ISSN 0819-2642
ISBN 978 0 7340 3726 8

THE UNIVERSITY OF MELBOURNE

DEPARTMENT OF ECONOMICS

RESEARCH PAPER NUMBER 1016

November 2007

The Fundamental Duality Theorem of Balanced Growth

by

Don Ferguson & Ian King

Department of Economics
The University of Melbourne
Melbourne Victoria 3010
Australia.

The Fundamental Duality Theorem of Balanced Growth

Don Ferguson[*] and Ian King[#]

November 2007

Abstract

In this paper we demonstrate that a simple duality relation underlies balanced growth models with non-joint production. Included in this class of models is the standard neoclassical growth model and endogenous growth models that admit balanced growth paths. In all of these models, the optimal transformation frontier and the factor price frontier take precisely the same mathematical formulation. Studying these identical frontiers in the context of the different models provides new insights into the relative structures of these models, the role of savings, and the nature of dynamic efficiency in each.

JEL Codes: O40, O41

[*] Department of Economics, University of Victoria, Victoria, B.C. V8W 2Y2 Canada

[#] Department of Economics, University of Melbourne, Victoria 3010, Australia

We would like to thank Merwan Engineer and Peter Kennedy for many helpful discussions, and Peter for help with the diagrams in this paper. We would also to thank participants in a seminar at the university of Victoria and at the Southern Workshop in Macroeconomics (SWIM 2007) for helpful comments.

1. INTRODUCTION

In this paper, we revisit an important theorem that was developed in the 1960's in the context of the neoclassical growth theory, which has been absent from recent discussions about growth, and which (appropriately modified) has significant relevance for modern endogenous growth theory – both in understanding the mechanics of the theory itself, and in understanding it's relationship with neoclassical theory. This theorem was originally developed by Bruno (1969) in the context of dynamic Leontief models, but was adapted to neoclassical models by Burmeister and Kuga (1970a,b) and further developed in Burmeister and Dobell (1970). The theorem concerns a duality result that pervades models with "neoclassical" production technologies – significantly, technologies with constant returns to scale. It identifies an equivalence between two frontiers that exist within these models: the "optimal transformation frontier" (OTF) and the "factor price frontier" (FPF). These two frontiers have very different interpretations, one normative and the other positive, but have precisely the same mathematical formulation. This equivalence, within the neoclassical model, was known 37 years ago.

Here, we consider the relevance of this theorem in endogenous growth models with neoclassical production technologies. This is an important class of endogenous growth models, and the generic model we consider here can include well-known models such as convex versions of Romer's (1986) model, Lucas' (1988) model, and Aghion and Howitt's (1992) model as special cases (albeit sometimes distorted).[1] We show that, as in neoclassical models, the equivalence of the OTF and FPF holds in this class of endogenous growth models. Moreover, the equivalence result extends further: *across* models. In particular, we show that the OTF and FPF in our generic endogenous growth model are not only identical to each other, but are also identical to

[1] See Ferguson (1994), or Jones and Manuelli (1997) for a discussion.

the OTF and FPF in the comparable neoclassical model. This equivalence holds for both one-sector models and multi-sector models.

The OTF and FPF frontiers themselves are also interesting objects to study and, in the context of the equivalence results across the models, shed light on the common features and distinctions between neoclassical and endogenous growth models. We present a simple diagrammatic analysis of these frontiers that illustrates the fundamental causal differences in these models, and (we believe) significantly clarifies the issue of dynamic efficiency in each case.

The remainder of the paper is structured as follows. Section 2 considers one-sector models. We start with a brief review of the one-sector neoclassical model, highlighting the roles (and equivalence) of the OTF and the FPF, using somewhat more modern notation than in, for example, Burmeister and Dobell (1970).[2] We then consider the role of savings in this model, allowing for two different alternatives (a fixed savings rate and optimal savings) and review the issue of dynamic efficiency. A one-sector endogenous growth model is then introduced, based on Barro and Sala-i-Martin's (2004) textbook. We generate the OTF and FPF in this model, and demonstrate the basic equivalence result, both within and across the models. Section 3 then extends the analysis to cover multi-sector models, both neoclassical and endogenous growth, and shows that the OTF and FPF equivalencies still hold and that they can also be represented simply on the same two-dimensional diagram as is used for the one-sector model. Our conclusions are then presented in Section 4.

---

[2] As much as possible, we follow the notation used in the Barro and Sala-i-Martin (2004) text, which provides an excellent overview of modern growth theory.

2. ONE SECTOR MODELS

Here, we consider models in which homogeneous output can be adapted instantaneously and costlessly into either a consumption good or a capital good. The key distinction between the neoclassical and endogenous growth models is the interpretation of the labour input. In neoclassical models, labour is a productive asset, but its accumulation is not determined by market forces. Typically, in these models, labour grows at an exogenous constant rate.[3] A key feature of endogenous growth models is that, in these models, the accumulation of the labour component (augmented by human capital) is influenced by market forces. We first review the known results of the neoclassical model, as discussed in Burmeister and Dobell (1970), but using more modern notation.

2.1 The Neoclassical Growth Model[4]

Output Y is produced, using the capital stock K and labour L, according to the neoclassical production function:[5]

$$Y = F(K, L) \tag{2.1.1}$$

This output can be used either for consumption C or gross investment in capital I_K:

$$Y = C + I_K \tag{2.1.2}$$

---

[3] A constant growth rate of labour is not necessary for the distinction. What is important is this growth rate is not determined by market forces.

[4] This model is identical to the one considered in Barro and Sala-i-Martin (2004), chapters 1 and 2.

[5] See Burmeister and Dobell (1970), p. 10, for a formal definition of the neoclassical production function. Importantly, the function is increasing in both arguments, has continuous second partial derivatives, is homogeneous of degree one and is strictly quasiconcave.

Net investment $\dot{K}$ is gross investment minus depreciation, where a constant fraction δ_K of capital depreciates at each moment in time:

$$\dot{K} = I_K - \delta_K K \tag{2.1.3}$$

The stock of labour grows exogenously at the constant rate $n > 0$:

$$\dot{L} = nL \tag{2.1.4}$$

Substitution of (2.1.1) and (2.1.3) into (2.1.2) yields:

$$F(K, L) = C + \dot{K} + \delta_K K$$

Now, defining $k \equiv K / L$, $c \equiv C / L$, $f(k) \equiv F(K / L, 1)$, $\gamma_K \equiv \dot{K} / K$, and $\gamma_L \equiv \dot{L} / L$, rearrangement of the above equation gives us:

$$c = f(k) - (\gamma_K + \delta_K)k \tag{2.1.5}$$

Along any balanced growth path, we have the additional condition:

$$\gamma_K = \gamma_L = n \tag{2.1.6}$$

2.1.1 The Optimal Transformation Frontier

With δ_K as a parameter, and γ_K pinned down by equation (2.1.6), we can consider the problem of choosing k to maximize per capita consumption c in (2.1.5) along a balanced growth path. The solution to this problem is, of course, the golden rule:

$$f'(k^*) = \gamma_K + \delta_K \tag{2.1.7}$$

Inverting, we have:

$$k^* = f'^{-1}(\gamma_K + \delta_K) \tag{2.1.8}$$

Substitution, then, of this balanced growth consumption-maximizing value of *k* into the objective function (2.1.5) leads to the value function:

$$c^* = f\left(f'^{-1}(\gamma_K + \delta_K)\right) - (\gamma_K + \delta_K) f'^{-1}(\gamma_K + \delta_K) \equiv G(\gamma_K + \delta_K) \tag{2.1.9}$$

This value function is known as the *optimal transformation frontier* (OTF). For any given values of γ_K and δ_K, it tells us the maximal value of per capita consumption available along any balanced growth path.

[Figure 1 about here.]

Figure 1 illustrates this frontier. The shape of the frontier is very easy to ascertain. From (2.1.9), using (2.1.8) and the envelope theorem, we have: $G'(\gamma_K + \delta_K) = -k^* < 0$. Using this and equation (2.1.7), one can find: $G''(\gamma_K + \delta_K) = -1/f''(k^*) > 0$.

A Cobb-Douglas Example

If the production technology takes the Cobb-Douglas form

$$F(K, L) = AK^{\alpha} L^{1-\alpha}, \quad A > 0, \quad \alpha \in (0,1) \tag{2.1.10}$$

then the golden rule value of k^* is given by

$$k^* = \left(\frac{\alpha A}{\gamma_K + \delta_K} \right)^{\frac{1}{1-\alpha}} \tag{2.1.11}$$

and the OTF is given by:

$$c^* = (1-\alpha) A \left(\frac{\alpha A}{\gamma_K + \delta_K} \right)^{\frac{\alpha}{1-\alpha}} \tag{2.1.12}$$

2.1.2 The Factor Price Frontier

We now add more structure to the economy by assuming that firms are competitive, facing a given real wage rate *w* and a gross rate of return on capital *r*. The net rate of return on capital, after depreciation, is given by $\rho_K = r - \delta_K$. Here, as is standard, we normalize the number of firms to unity, and each firm acts to maximize profits:

$$\Pi = F(K, L) - rK - wL$$

Expressed in terms of *k*, the first order conditions are:

$$r = f'(k) \tag{2.1.13}$$

$$w = f(k) - f'(k)k \tag{2.1.14}$$

For any given value of *r*, equation (2.1.13) can be inverted to find the profit maximizing value of *k:*

$$k = f'^{-1}(r) \tag{2.1.15}$$

Substitution of (2.1.13) and (2.1.15) in (2.1.14) then gives us the following relationship between *w* and *r*, consistent with profit maximization:

$$w = f\left(f'^{-1}(r)\right) - (r) f'^{-1}(r) \equiv V(r) \qquad (2.1.16)$$

This is known as the *factor price frontier* (FPF). In the Cobb-Douglas example, using (2.1.10), the FPF is given by:

$$w = (1-\alpha)A\left(\frac{\alpha A}{r}\right)^{\frac{\alpha}{1-\alpha}} \qquad (2.1.17)$$

This frontier tells us, for any given values of the parameters, the pairs of *w* and *r* that are consistent with competitive equilibrium. Figure 2 provides an illustration of the FPF.

