 3 says that for any realization of observations $y(t)$ the stochastic process for $x(t)$ is Khinchine continuous and is uniquely defined by its generator (Kanan (1979)). For example, it will be satisfied if conditional on the realization of $y(t)$ the process $x(t)$ is a Wiener process with drift. The second part is assumed for representational convenience.

Finally, I will assume:

Axiom 4 *$p(x, y, z, t)$ is four times continuously differentiable in x and y for any $t \geq 0$ and any realization of z.*

Under Axioms 1-4 it is possible to derive an expression for the generator of the Markov process with transition probability (1).

Theorem 1 *Assume Axioms 1-4 are satisfied. Then there exists a twice continuously differentiable vector function $\mu_1(x, t)$, and matrix-valued functions $\mu_2(x, y, t)$, $\Gamma_1(x, t)$, and $\Gamma_2(x, y, t)$, such that the matrices $\Gamma_1(x, t)$, and $\Gamma_2(x, y, t)$ are positive semidefinite and the generator of the Markov process with transition probabilities (1) is given by:*

$$\pounds = (\mu_1 + \int_\Omega \mu_2(y-x) f_Y dy)\nabla + \frac{1}{2} Tr(\Gamma_1 + \int_\Omega (y-x)^T \Gamma_2 (y-x) f_Y dy) D^2) \quad (2)$$

where

$$\nabla = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}), \ \{D^2\}_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad (3)$$

and Tr denotes the trace of a matrix.

Proof. By definition (Rogers and Williams (1994)) the generator of the Markov process with transition probabilities (1) is defined by:

$$\pounds = \lim_{\tau \to 0} \frac{P(x, \Gamma, t, \tau) - I}{\tau}, \quad (4)$$

where I is the identity operator. It can be shown (Kanan (1979)) that under the assumptions of the theorem

$$\pounds = a(x,t)\nabla + C(x,t) D^2, \quad (5)$$

where

$$
\begin{aligned}
a(x,t) &= \lim_{\tau\to 0}\frac{1}{\tau}E(x(t+\tau)-x(t)) \\
C(x,t) &= \lim_{\tau\to 0}\frac{1}{\tau}Var(x(t+\tau)-x(t)).
\end{aligned}
$$

Axiom 3 guarantees that the above limits exist. The law of iterated expectations implies that

$$
a(x,t) = \int_\Omega \mu(x,y,t) f_Y(y,t)dy,\ C(x,t) = \int_\Omega \Gamma(x,y,t) f_Y(y,t)dy \tag{6}
$$

where

$$
\mu(x,y,t) = \lim_{\tau\to 0}\frac{1}{\tau}\int_\Omega (z-x)pdz,\ \Gamma(x,y,t) = \lim_{\tau\to 0}\frac{1}{\tau}\int_\Omega (z-x)(z-x)^T pdz \tag{7}
$$

Using Axiom 4 one can write (omitting dependence on time to simplify notation):

$$
\mu(x,y) = \mu(x,x) + \mu'(x,y)(y-x) \tag{8}
$$

$$\Gamma_{ij}(x,y) = \Gamma_{ij}(x,x) + \sum_{k=1}^{n} \Gamma'_{ijk}(x,x)(y-x)_k + \sum_{k,\ell=1}^{n} \Gamma''_{ijk\ell}(x,y)(y-x)_k(y-x)_\ell \tag{9}$$

Axiom 3 implies that

$$\Gamma'_{ijk}(x,x) = 0.$$

Define

$$\mu_1(x,t) = \mu(x,x), \ \mu_2(x,y,t) = \mu'(x,y), \tag{10}$$

$$\Gamma_{1ij}(x) = \Gamma_{ij}(x,x), \ \Gamma_{2k\ell}(x,y,t) = \sum_{i.j=1}^{n} \Gamma''_{ijk\ell}(x,y) \tag{11}$$

Positive semidefinitness of matrices Γ_1 and Γ_2 follows from their definition and Axiom 3. Finally,

$$\pounds = (\mu_1 + \int_\Omega \mu_2(y-x) f_Y(y)dy)\nabla + \frac{1}{2} Tr(\Gamma_1 + \Gamma_3(z-x) + \int_\Omega (y-x)^T \Gamma_2 (y-x) f_Y(y)dy) D^2 \tag{12}$$

and the theorem is proven. ■

Theorem 1 allows us to derive an equation for the individual's choice density function. Denote this density by $f(x,t)$, then the function $f(\cdot,t)$ will satisfy the Kolmogorov forward equation for the stochastic process with transition probabilities (1). To write this equation we need the following definition:

Definition 2 *Operator $\pounds^*$ is called an adjoint operator for the operator $\pounds$ if*

1. *$dom\pounds = dom\pounds^* \subset C(\Omega)$*
2. *For any $g_1, g_2 \in dom\pounds$*

$$\int g_1(x)\pounds g_2(x)dx = \int g_2(x)\pounds^* g_1(x)dx.$$

Here $dom\pounds$ is the domain of the operator $\pounds$ and $C(\Omega)$ denotes the set of the continuous functions from Ω to R. If the generator of a stochastic process is $\pounds$, then the Kolmogorov forward equation has a form (Ito (1992)):

$$\frac{\partial f}{\partial t} = \pounds^* f, \tag{13}$$

where $\pounds^*$ is an adjoint operator for $\pounds$. Hence, the following theorem holds:

Theorem 2 *Let the distribution of choices in the population be described by a continuous density function $f(x, 0)$ at time $t = 0$. UnderAxioms 1-4, the time evolution of the function $f(x, t)$ is governed by the partial differential* *equation:*

$$\frac{\partial f(x,t)}{\partial t} + div(\mu(x,t)f(x,t)) = \frac{1}{2}Tr(D^2(\Gamma(x,t)f(x,t)) \tag{14}$$

and the boundary condition:

$$\langle \mu(x,t), n(x)\rangle f - \frac{1}{2}\langle \nabla Tr(\Gamma(x,t)f), n(x)\rangle \;\; = \;\; 0 \;\; on \;\partial\Omega, \tag{15}$$

where div denotes divergence of the vector field and, $n(x)$ is a unit vector normal to the boundary, $\langle\cdot,\cdot\rangle$ denotes inner product, and

$$\mu(x,t) = \mu_1(x,t) + \int_\Omega \mu_2(x,y,t)(y-x)f_Y(y,t)dy, \tag{16}$$

$$\Gamma(x,t) = \Gamma_1(x,t) + \int_\Omega (y-x)^T\Gamma_2(x,y,t)(y-x)f_Y(y,t)dy \tag{17}$$

