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Takashi Kamihigashi*
John Stachurski†

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* Research Institute for Economics and Business Administration, Kobe University, Rokkodai, Nada, Kobe 657-8501 Japan. Email: tkamihig@rieb.kobe-u.ac.jp

† Research School of Economics, Australian National University, ACT, Australia. Email: john.stachurski@anu.edu.au

Stability of Stationary Distributions in Monotone Economies*

Takashi Kamihigashi† and John Stachurski‡

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Abstract

This paper generalizes the sufficient conditions for stability of monotone economies and time series models due to Hopenhayn and Prescott (*Econometrica,* 60, p. 1387–1406, 1992). We introduce a new order-theoretic mixing condition and characterize stability for monotone economies satisfying this condition. We also provide a range of results that can be used to verify our mixing condition in applications, as well as the other components of our main stability theorem. Through this approach, we extend Hopenhayn and Prescott's method to a significantly larger class of problems, and develop new perspectives on the causes of instability and stability.

Keywords: Stochastic stability, order reversing, mixing conditions

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†Research Institute for Economics and Business Administration, Kobe University, Rokkodai, Nada, Kobe 657-8501 Japan. Email: tkamihig@rieb.kobe-u.ac.jp

‡Research School of Economics, Australian National University, ACT, Australia. Email: john.stachurski@anu.edu.au

1 Introduction

This paper is concerned with stochastic stability of dynamic economic systems. Stability analysis plays a role in modeling and estimation for a great variety of theoretical and quantitative problems. For example, in time series econometrics, stationarity and ergodicity are closely tied to the limit theorems required for consistency and asymptotic normality of estimators using correlated data. In calibration exercises, stationarity is used to draw comparisons between observed and simulated moments. In models of renewable resource exploitation, stability is associated with sustainable exploitation. In growth and development theory, stability may be identified with long-run convergence, or the absence of poverty traps.

For linear stochastic models, the stability problem is trivial. For nonlinear models, the same problem is much harder. Moreover, in almost all cases, linearization provides little insight, as stability or instability of the linearized system is largely irrelevant in the stochastic case. On the other hand, treating stability of the original nonlinear model is technically challenging. For example, the standard irreducibility-based approach used in Markov process theory (see, e.g., Meyn and Tweedie, 2009) meshes well with some but not all economic applications.

In response, economists have developed an alternative approach, initiated by the seminal contribution of Razin and Yahav (1979). Razin and Yahav introduced a new condition, now called the monotone mixing condition (MMC), and showed that the MMC implies global stability for monotone and suitably continuous Markov processes evolving on an interval of $\mathbb{R}$. Stokey and Lucas (1989) then generalized this result to multiple dimensions. Their result was in turn generalized by Hopenhayn and Prescott (1992), who showed, by an application the Knaster-Tarski fixed point theorem, that the continuity assumption can be dropped without changing the conclusion.

Their results were significant advances both to economic theory and to Markov process theory, and have been used to establish existence, uniqueness and stability of stochastic equilibria in a wide range of applications. Their techniques were applied to the classical one-sector stochastic optimal growth model by Hopenhayn and Prescott (1992), to a stochastic endogenous growth model by de Hek (1999), to a stochastic small open economy by Chatterjee and Shukayev (2010), and to a stochastic overlapping generations model with a nonconcave production function by Morand and Reffett (2007). They have been used to analyze wealth distributions in a variety of contexts which feature imperfection in credit, insurance, or capital markets. Huggett (1993) used their results to analyze the wealth distribution in an incomplete-market economy with infinite-lived agents.[1] The same methodology has been applied to variants of Huggett's model with features such as habit formation (Díaz et al., 2003), endogenous labor supply (Joseph and Weitzenblum, 2003; Pijoan-Mas, 2006), endogenous labor supply and capital accumulation (Marcet et al., 2007), and international trade (Portes, 2009). Their result has been used in a wide range of OLG models with features such as credit rationing (Aghion and Bolton, 1997; Piketty, 1997), human capital (Owen and Weil, 1998; Lloyd-Ellis, 2000; Cardak, 2004; Couch and Morand, 2005; Cabrillana, 2009), international trade (Ranjan, 2001; Das, 2006), and occupational choice (Lloyed-Ellis and Bernhardt, 2000; Antunes and Cavalcanti, 2007; Antunes et al., 2008)

Finally, other applications of the MMC in the literature include variants of Hopenhayn and Rogerson's (1993) model of job turnover (Cabrales and Hopenhayn, 1997; Samaniego, 2008) as well as variants of Hopenhayn's (1993) model of entry and exit (Cooley and Quadrini, 2001; Samaniego, 2006)

In this paper we provide a new theorem that generalizes the global stability result of Hopenhayn and Prescott (1992). We do this by first introducing a new mixing condition called "order reversing," which is considerably weaker than the MMC. We also relax the restriction that the state space be compact and order bounded. In this setting, our theorem provides conditions for monotone, order reversing processes to attain global stability. These conditions are also necessary, and hence we completely characterize global stability for monotone, order reversing processes.

