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**Providing Intuition to the Fieller Method  
with Two Geometric Representations  
using STATA and Eviews**

by

J. G. Hirschberg  
&  
J. N. Lye

Department of Economics  
The University of Melbourne  
Melbourne Victoria 3010  
Australia.

# Providing Intuition to the Fieller Method with two Geometric Representations using STATA and EVIEWS

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J. G. Hirschberg and J. N. Lye\*

Department of Economics, University of Melbourne, Parkville, Vic 3010, Australia

The Fieller Method for the construction of confidence intervals for ratios of the expected value of two normally distributed random variables has been shown by a number of authors to be a superior method to the delta approximation. However, it is not widely used due in part, to the tendency to present the intervals only in a formula context. In addition, potential users have been deterred by the potential difficulty in interpreting non-finite confidence intervals when the confidence level is less than 100%. In this paper we present two graphical methods which can be easily constructed using two widely used statistical software packages (Eviews and Stata) for the representation of the Fieller intervals. An application is presented to assess the results of a model of the non-accelerating inflation rate of unemployment (NAIRU).

**Key words:** Fieller method, ratios of parameters, confidence interval, confidence ellipsoid, 1<sup>st</sup> derivative function, NAIRU, EViews, STATA

**JEL:** C12, C20, E24

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\*Corresponding author. Email [jnlye@unimelb.edu.au](mailto:jnlye@unimelb.edu.au).

## I. Introduction

The Fieller method (Fieller 1932, 1954) provides a general procedure for constructing confidence limits for statistics defined as ratios of the expected values of normally distributed random variables. Zerbe (1978) defines a generalized version of Fieller's method in the regression context where the ratio is defined in terms of linear combinations of the regression parameters.

Drawing inferences from the ratio of regression coefficients is elemental in a number of statistical applications. Generally, the results of Monte Carlo simulations to compare the Fieller method with other methods for the construction of confidence intervals indicated that the Fieller method works reasonably well under a range of assumptions including departures from normality, whereas the widely used Delta method was a consistently poor performer and often underestimated the limits of the intervals (see Hirschberg and Lye 2004).

Applications in the economics literature are common. Dufour (1997) proposed that ratios of regression parameter problems be subject to confidence intervals based on the Fieller type methods. Fieller estimates have been used to calculate confidence bounds: for long-run elasticities in dynamic energy demand models (Bernard et al. 2005); mean elasticities obtained from linear regression models (Valentine 1979); non-accelerating inflation rate of unemployment, the NAIRU (Staiger et al. 1997); steady state coefficients in models with lagged dependent variables (Blomqvist 1973) and the extremum of a quadratic model (Hirschberg and Lye 2004).

However the Fieller method is not routinely used, partly due to its apparent non-intuitive form. In this paper we present two graphical representations of the Fieller method which we feel provides the intuition for the Fieller. Both these approaches can be implemented using existing routines in such widely used computer software such as Stata and Eviews.

## II. The Fieller method

The Fieller method (Fieller 1932, 1954) provides a general procedure for constructing confidence limits for statistics defined as ratios. In this paper we will concentrate on the case of the ratio of regression parameters. Zerbe (1978) defines a version of Fieller's method in the regression context, where the ratio  $\psi = \frac{\mathbf{K}'\beta}{\mathbf{L}'\beta}$  is defined in terms of linear combinations of the regression parameters from the same regression,  $\mathbf{Y}_{T \times 1} = \mathbf{X}_{T \times k}\beta_{k \times 1} + \varepsilon_{T \times 1}$ ,  $\varepsilon \sim (\mathbf{0}_{T \times 1}, \sigma^2 \mathbf{I}_{T \times T})$ , in which the OLS estimators for the parameters are  $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ ,  $\hat{\sigma}^2 = \hat{\varepsilon}'\hat{\varepsilon}/(T - k)$ , and

the vectors  $\mathbf{K}_{k \times 1}$  and  $\mathbf{L}_{k \times 1}$  are known constants. Under the usual assumptions, the parameter estimates are asymptotically normally distributed according to  $\hat{\beta} \sim N\left(\beta, \sigma^2 (\mathbf{X}'\mathbf{X})^{-1}\right)$ . The

ratio  $\psi$  is estimated as  $\hat{\psi} = \frac{\mathbf{K}'\hat{\beta}}{\mathbf{L}'\hat{\beta}}$ .

A  $100(1 - \alpha)\%$  confidence interval for  $\psi$  is determined by solving the quadratic equation  $a\psi^2 + b\psi + c = 0$ , where  $a = (\mathbf{L}'\hat{\beta})^2 - t_{\alpha/2}^2 \mathbf{L}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{L}\hat{\sigma}^2$ ,

$b = 2\left[t_{\alpha/2}^2 \mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{L}\hat{\sigma}^2 - (\mathbf{K}'\hat{\beta})(\mathbf{L}'\hat{\beta})\right]$  and  $c = (\mathbf{K}'\hat{\beta})^2 - t_{\alpha/2}^2 \mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{K}\hat{\sigma}^2$ .

When  $a > 0$ , the two roots of the quadratic equation,  $(\psi_1, \psi_2) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ , define the finite confidence bounds of the parameter value. The condition  $a > 0$ , is only applicable if the hypothesis test  $H_0 : \mathbf{L}'\beta = 0$  is rejected at the  $\alpha$  level of significance (Buonaccorsi 1979).

That the Fieller method does not always result in finite confidence bounds is a crucial feature of this technique because in a number of applications the denominator may have a distribution with significant mass around zero. The resulting confidence interval may be the complement of a finite interval ( $b^2 - 4ac > 0$ ,  $a < 0$ ) or of the whole real line ( $b^2 - 4ac < 0$ ,  $a < 0$ ). These conditions are discussed in Scheffé (1970) and Zerbe (1982).

### III. Confidence Bounds of the Linear Combination (CBLC) Graphical Representation of Fieller Method

Consider the line,

$$g = \left\{ (\mathbf{K}'\hat{\beta}) - (\mathbf{L}'\hat{\beta})\psi \right\} \quad (1)$$

where  $\mathbf{K}$ ,  $\mathbf{L}$  and the estimated regression parameters  $\hat{\beta}$  are as defined in Section II. This line can be plotted as a function of  $\psi$  along with a  $100(1 - \alpha)\%$  confidence interval. The confidence bounds for this line are given by:

$$CI(g) = \left\{ (\mathbf{K}'\hat{\beta}) - (\mathbf{L}'\hat{\beta})\psi \right\} \pm t_{\alpha/2} \sqrt{\left\{ \hat{\sigma}^2 \mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{K} - 2(\hat{\sigma}^2 \mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{L})\psi + (\hat{\sigma}^2 \mathbf{L}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{L})\psi^2 \right\}} \quad (2)$$

where  $t_{\alpha/2}$  is the value from the  $t$  distribution with an  $(\alpha/2)\%$  level of significance and  $T - k$  degrees of freedom.