Proof. The boundary condition ensures the conservation of the probability and follows from Axiom 2. Axiom 2 precludes the possibility of having a positive probability mass concentrated on the boundary and forces the flow of the probability to be zero at each point on the boundary. Take $dom\pounds$ to be the set of all twice continuously differentiable functions on Ω satisfying the boundary conditions. Then it is straightforward to check (using integration by parts) that the operator $\pounds^*$ defined by:

$$\pounds^* h = -div(\mu(x,t)h(x,t)) + \frac{1}{2}Tr(D^2(\Gamma(x,t)h(x,t))$$

is an adjoint operator for the operator $\pounds$. ■

I will symbolically write system (14)-(17) in a form:

$$\frac{\partial f}{\partial t} = L(f,\ f_Y).$$

For any twice continuously differentiable function f there exists a unique f satisfying (14)-(17) and a given initial condition (Ito (1992)).

It is worth noting that writing the Kolmogorov forward equation in the form (13) rather then (14) allows us to use it for description of cases where the population choices do not have densities (for example, are discrete). In this case f should be interpreted as a generalized function (distribution).[1] Let $\Im$ denote the space of infinitely differentiable functions with compact support. A generalized function is a continuous linear functional on $\Im$. Let K be a linear differential operator defined on functions from $\Im$, and K^* be its adjoint

[1] Although the term distribution is common in the literature I will use the term "generalized function" to prevent confusion with probability distribution.

operator. Then the generalized function g is said to solve the differential equation

$$Kg = 0$$

if

$$g(K^* h) = 0 \ \forall h \in \Im.$$

3 BEHAVIORAL INTERPRETATION OF A SOCIAL ADAPTATION RULE

To interpret Theorem 2, let us consider a specific behavioral model. The framework is similar to the general model except that time is assumed to be discrete, thus $t \in \{0, \tau, 2\tau, 3\tau, ...\}$, and the behavioral rule is given explicitly by a stochastic difference equation:

$$x_{t+\tau} - x_t = \kappa(x_t, y_t, t)\tau + B(\tau, x_t, y_t, t)(y_t - x_t)\varepsilon_t + \Lambda(x_t, t)\xi_t. \qquad (18)$$

Here $\kappa(x_t, y_t, t) = \lambda(x_t, t) + \delta(x_t, y_t, t)$, where $\delta(x_t, y_t) = \nu(x_t, y_t, t)(y_t - x_t)$ for some matrix ν. All functions are assumed to be twice continuously differentiable in x and y, and continuously differentiable in t. The random variables ε_t and ξ_t are assumed to be independently, identically distributed for each t, independent across time, have a compact support, $E(\varepsilon) = E(\xi) = 0$, and $Var(\varepsilon) = 1$, $Var(\xi) = I$.

In (18) the first term on the right hand side describes the deterministic adaptation, the second the direct imitation of choices, the third experimentation based on social information, which I will identify as "imitation of scope," and the fourth exogenous experimentation. Direct imitation means that the agent simply moves towards the observed choice. This interpretation suggests that $\nu_{kk} \geq 0$ though it is not important for the formal derivation of the model. Imitation of scope means that the individual opens a search window the width of which is determined by the degree of disagreement between her current choice and the observed choice of a randomly selected agent from the population with density $f_Y(y, t)$, that is by $(y_t - x_t)$. The value of such behavior can be intuitively explained: since the observation the agent makes is the choice of another boundedly rational agent, there is no good reason to imitate the observed choice directly. On the other hand, the spread of the choices in the population indicates that society as a whole does not know the optimal choice and, hence, that there may be returns to experimentation. The third term in (18) embodies the simple version of this intuition: $\left|y_t^k - x_t^k\right|$ increases probabilistically in the population spread for each k. Nondiagonal coefficients of matrices ν and B can be interpreted as similarity coefficients. They measure how much the choice of x_k is similar to the choice of x_ℓ. Further discussion of the social imitation rules will be given in Section 4.

I will study the continuous time limit of the stochastic process generated by (18). To pass to this limit I assume:

Condition 1 *There exists a twice continuously differentiable matrix valued function* $b : R^{2n} \to R^{n^2}$*, such that* $B(\tau, x_t, y_t, t) = b(x_t, y_t)\sqrt{\tau}$.

Let $f(x,t)$ denote the density of the individual's choices at time t. If $K \subset \Omega$ is a Borel set, denote by $G(\Delta t, x, K)$ the probability of getting to the set K from point x during the time interval Δt under the dynamics (18). For any $\eta > 0$ let $U_\eta(x)$ denote an $\eta-$neighborhood of the point x. The following result is well known in the theory of stochastic processes:

Theorem 3 *Suppose there are functions* $\zeta(x)$ *and* $\Xi(x,t)$ *twice continuously differentiable on the interior of* Ω *and continuously differentiable on* Ω *such* *that for any* $\eta > 0$ *:*

1. $G(\Delta t, x_t, U_\eta^c) = o(\Delta t)$

2. $\int_{U_\eta}(w_t - x_t)G(\Delta t, x_t, dw_t) = \zeta(x)\Delta t + o(\Delta t)$

3. $\int_{U_\eta}(w_t - x_t)(w_t - x_t)^T G(\Delta t, x_t, dw_t) = \Xi(x,t)\Delta t + o(\Delta t)$.

