Sequential parimutuel games

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Abstract: In a parimutuel betting system, a successful player's return depends on the number of other players who choose the same action. This paper examines a general solution for two-action sequential parimutuel games, and shows how the (unique) equilibrium of such games leads to simple pattern of behavior. In particular, we show that there is an advantage to being an early mover, that early players might choose actions with an *ex ante* low probability of success, and that player action choices can 'flip' with small changes in the parameters of the game.

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1. Introduction

Many investments involve returns that are dependent on the actions of others. A simple case is a parimutuel gambling system where bettors make a financial investment on the outcome of a sporting event. An example is the totalizer system at horse races where individuals place bets on horses. If their chosen horse wins, then the return to an individual is a fractional proportion of the entire amount wagered on the race. The return depends on both the total bets of all gamblers and the proportion who bet on the same horse.

Parimutuel gambling systems have long been of interest to economists because they capture important elements of more general investment decisions. They are analogous to simplified financial markets where the scale of the pricing problem has been reduced, with gamblers betting on horses rather than stocks and comparing odds rather than prices. Chadha and Quandt (1996) argue that betting markets can provide an excellent test-bed for examining market efficiency. Parimutuel gambling systems have also been used to test for risk attitudes and utility preferences, for example Asch, Malkiel and Quandt (1982) and more recently Hamid, Prakash and Smyser (1996).

Much of the theoretical work analysing parimutuel systems, however, makes a critical 'small player' assumption that one individual cannot influence the actions of others. In other words, there are always enough players in the system so that the effect of one player's actions on the information and returns of other players can be ignored. For example, Potter and Wit (1995) examine a parimutuel system where players independently choose actions after receiving an individual signal that is drawn from a common distribution. But each player ignores the consequences of their action on the odds of the gamble. Similarly, Watanabe (1997) analyses a parimutuel system with a continuum of players.[1]

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[1] Plott, Wit, and Yang (1997) consider various models of parimutuel betting markets, and compare some experimental results against models. They develop a model without the 'small player' assumption but are unable to solve for equilibrium in this model. Watanabe, Nonoyama, Mori (1994) consider a finite player game but where players have mutually inconsistent beliefs.

The small player assumption greatly simplifies the modeling of parimutuel systems, but it is very strong. For example, if there are few players or if some players wager relatively large sums of money then the interdependency of returns in a parimutuel system means that the 'small player' assumption is likely to be violated. Further, as has been observed in other contexts, when there is asymmetric information, the actions of one player can result in significant information transmission that affects the actions of other players even when returns are not directly linked. The literature on herding and information cascades analyses this phenomenon (Banerjee (1992), Bikhchandani, Hirscheifer, and Welch (1992, 1998), Welch (1992)).[2]

While much of the work in related areas has focussed on interdependency and the transmission of information, this paper focuses on interdependency of player *actions* as governed by the parimutuel form of return. We consider a simple model of a parimutuel system where all players have identical information. This means that each individual's action will affect the returns of all other players by (a) increasing the size of the prize pool and (b) raising the expected return associated with other actions. We consider a sequence of players who must choose between two actions. Expected returns depend on the number of players, the actions chosen by all other players, and an exogenous parameter such as the probability that one action is 'correct'. Players must determine their optimal action given their knowledge of the actions taken by all preceding players and their expectation of the choices of all succeeding players.

Our results highlight two important features of sequential decision making. First, players may cluster on choices, in that they will often make the same choice as the player who immediately precedes them. Such clustering does not reflect any information asymmetry or inference process as in the literature on information cascades. Rather, it reflects the inability of marginal returns over actions to be perfectly equated in a finite player game.

Secondly, players who must make relatively early action choices may choose an action that *ex ante* appears to have a low expected payoff. Their decision is driven by the rational belief that later players will cluster on the outcome that *ex ante* appears more

[2] As the classic paper by Grossman and Stiglitz (1980) points out, even if all players are price takers, in a rational expectations equilibrium, aggregate actions by otherwise small agents can convey

favorable. The interdependency of returns means that such clustering can result in an *ex ante* more favorable action having a lower expected return *ex post*.

To see an example of this second phenomenon, consider a situation where three players sequentially choose the outcome of a football game between the *Bears* and the *Bulls*. There is a total prize pool of $100, and the *ex ante* probability of the *Bears* winning is 60%. But returns are parimutuel, so that those players who correctly choose the winner of the game share the prize pool equally among themselves. Then in equilibrium, we would expect the first player to maximize his return by choosing to back the *Bulls* while the second and third players support the favored *Bears*. Given the first player's choice, the second and third players individually prefer to both support the *Bears* and gain an expected return of (at least) $30 rather than to back the *Bulls* and receive an expected return of no more than $20. The first player, who can accurately infer the behaviour of the other players, will back the *Bulls* and receive an expected return of $40. If, in contrast, the first player had chosen to support the favored *Bears*, then his expected return would only be $30 at best, as at least one of the other players would also support the *Bears*. The first player maximizes his expected return by isolating himself through his choice of the *ex ante* worse alternative.