An estimate of the value of the ratio ( $\hat{\psi}$ ) is found by solving:

$$(\mathbf{K}'\hat{\beta}) - (\mathbf{L}'\hat{\beta})\hat{\psi} = 0 \quad (3)$$

Similarly, the bounds defining the  $100(1-\alpha)\%$  on  $\hat{\psi}$  are found by solving:

$$\left\{ (\mathbf{K}'\hat{\beta}) - (\mathbf{L}'\hat{\beta})\psi \right\} \pm t_{\alpha/2} \sqrt{\left\{ \hat{\sigma}^2 \mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{K} - 2(\hat{\sigma}^2 \mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{L})\psi + (\hat{\sigma}^2 \mathbf{L}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{L})\psi^2 \right\}} = 0 \quad (4)$$

which is equivalent to solving the roots of the equation:

$$\left\{ (\mathbf{K}'\hat{\beta}) - (\mathbf{L}'\hat{\beta})\psi \right\}^2 - t_{\alpha/2}^2 \left\{ \hat{\sigma}^2 \mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{K} - 2(\hat{\sigma}^2 \mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{L})\psi + (\hat{\sigma}^2 \mathbf{L}'(\mathbf{X}'\mathbf{X})^{-1} \mathbf{L})\psi^2 \right\} = 0 \quad (5)$$

By rearranging the terms in (5), this can be written as the quadratic equation,

$a\psi^2 + b\psi + c = 0$ , where  $a$ ,  $b$  and  $c$  are defined as in the Fieller method described in Section

II. An advantage of this approach is that it can provide a graphical representation of the various possible Fieller-type confidence intervals. In addition, it is easily implemented in any existing computing software that saves the estimated coefficients and variance-covariance matrix of the estimated coefficients after running a regression. It is also easily implemented in existing computer programs that allow for the prediction of out-of-sample values along with the standard errors of the mean prediction. Stata programs adopting these approaches are described in Appendix A1.

#### IV. Confidence Ellipse (CE) Graphical Representation of Fieller Method

If we partition the regression equation  $\mathbf{Y} = \mathbf{X}\beta + \varepsilon$  as  $\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \varepsilon$  where  $\beta_1$  is a  $(\{k-2\} \times 1)$  vector and  $\beta_2$  is a  $(2 \times 1)$  vector containing the parameter coefficients  $\beta_{21}$  and  $\beta_{22}$

we can define a ratio  $\psi = \frac{\beta_{21}}{\beta_{22}}$ , which is estimated as  $\hat{\psi} = \frac{\hat{\beta}_{21}}{\hat{\beta}_{22}}$ . Define the  $100(1-\alpha)\%$

confidence ellipse as:

$$(\beta_2 - \hat{\beta}_2)' \hat{\sigma}^{-2} (\mathbf{X}_2' \mathbf{M}_1 \mathbf{X}_2) (\beta_2 - \hat{\beta}_2) \leq F_{\alpha}(1, T-k) \quad (6)$$

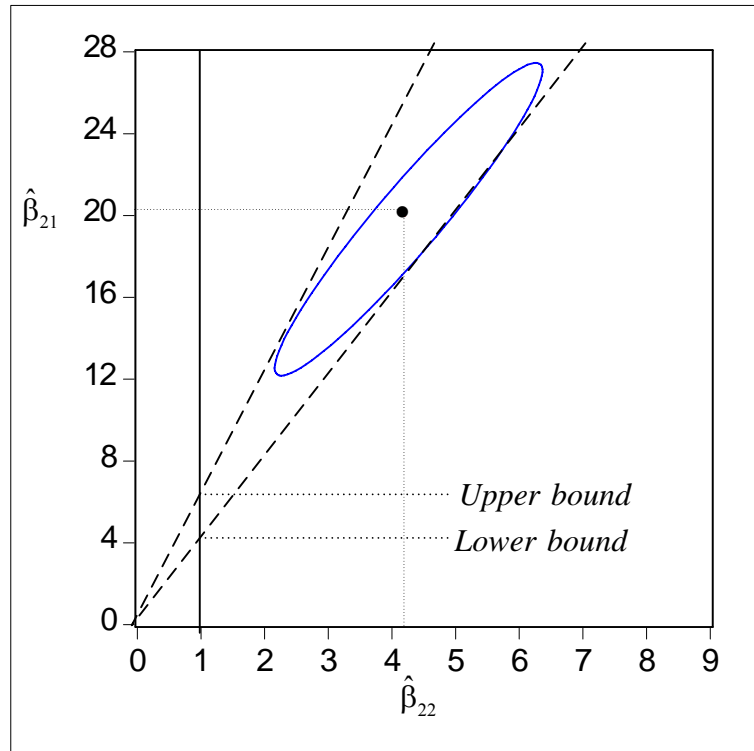
where  $\mathbf{M}_1 = \mathbf{I} - \mathbf{X}_1 (\mathbf{X}_1' \mathbf{X}_1)^{-1} \mathbf{X}_1'$ .

In Appendix B we show the solution to the constrained optimization problem defined as:

$$L = \psi - \lambda \left[ \begin{array}{cc} \left[ \left( \hat{\beta}_{21} - \psi \hat{\beta}_{22} \right) & \left( \hat{\beta}_{22} - \hat{\beta}_{22} \right) \right] \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} \begin{bmatrix} \left( \hat{\beta}_{21} - \psi \hat{\beta}_{22} \right) \\ \left( \hat{\beta}_{22} - \hat{\beta}_{22} \right) \end{bmatrix} - F_{\alpha}(1, T-k) \end{array} \right] \quad (7)$$

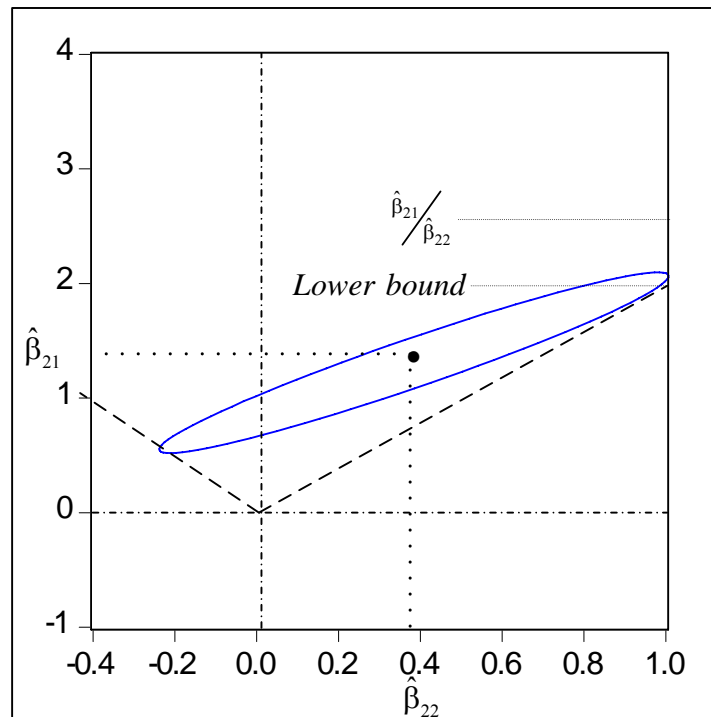
where  $\omega_{ij}$  are elements of  $\Omega = \hat{\sigma}^{-2} (X_2' M_1 X_2)$ . The solution to this optimization has two roots that are equivalent to the Fieller interval.