Then the function $f(x,t)$ *is governed by the following partial differential equation:*

$$\frac{\partial f}{\partial t} + div(\zeta(x)f) = \frac{1}{2}Tr(D^2(\Xi(x,t)f)). \tag{19}$$

For a proof see Kanan (1979). Applying theorem 3 to the behavioral rule (18) one obtains the following theorem:

Theorem 4 *Assume that the adaptation process satisfies Condition 1; then it is governed by:*

$$\frac{\partial f}{\partial t} + div(\mu(x,t)f) = \frac{1}{2}Tr(D^2(\Gamma(x,t)f)) \tag{20}$$

where

$$\mu(x,t) = \lambda(x,t) + \int_\Omega \nu(x,y)(y-x)f_Y(y,t)dy \tag{21}$$

and the matrix $\Gamma(x,t)$ *is defined by:*

$$\Gamma(x,t) = \int_{\Omega} b(y-x)(y-x)^T b^T f_Y(y,t)dy + \Lambda^T \Lambda \tag{22}$$

Proof. The proof of this theorem consists of checking all three hypothesis of Theorem 3, which can be done directly from the adaptation formula (18).

■

Comparing (13)-(16) with (19)-(21) one can arrive at the following result.

Theorem 5 *Any social adaptation rule can be decomposed into deterministic adaptation, direct imitation, experimentation and imitation of scope.*

4 AN ENDOGENOUS MODEL OF SOCIAL ADAPTATION

In this section I assume that there exists a population of individuals that follow the adaptation process described in Sections 1 and 2. I will assume that the individuals make their observations and adjustments independently. Under these assumptions I will be able to give a population interpretation of the function $f(x,t)$. Then I will show that it is possible to assume that the observations are drawn from the same population that is engaged in adaptation. This will complete my model of social adaptation.

Assume that there is a population of individuals engaged in the adaptation process described in Sections 1 and 2, and for any finite set of these individuals their observations and adaptations are independent. Let I denote the population. Suppose $A \in \Sigma$, and consider a sigma-algebra S on Ω^I generated by the sets:

$$\{x_i | x_i \in A\} = A^i.$$

Let μ be a measure on S consistent with the finite dimensional distributions of the individuals' choices. Select N individuals from the population at random and define the indicator variables:

$$X_i(t) = \begin{cases} 1 \text{ if } x_i(t) \in A, \\ 0 \text{ if } x_i(t) \notin A. \end{cases}$$

Let $B \subset \Omega^I$ be the set defined by:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{n} X_i(t) \neq \int_A f(x,t)dx.$$

Then there exists an extension $\overline{\mu}$ of measure μ such that B is $\overline{\mu}$ - measurable and $\overline{\mu}(B) = 0$. (See Judd 1985 for a detailed discussion). This result allows us to interpret $f(x,t)$ as the population density of choices. To be able to interpret the model with a continuum of agents as the limit of the models with finitely many agents some additional work has to be done. For example, one can use hyperfinite discrete models from the nonstandard analysis. These models have the advantage of simultaneously approximating the function theory in the Euclidean space and the probability theory of large discrete models. This argument is developed in Keisler (1984).

Now I want to allow the individuals to make observations from their own population. For this purpose define $f_Y(x,t)$ to be a solution to the equation

$$\frac{\partial f_Y}{\partial t} = L(f_Y,\ f_Y)$$

with the initial condition $f_Y(x,0) = f(x,0)$. I prove the existence of a solution to this problem in the Appendix. Since system (14)-(17) has a unique solution for any $f_Y(x,t)$, any such choice of $f_Y(x,t)$ implies that $f(x,t) = f_Y(x,t)$ at all $t \geq 0$. This implies that the two populations can be identified. Hence, I will assume below that there is one population with a density function evolving accordingly to

$$\frac{\partial f}{\partial t} = L(f,\ f).$$

This equation is nonlinear. Hence, it is interesting to investigate the uniqueness of a steady state for the population density. The following result holds.

Theorem 6 *Suppose the coefficients of a social adaptation rule do not depend on time. Then a steady state of Markov process (2.18) exists. If* $\exists\ \delta > 0$ *such that* $c^T\Gamma(x)c \geq \delta \left\| c \right\|^2$ *for* $\forall x \in \Omega$*, then the steady state is unique.*

Proof. Existence follows from Theorem 5 and Axiom 2. If $\exists\ \delta > 0$ such that $c^T\Gamma(x)c \geq \delta \left\| c \right\|^2$ for $\forall x \in \Omega$ then it is easy to check that the Markov chain (1) is irreducable; hence the steady state is unique. ■

Uniqueness will not generally hold in the absence of exogenous noise. It is easy to see that in this case any distribution concentrated at a point x^* such that $\mu_1(x^*) = 0$ is a steady state. Since there might be several such points, a steady state need not be unique. If $\mu_2(x) = 0$ then a probability distribution that assigns arbitrary weights to different critical points of $\mu_1(\cdot)$ is a steady state. However, it will not be a steady state if $\mu_2(\cdot)$ is not zero. Steady states characterized with continuous densities are also possible. Consider the following example.

Example. Let $\mu_1(x) = \mu_2(x) = \Gamma_1(x) = 0$ and $\Gamma_2(x) = 1$. Let the admissible set be $\Omega = [-a, a]$. It is easy to see that the distribution concentrated at any point is a steady state. It is easy to show that $f(x)$ is a density of a nondegenerate steady state if and only if the following conditions are satisfied.

$$
\begin{aligned}
f(x) &= \frac{A}{(x-w)^2+\sigma^2} \\
\int_{-a}^{a} f(x)dx &= 1 \\
w &= \int_{-a}^{a} x f(x)dx \\
\sigma^2 &= \int_{-a}^{a} (x-w)^2 f(x)dx
\end{aligned}
$$

Let σ be a positive solution of the equation

$$2\sigma = \frac{a}{\tan^{-1}(\frac{a}{\sigma})}.$$

Such a solution exists and is unique. Let

$$
\begin{aligned}
A &= \frac{\sigma^2}{a}, \quad w = 0 \\
f(x) &= \frac{\sigma^2}{a(x^2+\sigma^2)}
\end{aligned}
$$

Then $f(x)$ is a steady state density function.