[Figure 2 about here.]

Comparing the OTF in equation (2.1.9) (or equation 2.1.11) with the FPF in equation (2.1.16) (or equation (2.1.17) it is immediately apparent that they have precisely the same mathematical formulation. Since the profit maximization for firms can be cast, equivalently, as a cost minimization problem (as we consider below) then this is a duality result: the value function from maximizing per capita consumption is identical to the value function from minimizing costs.

At this point, it is worth noting that these two frontiers have very different interpretations. The OTF is a *normative* concept: it identifies maximal values of *c* obtainable in the economy with balanced growth. The content of FPF *positive*: it identifies pairs of *w* and *r* that are consistent with competitive equilibrium.

2.1.3 Closing the Model: the Savings Decision

Up to this point, no savings decision has been specified. In this paper, we consider two alternative savings regimes: the Solow savings function with a fixed savings rate, and optimal (Ramsey-Cass-Koopmans) savings.

Fixed Savings Rates

The consumption function consistent with a fixed savings rate is: $c = (1-s)f(k)$, where $s \in (0,1)$ is a parameter. Substitution of this function into equation (2.1.5), and collecting terms, we obtain, along the balanced growth path:

$$sf(k) = (\gamma_K + \delta_K)k \qquad (2.1.18)$$

Equation (2.1.18) determines a unique equilibrium value of *k,* call it $\tilde{k}$. With $\tilde{k}$ determined, all the other equilibrium values of the variables along the path are determined. With the Cobb-Douglas production technology (2.1.10), it is simple to use equation (2.1.18) to solve for:

$$\tilde{k} = \left(\frac{sA}{\gamma_K + \delta_K}\right)^{\frac{1}{1-\alpha}} \qquad (2.1.19)$$

Optimal Savings

Consider the following standard optimal savings problem. Identical infinitely-lived households, each growing in size at rate *n*, with marginal rate of time preference $\theta \in (0,1)$, and constant elasticity of substitution $1/\sigma > 0$, given a path of wages $\{w_t\}$ and a path of net returns on assets $\{\rho_{Kt}\}$, choose a path of consumption $\{c_t\}$, with an implied path of assets $\{a_t\}$ to solve:

$$\underset{\{c_t\}}{Max} \quad U_0 = \int_0^\infty \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right) e^{-(\theta - n)t} dt \qquad (2.1.20)$$

subject to: a) $\dot{a}_t = \rho_{Kt} a_t + w_t - n a_t - c_t$

b) $a_0 = \overline{a}_0$ (given)

c) $\lim_{t \to \infty} a_t e^{-\int_0^t (\rho_{Kv} - n) dv} \geq 0$

Necessary conditions for this problem imply the Euler equation:

$$\dot{c} / c = (\rho_K - \theta) / \sigma \qquad (2.1.21)$$

In a balanced growth equilibrium $\dot{c} = 0$, in which case equation (2.1.21) implies:

$$\rho_K = \theta \qquad (2.1.22)$$

Now, recalling that the gross rate of return is $r = \rho_K + \delta_K$, and using equation (2.1.13) to substitute out ρ_K in equation (2.1.22), we have, in equilibrium:

$$f'(k) = \theta + \delta_K \qquad (2.1.23)$$

This equation determines a unique value of *k*, call it $\hat{k}$, in the balanced growth equilibrium. Inverting the function in (2.1.23), we obtain:

$$\hat{k} = f'^{-1}(\theta + \delta_K) \qquad (2.1.24)$$

In the Cobb-Douglas example, equation (2.1.24) becomes:

$$\hat{k} = \left(\frac{\alpha A}{\theta + \delta_K} \right)^{\frac{1}{1-\alpha}} \qquad (2.1.25)$$

2.1.4 Dynamic Efficiency

Following Cass (1972), a balanced growth equilibrium allocation is said to be *dynamically efficient* if there exist no other feasible balanced growth paths where consumption is at least as high at all moments in time and strictly higher for at least one moment in time. In the context of the neoclassical growth model, it is easy to show that this implies that the value of *k* in the steady state equilibrium is no greater than the golden rule k^*, given in equation (2.1.8).[6] It is clear that, in these models, whether or not the balanced growth equilibrium is dynamically efficient depends crucially on the savings behaviour presumed.

With a fixed savings rate *s*, the equilibrium will be dynamically inefficient if *s* is too large. In the Cobb-Douglas example, comparing $\tilde{k}$ in (2.1.19) with k^* in (2.1.11), we can see that the equilibrium will be dynamically inefficient if and only if $s > \alpha$. That is, in the Solow-Swan model, the steady state equilibrium is dynamically inefficient if and only if the savings rate is greater than capital's share of income.

When agents choose their savings optimally, according to the standard formulation, a comparison of $\hat{k}$ in (2.1.24) with k^* in (2.1.11) reveals that the relative sizes of γ_K and θ matter. Recalling, from (2.1.6), that $\gamma_K = n$, we find $n > \theta$ as the critical condition for dynamic inefficiency. This, however, is typically ruled out in order to keep the objective function (2.1.20) bounded.

[6] See, for example, Proposition 2.4 in de la Croix and Michel (2002).

Most generally, k^* is defined by (2.1.7), where $f'(k^*) = \gamma_K + \delta_K$. Since $f''(k) < 0$ for all *k*, then an allocation is dynamically inefficient if and only if $f'(k^*) < \gamma_K + \delta_K$. Recalling (2.1.13): $r = f'(k)$, and the definition $r = \rho_K + \delta_K$, this condition can be re-written in the way expressed in Gale and Rockwell (1975), which applies to all models that admit balanced growth paths: an allocation is dynamically efficient if and only if [7]

$$\gamma_K \leq \rho_K \tag{2.1.26}$$

2.2 A Simple Endogenous Growth Model[8]

Here, output is produced using the capital stock *K* as before, but together with an accumulable factor *H* (which we will call "human capital"), using the neoclassical production function:

$$Y = F(K, H) \tag{2.2.1}$$

This output can be used either for consumption *C* or gross investment in capital I_K, or gross investment in human capital I_H:

$$Y = C + I_K + I_H \tag{2.2.2}$$

As in the previous subsection, net investment $\dot{K}$ is gross investment minus depreciation, where a constant fraction δ_K of capital depreciates at each moment in time:

$$\dot{K} = I_K - \delta_K K \tag{2.2.3}$$

[7] See King and Ferguson (1993) for a detailed discussion.
[8] This model is identical to the one presented in section 4.2 of Barro and Sala-i-Martin (2004).

The variable *H*, unlike the variable *L* in the neoclassical model, accumulates in a similar fashion:

$$\dot{H} = I_H - \delta_H H \tag{2.2.4}$$

Substitution of (2.2.1), (2.2.3), and (2.2.4) into (2.2.2) yields:

$$F(K,H) = C + \dot{K} + \delta_K K + \dot{H} + \delta_H H$$

Now, defining $k \equiv K/H$, $c \equiv C/H$, $f(k) \equiv F(K/H,1)$, $\gamma_K \equiv \dot{K}/K$, and $\gamma_H \equiv \dot{H}/H$, rearrangement of the above equation gives us:

$$c + \gamma_H + \delta_H = f(k) - (\gamma_K + \delta_K)k \tag{2.2.5}$$

2.2.1 The Optimal Transformation Frontier

Consider now the problem of choosing *k* to maximize $(c + \gamma_H + \delta_H)$ in (2.2.5). Given any particular values of γ_K and δ_K, this problem is *identical* to the one in the previous section, where we chose *k* to maximize *c* in (2.1.5). The solution is given in equations (2.1.7) and (2.1.8), and substitution of k^* from (2.1.8) back into (2.2.5) yields the following optimal transformation frontier.

$$(c + \gamma_H + \delta_H)^* = f\left(f'^{-1}(\gamma_K + \delta_K)\right) - (\gamma_K + \delta_K)\, f'^{-1}(\gamma_K + \delta_K) \equiv G(\gamma_K + \delta_K)$$
(2.2.6)

In the Cobb-Douglas example, we have:

$$(c+\gamma_H+\delta_H)^* = (1-\alpha)A\left(\frac{\alpha A}{\gamma_K+\delta_K}\right)^{\frac{\alpha}{1-\alpha}} \quad (2.2.7)$$

This is represented in Figure 3.