**Figure 1:** An Example of Finite Confidence Bounds



The constrained optimization problem can be investigated via an equivalent diagrammatic approach. Following von Luxburg and Franz (2004), the ratio  $\frac{\hat{\beta}_{21}}{\hat{\beta}_{22}}$  is the slope of the line through the points  $(0,0)$  and  $(\hat{\beta}_{21}, \hat{\beta}_{22})$ . If  $(0,0)$  is not within the ellipse, two tangents to the ellipse which go through the origin are constructed. If  $H_0 : \beta_{22} = 0$  is rejected at the  $\alpha\%$  level of significance the finite confidence bounds are defined at the points of intersection between each tangent and where the x-axis equals 1 (see Figure 1). However, if  $H_0 : \beta_{22} = 0$  cannot be rejected and the ellipse cuts either 0 axes lines, the complement of a finite interval is defined (see Figure 2). In Figure 2 the practical interpretation would be that the ratio has a lower bound but no upper bound. If  $(0,0)$  is within the ellipse the interval is then the whole real line.

**Figure 2:** An Example of A Complement of A Finite Interval



In many cases the ratio will be of the form  $\psi = d \frac{\beta_{21}}{\beta_{22}}$ . For example, when the regression equation includes a quadratic specification and  $\psi$  corresponds to the extremum, then  $d = -\frac{1}{2}$ . For ratios defined in this way, to determine the Fieller interval, the appropriate intersection is between each tangent and  $x = d$ . In the more general case when the ratio is defined as sums of regression parameters,  $\psi = \frac{\mathbf{K}'\beta}{\mathbf{L}'\beta}$ , define new parameters as  $\gamma = \mathbf{K}'\beta$  and  $\lambda = \mathbf{L}'\beta$ , and reformulate the regression equation in terms of the new parameters  $\gamma$  and  $\lambda$ . To define  $\gamma$  and  $\lambda$  stack the two linear independent combinations so that the matrix  $\mathbf{R}_{2 \times k} = [\mathbf{K}' \quad \mathbf{L}']'$  and  $\rho_{2 \times 1} = [\gamma \quad \lambda]'$ . Then we can show that the equivalent equation to  $\mathbf{Y} = \mathbf{X}\beta + \varepsilon$  is  $\mathbf{Y} = (\mathbf{X}\mathbf{A})\theta + (\mathbf{X}\mathbf{R}^+)\rho + \varepsilon$ , where  $\mathbf{R}^+$  is the generalized inverse of  $\mathbf{R}$  and  $\mathbf{A}$  is the matrix of eigenvectors corresponding to the zero valued eigenvalues of  $\mathbf{R}'\mathbf{R}$ . This then implies that  $\mathbf{X}_1 = \mathbf{X}\mathbf{A}$  and  $\mathbf{X}_2 = \mathbf{X}\mathbf{R}^+$ . For further details see Hirschberg, Lye and Slotte (2005)

This approach is easily implemented in any computer program that plots the confidence ellipse provided the ellipse has the correct dimensions. Note that the confidence ellipse produced as part of the OLS options in Eviews 5.0 is specified as:

$$(\beta_2 - \hat{\beta}_2)' \hat{\sigma}^{-2} (X_2' M_1 X_2) (\beta_2 - \hat{\beta}_2) \leq 2F_{\alpha} (2, T - k) \quad (8)$$

To obtain the appropriate confidence ellipse as in (6) specify the confidence level as  $(1 - \tilde{\alpha})$ , such that  $F_{\alpha} (1, T - k) = 2F_{\tilde{\alpha}} (2, T - k)$ . For example, to obtain the appropriate 95% Confidence Ellipse when  $T - k$  is large, specify the confidence level as 0.85 instead of the default 0.95. In Stata 8 when  $T - k$  is large, the 95% Confidence Ellipse can be obtained by using the program *ellip* (Alexandersson 2004) by specifying the boundary constant using *chi2* with 1 degree of freedom. Both of these approaches are described in detail in Appendix A2.

## V. Example application for the estimation of the NAIRU

The example is based on the estimating equation in Gruen et al. (1999). In this specification the rate of wage inflation, measured by the rate of change of unit labour costs, is a function of the level of unemployment, the change in the rate of unemployment and the expected rate of inflation. Gruen et al. (1999) also choose to model annual movements. Furthermore, a lagged dependent variable is added on the basis of its significance. We write this specification as:

$$\begin{aligned} \Delta_4 \ln ULC_t - \Delta_4 \ln P_{t-1} = & \alpha_1 (\Delta_4 \ln P_t^* - \Delta_4 \ln P_{t-1}) + \alpha_2 U_t + \alpha_3 \Delta U_{t-1} \\ & + \alpha_4 (\Delta_4 \ln ULC_{t-1} - \Delta_4 \ln P_{t-2}) + \alpha_5 (\Delta_4 \ln ULC_{t-1} - \Delta_4 \ln ULC_{t-4}) + \alpha_6 + \varepsilon \end{aligned} \quad (9)$$

Where  $ULC$  = unit labour costs per person, and is equal to wages per person divided by non-farm productivity per person;  $P$  = CPI,  $P^*$  = expected price level;  $U$  = rate of unemployment;  $\Delta$  = 1 period change; and  $\Delta_4$  = 4 period change. An estimate of the NAIRU is given as

$\hat{U}^* = \frac{\hat{a}_6}{-\hat{a}_2}$ , where  $\hat{a}_6, \hat{a}_2$  are the corresponding OLS estimates of (9).

**Table 1:** Phillips Curve Estimates for Australia 1985:1 – 2003:4

Parameter	Estimate	Standard Error	p-value
$\hat{a}_1$	0.16716	0.07790	0.0354
$\hat{a}_2$	-0.24589	0.11118	0.0303
$\hat{a}_3$	-0.28008	0.47844	0.5602
$\hat{a}_4$	0.58431	0.07351	0.0000
$\hat{a}_5$	0.55623	0.10064	0.0000
$\hat{a}_6$	1.32780	0.83375	0.1157
$\hat{\sigma}_{\hat{a}_2 \hat{a}_6} = -0.090$	$R^2 = 0.693$	Number of observations = 76	



Using quarterly Australian data from Lye and McDonald (2006) for the period 1985:1 – 2003:4, (9) is estimated and the results are reported in Table 1. Using these estimates the estimated NAIRU is  $\hat{U}^* = 1.328 / -0.246 = 5.40\%$ .

The estimated variance of  $\hat{U}^*$  based on the Delta method is given by (see, for example, Rao 1973, pp 385-389):

$$\frac{\hat{a}_2^2 \sigma_{\hat{a}_6}^2 - 2\hat{a}_2 \hat{a}_6 \sigma_{\hat{a}_2 \hat{a}_6} + \hat{a}_6^2 \sigma_{\hat{a}_2}^2}{\hat{a}_2^4} \quad (10)$$

where  $\sigma_{\hat{a}_6}^2$  is the variance of  $\hat{a}_6$ ,  $\sigma_{\hat{a}_2}^2$  is the variance of  $\hat{a}_2$  and  $\sigma_{\hat{a}_2 \hat{a}_6}$  is the covariance between  $\hat{a}_2$  and  $\hat{a}_6$ . A  $100(1 - \alpha)\%$  confidence interval for  $U^*$  based on the Delta method is given by:

$$\left( \frac{\hat{a}_6}{-\hat{a}_2} \right) \pm t_{\alpha/2} \sqrt{\frac{\hat{a}_2^2 \hat{\sigma}_{\hat{a}_6}^2 - 2\hat{a}_2 \hat{a}_6 \hat{\sigma}_{\hat{a}_2 \hat{a}_6} + \hat{a}_6^2 \hat{\sigma}_{\hat{a}_2}^2}{\hat{a}_2^4}} \quad (11)$$

where  $\hat{\sigma}_{\hat{a}_2}^2$ ,  $\hat{\sigma}_{\hat{a}_6}^2$  and  $\hat{\sigma}_{\hat{a}_2 \hat{a}_6}$  are the estimated values of  $\sigma_{\hat{a}_2}^2$ ,  $\sigma_{\hat{a}_6}^2$  and  $\sigma_{\hat{a}_2 \hat{a}_6}$  respectively. The 95% Delta confidence bounds are then [3.120%, 7.682%] from which one would conclude that the NAIRU is significantly different from zero.