Even though Axioms 1-4 put no restrictions on the coefficients in (18), the intuitive interpretation of this rule allows us to lay down some restrictions. For example, interpretation of μ_2 as direct imitation allows us to impose restrictions $\mu_{2kk} \geq 0$, $\mu_{2k\ell} = 0$ for $k \neq \ell$. Such restrictions may prove very useful if one attempts to estimate equation (18). In the next subsection I will argue that an interpretation of (18) as a form of learning allows to put rather strong restrictions on its deterministic part, μ_1.

5 THE DETERMINISTIC COMPONENT OF A SOCIAL ADAPTATION RULE.

5.1 LOCALLY IMPROVING ADAPTATION RULES

In the previous section I showed that any social adaptation rule can be decomposed into deterministic adaptation, exogenous experimentation, direct imitation, and imitation of scope. However, no restrictions apart from some regularity conditions were obtained for the coefficients of the process. In this section, I will define a class of stochastic adaptation rules called *lo-*

cally improving adaptation rules and get for them some restrictions on the deterministic part of the stochastic process. To do this I will introduce the notion of a deterministic agent. The deterministic agent adjusts her choices based only on her current position and does not engage in any kind of experimentation. Intuitively, a rule is locally improving if a deterministic agent does not have cycles in her choices.

One natural deterministic learning rule is the gradient dynamics, where the time derivative of choices equals the gradient of some scalar function which can be naturally interpreted as a utility function. Intuitively, it means that individuals adjust their choices in the direction of the fastest increase of utility. The gradient dynamics was thoroughly studied by Arrow and Hurwicz (1960). Below I will show that a slight generalization of the gradient dynamics covers all reasonable continuous deterministic learning rules.

In this subsection I will restrict myself to time homogenous stochastic adaptation rules, that is I will assume that μ_1 and Γ_1 do not depend on time, while μ_2 and Γ_2 may depend on time only through $f(y,t)$. I will show that under some regularity conditions a time homogenous stochastic behavioral rule is locally improving if and only if its deterministic part, $\mu_1(x)$, determines a generalized gradient dynamics, that is, it always points in the direction of an increase of some real-valued function. This function is a natural candidate for a utility function of the individual. To proceed further I will need some definitions.

Definition 3 *A population is called deterministic if* $\mu_2(x, y) = \Gamma_1(x) = \Gamma_2(x, y) = \Gamma_3(x) = 0$.

Theorem 5 and equation (18) imply that the choices of an individual of such a population are determined according to

$$\frac{dx}{dt} = \mu_1(x). \tag{23}$$

The evolution of the density of choices in a deterministic population is governed by the continuity equation:

$$\frac{\partial f}{\partial t} + div(f\mu_1) = 0. \tag{24}$$

The term "continuity equation" comes from fluids mechanics. If one assumes that $f(x, t)$ is a density of a moving fluid at point x at time t and μ_1 is its velocity, then equation (24) simply says that the flow of fluid is continuous.

Definition 4 *I will say that the choice* x_1 *is revealed-strictly-preferred by a deterministic agent to the choice* x_2 $(x_1 \neq x_2)$*, denoted* $x_1 R x_2$*, if there* *exist* $t_1 > t_2 \geq 0$*, such that* $x(t)$ *is a solution of (23) with* $x(t_1) = x_1$ *and* $x(t_2) = x_2$*.*

This definition says that, assuming the deterministic agent follows a local improvement rule, choices which are made later can be interpreted as better choices for a deterministic agent[2]. However, to interpret later choices as better choices R should be rationalizable by a preference relation. A sufficient condition for this is that R satisfies the Strong Axiom of Revealed Preferences (Mas-Colell, Whinston, and Green (1995)). To determine the conditions under which this preference relation is representable by a continuously differentiable utility function, we need the following definition.

Definition 5 *A social behavioral rule with the deterministic part* $\mu_1(x)$ *is called a locally improving rule if there exists a continuously differentiable function* $U(x)$ *such that*

1. $\nabla U(x) = 0$ *if and only if* $\mu_1(x) = 0$*;*

2. $x_1 R x_2$ *implies* $U(x_1) > U(x_2)$*.*

Finally, I will define a generalized gradient dynamics.

[2] This need not be the case when an agent uses a stochastic algorithm since she might experiment with choices.

Definition 6 *A vector field $\mu_1(x)$ is said to induce a generalized gradient dynamics if there exists a continuously differentiable function $\Pi(x)$ such that:*

1. $\mu_1(x) = 0$ if and only if $\nabla\Pi(x) = 0$;

2. $\langle \mu_1(x), \nabla\Pi(x) \rangle \geq 0$ for any $x \in \Omega$ and $\langle \mu_1(x), \nabla\Pi(x) \rangle = 0$ implies $\mu_1(x) = 0$.

Here and throughout the dissertation $\langle \mu_1(x), \nabla\Pi(x) \rangle$ denotes the inner product of two vectors. This definition implies that the function $\Pi(x)$ increases across the solutions of system (23), and stable steady states of the system (23) correspond to regular local maxima of the function $\Pi(\cdot)$.

I will make the following regularity assumption.

Condition 2 *The Jacobian matrix $D\mu_1/Dx$ has full rank at every point of Ω.*

Now I am ready to state the following theorem.

Theorem 7 *Assume Axioms 1-4 and Condition 2 are satisfied. A social adaptation rule is a locally improving rule if and only if its deterministic part $\mu_1(x)$ induces a generalized gradient dynamics.*

Proof. Under Axioms 1-4, Theorem 1 implies that the deterministic part of a social adaptation rule is well defined. Suppose the vector $\mu_1(x)$ induces a generalized gradient dynamics, then there exists a continuously differentiable function $\Pi(x)$ such that $\langle \mu_1(x), \nabla\Pi(x)\rangle \geq 0$ for any $x \in \Omega$ and $\langle \mu_1(x), \nabla\Pi(x)\rangle = 0$ implies $\mu_1(x) = 0$. Let $x_1 R x_2$. Then there exists a solution $x(t)$ of system (23) beginning at x_2 and ending at x_1. Then

$$\Pi(x_1) - \Pi(x_2) = \int_{t_2}^{t_1} \langle \mu_1(x(t)), \nabla\Pi(x(t))\rangle dt \tag{25}$$

Since $x_2 \neq x_1$ implies that x_2 is not a steady state of (23) $\mu_1(x_2) \neq 0$. Hence, Definition 5 implies that $\langle \mu_1(x_2), \nabla\Pi(x_2)\rangle > 0$. By continuity, $\langle \mu_1(x), \nabla\Pi(x)\rangle > 0$ in some neighborhood of point x_2, and is nonnegative everywhere, hence $\Pi(x_1) > \Pi(x_2)$. Since $\Pi(x)$ is continuously differentiable, take $U(x) = \Pi(x)$. Then Definition 4 will be satisfied, hence the social adaptation rule is a locally improving rule.

