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Abstract. We consider discrete time Markov chains on general state space. It is shown that a certain property referred to here as nondecomposability is equivalent to irreducibility, and that a Markov chain with invariant distribution is irreducible if and only if the invariant distribution is unique and assigns positive probability to all absorbing sets.

1. Introduction

Let $(S, \mathscr{S})$ be any measurable space, let $\mathscr{M}$ be the finite signed measures on same, and let $\mathscr{P}$ be all $\mu \in \mathscr{M}$ with $\mu \geq 0$ and $\mu(S) = 1$. Let $P\colon S \times \mathscr{S} \to [0, 1]$ be a Markov kernel on $(S, \mathscr{S})$. In other words, $x \mapsto P(x, B)$ is $\mathscr{S}$-measurable for all $B \in \mathscr{S}$, and $B \mapsto P(x, B)$ is an element of $\mathscr{P}$ for all $x \in S$. For $\mu \in \mathscr{M}$ define $P\mu$ by $P\mu(B) := \int P(x, B)\mu(dx)$. Also $P^{n+1}(x, B) := \int P(x, dy)P^n(y, B)$, with $P^1 := P$.

A $\pi \in \mathscr{M}$ satisfying $P\pi = \pi$ is called P-invariant. A set $B \in \mathscr{S}$ is called P-absorbing if it is nonempty and $P(x, B) = 1$ for all $x \in B$. Let $\psi \in \mathscr{P}$. We call P ψ-nondecomposable when every P-absorbing set satisfies $\psi(B) = 1$. As usual, P is called ψ-irreducible if for every $x \in S$ and $B \in \mathscr{S}$ with $\psi(B) > 0$ we have $\sum_{n=1}^{\infty} P^n(x, B) > 0$. Finally, let us agree to call P irreducible if it is ψ-irreducible for some $\psi \in \mathscr{P}$.

The notion of irreducibility is fundamental to the modern theory of Markov chains (cf., e.g., Meyn and Tweedie, 1993). In this paper we show that ψ-nondecomposability is equivalent to ψ-irreducibility. In addition, we show that when a P-invariant distribution π exists, P is irreducible if and only if π is the only invariant distribution in $\mathscr{P}$ and $\pi(B) = 1$ for every P-absorbing $B \in \mathscr{S}$.

2. Results

The first result from which many of our conclusions follow is

Theorem 2.1. *Let $\psi \in \mathscr{P}$. The Markov kernel P is ψ-nondecomposable if and only if it is ψ-irreducible.*

In Theorem 2.1 necessity is well known and rather obvious, but sufficiency is not. The proof is given in the next section.

Theorem 2.2. *Let a P-invariant $\pi \in \mathscr{P}$ exist. The following statements are all equivalent.*

(i) *P is irreducible.*
(ii) *P is ψ-nondecomposable for some $\psi \in \mathscr{P}$.*
(iii) *P is π-nondecomposable.*
(iv) *π is unique, and P is π-nondecomposable.*
(v) *π is unique, and every P-absorbing set has positive π-measure.*[1]

Proof. From Theorem 2.1 (i) and (ii) are equivalent. Evidently (iv) implies (iii) implies (ii). Now suppose that (i) holds for $\psi \in \mathscr{P}$. We show that (iv) holds. By Meyn and Tweedie (1993, Theorem 10.4.9) π is unique, and also absolutely continuous with respect to ψ. From the latter it follows that P is also π-irreducible, and hence π-nondecomposable. Thus (iv) is established. Finally, (iv) implies (v) is clear, and it remains only to show that (v) implies (iv). This is established by the following lemma. □

Lemma 2.1. *If π is the only P-invariant distribution in $\mathscr{P}$ and B is P-absorbing, then $\pi(B) > 0$ implies $\pi(B) = 1$.*

[1] In (iv) and (v), uniqueness means of course that there is no other P-invariant measure in $\mathscr{P}$.

For completeness we give a direct proof of Theorem 2.2 in the appendix. Some of the necessary lemmas are of independent interest.

3. Proofs

To begin we state and prove the following simple lemma.

Lemma 3.1. *Let $B \in \mathscr{S}$ and define $(B_n)_{n=1}^{\infty}$ by $B_1 := B$ and*

$$B_{n+1} := \{x \in B_n : P(x, B_n) = 1\}.$$

If $B_\infty := \cap_{n=1}^{\infty} B_n$ is nonempty, then it is P-absorbing.