[1] See Kam and Lee (2011) for a recent extension of Huggett's (1993) analysis.

One reason that Hopenhayn and Prescott's 1992 theorem has not been extended until now is that their proof of the existence of a stochastic steady state uses the Knaster-Tarski fixed point theorem, and for many kinds of more general state spaces this theorem cannot be applied, since a chain in the space of distributions need not have a supremum (even in the one dimensional case). Our fixed point argument is new, combining order-theoretic and topological results to obtain existence of the stochastic steady state.[2]

Our results can be used to establish stability in a wider set of applications. In addition, our results provide new perspectives on the problem of stability. For example, one feature of the previous literature was a restriction to compact state spaces, which in turn requires that shocks are bounded. This seems to suggest that small shocks are necessary for stability, or, conversely, that large shocks are destabilizing. Our results suggest that the opposite is true. Large shocks tend to be stabilizing, in the sense that global stability becomes a more likely outcome when large shocks are present. Although this appears counterintuitive, the reason is that, provided that the fundamental forces acting on the state vector are inherently stabilizing (e.g., low discount rates, diminishing returns, etc.), large shocks generate mixing, and mixing is a key component of stochastic stability.

Concerning related literature, other important contributions to the dynamic properties of monotone Markov models were provided by Dubins and Freedman (1966), Bhattacharya and Lee (1988) and Bhattacharya and Majumdar (2001), who studied stability in the monotone setting via a "splitting condition," defined in terms of an ordering on the state space. As shown in section 2.1, this condition is stricter than order reversing. At the same time, the literature on splitting contains important results not treated in this paper.

The rest of the paper is structured as follows. Section 2 reviews basic definitions concerning Markov processes, and introduces the concept of order reversing. Section 3 states the main result (theorem 3.1), and compares it to

[2]To verify the conditions of our fixed point argument, we draw on a theorem recorded in the technical note of Kamihigashi and Stachurski (2011).

earlier results based on the MMC. Section 4 provides sufficient conditions for order reversing, and other results useful for checking the conditions of theorem 3.1. Section 5 gives applications and section 6 concludes.

2 Preliminaries

At each point in time $t = 0, 1, \ldots$, the state of the economy is described by a vector X_t in state space $S \subset \mathbb{R}^n$. An order interval of S is a set of the form $[a, b] := \{x \in S : a \leq x \leq b\}$, where $\leq$ is the standard partial order on $\mathbb{R}^n$. A subset A of S is called *order bounded* if there exists points a and b in S such that $A \subset [a, b]$. As usual, A is called *precompact* there exists a compact $K \subset S$ such that $A \subset K$.

In this paper, we take S to be such that the order bounded subsets of S and the precompact subsets of S coincide. For example, this is the case if $S = \mathbb{R}^n$, because, for a subset of $\mathbb{R}^n$, precompactness and order boundedness are both equivalent to boundedness. Furthermore, the order bounded subsets of S and the precompact subsets of S will coincide when S is $\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$, an order interval of $\mathbb{R}^n$, or, indeed, any convex sublattice of $\mathbb{R}^n$.

Remark 2.1. The main results of this paper remain valid for more general state spaces and partial orders. The details are left to the proofs (section 7) in order to simplify the exposition.

Let $\mathscr{B}$ be the Borel subsets of S, and let $\mathscr{P}$ be the set of probability measures on $(S, \mathscr{B})$. Furthermore, let

- cbS be the continuous bounded functions from S to $\mathbb{R}$, and
- ibS be the set of increasing[3] bounded measurable functions from S to $\mathbb{R}$.

We use inner product notation to represent integration, so that, for example, if $\mu \in \mathscr{P}$ and $h \in ibS \cup cbS$, then

$$\langle \mu, h \rangle := \int h(x) \mu(dx).$$

[3]Function $f \colon S \to \mathbb{R}$ is increasing if $f(x) \leq f(y)$ whenever $x \leq y$.

We use the standard definitions of convergence in distribution and stochastic domination: For $\{\mu_n\}_{n=0}^{\infty} \subset \mathscr{P}$, the statement $\mu_n \to \mu_0$ means that $\langle \mu_n, h \rangle \to \langle \mu_0, h \rangle$ for all $h \in cbS$; while $\mu_1 \preceq \mu_2$ means that $\langle \mu_1, h \rangle \leq \langle \mu_2, h \rangle$ for all $h \in ibS$.