Now suppose that a social adaptation rule is a locally improving rule. Then there exists a continuously differentiable function $U(x)$ such that

1. $\nabla U(x) = 0$ if and only if $\mu_1(x) = 0$;
2. $x_1 R x_2$ implies $U(x_1) > U(x_2)$.

Define $\Pi(x) = U(x)$. Then the function $\Pi(x)$ is continuously differentiable and satisfies the first part of the Definition 5. To check the second part of Definition 5, first, assume that there exists x_2 such that $\langle \mu_1(x_2), \nabla\Pi(x_2)\rangle$

< 0. Then there exists a neighborhood $V(x_2)$ of x_2 such that $\langle\mu_1(w), \nabla\Pi(w)\rangle < 0$ for any $w \in V(x_2)$. Let $t_2 = 0$, $t_1 > 0$ and denote $x_1 = x(t_1)$, where $x(t)$ is the solution of system (24) with initial condition $x(0) = x_2$. For a sufficiently small t_1 the inclusion $x_1 \in V(x_2)$ will be satisfied. Then $x_1 R x_2$ and $U(x_1) - U(x_2) = \Pi(x_1) - \Pi(x_2) = \int_{t_2}^{t_1} \langle\mu_1(x(t)), \nabla\Pi(x(t)\rangle dt < 0$, which is a contradiction, hence $\langle\mu_1(x), \nabla\Pi(x)\rangle \geq 0$.

Now assume that $\langle\mu_1(x), \nabla\Pi(x)\rangle = 0$, but $\mu_1(x) \neq 0$. Then, by construction, $\nabla\Pi(x) \neq 0$. Then there exists a neighborhood $W(x)$ of point x such that $\nabla\Pi(w) \neq 0$ for any $w \in W(x)$. Let $J(x)$ denote the Jacoby matrix $D\mu_1(x)/Dx$. Then there exists $w^* \in W(w)$ such that $\langle\mu_1(w), \nabla\Pi(w)\rangle = (w - x)^T J(w^*)\nabla\Pi(w)$. Since matrix J is nondegenerate one can always find w such that $\langle\mu_1(w), \nabla\Pi(w)\rangle < 0$, which was already proven to be impossible. This completes the proof. ■

The above theorem restricts the deterministic part of a social adaptation rule, assuming that in the absence of the stochastic and social components the adaptation rule corresponds to a gradual increase of some real valued function which can be interpreted as a utility function. I will summarize the results obtained so far in a theorem.

Theorem 8 *Any time-homogeneous locally improving social adaptation rule* *can be decomposed into the generalized gradient dynamics, exogenous experimentation, direct imitation, and imitation of scope.*

The above theorem allows us to parametrize general adaptation rules. Such a parametrization is useful for estimating these rules. For a discussion of possible estimation techniques see Basov (2001).

5.2 PREFERENCES

In the preceding subsection I showed that the deterministic part, $\mu_1(x)$, of any adaptation rule defines a generalized gradient dynamics. In other words, there exists a continuously differentiable function $\Pi(\cdot)$ such that:

1. $\mu_1(x) = 0$ if and only if $\nabla\Pi(x) = 0$;

2. $\langle \mu_1(x), \nabla\Pi(x)\rangle \geq 0$ for any $x \in \Omega$ and $\langle \mu_1(x), \nabla\Pi(x)\rangle = 0$ implies $\mu_1(x) = 0$.

The function $\Pi(\cdot)$ can be naturally interpreted as the utility function of an individual. However, the function $\Pi(\cdot)$ need not be unique. Different functions satisfying the above conditions need not even be increasing transformations of one another. In this section I will deal with the uniqueness problem. At a conceptual level, I am asking the question: Given an adaptation rule, what can we conclude about the preferences of an individual? To answer this question let us first consider the case when the vector $\mu_1(\cdot)$ can be represented as the gradient of a scalar function $\Pi(\cdot)$. In this case, the function $\Pi(\cdot)$ is determined uniquely up to an additive constant. If one interprets the function $\Pi(\cdot)$ as a utility function of the individual, then the deterministic part of the learning rule describes the gradient dynamics; namely, the individual will adjust her choices in the direction of the fastest increase of her utility. This makes the interpretation of $\Pi(\cdot)$ as a utility function natural. In general, the vector field $\mu_1(\cdot)$ cannot be represented as the gradient of a scalar function. In this case, I will define a utility function $\Pi(\cdot)$ in a way such that $\mu_1(\cdot)$ is a generalized gradient dynamics and the discrepancy between the gradient of $\Pi(\cdot)$ and $\mu_1(\cdot)$ is minimal. Formally, define a set $\Omega_1 = \{x \in \Omega : \mu_1(x) \neq 0\}$, and an arbitrary number M,[3] then $\Pi(\cdot)$ solves:

$$\min_{\Pi \in C^1(\Omega)} \int_\Omega \|\mu_1(x) - \nabla\Pi(x)\|^2 \, d\lambda \tag{26}$$

$$s.t. \langle \mu_1(x), \nabla\Pi(x) \rangle \geq 0, \ \Pi(x) \leq M \text{ on } \Omega, \ \nabla\Pi(x) = 0 \text{ on } \Omega/\Omega_1, \tag{27}$$

where $\|\cdot\|$ denotes the Euclidean norm and λ is the Lebesque measure on Ω.

[3] This number is defined for technical reasons which will become clear later.