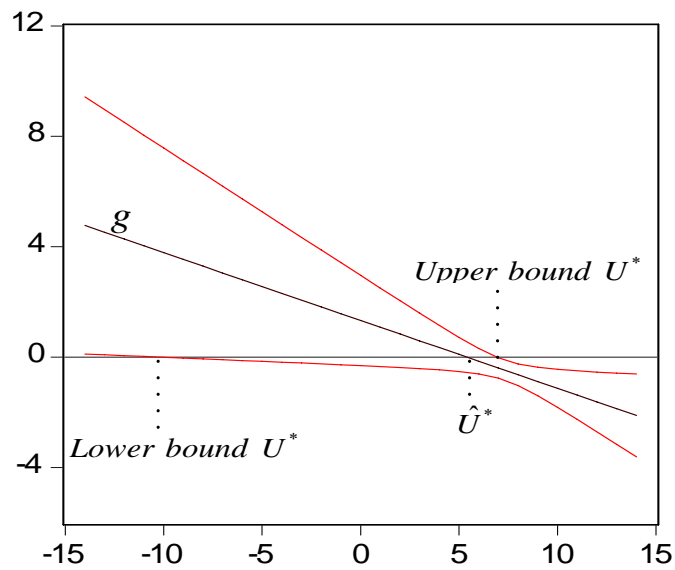
To obtain the 95% Fieller confidence bounds using the CBLE approach, in Figure 3a, we plot

$$g = \{\hat{a}_6 + \hat{a}_2 U^*\} \quad (12)$$

along with the 95% confidence bounds of  $LY$  given by,

$$CI(g) = \{\hat{a}_6 + \hat{a}_2 U^*\} \pm t_{\alpha/2} \sqrt{\hat{\sigma}_{\hat{a}_6}^2 + 2\hat{\sigma}_{\hat{a}_2 \hat{a}_6} U^* + \hat{\sigma}_{\hat{a}_2}^2 (U^*)^2} \quad (13)$$

**Figure 3a:** The Estimated NAIRU and its Fieller Confidence bounds using Confidence Bounds Approach



The points where the confidence bounds are equal to 0 define an interval that is equivalent to the confidence interval associated with the 95% Fieller interval. From Figure 3a, the 95% Fieller Interval is  $[-10.11\%, 6.91\%]$ , quite a different result from above which indicates that we have not found a lower bound that is consistent with economic theory. Figure 3b illustrates the same interval but obtained using the CE approach. In Figure 3b two tangents to the ellipse which go through the origin are plotted. The upper and lower limits of the Fieller interval in this case are finite and are defined at the points of intersection between each tangent and where the  $x$ -axis equals  $-1$ .

**Figure 3b:** The Estimated NAIRU and its Fieller Confidence bounds using Confidence Ellipse Approach

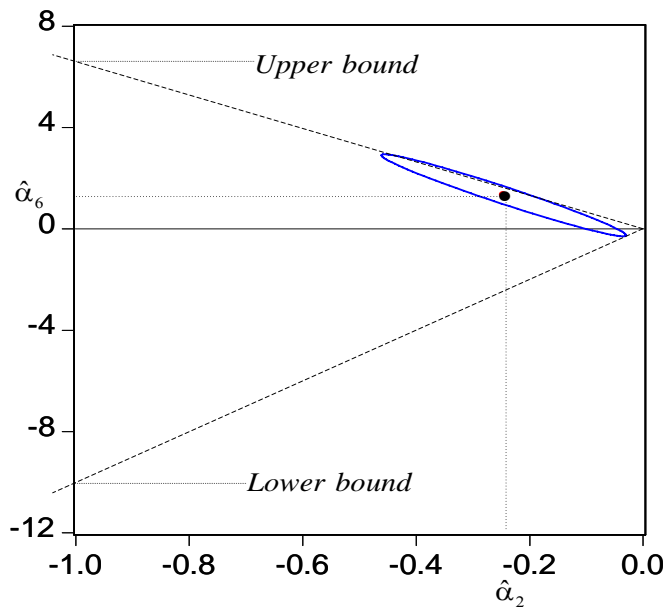
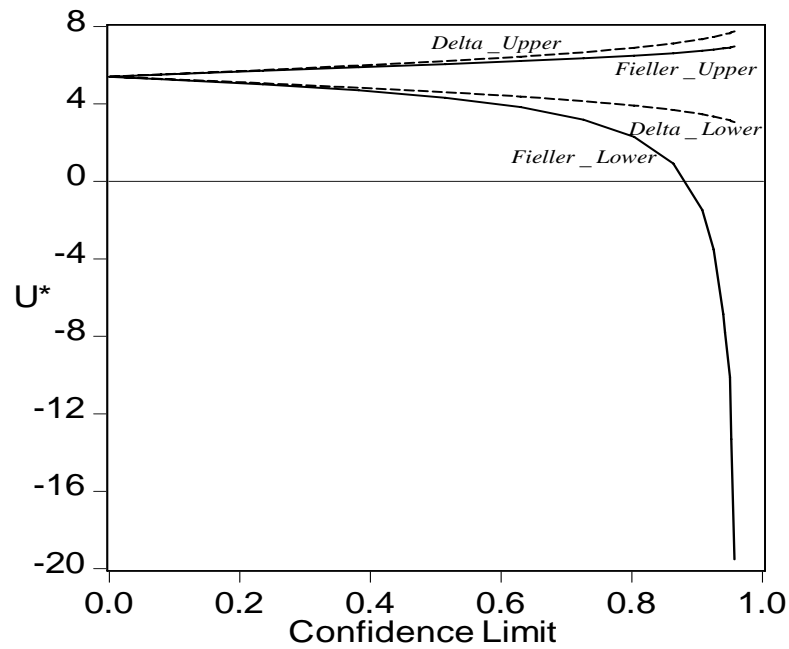


Figure 4 shows the bounds of the Fieller and Delta intervals for a range of confidence limits. At around the 97% Confidence Interval the lower bound of the Fieller interval becomes unbounded whereas the Delta and Fieller upper bounds are similar. In this case the Fieller interval is a complement of a finite interval. At around the 99% Confidence level both bounds of the Fieller interval become unbounded and the Fieller interval is the whole real line.

**Figure 4:** A Comparison of Fieller and Delta for a range of Confidence Limits



## VI. Conclusions

In this paper we demonstrate two graphical methods for the demonstration of the Fieller interval. From these graphical representations one can see how the distribution of the two variables which form the ratio influence the nature of the confidence interval. In the example we find that the Fieller although providing a finite upper bound does not necessarily result in a bounded lower bound.