Our results can be applied to a variety of investment situations. For example, suppose that investors sequentially choose between two towns in which to locate similar retail stores. We should not be surprised if early investors all choose the same town and later investors all choose the alternative location. Further, if one town has a larger population and, as such, would appear *ex ante* to provide a preferred retail location, we should not be surprised if early investors all choose to locate in the smaller town. This behavior does not reflect any information asymmetry or market-driven herding. Rather, it simply reflects the parimutuel form of the payoffs associated with the investments.

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information.

2. The Model

N individuals sequentially choose one of two possible actions, $a \in \{A, B\}$. Each individual receives a payoff that depends on both the specific action that they choose and the number of other players that have chosen the identical action. Denote the number of players choosing *A* as *n*. The payoff to player *i* from choosing *A* is given by $pM\frac{N}{n}$ while the payoff from choosing *B* is $(1-p)M\frac{N}{N-n}$ where p and M are exogenous parameters. All players can observe the history of the game. We wish to examine the subgame perfect equilibria of this game.

This game can be interpreted in (at least) two ways:

Dividing a fixed monetary pool: each player knows *ex ante* the total payoff associated with any action. This is given by pMN for action *A* and by (1-*p*)*MN* for action *B*. The payoff to a player *i* from action *a* is equal to the total payoff associated with action *a* divided by the number of players who choose that action. In other words, an individual's return from choosing *A* or *B* is $pM\frac{N}{n}$ and $(1-p)M\frac{N}{N-n}$ respectively.

A simple parimutuel game: There exists a true state of the world Ω where $\Omega \in \{\alpha, \beta\}$. The probability that $\Omega = \alpha$ is *p* and the probability that $\Omega = \beta$ is $1-p$ where $p \in [0,1]$. Individuals are unaware of the true state *ex ante* but they do know the value of *p*. Action *A* is 'correct' when the true state is α, while action *B* is 'correct' when the true state is β. Each individual pays one dollar to play the game. If an individual *i* chooses action *A* and this is the correct action then they receive a gross return equal to $M\frac{N}{n}$. If they choose *B* and this is correct then they receive a gross return of $M\frac{N}{N-n}$. So an individual's expected return from choosing *A* or *B* given *n* is $pM\frac{N}{n}$ and $(1-p)M\frac{N}{N-n}$ respectively.

3. General Results

The following two theorems apply to the model presented above. The proofs are given in the appendix.

Theorem 1

Given *N and p*, consider $n \in \mathbb{N}$ such that $\frac{n}{N+1} \le p \le \frac{n+1}{N+1}$.[3] Then:

a) If $p\frac{N}{n} \ge (1-p)\frac{N}{N-n}$ then there is a subgame perfect Nash equilibrium where players 1,2,…,*n* choose *A* and the remainder choose *B*. i.e. an equilibrium of the form *AAA…AB…BBB*.

b) If $p\frac{N}{n} \le (1-p)\frac{N}{N-n}$ then there is a subgame perfect Nash equilibrium where players 1,2,…,*N-n* choose *B* and the remainder choose *A*. i.e. an equilibrium of the form *BBB…BA…AAA*.

Theorem 2

Given *N* and *p*, consider $n \in \mathbb{N}$ such that $\frac{n}{N+1} < p < \frac{n+1}{N+1}$ and $p\frac{N}{n} \ne (1-p)\frac{N}{N-n}$. Then the equilibrium defined in *Theorem 1* is unique.

These two theorems show that the equilibrium outcomes of the sequential parimutuel investment game will follow a simple pattern. The players break into two simple groups according to their order in the sequence and their action choice. Either the first *n* players will all choose action *A* with the remainder choosing action *B*, or the first $N-n$ players will all choose *B* with the remainder choosing *A*. Further the subgame perfect equilibrium is generally unique in the sense that given a finite value of *N*, multiple equilibria only exist for a finite number of values of *p*.

[3] Note that such an n will always exist but need not be unique.

The equilibrium involves an ‘early mover’ advantage. While the first group (either *n* or *N*-*n* players) all receive the same payoff, this payoff is greater than that received by the remainder of the players. But this advantage does not mean that early movers choose the *ex ante* more likely outcome or the action associated with the *ex ante* larger pool of funds. With $p > \frac{1}{2}$, the first players may still all choose action *B* in equilibrium and receive a higher payoff than those player who choose action *A*.