The following theorem holds:

Theorem 9 *Assume that $\mu_1(x)$ induces a generalized gradient dynamics. A solution to the problem (26)-(27) exists and is unique up to an additive constant. Moreover, if $\Pi(\cdot)$ is a solution to (26)-(27) and at some $x_0 \in \Omega$ either $\nabla\Pi(x_0) = 0$ or $\langle \mu_1(x_0), \nabla\Pi(x_0)\rangle = 0$, then $\mu_1(x_0) = 0$.*

Proof. First, note that $\Pi \in C^1(\Omega)$ and the fact that Ω is compact implies that $\Pi \in L^2(\Omega)$ and $\nabla\Pi \in L^2(\Omega)$. Define

$$
\begin{aligned}
H^1(\Omega) &= \{\phi : \phi \in L^2(\Omega),\ \nabla\phi \in L^2(\Omega)\} && (28)\\
|\phi|_{H^1} &= \int_\Omega (\phi^2 + \|\nabla\phi\|^2). && (29)
\end{aligned}
$$

Define a functional V by the formula:

$$
V(\Pi) = -\int_\Omega \|\mu_1(x) - \nabla\Pi(x)\|^2 \, d\lambda \qquad (30)
$$

Let $K = \{\Pi \in H^1(\Omega) : \langle \mu_1(x), \nabla\Pi(x)\rangle \geq 0$ on Ω_1, $\Pi(x) \leq M$ on Ω, $\nabla\Pi(x) = 0$ on $\Omega/\Omega_1\}$.

To prove that the functional $V(\cdot)$ achieves a maximum at K it is sufficient to prove that $V(\cdot)$ is coercive and K is nonempty, closed, and convex (see, e.g., Kinderlehrer and Stampacchia 1980). It is straightforward to see that the set K satisfies the above requirements.

Recall that a functional $V(\cdot)$ is called coercive if $V(\Pi)$ tends to $-\infty$ when $|\Pi|_{H^1}$ tends to $+\infty$. For all $\Pi \in H^1(\Omega)$, denote by $\underline{\Pi}$ the mean value of Π over Ω:

$$\underline{\Pi} = \frac{1}{|\Omega|} \int_{\Omega} \Pi(x) d\lambda. \tag{31}$$

By Poincare's inequality (see, e.g., Kinderlehrer and Stampacchia 1980) there exists a constant $N(\Omega)$ such that for all $\Pi \in H^1(\Omega)$, $|\Pi - \underline{\Pi}|_{L^2} \leq N(\Omega) \, |\nabla \Pi|_{L^2}$. Since the function Π is assumed to be bounded from above ($\Pi(x) \leq M$ on Ω), this implies that

$$|\Pi|_{H^1} \to +\infty \;\Leftrightarrow\; |\nabla \Pi|_{L^2} \to +\infty\ . \tag{32}$$

Now it is straightforward from (30) that $V(\Pi)$ tends to $-\infty$ when $|\Pi|_{H^1}$ tends to $+\infty$, i. e. $V(\cdot)$ is coercive.

To prove the uniqueness note that the function $v : R^n \to R$ defined as

$$v(z) = -(\mu_1(x) - z)^2 \tag{33}$$

is strictly concave. Hence, for any $\alpha \in [0, 1]$

$$v(\alpha z_1 + (1 - \alpha)z_2) \geq \alpha v(z_1) + (1 - \alpha)v(z_2) \tag{34}$$

with equality only if $\alpha \in \{0, 1\}$. Suppose there are two solutions to the maximization problem, Π_1 and Π_2 and $V(\Pi_1) = V(\Pi_2) = V^*$. Since the set K is convex, the function

$$\Pi_{1/2} = \frac{\Pi_1 + \Pi_2}{2} \tag{35}$$

is also in K.

$$V(\Pi_{1/2}) = -\int_{\Omega} v(\nabla\Pi_{1/2})d\lambda \geq \frac{1}{2}\int_{\Omega} v(\nabla\Pi_1)d\lambda + \frac{1}{2}\int_{\Omega} v(\nabla\Pi_2)d\lambda = V^*. \tag{36}$$

Since V^* is defined to be the maximum value of the functional $V(\cdot)$ the inequality (36) is satisfied as an equality, which implies $\nabla\Pi_1 = \nabla\Pi_2$, and, hence, the function $\Pi(\cdot)$ is defined uniquely up to an additive constant.

Now suppose that either $\nabla\Pi(x_0) = 0$ or $\langle \mu_1(x_0), \nabla\Pi(x_0)\rangle = 0$ but $\mu_1(x_0) \neq 0$. Since $\mu_1(\cdot)$ induces a generalized gradient dynamics, there exists a continuously differentiable function Π^* such that $\langle \mu_1(x_0), \nabla\Pi^*(x_0)\rangle > 0$. Since the set Ω is compact, Π^* is bounded, and hence one can always assure that $\Pi^*(x) \leq M$ on Ω with an additive constant. Let $x_0 \in W_1 \subset W_2 \subset \Omega$, where $\subset$ denotes strict inclusion and W_1 and W_2 are open sets. One can always construct a function $\Pi^{**} \in K$ which coincides with $\varepsilon\Pi^*$ on W_1 and with Π outside W_2. For an appropriate choice of $\varepsilon > 0$, W_1, and W_2, it will decrease the objective functional given by (30), which contradicts the hypothesis that Π maximizes V. ■

The theorem states that the utility function is determined up to an additive constant. This means that the difference in utility between any two choices is well defined. This representation is more precise than in conventional utility maximization theory under certainty, where the utility is defined up to a continuous increasing transformation; or under uncertainty, where the utility is defined up to a positive affine transformation. In the last case, one can choose arbitrarily both the origin and the scale. In the present case one is still free to choose the origin, but the level is fixed. This happens because in my framework the utility function determines not only the preference ordering but also the speed of adaptation. Since the speed of adaptation is observable, the change in the level of utility is observable as well.

Theorem 8 also states that if the vector field $\mu_1(\cdot)$ is not identically zero, then the underlying preferences are not trivial, that is there are at least two choices with different levels of utility.