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## Appendix A: Computer Programs

All of the programs discussed in this Appendix involve estimating Equation (8) defined as:

$$\begin{aligned}\Delta_4 \ln ULC_t - \Delta_4 \ln P_{t-1} &= \alpha_1 (\Delta_4 \ln P_t^* - \Delta_4 \ln P_{t-1}) + \alpha_2 U_t + \alpha_3 \Delta U_{t-1} \\ &+ \alpha_4 (\Delta_4 \ln ULC_{t-1} - \Delta_4 \ln P_{t-2}) + \alpha_5 (\Delta_4 \ln ULC_{t-1} - \Delta_4 \ln ULC_{t-4}) + \alpha_6 + \varepsilon\end{aligned}$$

In the Appendix we refer to the variables  $\Delta_4 \ln ULC_t - \Delta_4 \ln P_{t-1}$  as  $y$ ;  $(\Delta_4 \ln P_t^* - \Delta_4 \ln P_{t-1})$  as  $x_1$ ;  $U_t$  as  $x_2$ ;  $\Delta U_{t-1}$  as  $x_3$ ;  $(\Delta_4 \ln ULC_{t-1} - \Delta_4 \ln P_{t-2})$  as  $x_4$  and  $(\Delta_4 \ln ULC_{t-1} - \Delta_4 \ln ULC_{t-4})$  as  $x_5$ .

### Appendix A1: Confidence Bounds Approach

Program 1: Using out-of-sample predictions approach

*Programs: phillips\_stata1.dta; phillips\_regress1.do*<sup>1</sup>

In the program *phillips\_stata1.dta* we want to plot

$$LY = \{ \hat{a}_6 + \hat{a}_2 U^* \}$$

where  $\hat{a}_6$  is the estimated constant and  $\hat{a}_2$  is the estimated coefficient on the variable  $x_2$  and, in addition to also plot the corresponding confidence interval.

In the data file (*phillips\_stata1.dta*) the first observations are the observations for estimation (the first 76 observations). The rest of the observations are to be used for the out-of-sample predictions to obtain the values for  $LY$  and its corresponding confidence intervals (called *up* and *low* in the program). Set these observations up such that the dependent variable is missing (ie use *na*); the observations on  $x_1$ ,  $x_3$ ,  $x_4$ , and  $x_5$  are all 0 and the observations for  $x_2$  correspond to the values over which  $LY$  is to be plotted. In this example we used values from -14 to 14. A few lines from the data file are shown below which show the last two rows of data for the estimation period and the first two rows of observations to be used to obtain the out-of-sample predictions and corresponding confidence intervals:

3.19777 0.11832 5.98071 0.06158 0.9285 1.95527

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<sup>1</sup> All programs mentioned in Appendix A are available from the authors on request.

	0.91987	0.50611	5.74786	-0.2045	3.19777	1.00789
Na		0	-14	0	0	0
Na		0	-13	0	0	0

In calculating the confidence bounds *up* and *low* using predict the appropriate standard error to use is the standard error of the mean prediction using predict's stdp command.

```
use "d:\stata\phillips_stata1.dta"
gen zero = 0
regress y x1 x2 x3 x4 x5 if _n<77
predict double ly, xb
predict double stdly, stdp
scalar tval = invttail(e(df_r),0.025)
generate double up = ly + tval*stdly
generate double low = ly - tval*stdly
tabdisp x2 if _n>76, cellvar(ly stdly low up)
scatter zero ly low up x2 if _n>76, connect(l) msymbol(i) xlab(-12 -8 -4 0
4 8)
```

## Program 2: Using saved estimates and variance-covariance matrix

*Programs:phillips\_saved1.do; phillips\_aus1.dta; us.dta*

The 76 observations to estimate the model are in *Phillips\_aus1.dta* and *us.dta* has a variable *us* which contains the values over which *LY* is to be calculated. We use 29 values of *us* ranging from -14 to 14 in steps of 1. In the program *phillips\_saved1.do*, *LY* and the confidence bounds of *LY*, named *up* and *low*, are generated using saved values of the regression estimates (*\_b[\_cons]*, *\_b[x2]*) and saved values of the standard errors (*\_se[\_cons]*, *\_se[x2]*). To obtain the covariance the appropriate element is accessed from the saved variance-covariance matrix (VCE). In the program this is named C26.

```
use "d:\stata\phillips_aus1.dta"
regress y x1 x2 x3 x4 x5
use "d:\stata\us.dta"
gen ly = _b[_cons]+_b[x2]*us
gen zero = 0
matrix V = get(VCE)
matrix C26 = (vecdiag((V[2,6])*I(29)))'
svmat C26, name(C26)
gen varly = (_se[_cons])^2+(us^2)*(_se[x2]^2)+2*us*C26
gen stdly = sqrt(varly)
tabdisp us, cellvar(ly stdly)
gen low = ly-invttail(e(df_r),.025)*stdly
gen up = ly+invttail(e(df_r),.025)*stdly
tabdisp us, cellvar(ly stdly low up)
```

```
scatter zero low up us, connect(l) msymbol(i) xlab(-12 -8 -4 0 4 8)
```

## Appendix A2: Confidence Ellipse Approach

Stata Program

*Program: phill1.dta; ellipse1.do*

The data file *phill1.dta* contains the regression data including data on a variable  $x_6$  which takes values 1 corresponding to the constant term. The program *ellipse1.do* contains 3 lines to generate a confidence ellipse although it calls upon the program “*ellip*” from Alexandersson (2004).

First the regression is estimated. In this case  $x_6$  takes values 1 as corresponds to the constant term so a regression is run with no constant. For large samples, to obtain the appropriate dimensions of the confidence ellipse the appropriate boundary constant is a chi square with 1 degree of freedom. The options *xlab* and *ylob* are used to plot the ellipse over appropriate values.

```
use "d:\stata\phill1.dta"  
regress y x6 x2 x1 x3 x4 x5, noc  
ellip x6 x2 ,coefs c(chi2 1) ylab(-12 -8 -4 0 4 8) xlab(0 -0.2 -0.4 -0.6 -  
0.8 -1.0)
```

Eviews Program (Eviews 2005).

The confidence ellipse produced as part of the OLS options in Eviews 5.1 (see Eviews 2005 p 572) is specified as:

$$(\beta_2 - \hat{\beta}_2)' \hat{\sigma}^{-2} (X_2' M_1 X_2) (\beta_2 - \hat{\beta}_2) \leq 2F_\alpha(2, T - k)$$

However, the appropriate confidence ellipse is

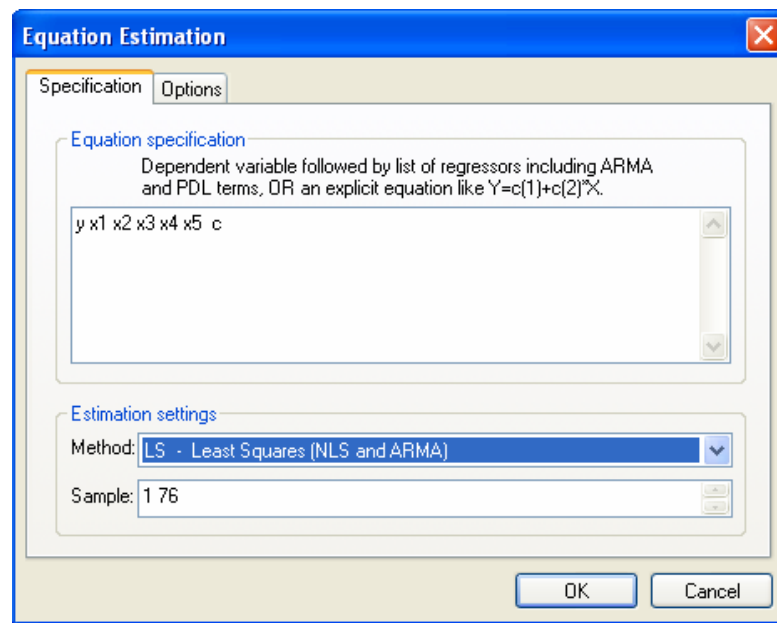
$$(\beta_2 - \hat{\beta}_2)' \hat{\sigma}^{-2} (X_2' M_1 X_2) (\beta_2 - \hat{\beta}_2) \leq F_\alpha(1, T - k)$$