[Figure 3 about here.]

2.2.2 The Factor Price Frontier

Firms, once again, are assumed to be competitive, and maximize profits. As in the previous section, *r* denotes the gross rate of return on *K*: $r = \rho_K + \delta_K$, where ρ_K is the net return on capital. A similar distinction now applies to the return on human capital. Let *w* denote the gross rate of return, and ρ_H denote the net return to human capital. Thus: $w = \rho_H + \delta_H$. Each firm chooses *K* and *H* to maximize profits:

$$\Pi = F(K,H) - rK - wH$$

Expressed in terms of $k \equiv K/H$, the first order conditions are precisely the same as in the previous section, given by equations (2.1.13) and (2.1.14). Hence, this leads to precisely the same factor price frontier (2.1.16). We can express this, though, in terms of the net returns:

$$\rho_H + \delta_H = f\left(f'^{-1}(\rho_K+\delta_K)\right) - (\rho_K+\delta_K)f'^{-1}(\rho_K+\delta_K) \equiv V(\rho_K+\delta_K)$$
(2.2.8)

In the Cobb-Douglas example, using (2.1.10), this is given by:

$$\rho_H + \delta_H = (1-\alpha)A\left(\frac{\alpha A}{\rho_K + \delta_K}\right)^{\frac{\alpha}{1-\alpha}} \quad (2.2.9)$$

This frontier is illustrated in Figure 4.

[Figure 4 about here.]

Clearly, as in the neoclassical model, in this endogenous growth model, the optimal tranformation frontier (given in (2.2.6) and (2.2.7)) is mathematically identical to the factor price frontier (given in (2.2.8) and (2.2.9)). Moreover, this same mathematical formulation for the two frontiers is common across the two models.

2.2.3 Balanced Growth

As in the neoclassical model, balanced growth implies that the growth rates of the two assets are the same:

$$\gamma_K = \gamma_H = \gamma^* \quad (2.2.10)$$

However, the endogenous growth model requires an extra condition that is not present in the neoclassical model – a no-arbitrage condition: the net rates of return on the two assets must also be equalized:

$$\rho_K = \rho_H = \rho^* \quad (2.2.11)$$

The introduction of this condition has profound implications, both positively (in terms of the causal structure of the equilibrium) and normatively (in terms of dynamic efficiency). To see these effects most clearly, it is useful to consider two different cases. The first case shuts down consumption (and thus corresponds closely with the growth model considered by von Neumann (1936)) and second allows for positive consumption.

Case 1: Zero Consumption (the von Neumann Case)

In this case, the OTF simplifies down to:

$$(\gamma_H + \delta_H)^* = f\left(f'^{-1}(\gamma_K + \delta_K)\right) - (\gamma_K + \delta_K) f'^{-1}(\gamma_K + \delta_K) \equiv G(\gamma_K + \delta_K) \quad (2.2.6')$$

with the Cobb-Douglas example:

$$(\gamma_H + \delta_H)^* = (1-\alpha)A\left(\frac{\alpha A}{\gamma_K + \delta_K}\right)^{\frac{\alpha}{1-\alpha}} \quad (2.2.7')$$

Figure 5 illustrates the economy with no consumption.

[Figure 5 about here.]

In this diagram, both the OTF and the FPF are represented. The horizontal axis measures both $\gamma_K + \delta_K$, for the OTF, and $\rho_K + \delta_K$ for the FPF. Similarly, the vertical axis measures both $\gamma_H + \delta_H$ and $\rho_H + \delta_H$ for the OTF and FPF, respectively. The strictly convex line represents both the OTF and the FPF – they coincide on this diagram. To make the diagram as clear as possible, we have picked particular values for the depreciation rates δ_K and δ_H. Having specified these values, we can draw additional axes, with the origin at (δ_K, δ_H), representing γ_K and γ_H for the OTF and ρ_K and ρ_H for the FPF. Assuming that both δ_K and δ_H are positive, as we have in the diagram, this second set of axes lies above and to the right of the originals.

The equilibrium conditions (2.2.10) and (2.2.11) can both be represented as 45 degree lines from the origin of this second set of axes. The intersection, then, of this 45 degree line with the OTF and FTF represents the unique equilibrium balanced growth point on this diagram. This then *determines* the equilibrium rate of return on both factors ρ^*, and the model is solved. Algebraically, substitution of (2.2.11) into (2.2.8) yields one equation in one unknown: ρ^*:

$$\rho^* + \delta_H = f\left(f'^{-1}(\rho^* + \delta_K)\right) - (\rho^* + \delta_K) f'^{-1}(\rho^* + \delta_K) \qquad (2.2.12)$$

With ρ^* determined, then k^* can be determined through equation (2.1.13):

$$\rho^* + \delta_K = f'(k^*) \qquad (2.2.13)$$

Similarly, substitution of (2.2.10) into the OTF (2.2.6') yields a unique solution for γ^*:

$$\gamma^* + \delta_H = f\left(f'^{-1}(\gamma^* + \delta_K)\right) - (\gamma^* + \delta_K) f'^{-1}(\gamma^* + \delta_K) \qquad (2.2.14)$$

Thus, the equilibrium growth rate is now determined. This is the von Neumann growth rate and, in general, it represents the maximal balanced growth rate possible in balanced endogenous growth models.

One final point to notice in this case, which can be seen clearly by comparing equations (2.2.12) and (2.2.14), is that the equilibrium growth rate and net return to assets are equal:

$$\gamma^* = \rho^* \qquad (2.2.15)$$

Case 2: Positive Consumption

We now consider the more common case where consumption is not set equal to zero, but is determined by a savings decision (yet to be specified). Figure 6 illustrates this case.

[Figure 6 about here.]

This figure is identical to Figure 5 except for one crucial detail: the OTF and the FPF no longer coincide. The axes remain the same as in Figure 5, so the introduction of a positive consumption value shifts the OTF downwards by the amount *c*.

As in Case 1, the intersection of the 45 degree line with the FPF (i.e., equations (2.2.8) and (2.2.11)) uniquely determines ρ^* and, through (2.2.13), k^*. That is, *the equilibrium values of the rates of return and the mix of capital to human capital are independent of the savings decision*. These are determined purely by the production technology, the firm's optimization conditions, and the no-arbitrage condition, as in the von Neumann model.

Here, however, while the intersection of the 45 degree line and the OTF does play a crucial role in the determination of the growth rate, it does not fully determine this rate. We can see this by substitution of (2.2.10) into (2.2.6):

$$(c+\gamma^* + \delta_H)^* = f\left(f'^{-1}(\gamma^* + \delta_K)\right) - (\gamma^* + \delta_K) f'^{-1}(\gamma^* + \delta_K) \equiv G(\gamma^* + \delta_K) \quad (2.2.16)$$

Equation (2.2.16) is one equation in two unknowns: γ^* and c. To close this model, we need to specify some sort of consumption/savings decision. Thus, *in this case*, *the consumption/savings decision influences the growth rate*. This is illustrated, in Figure 6, by the fact that γ^* is determined by the intersection of the OTF and the 45 degree line, where the OTF is drawn for some positive value of c.

If, for example, the savings rate is fixed by some parameter *s*, consumption per unit of *H* is given, as in the previous section, by $c = (1-s)f(k)$. With k^* determined in (2.2.12), we then determine:

$$c^* = (1-s)f(k^*) \qquad (2.2.17)$$

Substitution of c^* in (2.2.17) into (2.2.16) then determines the growth rate γ^*.

Alternatively, using the Ramsey-Cass-Koopmans optimal savings structure, we need to re-specify the problem introduced in the previous section. Here, population growth is zero, and we normalize the size of each family to unity. Households can own two assets: *K* or *H*. Each of these assets earns the same net return ρ^*, fixed by (2.2.12). Hence, households are indifferent about the mix of assets, and we can define the amount of assets per household as $Z = K + H$. Household choose a path of consumption $\{C_t\}$, with an implied path of assets $\{Z_t\}$ to solve:

$$\underset{\{Z_t\}}{Max} \quad U_0 = \int_0^\infty \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} \right) e^{-\theta t} dt \qquad (2.2.18)$$

subject to: a) $\dot{Z}_t = \rho_t^* Z_t - C_t$

d) $Z_0 = \overline{Z}_0$ (given)

e) $\lim_{t\to\infty} Z_t e^{-\int_0^t \rho_v^* dv} \geq 0$

Necessary conditions for this problem imply the Euler equation:

$$\dot{C}/C = (\rho^* - \theta)/\sigma \qquad (2.2.19)$$

With balanced growth, we have the additional condition:

$$\dot{C}/C \equiv \gamma_C = \gamma^* \tag{2.2.20}$$

Thus, substituting (2.2.20) into (2.2.19), we have determined the balanced growth rate:

$$\gamma^* = (\rho^* - \theta)/\sigma \tag{2.2.21}$$

Finally, substitution of (2.2.21) into (2.2.16) determines consumption c^*, and the model is entirely solved.