Throughout the paper, we suppose that the model under consideration is time-homogeneous and Markovian. The dynamics of such a model can be summarized by a stochastic kernel (or transition probability function) Q, where $Q(x, B)$ represents the probability that the state moves from $x \in S$ to $B \in \mathscr{B}$ in one unit of time. As usual, we require that $Q(x, \cdot) \in \mathscr{P}$ for each $x \in S$, and that $Q(\cdot, B)$ is measurable for each $B \in \mathscr{B}$.

Given $\mu \in \mathscr{P}$ and stochastic kernel Q, an S-valued stochastic process $\{X_t\}$ is called *Markov-*(Q, μ) if X_0 has distribution μ and $Q(x, \cdot)$ is the conditional distribution of X_{t+1} given $X_t = x$.[4] If μ is the probability measure $\delta_x \in \mathscr{P}$ concentrated on $x \in S$, we call $\{X_t\}$ Markov-(Q, x). Finally, we call $\{X_t\}$ Markov-Q if $\{X_t\}$ is Markov-(Q, μ) for some $\mu \in \mathscr{P}$.

Example 2.1. Many economic models result in processes for the state variables represented by nonlinear, vector-valued stochastic difference equations. As a generic example, consider the S-valued process

$$X_{t+1} = F(X_t, \xi_{t+1}), \qquad \{\xi_t\} \overset{\text{IID}}{\sim} \phi, \qquad X_0 = x \in S, \tag{1}$$

where $\{\xi_t\}$ takes values in $Z \subset \mathbb{R}^m$, the function $F\colon S \times Z \to S$ is measurable, and ϕ is a probability measure on the Borel sets of Z. Let Q_F be the kernel

$$Q_F(x, B) := \mathbb{P}\{F(x, \xi_t) \in B\} = \phi\{z \in Z : F(x, z) \in B\}. \tag{2}$$

Then $\{X_t\}$ in (1) is Markov-(Q_F, x).[5]

For each $t \in \mathbb{N}$, let Q^t be the t-th order kernel, defined by

$$Q^1 := Q, \quad Q^t(x, B) := \int Q^{t-1}(y, B) Q(x, dy) \qquad (x \in S,\ B \in \mathscr{B}).$$

[4] More formally, $\mathbb{P}[X_{t+1} \in B \mid \mathscr{F}_t] = Q(X_t, B)$ almost surely for all $B \in \mathscr{B}$, where $\mathscr{F}_t$ is the σ-algebra generated by the history $X_0, \ldots, X_t$. Here and below, we take an underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ as given.

[5] Although the process (1) is only first order, models including higher order lags of the state and shock process can be rewritten in the form of (1) by redefining the state variables.

Here $Q^t(x, B)$ represents the probability of transitioning from x to B in t steps.

A sequence $\{\mu_n\} \subset \mathscr{P}$ is called *tight* if, for all $\epsilon > 0$, there exists a compact $K \subset S$ such that $\mu_n(K) \geq 1 - \epsilon$ for all n. A stochastic kernel Q is called *bounded in probability* if $\{Q^t(x, \cdot)\}_{t \in \mathbb{N}}$ is tight for all $x \in S$. (Intuitively, for any initial condition, the entire sequence of distributions is almost supported on a single compact set, and probability mass does not diverge as $n \to \infty$.)

Given $\mu \in \mathscr{P}$, we let $\mu Q \in \mathscr{P}$ be the probability measure

$$(\mu Q)(B) := \int Q(x, B)\mu(dx) \qquad (B \in \mathscr{B}). \tag{3}$$

We regard (3) as defining an operator $\mu \mapsto \mu Q$ from $\mathscr{P}$ to itself. The interpretation of the operation $\mu \mapsto \mu Q$ is that it shifts the distribution for the state forward by one time period. In particular, if $\{X_t\}$ is Markov-(Q, μ), then μQ^t is the distribution of X_t.

For any bounded measurable function $h \colon S \to \mathbb{R}$ we define

$$Qh(x) := \int h(y)Q(x, dy) \qquad (x \in S).$$

It is known that this operator $h \mapsto Qh$ and the operator $\mu \mapsto \mu Q$ are adjoint, in the sense that, for any such h and any $\mu \in \mathscr{P}$, we have $\langle \mu, Qh \rangle = \langle \mu Q, h \rangle$ (see, e.g., Stokey and Lucas, 1989, p. 219). Also, it can be shown that if Q_F is the kernel in (2), then $Q_F h(x) = \int h[F(x, z)]\phi(dz)$.