If we interpret *p* as a probability, then the unique equilibrium can involve early movers backing an extreme ‘long-shot’. For example, suppose that $p = 0.99$ and $N = 101$. Then the two theorems imply that the unique equilibrium involves the first player choosing *B* and the remaining 100 choosing *A*.

The theorems show how the equilibrium behavior of the players can be very sensitive to the parameters of the game. The addition of an extra player, for example, can cause the equilibrium to ‘flip’ so that almost all players alter their choice. To see this, suppose that $p = \frac{11}{20}$. If there are initially eight players then the unique equilibrium outcome is for the first four players to choose action *A* and the remaining players to choose action *B*. But if we add an extra player to the game, then the unique equilibrium involves the first four players choosing action *B* while the remaining players choose *A*. If the extra player moves last, then the addition of this player has led each of the original players to change their choice.

Similarly, as *p* changes, the equilibrium actions will alter, following a predictable pattern. For example, as *p* falls from one to one-half, successively more players will choose action *B* rather than *A*. Further, the choice of early movers will flip between *A* and *B* as *p* falls. To see this, suppose that $N = 3$. It is easy to confirm that if $p \in (\frac{3}{4}, 1]$ then the unique equilibrium outcome involves all three players choosing *A*. If *p* falls so that $p \in \left(\frac{2}{3}, \frac{3}{4}\right)$ then the unique outcome involves the first two players choosing *A* and the last choosing *B*. But if *p* falls further, so that $p \in \left(\frac{1}{2}, \frac{2}{3}\right)$ then the first player will choose *B* and the latter two players will choose *A*.

Multiple equilibria only exist for specific values of *N* and *p*. When $p\frac{N}{n}$ = $(1-p)\frac{N}{N-n}$ there are ${}^{N}C_{n}$ equilibrium outcomes. In each, n players choose action *A* and $(N-n)$ players choose action *B* but all possible combinations of orderings can arise as equilibria. For example, with $N=3$ and $p=\frac{2}{3}$ any outcomes where one player chooses action *B* and the other two players choose action *A* can occur in equilibrium. When $p\frac{N}{n+1}$ = $(1-p)\frac{N}{N-n}$ there are ${}^{N}C_{n+1}$ + ${}^{N}C_{n}$ equilibrium outcomes. In equilibrium either $n+1$ players choose *A* and $N-n-1$ players choose *B* or *n* players choose *A* and $(N-n)$ choose *B*. In either case, all possible combinations of orderings can arise.

4. Conclusion

In this paper we have analyzed a sequential parimutuel gambling game and have characterized the equilibria of this game. While our model is relatively simple, our results provide insight into the behavior of other systems that involve interdependent investment decisions. For example, our results can explain why an early retailer of a new product might prefer to locate in a relatively small town rather than close to a larger market, or why sequential investment decisions might appear to be characterized by clustering subject to sudden switches in choice.

Our results do not depend on information asymmetries or inconsistent beliefs, but simply reflect the interdependent nature of investment returns. In this sense, our results provide a simple explanation for observed phenomenon, like clustering. This said, the model could obviously be extended to allow for information asymmetry and the aggregation of information as the game progresses. As noted in the literature on information cascades, even with independent returns, sequential information aggregation can involve significant imperfections. This will be complicated by parimutuel returns as any cascade dilutes the return to early players. However, this remains the topic of future research.

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Appendix.

For the proofs of theorems 1 and 2 it is convenient to introduce the following notation. Let $P\left[a|\eta\right]$ be the return to a player choosing action a when η players in total choose action a.

1. Proof of Theorem 1

Consider the putative equilibrium where $p\frac{N}{n} \geq (1-p)\frac{N}{N-n}$ and the first *n* players choose *A*, and the remainder choose *B*, i.e. an equilibrium of the form *AAA…AB…BBB*. (The proof of the symmetric case where $p\frac{N}{n} \leq (1-p)\frac{N}{N-n}$ is analogous.)

From our parameter restrictions it is easy to show that the following inequalities hold:

$P\left[A|n\right] \geq P\left[B|N-n\right]$

$P\left[A|n+1\right] \leq P\left[B|N-n-1\right]$ $\qquad$ $P\left[A|n-1\right] \geq P\left[B|N-n+1\right]$

$P\left[A|n+x\right] < P\left[B|N-n-x\right]$ $\qquad$ $P\left[A|n-x\right] > P\left[B|N-n+x\right]$

We now show that no player will find it desirable to unilaterally deviate from the putative equilibrium. First, consider the players who choose *A* in the putative equilibrium. Suppose a subset of Z of these players deviate and choose *B*. From the above inequalities, these players are only strictly better off if at least $Z+1$ players who would have chosen *B* in the putative equilibrium now choose *A*.