2.2.4 Dynamic Efficiency

However c^* is determined, *any* positive value of c^* will reduce the growth rate γ^* below it's von Neumann rate of ρ^*. As in the case with zero consumption, ρ^* is determined by equation (2.2.12), rearranged here:

$$\rho^* = f\left(f'^{-1}(\rho^* + \delta_K)\right) - (\rho^* + \delta_K) f'^{-1}(\rho^* + \delta_K) - \delta_H \tag{2.2.22}$$

For any given c^*, the growth rate γ^* is determined by equation (2.2.16), which we re-write as:

$$\gamma^* = f\left(f'^{-1}(\gamma^* + \delta_K)\right) - (\gamma^* + \delta_K) f'^{-1}(\gamma^* + \delta_K) - \delta_H - c^* \tag{2.2.23}$$

Using equations (2.2.22) and (2.2.23), it is straightforward to show that, for any $c^* > 0$, we have:

$$\gamma^* < \rho^* \tag{2.2.24}$$

This point is also comes out very clearly in Figure 6: any positive level of consumption implies $\gamma^* < \rho^*$. This condition, together with equation (2.2.15), covering the case where $c^* = 0$, implies the equilibrium satisfies condition (2.1.26). Hence, *the balanced growth equilibrium in this model is dynamically efficient, regardless of savings behaviour*.[9]

2.3 The Cost Function Approach

Before moving on to consider multi-sector models, we first take a detour by redrafting the one sector models, and deriving the FPF, using cost functions. This simplifies the analysis considerably.

The Neoclassical Model

For a representative firm, define the unit cost function as $m(w,r)$. With competitive pricing, and with goods at the numeraire, we have:

$$m(w,r) = 1 \tag{2.3.1}$$

This is exactly the factor price frontier. In the Cobb-Douglas example, the cost function is given by:

$$m(w,r) = \tilde{A} w^{1-\alpha} r^{\alpha} \tag{2.3.2}$$

Where $\tilde{A} = A^{-1}\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$. Now, using (2.3.2) in (2.3.1) gives us an explicit formulation for the FPF:

$$w = \tilde{A}^{\frac{-1}{1-\alpha}} r^{\frac{-\alpha}{1-\alpha}}$$

[9] This last point was made in King and Ferguson (1993), in a somewhat different setting.

Rearranging:

$$w = (1-\alpha)A\left(\frac{\alpha A}{r}\right)^{\frac{\alpha}{1-\alpha}}$$

This is precisely the same equation as derived above, in (2.1.17).

The Endogenous Growth Model

In this case, the unit cost function is given by

$$m(w,r) = m(\rho^* + \delta_H , \rho^* + \delta_K) \tag{2.3.3}$$

The growth rate ρ^* can be solved immediately from the competitive condition

$$m(\rho^* + \delta_H , \rho^* + \delta_K) = 1 \tag{2.3.4}$$

Similarly, the equilibrium k^*, (known as the “von Neumann ray”) can be found from the following condition:

$$k^* = \frac{m_{r_K}(r_H^*, r_K^*)}{m_{r_H}(r_H^*, r_K^*)} \tag{2.3.5}$$

Where $r_K^* = \rho^* + \delta_K$ and $r_H^* = \rho^* + \delta_H$. In the Cobb-Douglas example, equations (2.3.4) and (2.3.5) become, respectively:

$$\tilde{A}(\rho^* + \delta_H)^{1-\alpha}(\rho^* + \delta_K)^{\alpha} = 1 \tag{2.3.6}$$

$$k^{*} = \frac{\alpha}{1-\alpha}\frac{\rho^{*}+\delta_{H}}{\rho^{*}+\delta_{K}} \quad (2.3.7)$$

In the special case where $\delta_H = \delta_K = \delta$, equations (2.3.6) and (2.3.7) collapse down to, respectively, $\rho^* = \tilde{A}^{-1} - \delta$ and $k^* = \alpha/(1-\alpha)$. This is the case considered in Barro and Sala-i-Martin (2004).

3. MODELS WITH TWO OR MORE SECTORS

With a little modification in the manner of presentation, the above results carry over to neoclassical and endogenous growth models with two or more sectors. The modifications are of two sorts. First, in the two sector endogenous growth models in the literature, the convention has been to treat consumption and investment in physical capital as the joint product of a single *goods* sector. Hence, in the diagrams, in contrast to figures 3-6 (which combine consumption with growth in *H*) consumption will henceforth be combined with growth in *K*. Second, since these are now outputs of different production processes, it is no longer the case that the production of both assets requires the usage of both. In particular, the Uzawa-Lucas model, and all of the other models that map into it, suppose that only *H* is used to produce I_H. As we shall see, these modifications have implications for the shape of the OTF and the FPF, and for the labelling of the axes, but not for the duality relation itself.

3.1 The Neoclassical Two Sector Model

In this section, we consider a standard two-sector neoclassical model, based on Uzawa (1964), and covered in Burmeister and Dobell (1970), Chapter 4. In this model, consumption goods are produced in in one sector (which we give the index 0), investment goods are produced in the other sector (given the index 1), and labour grows exogenously. Both production technologies are neoclassical, and both factors (capital and labour) are able to move costlessly and instantaneously across the sectors.

Expressed in inequality form (to be consistent with the general statement of the duality result presented below) the technology for producing consumption goods is given by:

$$C \leq F_0((1-v)K,(1-u)L) \tag{3.1.1}$$

Where v and u respectively denote the fractions of capital and labour allocated to investment goods production. The technology for producing investment goods is given by:

$$I_K \leq F_1(vK,uL) \tag{3.1.2}$$

As in the single sector model, a constant fraction δ_K of capital depreciates at each moment in time:

$$\dot{K} = I_K - \delta_K K \tag{3.1.3}$$

And the stock of labour grows exogenously at the constant rate $n > 0$:

$$\dot{L} = nL \tag{3.1.4}$$

Now, defining $k \equiv K/L$, $c \equiv C/L$, $\gamma_K \equiv \dot{K}/K$, and $\gamma_L \equiv \dot{L}/L$, substitution of (3.1.3) into (3.1.2) and rearrangement yields the following two equations in per capita terms:

$$c \leq F_0((1-v)k,(1-u)) \tag{3.1.5}$$

$$(\gamma_K + \delta_K)k \leq F_1(vk, u) \tag{3.1.6}$$

Where it is understood that, with balanced growth:

$$\gamma_K = \gamma_L = n \tag{3.1.7}$$

Let *p* denote the price of the investment good relative to the consumption good (which is the numeraire). The gross real rate of return on capital is $r = \rho_K + \delta_K$. Expressed in terms of the consumption good, this gross return is pr. Thus, the competitive price – unit cost relations, for consumption and investment goods respectively, are:

$$1 \leq m_0(w, pr) \tag{3.1.8}$$

$$p \leq m_1(w, pr) \tag{3.1.9}$$

The *fundamental duality theorem of neoclassical growth theory*[10] states that, with suitable qualifications about the technologies, the optimal transformation frontier is mathematically identical to the factor price frontier where:

- Given $\gamma_K + \delta_K$, the optimal transformation $c = G(\gamma_K + \delta_K)$ is the value function for the problem of choosing (k, u, v) to maximize *c* subject to (3.1.5) and (3.1.6), with the inequality constraints $0 \leq (u, v) \leq 1$.

- Given *r*, the factor price frontier $w = V(r)$ is the value function for the problem of choosing *p* to minimize *w* subject to (3.1.8) and (3.1.9).

[10] For a general statement with arbitrary finite numbers of sectors, see *Theorem 10* in Chapter 9 of Burmeister and Dobell (1970).

This theorem implies that the diagrams used in, and the discussion surrounding, the one sector neoclassical model can also be carried over to the two sector model (and, in fact, any finite number of sectors). That is, the OTF and the FPF for this two-sector model are represented in Figures 1 and 2 respectively.[11] To make this as clear as possible, we now consider an example.

The Cobb-Douglas Example

Let the two production technologies take the following respective forms:

$$F_0((1-v)K,(1-u)L) = A((1-v)K)^{\alpha}((1-u)L)^{1-\alpha}, \quad A > 0, \quad \alpha \in (0,1) \qquad (3.1.10)$$

$$F_1(vK,uL) = B(vK)^{\beta}(uL)^{1-\beta}, \quad B > 0, \quad \beta \in (0,1) \qquad (3.1.11)$$

Choosing (k,u,v) to maximize c subject to (3.1.5) and (3.1.6), using (3.1.10) and (3.1.11) with the inequality constraints $0 \le (u,v) \le 1$, we find:

$$u^* = \alpha, \quad v^* = \beta \qquad (3.1.12)$$

And the golden rule *k*:

$$k^* = \left(\frac{B\beta^{\beta}\alpha^{1-\beta}}{\gamma_K + \delta_K}\right)^{\frac{1}{1-\beta}} \qquad (3.1.13)$$

Substitution of (3.1.12) and (3.1.13) into the objective function

$$c = A((1-v)k)^{\alpha}(1-u)^{1-\alpha}$$

yields the OTF:

[11] Although, of course, the precise functional forms for the OTF are different across the one and two sector models. Similarly for the FPF.

$$c = A\alpha^{\alpha}(1-\alpha)^{1-\alpha}B^{\frac{\alpha}{1-\beta}}\beta^{\frac{\alpha\beta}{1-\beta}}(1-\beta)^{\alpha}(\gamma_K+\delta_K)^{\frac{-\alpha}{1-\beta}} \quad (3.1.14)$$

Similarly, to minimize costs, firms choose p so that (3.1.8) and (3.1.9) hold with equality. Using (3.1.10) and (3.1.11), this implies:

$$1 = \tilde{A}w^{1-\alpha}(pr)^{\alpha} \quad (3.1.15)$$

and

$$p = \tilde{B}w^{1-\beta}(pr)^{\beta} \quad (3.1.16)$$

where $\tilde{A} = A^{-1}\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$ and $\tilde{B} = B^{-1}\beta^{-\beta}(1-\beta)^{-(1-\beta)}$. Substitution of (3.1.16) into (3.1.15), and solving for w, gives us the FPF:

$$w = A\alpha^{\alpha}(1-\alpha)^{1-\alpha}B^{\frac{\alpha}{1-\beta}}\beta^{\frac{\alpha\beta}{1-\beta}}(1-\beta)^{\alpha}r^{\frac{-\alpha}{1-\beta}} \quad (3.1.17)$$

Clearly, the OTF in (3.1.14) and the FPF in (3.1.17) have the same formulation.