6 SOCIAL IMITATION

Learning, and adaptation in general, is often a social process. Equation (18) expresses this. It can be considered as a numerical rule for adaptation that utilizes social information. The agents using this rule can be considered to be procedurally rational; under some conditions this process may converge to the rational outcome[4]. The stochastic component of the rule has the virtue of reducing the probability of getting stuck at a local maximum. Endogenising the random component by making it a function of others' actions can facilitate convergence. But this comes at a cost: a population using *social* adaptation instead of, say, individual experimentation, may not reach the ra-

[4]By the rational outcome I mean the choice that maximizes the function Π defined in the Theorem 8. This function can naturally be interpreted as a utility function of the individual.

tional outcome. In contrast, there are individual experimentation algorithms that guarantee the rational outcome will be reached provided noise is reduced sufficiently slowly. An example of such a rule is "simulated annealing." For a discussion of theoretical and computational aspects of simulated annealing, see Laarhoven (1988). The problem with individual adaptation rules is that they guarantee convergence to an optimum only if noise is reduced very slowly, and hence, might perform rather poorly if time is valuable. Social adaptation speeds up convergence when there is consensus in society about the optimal choice, due to the fact that information from the entire society about the payoff structure is used.

In the discussion after equation (18), I identified two types of social adaptation, direct imitation and imitation of scope. The first is rather easy to understand: it simply says that the choice of an agent at time $t+1$ is a convex combination of her choice and the choice of a randomly selected member of the population at time t. Now I discuss imitation of scope. Imitation of scope necessarily arises as a part of the general adaptation rule as long as the rate of experimentation is affected by the social information. A possible intuition behind this is that: if the observed choice is close to the observer's choice then there is a good chance they are both close to the optimum. Hence, the incentive to experiment with a more distant choice is small. On the other hand, if the two choices are far apart, then at least one of them is far away from the optimum. Symmetry implies that the chances are about even that it is the observer or the observed who is farther from the optimum. In this case, the incentive to experiment with a choice farther away is higher.

Under imitation of scope, noisy choices may persist in the long-run even if there is no exogenous experimentation and the objective function is strictly concave. Intuitively, while the gradient dynamics shrinks the population variance, each act of experimentation injects a noise into the system, and leads to an increase in the population variance. Under some conditions these two effects can exactly balance each other. This may have profound economic implications. For example, low-powered incentives and compensation for luck rather than effort can be explained by this behavior. For a detailed discussion see Basov (2001). It is also important to mention that due to the social imitation, the equation for the population choice density function is nonlinear. One of the consequences of this nonlinearity, possibility of multiple steady states, was discussed earlier in this paper.

6.1 DOES THE OPTIMAL ADAPTATION RULE EXIST?

In this subsection, I consider the question of whether there exists an adaptation rule which behaves sufficiently well in all environments. To put the question more formally: Is there an adaptation rule which, given any payoff function and any initial distribution, will lead to a rational outcome in the steady state? A weaker version of this question would be: is there an adaptation rule that for any payoff function will result in a stationary distribution that assigns small probability to being far from the global maximum? If the answer to at least one of these questions were "yes," then one might expect that such an adaptation rule would have been selected by evolution, since the individuals who had been genetically programmed or indoctrinated to follow such a rule would have achieved high payoffs in a wide variety of environments. This would have allowed us to specify a priori the coefficients of the adaptation rule and make sharper predictions about the economic outcomes than is possible for a general adaptation rule.

The answer to the both of these questions is "no," however. More precisely, I am going to prove that, no matter what adaptation rule is used, in some environment it will be weakly out-performed by just making random choices from the set of alternatives.

Theorem 10 *Given any stationary adaptation rule specified by the matrices* Γ_1, Γ_2, $\mu_2 \in C^2(\Omega \times \Omega)$, *there exists a payoff function* $\Pi(x)$ *such that*

1. $\mu_1(x) = \nabla \Pi(x)$

2. the uniform distribution on Ω *is a steady state of the adaptation rule, defined by* Γ_1, Γ_2, μ_2 *and* μ_1.

Proof. Define the matrix $\Gamma(x)$ and the vector $m(x)$ by the formulae

$$\Gamma(x,t) = \Gamma_1 + \frac{1}{|\Omega|}\int_\Omega (y-x)^T \Gamma_2 (y-x) dy \quad (37)$$

$$m(x) = \frac{1}{|\Omega|}\int_\Omega \mu_2(y-x) dy \quad (38)$$

Define $\Pi(x)$ to be a solution to the boundary problem:

$$\Delta \Pi(x) = \frac{1}{2} Tr(D^2 \Gamma(x)) - div[m(x)] \quad (39)$$

$$< \nabla \Pi(x), n > \; = \; < \nabla Tr \Gamma(z,x) - m(x), n >, \quad (40)$$

where

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{41}$$

is the Laplace operator.

The system (39)-(40) defines $\Pi(x)$ as a solution to the von-Neumann boundary problem for the Poisson equation. Since the compatibility condition

$$\frac{1}{2}\int_{\Omega}(Tr(D^2\Gamma(z,x))-div[m(x)]dx = \int_{\partial\Omega} < \nabla Tr\Gamma(z,x)-m(x), n > dx \tag{42}$$

is satisfied, the solution to this problem exists and is unique up to an additive constant. It is straightforward to check that if $\Pi(x)$ is defined in this way then

$$f = \frac{1}{|\Omega|} \tag{43}$$

is a steady state of the stochastic adaptation rule defined by Γ_1, Γ_2, μ_2 and μ_1. ■

The above result suggests that any adaptation rule can fail in some environment. We might think of the adaptation rule as genetically transmitted and changing only due to mutations and natural selection, where tastes (represented by $\Pi(x)$ here) may vary significantly due to exogenous forces from generation to generation. In this case, the previous result suggests that a broad range of learning rules would be present in the population in the long run. This would result in rather diverse social behaviors, and this diversity should not be expected to go away with time.

7 DISCUSSION

Bounded rationality and learning models provide important insights which allow one to understand better both some regularities observed in the laboratory and real economic phenomena. Anderson, Goeree and Holt (1998) argued that a model of rent seeking based on boundedly rational behavior describes reality better than the conventional model. Maskin and Tirole (1999) and Tirole (1999) suggest that bounded rationality might prove to be important in providing the foundations for incomplete contracts. For a discussion of the interaction between bounded rationality, reciprocity, and the structure of the optimal contracts see Basov (2001).