To obtain the appropriate confidence ellipse, specify the confidence level as  $(1 - \tilde{\alpha})$ , such that  $F_\alpha(1, T - k) = 2F_{\tilde{\alpha}}(2, T - k)$ . The Table below lists the appropriate confidence levels to use in the Eviews program

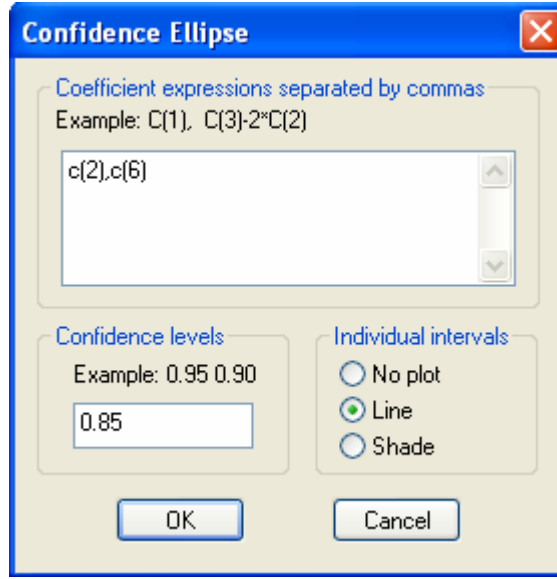


$df$	$(1-\tilde{\alpha})$	$(1-\alpha)$	$(1-\tilde{\alpha})$	$(1-\alpha)$
10	0.867	0.950	0.758	0.900
20	0.860	0.950	0.750	0.900
30	0.858	0.950	0.747	0.900
40	0.856	0.950	0.746	0.900
50	0.856	0.950	0.745	0.900
60	0.856	0.950	0.744	0.900
70	0.855	0.950	0.744	0.900
80	0.855	0.950	0.744	0.900
90	0.855	0.950	0.743	0.900
100	0.855	0.950	0.743	0.900
120	0.855	0.950	0.743	0.900
150	0.854	0.950	0.743	0.900
200	0.854	0.950	0.742	0.900
300	0.854	0.950	0.742	0.900
500	0.854	0.950	0.742	0.900
1000	0.854	0.950	0.742	0.900
5000	0.854	0.950	0.742	0.900

The first step is to estimate the regression using Eviews as follows:



Then to obtain the 95% Confidence Ellipse, we use *Confidence Ellipse* available under *the View/Coefficient Tests* option and specify  $(1-\tilde{\alpha}) = 0.85$  as the Confidence level and the estimated coefficients correspond to  $c(2)$ , the estimated coefficient of  $x_2$  and  $c(6)$ , the estimated coefficient of  $c$ .



## Appendix B The equivalence between the ratio restricted by the confidence ellipse and the Fieller Method.

The bounds of the ratio of the regression parameters where the restriction is defined by the confidence ellipsoid of the two parameters can be found from the solution to the following constrained optimization problem:

$$L = \frac{\beta_1}{\beta_2} - \lambda \left( \begin{bmatrix} (b_1 - \beta_1) & (b_2 - \beta_2) \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} \begin{bmatrix} (b_1 - \beta_1) \\ (b_2 - \beta_2) \end{bmatrix} - t^2 \right) \quad (\text{B.1})^2$$

where  $\lambda$  is the Lagrange multiplier,  $b_1$  and  $b_2$  are the estimated parameters,  $\omega_{ij}$  are elements of the inverse of the covariance of the regression parameters, and  $t^2$  is the critical value of the t-distribution for a two tailed test. Alternatively writing this Lagrangian in terms of  $\psi$  where  $\psi = \beta_1/\beta_2$  allows us to write  $\beta_1 = \beta_2\psi$ , and we obtain:

$$L = \psi - \lambda \left( \begin{bmatrix} (b_1 - \psi\beta_2) & (b_2 - \beta_2) \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} \begin{bmatrix} (b_1 - \psi\beta_2) \\ (b_2 - \beta_2) \end{bmatrix} - t^2 \right) \quad (\text{B.2})$$

or:

$$L = \psi + \lambda \left( \begin{array}{l} -t^2 + (b_2 - \beta_2)(\omega_{22}(b_2 - \beta_2) + \omega_{12}(b_1 - \psi\beta_2)) \\ + (b_1 - \psi\beta_2)(\omega_{12}(b_2 - \beta_2) + \omega_{11}(b_1 - \psi\beta_2)) \end{array} \right)$$

The first order partial derivatives are then defined as:

$$\frac{\partial L}{\partial \psi} = 2\lambda\beta_2^2\omega_{12} - 2\lambda b_2\beta_2\omega_{12} - 2\lambda b_1\beta_2\omega_{11} + 2\lambda\psi\beta_2^2\omega_{11} + 1$$

<sup>2</sup> Note that the notation  $t^2$  is used. It is equivalent to  $F_\alpha(1, T - k)$ .

$$\frac{\partial L}{\partial \lambda} = 2b_1b_2\omega_{12} - 2b_1\beta_2\omega_{12} - 2b_2\beta_2\omega_{22} - 2\psi b_1\beta_2\omega_{11} - 2\psi b_2\beta_2\omega_{12} - t^2 + b_1^2\omega_{11} + b_2^2\omega_{22} + \beta_2^2\omega_{22} + 2\psi\beta_2^2\omega_{12} + \psi^2\beta_2^2\omega_{11}$$

$$\frac{\partial L}{\partial \beta_2} = 2\lambda\beta_2\omega_{22} - 2\lambda b_2\omega_{22} - 2\lambda b_1\omega_{12} - 2\lambda\psi b_1\omega_{11} - 2\lambda\psi b_2\omega_{12} + 4\lambda\psi\beta_2\omega_{12} + 2\lambda\psi^2\beta_2\omega_{11}$$

The first order conditions for an optimum are given as:

$$b_1\beta_2\omega_{11} + b_2\beta_2\omega_{12} - \beta_2^2\omega_{12} - \psi\beta_2^2\omega_{11} + \frac{1}{2\lambda} = 0 \quad (\text{B.3})$$

$$2\left(b_1\beta_2\omega_{12} - b_1b_2\omega_{12} + b_2\beta_2\omega_{22} + \psi b_1\beta_2\omega_{11} + \psi b_2\beta_2\omega_{12} - \psi\beta_2^2\omega_{12}\right) + t^2 - b_1^2\omega_{11} - b_2^2\omega_{22} - \beta_2^2\omega_{22} - \psi^2\beta_2^2\omega_{11} = 0 \quad (\text{B.4})$$

$$\beta_2\omega_{22} - b_2\omega_{22} - b_1\omega_{12} + \psi\left(2\beta_2\omega_{12} - b_1\omega_{11} - b_2\omega_{12}\right) + \psi^2\beta_2\omega_{11} = 0 \quad (\text{B.5})$$

First we can solve (B.5) for  $\beta_2$  as

$$\beta_2 = \left(\omega_{22} + 2\psi\omega_{12} + \psi^2\omega_{11}\right)^{-1} \left(b_1\omega_{12} + b_2\omega_{22} + \psi b_1\omega_{11} + \psi b_2\omega_{12}\right) \quad (\text{B.6})$$