Proof. Let $x \in B_\infty$ and suppose that $P(x, B_\infty) < 1$. Then $P(x, B_k) < 1$ for some k, and hence $x \notin B_{k+1}$. This contradicts $x \in B_\infty$. □

Proof of Theorem 2.1. It is easy to check that ψ-irreducibility implies ψ-nondecomposability (Meyn and Tweedie, 1993, Proposition 4.2.3). Regarding the converse, suppose instead that there is an $x_0 \in S$ and $A \in \mathscr{S}$ with $\psi(A) > 0$ and $P^n(x_0, A) = 0$ for all $n \in \mathbb{N}$. Let

$$B_1 := S \setminus A, \qquad B_{n+1} := \{x \in B_n : P(x, B_n) = 1\}.$$

As $B_n \subset S \setminus A$ it can never be P-absorbing, for this would contradict ψ-nondecomposability. Therefore $B_n \setminus B_{n+1}$ is never empty.

Observe also that if $x \in B_n \setminus B_{n+1}$, then by definition $P(x, B_n) < 1$ and $P(x, B_{n-1}) = 1$. Hence $P(x, B_{n-1} \setminus B_n) > 0$. Similarly, if $x \in B_1 \setminus B_2$, then $P(x, A) > 0$.

We claim that $P(x_0, B_n) = 1$ for all $n \in \mathbb{N}$. Clearly this is true for $n = 1$. Now let it hold for B_n. If $P(x_0, B_{n+1}) < 1$, then $P(x_0, B_n \setminus B_{n+1}) > 0$, in which case it can be deduced that

$$(1) \quad \int_{B_n \setminus B_{n+1}} P(x_0, dx_1) \int_{B_{n-1} \setminus B_n} P(x_1, dx_2) \int_{B_{n-2} \setminus B_{n-1}} P(x_2, dx_3)$$
$$\cdots \int_{B_2 \setminus B_3} P(x_{n-2}, dx_{n-1}) \int_{B_1 \setminus B_2} P(x_{n-1}, dx_n) P(x_n, A)$$

must be strictly positive. But $P^{n+1}(x_0, A)$ is not less than the term in (1), leading to a contradition. Hence $P(x_0, B_{n+1}) = 1$ as claimed.

Now $B_\infty := \cap_{n=1}^\infty B_n$ must be nonempty, because $P(x_0, B_\infty) = 1$ clearly holds. Hence B_∞ is P-absorbing (Lemma 3.1), which contradicts ψ-nondecomposability. □

Proof of Lemma 2.1. Suppose instead that $\pi(S \backslash B) > 0$. Define $\mu \in \mathscr{P}$ by $\mu(A) = \pi(A \cap B)/\pi(B)$. Since $\mu(B) = 1$ we have

$$P\mu(A) = \int_B P(x, A)\mu(dx).$$

Since B is P-absorbing,

$$\int_B P(x, A)\mu(dx) = \int_B P(x, A \cap B)\mu(dx).$$

$$\therefore \quad P\mu(A) = \int_B P(x, A \cap B)\mu(dx) = \frac{1}{\pi(B)} \int_B P(x, A \cap B)\pi(dx).$$

But in fact

$$\int_B P(x, A \cap B)\pi(dx) = \int_S P(x, A \cap B)\pi(dx),$$

because

$$\begin{aligned} \pi(B) &= \int_B P(x, B)\pi(dx) + \int_{S \backslash B} P(x, B)\pi(dx) \\ &= \pi(B) + \int_{S \backslash B} P(x, B)\pi(dx). \end{aligned}$$

$$\therefore \quad P\mu(A) = \frac{\pi(A \cap B)}{\pi(B)} = \mu(A).$$

But then $\mu = \pi$, because π is the only invariant distribution, and hence $0 = \mu(S \setminus B) = \pi(S \setminus B) > 0$. Contradiction. □

Appendix A

In the appendix we establish Theorem 2.2 directly. Some lemmas are of independent interest.

Lemma A.1. *If P is ψ-nondecomposable for some $\psi \in \mathscr{P}$, then P has at most one invariant distribution.*

Proof. Suppose instead that π and π' are invariant distributions. We can take decompositions $\pi = \varrho + \alpha$ and $\pi' = \varrho + \alpha'$, where α and α' are nontrivial, mutually singular and nonnegative (c.f., e.g., Stokey et al., 1989, p. 195). Note that

$$\alpha - \alpha' = P(\alpha - \alpha'). \tag{2}$$

Let B and C be disjoint sets in $\mathscr{S}$ satisfy $\alpha(B) = \alpha(S) > 0$ and $\alpha'(C) = \alpha'(S) > 0$, where existence is by the Hahn decomposition.