The Savings Decision and Dynamic Efficiency

Up to this point, we have not specified the form of the savings behaviour in the two-sector model. The above results hold true regardless of the specifics of this behaviour. As in the one-sector neoclassical model, however, closing the model and determining the equilibrium values of the variables requires some sort of savings specification. Analysis of equilibrium behaviour is complicated in multi-sector models such as this, though, by the potential (not present in the one-sector model) for multiple steady states, and even cyclical equilibria.

If all agents save a constant fraction s of their income, a simple sufficient condition for a unique steady state equilibrium is that the capital-labour ratio in the investment good sector be no greater than the its counterpart in the consumption goods sector.[12] In this model, this condition implies: $v \leq u$. Assuming this condition holds, as in the one-sector model, a steady state allocation is inefficient if and only if

$$\rho_K < \gamma_K. \tag{3.1.18}$$

3.1 The Two Sector Endogenous Growth Model

The corresponding two sector endogenous growth model, as presented in Chapter 5 of Barro and Sala-i-Martin (2004), has the same mathematical structure, but with different labels for the outputs of the two sectors. Specifically, human capital (rather than consumption) is the good produced in sector 0, and consumption now comes out of the output produced in sector 1 – which also produces physical capital. The technology for producing gross investment in human capital is given by:

$$I_H \leq F_0((1-v)K,(1-u)H) \tag{3.2.1}$$

Where v and u denote the fractions of physical and human capital allocated to the production of physical capital and consumption. Human capital also depreciates at the rate δ_H . Thus, net investment in human capital is given by:

$$\dot{H} = I_H - \delta_H H \tag{3.2.2}$$

[12] See Burmeister (1980), p. 79. For necessary and sufficient conditions, see Burmeister and Dobell (1970), section 4.4.

Production of physical capital and consumption comes occurs according to the technology:

$$Y \leq F_1(vK, uH) \tag{3.2.3}$$

This output can be used either for consumption C or gross investment in physical capital I_K:

$$Y = C + I_K \tag{3.2.4}$$

Net investment in physical capital $\dot{K}$ is gross investment minus depreciation, where a constant fraction δ_K of capital depreciates at each moment in time:

$$\dot{K} = I_K - \delta_K K \tag{3.2.5}$$

The population is constant, and normalized to unity. Now, defining $k \equiv K/H$, $\tilde{c} \equiv C/K$, $\gamma_K \equiv \dot{K}/K$, and $\gamma_H \equiv \dot{H}/H$, substitution of (3.2.2) into (3.2.1) together with substitution of (3.2.4) and (3.2.5) into (3.2.3) yields the following two equations:

$$\gamma_H + \delta_H \leq F_0((1-v)k, (1-u)) \tag{3.2.6}$$

$$(\gamma_K + \delta_K + \tilde{c})k \leq F_1(vk, u) \tag{3.2.7}$$

With respect to pricing, as in the neoclassical model, we set output from sector 0 (in this case, human capital) to be the numeraire. Let $r_K = p(\rho_K + \delta_K)$ and $r_H = \rho_H + \delta_H$ denote the gross rentals on the two assets. Hence, the competitive price – unit cost relations for the two goods are:

$$1 \le m_0(\rho_H + \delta_H, p(\rho_K + \delta_K)) \tag{3.2.8}$$

$$p \le m_1(\rho_H + \delta_H, p(\rho_K + \delta_K)) \tag{3.2.9}$$

Comparing equations (3.2.6) and (3.2.7) with equations (3.1.5) and (3.1.6), respectively, reveals that they have the precisely the same mathematical structure, but where $(\gamma_H + \delta_H)$ replaces *c*, and $(\gamma_K + \delta_K + \tilde{c})$ replaces $(\gamma_K + \delta_K)$. Similarly, equations (3.2.8) and (3.2.9) have the same structure as (3.1.8) and (3.1.9), but where $(\rho_H + \delta_H)$ replaces *w* and $p(\rho_K + \delta_K)$ replaces pr. This equivalence leads us to the following result.

Theorem: The optimal transformation frontier in this model is mathematically identical to the factor price frontier, where

- Given $\gamma_K + \delta_K + \tilde{c}$, the optimal transformation $\gamma_H + \delta_H = G(\gamma_K + \delta_K + \tilde{c})$ is the value function for the problem of choosing (k, u, v) to maximize $(\gamma_H + \delta_H)$ subject to (3.2.6) and (3.2.7), with the inequality constraints $0 \le (u, v) \le 1$.

- Given $(\rho_K + \delta_K)$, the factor price frontier $\rho_H + \delta_H = V(\rho_K + \delta_K)$ is the value function for the problem of choosing *p* to minimize $(\rho_H + \delta_H)$ subject to (3.2.8) and (3.2.9).

This theorem implies that the diagrams used in, and the discussion surrounding, the one sector endogenous growth model can also be carried over to the two sector model (and, in fact, any finite number of sectors). That is, the OTF and the FPF for this two-sector model are represented in Figures 3 and 4 respectively.

The Cobb-Douglas Example

Let the two production technologies take the following respective forms:

$$F_0((1-v)K,(1-u)H) = A((1-v)K)^{\alpha}((1-u)H)^{1-\alpha}, \quad A>0, \quad \alpha \in (0,1) \qquad (3.2.10)$$

$$F_1(vK,uH) = B(vK)^{\beta}(uH)^{1-\beta}, \quad B>0, \quad \beta \in (0,1) \qquad (3.2.11)$$

Choosing (k,u,v) to maximize $(\gamma_H + \delta_H)$ subject to (3.2.6) and (3.2.7), using (3.2.10) and (3.2.11) with the inequality constraints $0 \le (u,v) \le 1$, we find:

$$u^* = \alpha, \quad v^* = \beta \qquad (3.2.12)$$

and:

$$k^* = \left(\frac{B\beta^{\beta}\alpha^{1-\beta}}{\gamma_K + \delta_K}\right)^{\frac{1}{1-\beta}} \qquad (3.2.13)$$

Substitution of (3.2.12) and (3.2.13) into the objective function

$$\gamma_H + \delta_H = A((1-v)k)^{\alpha}(1-u)^{1-\alpha}$$

yields the OTF:

$$\gamma_H + \delta_H = A\alpha^{\alpha}(1-\alpha)^{1-\alpha}B^{\frac{\alpha}{1-\beta}}\beta^{\frac{\alpha\beta}{1-\beta}}(1-\beta)^{\alpha}(\gamma_K + \delta_K + \tilde{c})^{\frac{-\alpha}{1-\beta}} \qquad (3.2.14)$$

Similarly, to minimize costs, firms choose *p* so that (3.2.8) and (3.2.9) hold with equality. Using (3.2.10) and (3.2.11), this implies:

$$1 = \tilde{A}(\rho_H + \delta_H)^{1-\alpha} (p(\rho_K + \delta_K))^{\alpha}$$

(3.2.15)

and

$$p = \tilde{B}(\rho_H + \delta_H)^{1-\beta} (p(\rho_K + \delta_K))^{\beta} \qquad (3.2.16)$$

where $\tilde{A} = A^{-1}\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$ and $\tilde{B} = B^{-1}\beta^{-\beta}(1-\beta)^{-(1-\beta)}$. Substitution of (3.2.16) into (3.2.15), and solving for $(\rho_H + \delta_H)$, gives us the FPF:

$$\gamma_H + \delta_H = A\alpha^{\alpha}(1-\alpha)^{1-\alpha} B^{\frac{\alpha}{1-\beta}} \beta^{\frac{\alpha\beta}{1-\beta}} (1-\beta)^{\alpha} (\rho_K + \delta_K)^{\frac{-\alpha}{1-\beta}} \qquad (3.2.17)$$

Clearly, the OTF in (3.2.14) and the FPF in (3.2.17) have the same formulation.

Balanced Growth

As in the neoclassical model, balanced growth implies that the growth rates of the two assets are the same:

$$\gamma_K = \gamma_H = \gamma^* \qquad (3.2.18)$$

We also have the no-arbitrage condition: the net rates of return on the two assets must also be equalized:

$$\rho_K = \rho_H = \rho^* \qquad (3.2.19)$$

As with the one-sector endogenous growth model, here we consider two different cases: with zero consumption (the von Neumann case), and with positive consumption.