8 APPENDIX

This Appendix is devoted to the proof of the existence result needed in Section 3. A continuous solution of the system

$$\frac{\partial f(x,t)}{\partial t} + div(\mu(x,t)f(x,t)) = \frac{1}{2}Tr(D^2(\Gamma(x,t)f(x,t)) \tag{44}$$

$$\langle \mu(x,t), n(x)\rangle f - \frac{1}{2}\langle \nabla Tr(\Gamma(x,t)f), n(x)\rangle \ = \ 0 \text{ on } \partial\Omega, \tag{45}$$

$$\mu(x,t) = \mu_1(x,t) + \int_\Omega \mu_2(x,y,t)(y-x)f(y,t)dy, \tag{46}$$

$$\Gamma(x,t) = \Gamma_1(x,t) + \int_\Omega (y-x)^T\Gamma_2(x,y,t)(y-x)f(y,t)dy \tag{47}$$

exists for any $(x,t) \in \Omega \times [0,\tau]$ for any positive τ and any twice differentiable initial condition $f(x,0) = g(x)$. For the choice set Ω, denote by $C(\Omega)$ the Banach space of the functions continuous on Ω with the *sup* norm. Define the operator $T : C(\Omega) \to C(\Omega)$ in the following way. For any continuous density function $f(x,t)$, use (47)-(48) to evaluate the matrix $\Gamma(x,t)$ and vector $\mu(x,t)$, and plug them in (45)-(46). Let Tf be the solution of (45)-(46) with the initial condition $f(x,0) = g(x)$. The existence of a solution for the problem (45)-(48) is then reduced to the existence of the fixed point of the operator T. The proof of the existence of the fixed point is based on the Schauder Fixed Point Theorem (see Stokey and Lucas 1993).

Note, first, that the set of continuous probability distribution functions F is closed and convex. Let K be such that the maximum of the $\|\mu_1(x)\|$ and $\|\Gamma_1(x,t)\|$ on $\Omega\times[0,\tau]$, as well as maximum of the $\|\mu_2(x,y)\|$and $\|\Gamma_2(x,y,t)\|$on $\Omega\times\Omega\times[0,\tau]$, is less then K. Define $d = \sup_{\zeta,z\in\Omega}\|\zeta-\xi\|$ to be the diameter of the set Ω. Since the set Ω is compact, d is finite. Then there exist positive constants $C_1(K,d,\tau)$ and $C_2(K,d)$ such that $\|Tf(x)\| \leq C_2(K,d)$ and $\|Tf_j(x)\| \leq C_1(K,d,\tau)$ for all j. Here $Tf_j(x)$ denotes the partial derivative of $Tf(x)$ with respect to x_j. For a proof see Ladyzhenskaia, Solonnikov, and Uralceva (1968). The first of these inequalities proves that the family $T(F)$ is uniformly bounded, while the second, together with the formula for finite differences, proves its equicontinuity.

It remains to prove that the operator T is continuous. Consider T to be a composition of two operators: operator A that transforms a probability density function into the matrix valued function $\Gamma(x,t)$ and the vector valued function $\mu(x,t)$ according to (47)-(48), and operator B that transforms them into the solution (45)-(46) with the initial condition $f(x,0) = g(x)$. (The existence of f *given* μ and Γ was discussed in Section 3). Then $T = BA$. Operator A is obviously continuous. The continuity of operator B follows from the fact that solution of the initial-boundary problem for the diffusion equation depends continuously on its coefficients (Ladyzhenskaia, Solonnikov, and Uralceva 1968). Thus all the conditions of the Shauder Fixed Point Theorem are satisfied and hence system (45)-(48) has a solution.

REFERENCES

Anderson S. P., J. K Goeree, and C. A. Holt. "Rent Seeking with Bounded Rationality: An Analysis of All-Pay Auction," *Journal of Political Economy* 106, (1998), 828-853.

Arrow, K. J. and L. Hurwicz "Stability of Gradient Process in n-Person Games", *Journal of Society of Industrial and Applied Mathematics* 8, (1960), 280-94.

Basov, S. "Bounded Rationality, Reciprocity, and Their Economic Consequences," Ph.D. Dissertation, Graduate School of Arts and Sciences, Boston University, 2001.

Bush R., and F. Mosteller. *Stochastic Models of Learning*, (New York: Wiley, 1955).

Ito, S. *Diffusion Equations*, (Providence: American Mathematical Society, 1992).

Judd, K. L. "Law of Large Numbers," *Journal of Economic Theory* 45, (1985), 19-25.

Kanan, D. *An Introduction to Stochastic Processes*, (New York: Limerick, Ire., 1979).

Keisler, H. J. *An Infinitesimal Approach to Stochastic Analysis*, (Providence:

American Mathematical Society, 1984).

D. Kinderlehrer, and G. Stampacchia. *An Introduction to Variational Inequalities*, (Boston: Academic Press, 1980).

Laarhoven, P. J. M. *Theoretical and Computational Aspects of Simulated Annealing*, (Amsterdam: Centrum voor Wiskunde en Informatica, 1988).

Ladyzhenskaia O. A., Solonnikov, V. A., Uralceva N. N. *Linear and Quasilinear Equations of Parabolic Type*, (Providence: American Mathematical Society, 1968).

Luce, R. D. *Individual Choice Behavior*, (New York: Wiley, 1959).

Mas-Colell, A., M. D. Whinston, and J. R. Green. *Microeconomic Theory*, (Oxford: Oxford University Press, 1995).

Maskin, E., and J. Tirole "Unforeseen Contingencies and Incomplete Contracts," *Review of Economic Studies* 66, (1999), 83-114.

Rogers, L. C. G., and D. Williams *Diffusions, Markov Processes and Martingales,* (New York: John Wiley & Sons, 1994).

Stokey, N. L. and R. E. Lucas, Jr. *Recursive Methods in Economic Dynamics*, (Cambridge: Harvard University Press, 1993).

Tirole, J. "Incomplete Contracts: Where Do We Stand?" *Econometrica*, 1999, 67, pp.741-782.