Then we can use (B.6) to substitute for  $\beta_2$  in (B.4) to get an expression only in  $\psi$  and the other given values.

$$\begin{aligned} & t^2 - 2b_1b_2\omega_{12} - b_1^2\omega_{11} - b_2^2\omega_{22} \\ & + \frac{1}{\omega_{22} + 2\psi\omega_{12} + \psi^2\omega_{11}} \left( \begin{aligned} & 4b_1b_2\omega_{12}\omega_{22} + 4\psi b_1b_2\omega_{11}\omega_{22} + 4\psi b_1b_2\omega_{12}^2 + 4\psi b_1^2\omega_{11}\omega_{12} \\ & + 4\psi b_2^2\omega_{12}\omega_{22} + 4\psi^2 b_1b_2\omega_{11}\omega_{12} + 2b_1^2\omega_{12}^2 + 2b_2^2\omega_{22}^2 + \\ & 2\psi^2 b_1^2\omega_{11}^2 + 2\psi^2 b_2^2\omega_{12}^2 \end{aligned} \right) \\ & + \frac{1}{4\psi\omega_{12}\omega_{22} + \omega_{22}^2 + 2\psi^2\omega_{11}\omega_{22} + 4\psi^3\omega_{11}\omega_{12} + 4\psi^2\omega_{12}^2 + \psi^4\omega_{11}^2} \left( \begin{aligned} & -2b_1b_2\omega_{12}\omega_{22}^2 - 2\psi b_1b_2\omega_{11}\omega_{22}^2 - 2\psi b_1b_2\omega_{12}^2\omega_{22} - 2\psi b_1^2\omega_{11}\omega_{12}\omega_{22} \\ & - 4\psi^2 b_1b_2\omega_{11}\omega_{12}\omega_{22} - b_2^2\omega_{22}^3 - b_1^2\omega_{12}^2\omega_{22} - 2\psi b_2^2\omega_{12}\omega_{22}^2 \\ & - 2\psi^3 b_1b_2\omega_{11}\omega_{12}^2 - 2\psi^3 b_1b_2\omega_{11}^2\omega_{22} - 2\psi^4 b_1b_2\omega_{11}^2\omega_{12} \\ & - 2\psi^3 b_2^2\omega_{11}\omega_{12}\omega_{22} - \psi^4 b_1^2\omega_{11}^3 - \psi^2 b_1^2\omega_{11}\omega_{12}^2 - \psi^2 b_1^2\omega_{11}^2\omega_{22} \\ & - 2\psi^3 b_1^2\omega_{11}^2\omega_{12} - \psi^2 b_2^2\omega_{11}\omega_{22}^2 - \psi^2 b_2^2\omega_{12}^2\omega_{22} - \psi^4 b_2^2\omega_{11}\omega_{12}^2 \end{aligned} \right) \quad (\text{B.7})^3 \end{aligned}$$

A set of solutions is given by:

$$\psi_i = \frac{\left( \begin{aligned} & -b_1b_2\omega_{11}\omega_{22} - t^2\omega_{12} + b_1b_2\omega_{12}^2 \pm \sqrt{\begin{aligned} & -t^4\omega_{11}\omega_{22} + 2t^2b_1b_2\omega_{11}\omega_{12}\omega_{22} + t^4\omega_{12}^2 \\ & - 2t^2b_1b_2\omega_{12}^3 - t^2b_1^2\omega_{11}\omega_{12}^2 + t^2b_1^2\omega_{11}^2\omega_{22} \\ & + t^2b_2^2\omega_{11}\omega_{22}^2 - t^2b_2^2\omega_{12}^2\omega_{22} \end{aligned}} \right)}{t^2\omega_{11} - b_2^2\omega_{11}\omega_{22} + b_2^2\omega_{12}^2} \quad (\text{B.8})^4$$

<sup>3</sup> This expression was expanded using Scientific Workplace 5.0.

<sup>4</sup> This set of solutions requires  $\omega_{11} \neq 0$  which would always be the case when we estimate the regression parameters.

Alternatively, the Fieller method is defined as the solution for the values of  $\psi$  from this case:

$$(b_1 - \psi b_2)^2 - t^2 \begin{bmatrix} 1 & -\psi \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -\psi \end{bmatrix} = 0 \quad (\text{B.9})$$

where  $\sigma_{ij}$  are elements of the covariance matrix for the parameters. This can also be written as the quadratic equation in  $\psi$  defined as:

$$b_1^2 - t^2 \sigma_{11} + \psi (2t^2 \sigma_{12} - 2b_1 b_2) + \psi^2 (b_2^2 - t^2 \sigma_{22}) = 0 \quad (\text{B.10})$$

Again, in the case where we find two real roots the solution is given as:

$$\frac{1}{-b_2^2 + t^2 \sigma_{22}} \left( -b_1 b_2 + t^2 \sigma_{12} \pm \sqrt{-t^4 \sigma_{11} \sigma_{22} - 2t^2 b_1 b_2 \sigma_{12} + t^4 \sigma_{12}^2 + t^2 b_2^2 \sigma_{11} + t^2 b_1^2 \sigma_{22}} \right) \quad (\text{B.11})$$

In order to demonstrate the equivalence between these two methods we need to use the correspondence between the covariance matrix and its inverse defined as:

$$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{\sigma_{22}}{-\sigma_{11}\sigma_{22} + \sigma_{12}^2} & \frac{\sigma_{12}}{-\sigma_{11}\sigma_{22} + \sigma_{12}^2} \\ \frac{\sigma_{12}}{-\sigma_{11}\sigma_{22} + \sigma_{12}^2} & -\frac{\sigma_{11}}{-\sigma_{11}\sigma_{22} + \sigma_{12}^2} \end{bmatrix} \quad (\text{B.12})$$

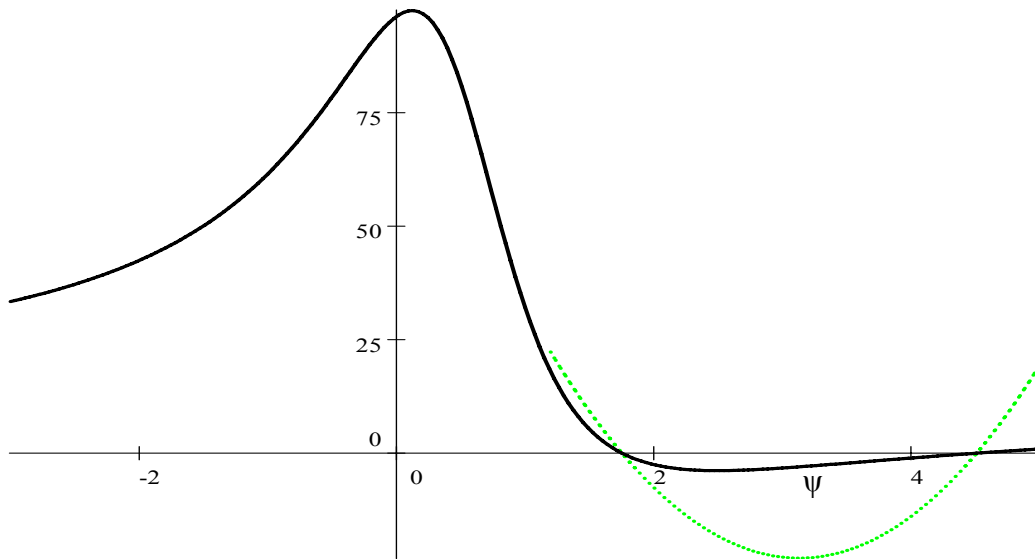
We can show that the roots for the constrained optimization problem are then defined as shown below when written in terms of the elements of the covariance matrix:

$$\psi_i = \frac{-1}{b_2^2 - t^2 \sigma_{22}} \left( \begin{aligned} & \left( b_1 b_2 \frac{\sigma_{11} \sigma_{22}}{\sigma_{12}^2 - \sigma_{11} \sigma_{22}} - b_1 b_2 \frac{\sigma_{12}^2}{\sigma_{12}^2 - \sigma_{11} \sigma_{22}} + t^2 \sigma_{12} \right) \\ & \pm \sqrt{t^2 \left( b_2^2 \sigma_{11} - 2b_1 b_2 \sigma_{12} + b_1^2 \sigma_{22} - t^2 \sigma_{11} \sigma_{22} + t^2 \sigma_{12}^2 \right)} \end{aligned} \right) \quad (\text{B.13})$$

It can then be shown that this is equal to the corresponding root for the quadratic as found from the Fieller method.

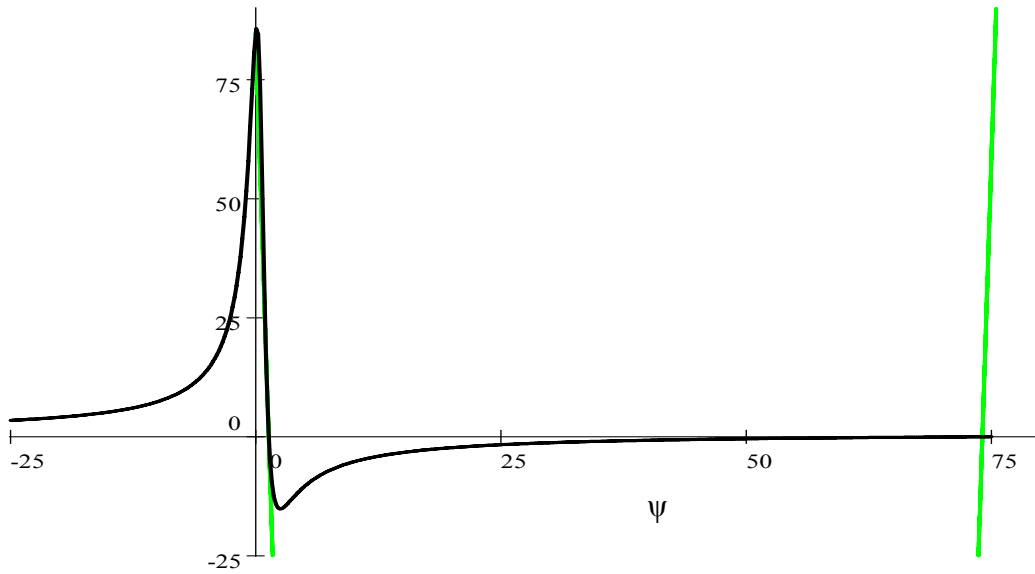
$$\Psi_i = \frac{1}{-b_2^2 + t^2 \sigma_{22}} \left( -b_1 b_2 + t^2 \sigma_{12} \pm \sqrt{-t^4 \sigma_{11} \sigma_{22} - 2t^2 b_1 b_2 \sigma_{12} + t^4 \sigma_{12}^2 + t^2 b_2^2 \sigma_{11} + t^2 b_1^2 \sigma_{22}} \right) \quad (\text{B.14})$$

Although the roots of these two problems are the same the functions they satisfy are quite different. To demonstrate this we have assumed the following values:  $b_1 = 10$ ,  $b_2 = 4$ ,  $\omega_{11} = 1 + \frac{1}{3}$ ,  $\omega_{12} = -\frac{2}{3}$ ,  $\omega_{22} = 1 + \frac{1}{3}$ ,  $\sigma_{11} = 1$ ,  $\sigma_{12} = \frac{1}{2}$ ,  $\sigma_{22} = 1$ , and  $t = 2$ . Using these parameter values we find that  $\frac{b_1}{b_2} = 2.5$  and the limits are defined as 1.7427 and 4.5907. Figure B.1 shown below has plotted the two functions to be solved the Fieller results in the parabola denoted by the dashed line and the solution to the constrained optimization is given by the thicker line.



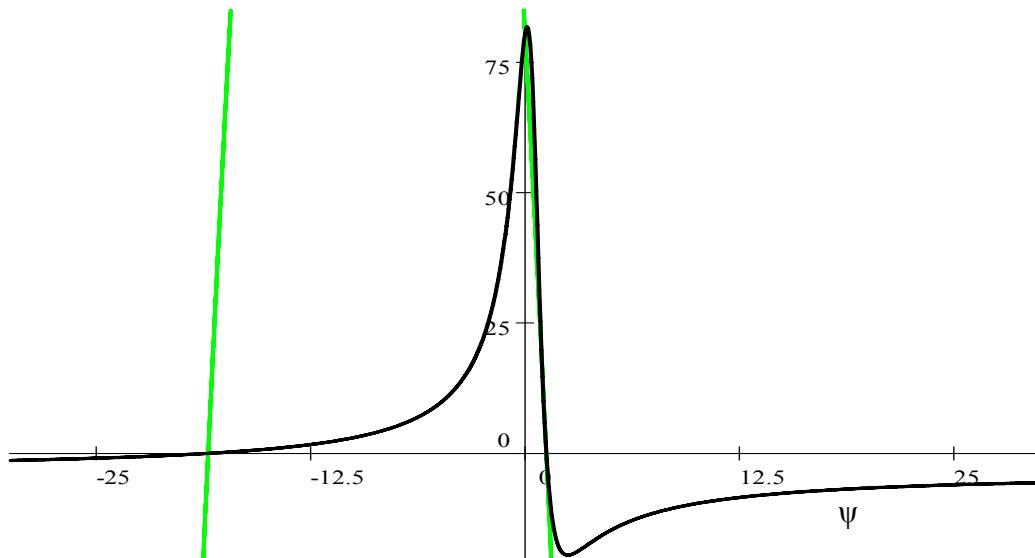
**Figure B1** The function defined by (B.7) (dark line) and the function (B.10) (dashed line) when the t-statistic is 2.

If we set the t-statistic for the case of a probability of a type one error to .0001 the  $t = 3.891$  we find the function in (B.7) as plotted in Figure B.2 below. Note that the limits in this case are 1.3319 to 74.116 and the Fieller intervals parabola appears to be almost vertical lines.



**Figure B2** The function defined by (B.7) (dark line) and the function (B.10) (thin line) when the  $t$ -statistic is 3.891.

In Figure B.3 the two functions are plotted assuming a  $t$ -statistic = 4.417 for a  $p$ -value of .00001. Note that the (B.7) function remains negative as  $\psi$  goes to infinity.



**Figure B3** The function defined by (B.7) (dark line) and the function (B.10) when the  $t$ -statistic is 4.417 for a  $p$ -value of .00001.

In this case the two roots are -18.475 and 1.2417. Recall that these are confidence intervals about the value of 2.5. For practical purposes, the appropriate interpretation is that the upper bound goes to  $\infty$  and the lower bound is 1.2412.