Case 1: Zero Consumption (the von Neumann Case)

In this case, the OTF simplifies down to: $\gamma_H + \delta_H = G(\gamma_K + \delta_K)$. The diagrammatic analysis of this case is identical to the one given, for the one-sector endogenous growth model in Section 2.2.3 above. That is, Figure 5 illustrates both the OTF and the FPF. The intersection of the 45 degree line, (representing conditions (3.2.18) and (3.2.19)) with the FPF ($\rho_H + \delta_H = V(\rho_K + \delta_K)$) determines the equilibrium net rate of return ρ^*, and with the OTF ($\gamma_H + \delta_H = G(\gamma_K + \delta_K)$) determines the equilibrium (von Neumann) growth rate γ^*. Moreover:

$$\gamma^* = \rho^* \tag{3.2.20}$$

With ρ^* determined, then p^* can be determined through either cost equation (3.2.15) or (3.2.16).

Case 2: Positive Consumption

We now consider the more general case where consumption is not set equal to zero, but is determined by a savings decision. As in the one-sector endogenous growth model of Section 2.2, Figure 6 illustrates this case. The axes remain the same as in Figure 5, so the introduction of a positive consumption value shifts the OTF leftwards by the amount $\tilde{c}$.

The intersection of the 45 degree line with the FPF uniquely determines ρ^* and k^*. That is, the equilibrium values of the rates of return and the mix of capital to human capital are independent of the savings decision. These are determined purely by the production technology, the firm's optimization conditions, and the no-arbitrage condition. As in the one-sector model, however, the intersection of the 45 degree line and the OTF does not entirely determine the growth rate. We are left with one equation in two unknowns (γ and $\tilde{c}$):

$$\gamma + \delta_H = G(\gamma + \delta_K + \tilde{c}) \tag{3.2.21}$$

To close this model, we need to specify some sort of consumption/savings decision. Thus, in this case, the consumption/savings decision influences the growth rate.

Regardless of the precise specification of the savings decision, however, it is clear from Figure 6 (and equation (3.2.21)) that any positive level of consumption will reduce the growth rate below it's von Neumann value of ρ^*. Thus, we have, with positive consumption:

$$\gamma^* < \rho^* \tag{3.2.20}$$

Hence, the balanced growth equilibrium is dynamically efficient.

4. CONCLUSIONS

The fundamental duality theorem, concerning the optimal transformation and factor price frontiers, identified in the neoclassical models many years ago, extends quite naturally to endogenous growth models that admit balanced growth paths. As such, this theorem applies to a very wide and important class of models, and helps us to understand the relationships between key variables in the models. Interestingly, the equivalence identified in the theorem also extends across models and helps us to recognize both common features and distinctions across these models.

One key distinction between neoclassical and endogenous growth models that comes out very clearly from this analysis is the role of savings in these different models. In the neoclassical model, with long run growth given exogenously, savings plays a role in the determination of the capital labour ratio, and hence, factor prices and dynamic efficiency. In endogenous growth models, savings play no role at all in the determination of the capital labour ratio, or factor prices, or the dynamic efficiency of the balanced growth equilibrium – these variables are determined purely by the production side of the economy. In these models, the macroeconomic role of savings is to determine the rate of growth.

Perhaps the most important result that comes out of this analysis is that the key source of inefficiency in balanced growth models is the mispricing of one or more factors of production. In the neoclassical model, this mispricing arises because labour is not produced according to competitive market forces. In endogenous growth models, externalities associated with knowledge or human capital can play a similar role. In the context of these models, a role for policy may exist to correct for these distortions. However, particularly in the case of the neoclassical model, it may be worthwhile to reconsider if any important ingredients are missing from the model itself which, in a more general model, would move us away from what might be seen as an odious conclusion.

REFERENCES

Aghion, P., and P. Howitt (1992), "A Model of Growth Through Creative Destruction", *Econometrica*, 60, 323-51.

Barro, R., and X. Sala-i-Martin (2004), *Economic Growth* (2^{nd} Ed.), MIT Press, Cambridge.

Bruno, M., (1969), "Fundamental Duality Relations in the Pure Theory of Capital and Growth", *Review of Economic Studies*, 36(1), 39-53.

Burmeister, E., (1980), *Capital Theory and Dynamics*, Cambridge University Press, Cambridge.

Burmeister, E., and R. Dobell (1970), *Mathematical Theories of Economic Growth*, MacMillan, London.

Burmeister, E., and K. Kuga (1970a), "The Factor-Price Frontier in a Neoclassical Multisector Model", *International Economic Review*, 11, 162-174.

Burmeister, E., and K. Kuga (1970b), "The Factor-Price Frontier, Duality and Joint Production", *Review of Economic Studies*, 37, 11-19.

Cass, D., (1965) "Optimum Growth in an Aggregative Model of Capital Accumulation", *Review of Economic Studies*, 32, 233-240.

Cass, D., (1972) "On Capital Overaccumulation in the Aggregative Neoclassical Model of Economic Growth: A Complete Characterization", *Journal of Economic Theory*, 4, 200-223.

De la Croix, D., and P. Michel (2002), *A Theory of Economic Growth*, Cambridge University Press, Cambridge.

Ferguson, D., (1994) "Back to the Future of Growth", University of Victoria manuscript.

Gale, D., and R. Rockwell (1975) "Note and Comment of the Interest Rate Theorem of Malinvaud and Starrett", *Econometrica*, 43, 347-359.

Jones, L., and R. Manuelli (1997) "The Sources of Growth", *Journal of Economic Dynamics and Control*, 21, 75-114.

King, I., and D. Ferguson (1993), "Dynamic Inefficiency, Endogenous Growth, and Ponzi Games", *Journal of Monetary Economics*, 32, 79-104.

Koopmans, T., (1965) "On the Concept of Optimal Economic Growth", in *The Econometric Approach to Development Planning*, North Holland, Amsterdam.

Kydland, F., and E. Prescott (1982) "Time to Build and Aggregate Fluctuations", *Econometrica*, 50, 1345-1370.

Lucas, R., (1988) "On the Mechanics of Economic Development", *Journal of Monetary Economics*, 22, 3-42.

Ramsey F., (1928) "A Mathematical Theory of Saving", *Economic Journal*, 38, 543-559.

Romer, P., (1986) "Increasing Returns and Long-Run Growth", *Journal of Political Economy*, 94, 1002-1037.

Solow, R., (1956) "A Contribution to the Theory of Economic Growth", *Quarterly Journal of Economics*, 70, 65-94.

Swan, T., (1956) "Economic Growth and Capital Accumulation", *Economic Record*, 32, 334-361.

Uzawa, H., (1964) "Optimal Growth in a Two-Sector Model of Capital Accumulation", *Review of Economic Studies*, 31, 1-24.

Uzawa, H., (1965) "Optimal Technical Change in an Aggregative Model of Economic Growth", *International Economic Review*, 6, 18-31.

von Neumann, J., (1937) "Über ein ökonomisches Gleichungs-System und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes" in *Ergebnisse eines Mathmatischen Kolloquiums*, ed. by K. Menger, no. 8, (translated as "A Model of General Equilibrium" in *Review of Economic Studies*, 13(1), 1945-1946, 1-9.)