Now, consider the players who choose *B* in the putative equilibrium. Suppose a subset of Z' of these players deviate and choose *A*. From the above inequalities, these players are only strictly better off if at least $Z'+1$ players who would have chosen *A* in the putative equilibrium now choose *B*.

Note that this implies that any deviations must make some players worse off than in equilibrium as it cannot be the case that $Z \geq Z'+1$ and $Z' \geq Z+1$ simultaneously. It remains to show that the first player who unilaterally deviates cannot be made strictly better off.

Let the first player deviate by choosing action a' rather than action a''. The first player will only strictly gain if all the players who choose action a'' are strictly worse off. But this cannot occur. To see this, suppose the converse and consider the last player that chooses a''. Noting that $P\left[a|\eta\right]$ is decreasing in η and from the above inequalities, this player will always gain by choosing a' rather than a'' regardless of the behavior of subsequent players. So this player will not choose action a'' and we have a contradiction.

As no player in the putative equilibrium can ever strictly gain by deviating, the putative equilibrium is a subgame perfect Nash equilibrium.

Q.E.D.

2. Proof of Theorem 2

The following lemmas are useful in the proof of theorem 2.

Lemma 1

Consider N, p, and $n \in \mathbb{N}$ *such that* $\frac{n}{N+1} < p < \frac{n+1}{N+1}$. *Then, in any subgame perfect Nash equilibrium, n players will choose A and N-n players will choose B.*

Proof: From theorem 1 we know that there exists an equilibrium where *n* players choose *A* and $N-n$ players choose *B*. Consider any other putative equilibrium. Suppose that in this putative equilibrium $n+x$ players choose *A*, and $N-n-x$ players choose *B*. Consider the last player that chose *A* and call this player *i*. If *i* deviates and chooses *B*, then *i* is strictly better off regardless of the actions of any subsequent players $i+1,\ldots,N$ since $P\left[B\middle|N-n-x+1\right] > P\left[A\middle|n+x\right]$. Therefore *i* will deviate and the putative equilibrium where more than *n* players choose *A* cannot be an actual equilibrium. Similarly if *n-x* players choose *A*.

Therefore, in equilibrium, *n* players choose *A* and *N-n* players choose *B*.

Q.E.D.

Lemma 2.

Consider any subgame of the whole game, beginning with the i^{th} *player and any subgame perfect Nash equilibria of this subgame given the choices of players* $1,\ldots,i-1$. *This equilibrium only depends on the number of players before i who choose A and B, not their specific order.*

Proof: This trivially follows as the payoffs for each player only depend on the number of other players associated with each action and not their order.

Q.E.D.

Proof of Theorem 2.

From lemma 1 we can restrict attention to equilibria where *n* players choose action *A* and the remainder choose action *B.* We begin with the case where $p\frac{N}{n} > (1-p)\frac{N}{N-n}$. From theorem 1 we know that the relevant equilibrium exists. To show that it is unique, consider any other putative equilibrium. In this putative equilibrium, there must exist two players, *i* and *i*+1, where *i* plays *B* and *i*+1 plays *A.*

Consider the last such pair of players. We show that such a player *i* will always find it optimal to deviate so that the putative equilibrium is not an actual equilibrium.

To see this, if *i* deviates and chooses *A*, then player *i*+1 chooses either *B* or *A*. Suppose *i*+1 chooses *B*, then by lemma 2 no player $i+2,\ldots,N$ will find it desirable to deviate. But then player *i* receives $p\frac{N}{n}$ rather than $(1-p)\frac{N}{N-n}$ and is better off and will therefore deviate.

Alternatively, suppose player *i*+1 chooses *A* after player *i* deviates. Player *i*+1 will only find this optimal if $p\frac{N}{n-x+1} > (1-p)\frac{N}{N-n}$, where *x* is the number of players $i+2,\ldots,N$ that deviate from *B* to *A* in the subgame perfect Nash equilibrium after *i*+1 chooses *A*. This is because player *i*+1 could choose *B* and get $(1-p)\frac{N}{N-n}$. But player *i* also gets $p\frac{N}{n-x+1} > (1-p)\frac{N}{N-n}$ and will therefore deviate.

As player *i* always deviates, there is no equilibrium with $p\frac{N}{n} > (1-p)\frac{N}{N-n}$, and a player *i* that plays *B* where *i*+1 plays *A*.

The analogous proof holds for the symmetric case where $(1-p)\frac{N}{N-n} > p\frac{N}{n}$.

Therefore, the equilibrium characterized in theorem 1 is unique.

Q.E.D.