Claim A.1. If $A \subset B$ and $\alpha(A) = \alpha(B)$, then $\alpha(A) = \int_A P(x, A)\alpha(dx)$.

By (2), $\alpha(A) = P\alpha(A) - P\alpha'(A)$. Since $P\alpha(A) \leq \alpha(A)$, it follows that $0 \leq P\alpha'(A) = P\alpha(A) - \alpha(A) = 0$. Therefore $P\alpha'(A) = 0$, and hence

$$\alpha(A) = P\alpha(A) = \int P(x, A)\alpha(dx) = \int_A P(x, A)\alpha(dx). \tag{3}$$

Now let $B_1 := B$ and $B_{n+1} := \{x \in B_n : P(x, B_n) = 1\}$.

Claim A.2. This construction yields a decreasing sequence $(B_n)_{n=1}^{\infty}$ such that (a) $\alpha(B_n) = \alpha(B)$; (b) $\alpha'(B_n) = 0$; and (c) $\alpha(B_n) = \int_{B_n} P(x, B_n)\alpha(dx)$.

That (a)—(c) hold for $n = 1$ is trivial. Suppose now that they hold for fixed $n \in \mathbb{N}$, and consider $n + 1$. Clearly (b) must always hold, and (a) implies (c) by Claim A.1. Regarding (a), we have $\alpha(B_n) = \int_{B_n} P(x, B_n)\alpha(dx)$ by (a) of the induction hypothesis and Claim A.1. This implies (a) for B_{n+1}, since $P(x, B_n) \leq 1$ and $P(x, B_n) < 1$ on $B_n \setminus B_{n+1}$, in which case $B_n \setminus B_{n+1}$ must be α-null.

If we define $B_\infty := \cap_{n=1}^{\infty} B_n$, then B_∞ is nonempty, because $\alpha(B_\infty) = \alpha(B) > 0$. Hence B_∞ is P-absorbing by Lemma 3.1. Also, $\alpha'(B_\infty) = \lim_n \alpha'(B_n) = 0$.

After a similar construction using α', we find a C_∞ which is absorbing and a subset of C. Since B_∞ and C_∞ are disjoint and absorbing no ψ as in the statement of the lemma can exist. □

Lemma A.2. *Let π be P-invariant. If $\pi(B) = 0$ for some P-absorbing set B, then P is not ψ-nondecomposable for any $\psi \in \mathscr{P}$.*

Proof. If $B \in \mathscr{S}$ and $\pi(B) = 0$, then

$$\int_{S \setminus B} P(x, S \setminus B)\pi(dx) = \int P(x, S \setminus B)\pi(dx) = \pi(S \setminus B) = 1. \tag{4}$$

Now let $D_1 := S \setminus B$ and $D_{n+1} := \{x \in D_n : P(x, D_n) = 1\}$. We claim that $\pi(D_n) = 1$ for all n. This is clear for D_1. Suppose it is true for D_n. If $\pi(D_n \setminus D_{n+1}) > 0$, then

$$\begin{aligned}\pi(D_n) &= \int_{D_{n+1}} P(x, D_n)\pi(dx) + \int_{D_n \setminus D_{n+1}} P(x, D_n)\pi(dx) \\ &< \pi(D_{n+1}) + \pi(D_n \setminus D_{n+1}) = \pi(D_n).\end{aligned}$$

Therefore $\pi(D_{n+1}) = 1$, and $\pi(D_n) = 1$ for all n as claimed.

Now let $D_\infty := \cap_{n=1}^{\infty} D_n$. Evidently D_∞ is nonempty, and hence (by Lemma 3.1) P-absorbing. Since B and D_∞ are disjoint the statement of the lemma immediately follows. □

We can now complete the proof of Theorem 2.2. In light of Theorem 2.1 and Lemma 2.1, the nontrivial component which remains to be proved is that either of (i) or (ii) implies (iv). We show (ii) implies (iv). By Lemma A.1, π is unique. Now let B be P-absorbing. By Lemma A.2, $\pi(B) > 0$, whence, by Lemma 2.1, $\pi(B) = 1$. This proves (iv).

References

[1] Meyn, S. P. and Tweedie, R. L. (1993): *Markov Chains and Stochastic Stability,* Springer-Verlag: London.

[2] Stokey, N. L., R. E. Lucas and E. C. Prescott (1989): "Recursive Methods in Economic Dynamics (Harvard University Press, Massachusetts).

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