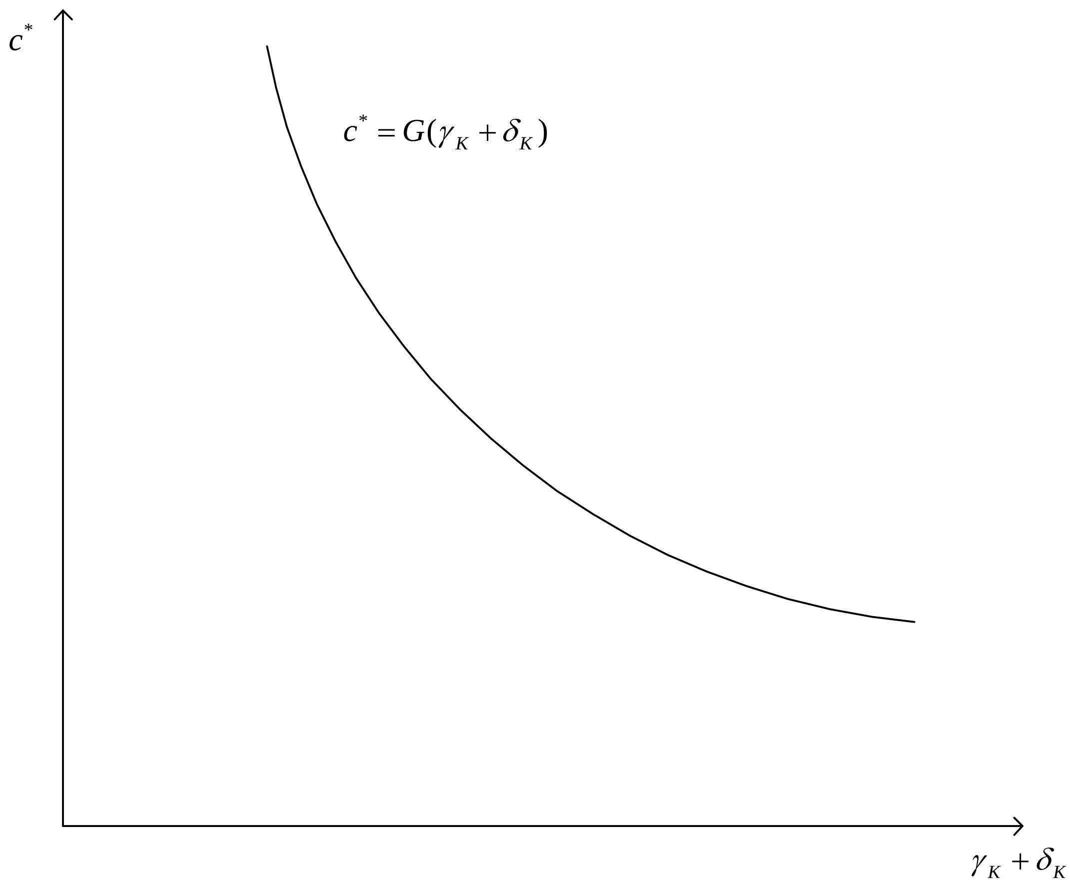

FIGURE 1

The OTF in the One-Sector Neoclassical Model

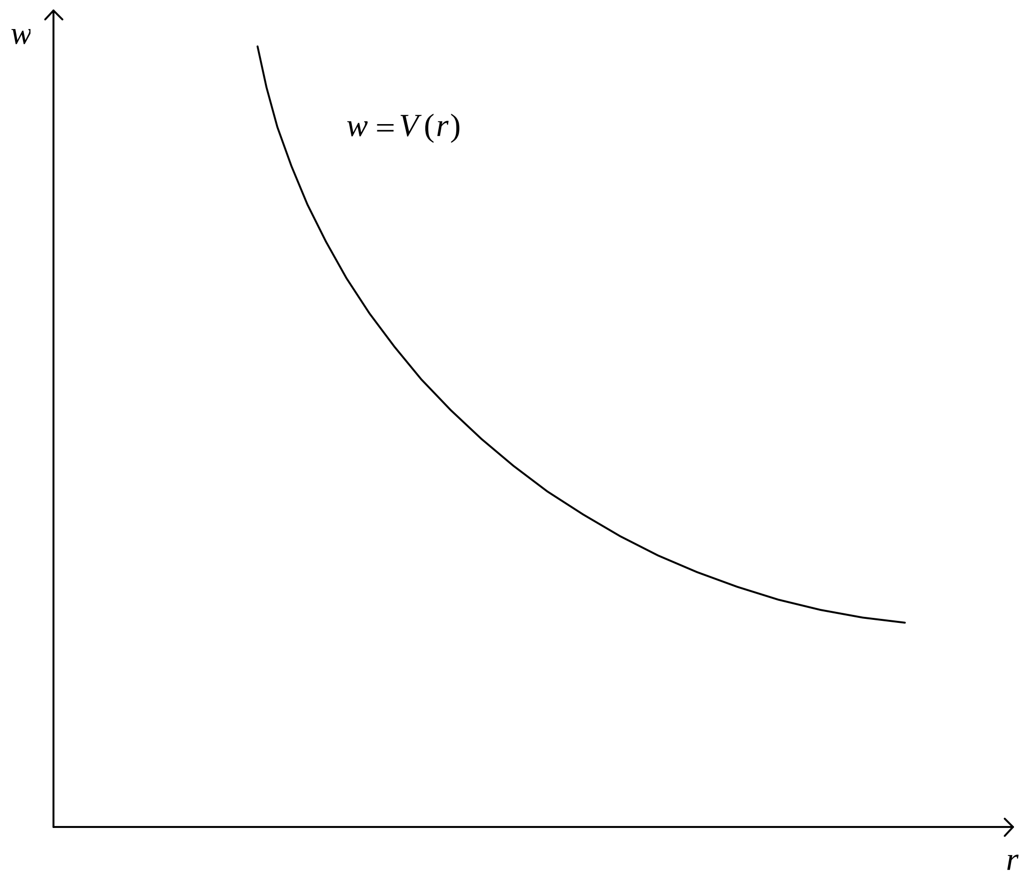

FIGURE 2

The FPF in the One-Sector Neoclassical Model

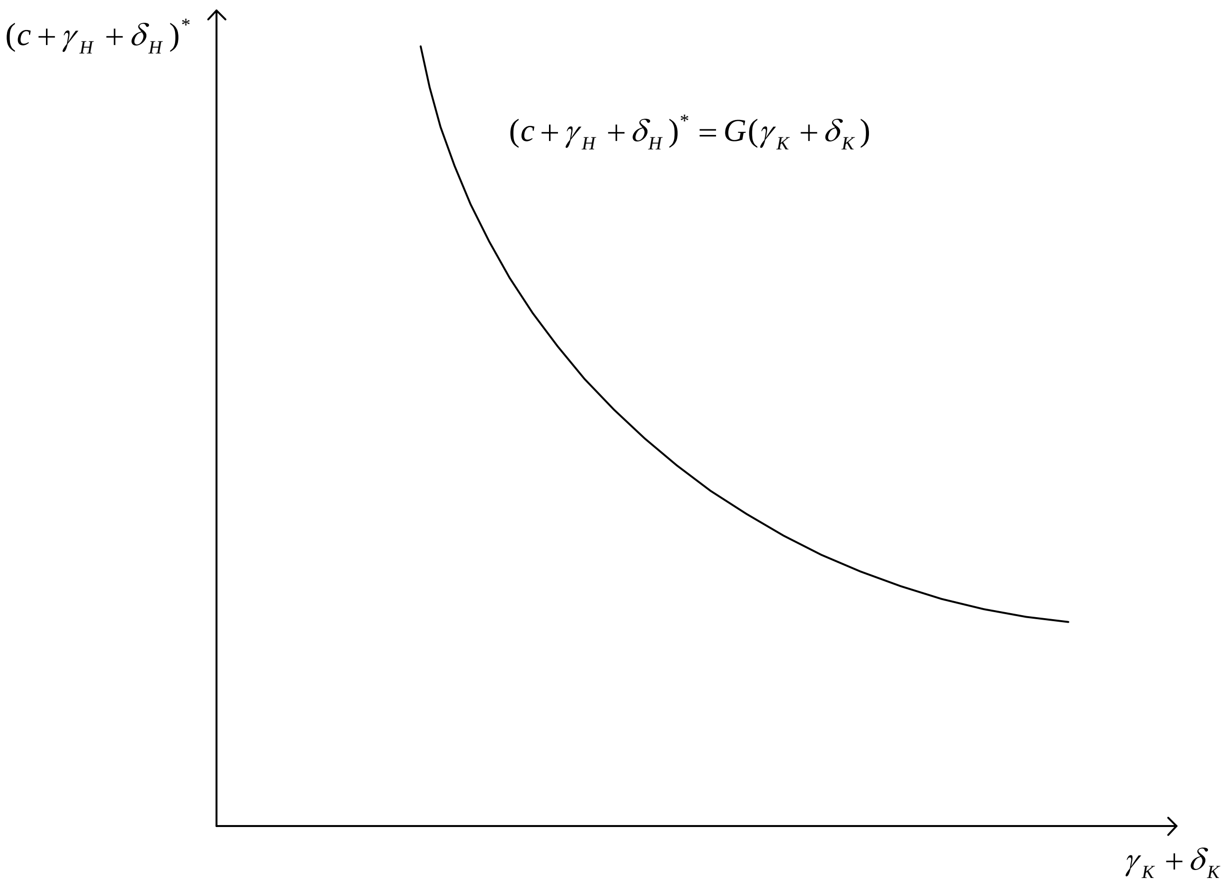

FIGURE 3

The OTF in the Simple Endogenous Growth Model

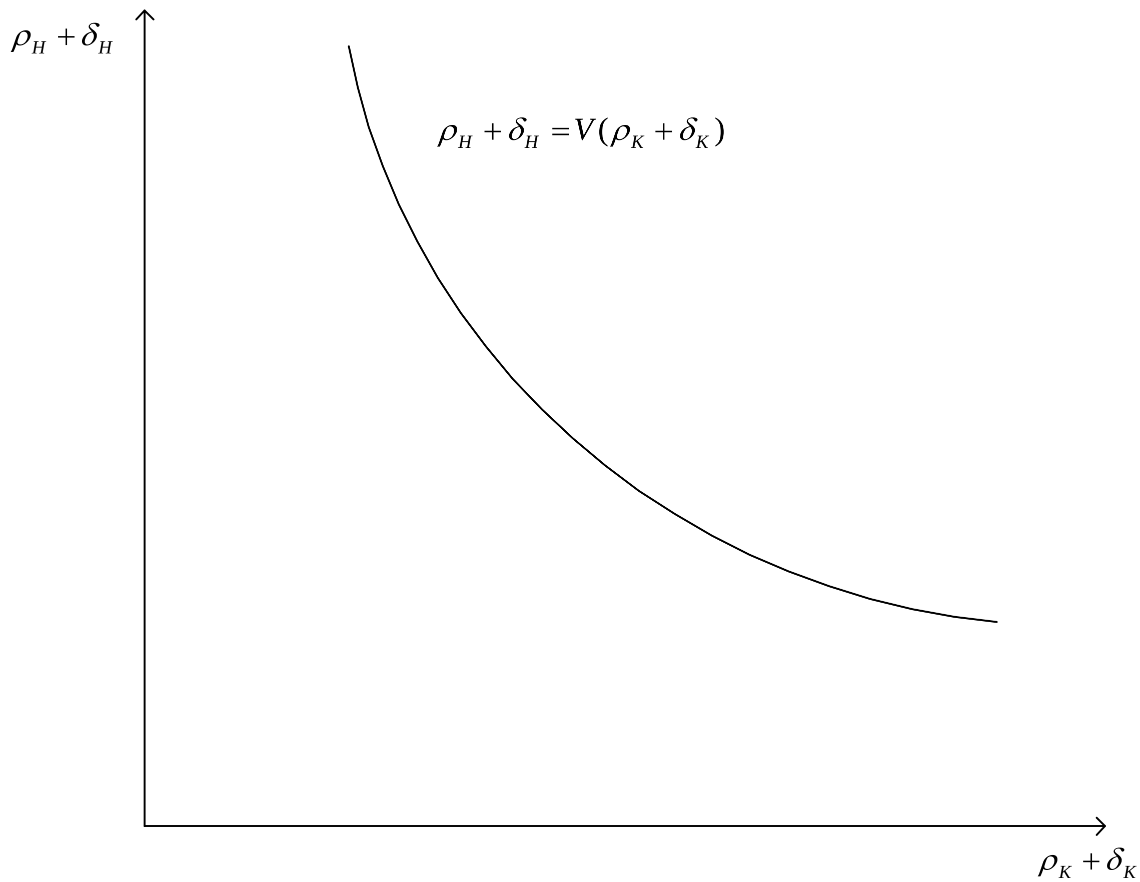

FIGURE 4
The FPF in the Simple Endogenous Growth Model

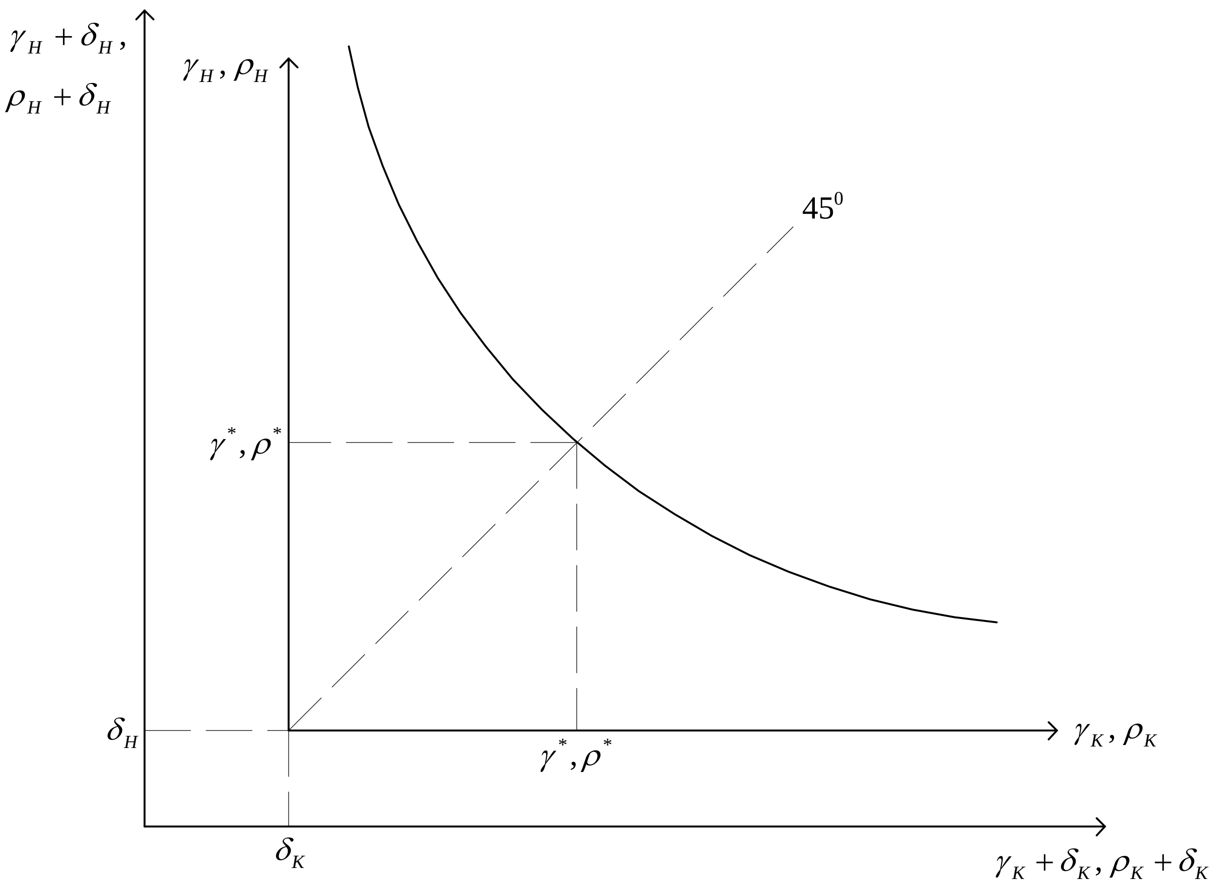

FIGURE 5

Balanced Growth Equilibrium in the Von Neumann Model

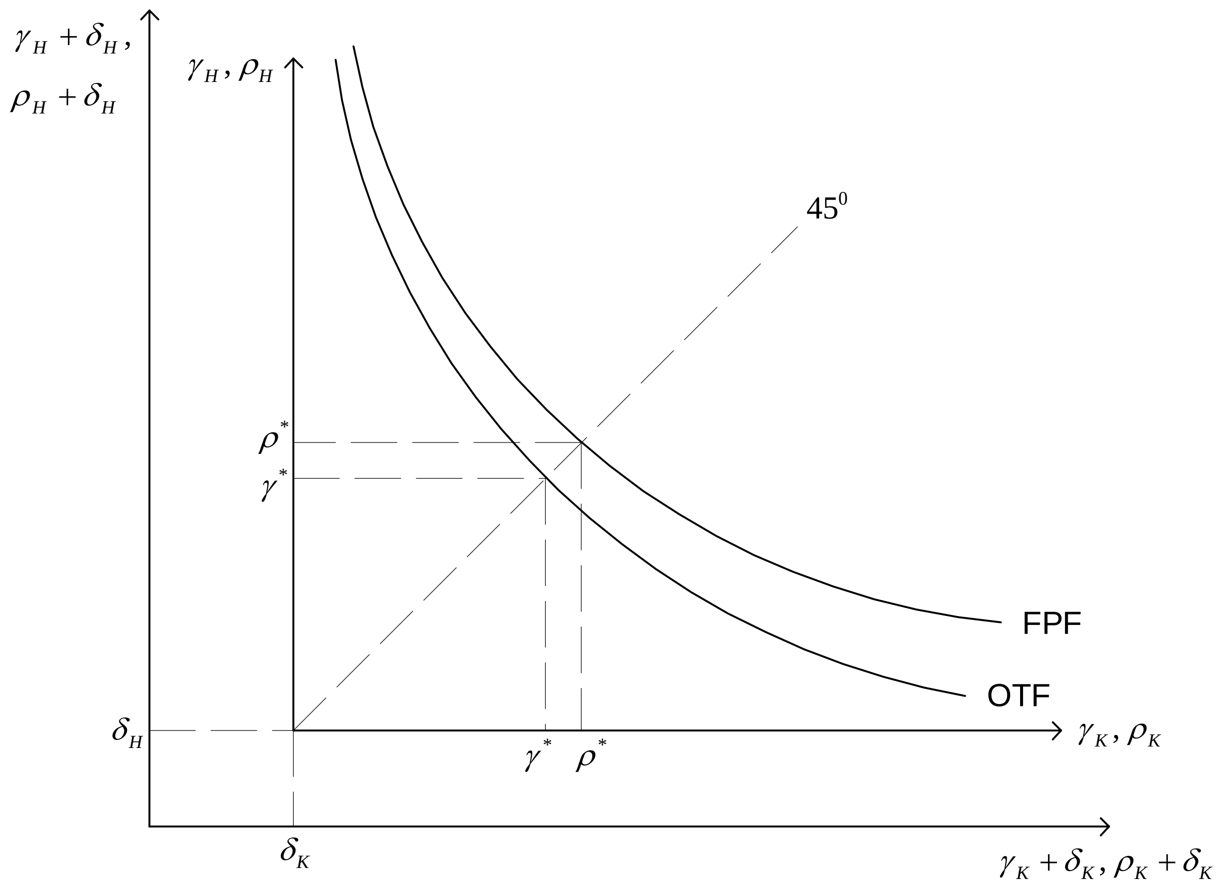

FIGURE 6

Balanced Growth Equilibrium in the Von Neumann Model with Positive Consumption