If $\mu^* \in \mathscr{P}$ and $\mu^* Q = \mu^*$, then μ^* is called *stationary* for Q. If Q has a unique stationary distribution μ^* in $\mathscr{P}$, and, moreover, $\mu Q^t \to \mu^*$ as $t \to \infty$ for all $\mu \in \mathscr{P}$, then Q is called *globally stable*. In this case, μ^* is naturally interpreted as the long-run equilibrium of the economic system in question. If μ^* is stationary, then any Markov-(Q, μ^*) process $\{X_t\}$ is strict-sense stationary with $X_t \sim \mu^*$ for all t.[6]

If Q satisfies $\mu Q \preceq \mu' Q$ whenever $\mu \preceq \mu'$, then Q is called *increasing*. Two equivalent conditions are that $Qh \in ibS$ whenever $h \in ibS$, and that $Q(x, \cdot) \preceq Q(x', \cdot)$ whenever $x \leq x'$. Typically, Q will be increasing when

[6] In addition, if μ^* is also unique, then $\{X_t\}$ is ergodic in the sense that $\frac{1}{n}\sum_{t=1}^{n} h(X_t) \to \int h \, d\mu^*$ as $n \to \infty$ almost surely whenever $\int |h| \, d\mu^*$ exists.

equilibrium actions are increasing in the state. Many examples of models with increasing kernels were discussed in the introduction. Other examples not discussed there include various infinite horizon optimal growth models, with features such as irreversible investment, renewable resources, distortions, and capital-dependent utility. Increasing kernels are also found in stochastic OLG models besides those mentioned previously, such as models with with limited committment, and in a variety of stochastic games.[7] For a general discussion of increasing kernels in the context of dynamic optimizing models, see Hopenhayn and Prescott (1992). For an empirical test of the same property, see Lee et al. (2009).

Remark 2.2. A set $A \in \mathscr{B}$ is called increasing (resp., decreasing) if its indicator function is increasing (resp., decreasing). If Q is an increasing kernel and A is an increasing (resp., decreasing) set, then the function $x \mapsto Q(x, A)$ is increasing (resp., decreasing).

Example 2.2. Let F and Q_F be as in example 2.1. If $x \mapsto F(x, z)$ is increasing, then Q_F is increasing.[8]

If $\mu \in \mathscr{P}$ and $\mu Q \preceq \mu$, then μ is called *excessive*. If $\mu \preceq Q\mu$, then μ is called *deficient*.

Remark 2.3. If S has a least element a, then δ_a is deficient for any kernel Q, because $\delta_a \preceq \mu$ for every $\mu \in \mathscr{P}$, and hence $\delta_a \preceq \delta_a Q$. Similarly, if S has a greatest element b, then δ_b is excessive for Q.

[7] See, for example, Amir (2002, 2005), Gong et al. (2010), Balbus et al. (2010), Olson (1989), Olson and Roy (2000), Datta et al. (2002) and Mirman et al. (2008).

[8] The statement that $F(\cdot, z)$ is increasing means that $x, x' \in S$ with $x \leq x'$ and $z \in Z$ implies $F(x, z) \leq F(x', z)$. Since $Q_F h(x) = \int h[F(x, z)]\phi(dz)$, to prove that Q_F is increasing, it suffices to show that if $x \leq x'$ and $h \in ibS$, then $\int h[F(x, z)]\phi(dz) \leq \int h[F(x', z)]\phi(dz)$. Since $h \in ibS$ and $F(\cdot, z)$ is increasing for each z, this follows from monotonicity of the integral.

2.1 Order Reversing Processes

In this paper we introduce a new order-theoretic mixing condition and illustrate its close relationship to stability. To state the condition, let

$$\mathbb{G} := \operatorname{graph}(\leq) := \{(y, y') \in S \times S : y \leq y'\},$$

so that $y \leq y'$ iff $(y, y') \in \mathbb{G}$. Also, let Q be a stochastic kernel on S, and consider the product kernel $Q \times Q$ on $S \times S$ defined by

$$(Q \times Q)((x, x'), A \times B) = Q(x, A) Q(x', B) \tag{4}$$

for $(x, x') \in S \times S$ and $A, B \in \mathscr{B}$.[9] The product kernel represents the stochastic kernel of the $S \times S$-valued process $\{(X_t, X_t')\}$ when $\{X_t\}$ and $\{X_t'\}$ are independent Markov-Q processes.

Using this notation, we say that Q is *order reversing* if

$$\forall\, x, x' \in S \text{ with } x \geq x', \;\; \exists\, t \in \mathbb{N} \;\text{ such that }\; (Q \times Q)^t((x, x'), \mathbb{G}) > 0.$$

Here the definition is presented in a way that emphasizes the fact that order reversing is a property of the kernel Q alone (taking S as given). It can be stated more intuitively using different notation. In particular, Q is order reversing if, given any x and x' in S with $x \geq x'$, and given two independent Markov-Q processes $\{X_t\}$ and $\{X_t'\}$ starting at the higher state x and the lower state x' respectively, the initial ordering is reversed at some point in time with positive probability. That is, there exists a t with $\mathbb{P}\{X_t \leq X_t'\} > 0$ for some t.

Remark 2.4. In verifying order reversing, it is clearly sufficient to check the existence of a t with $(Q \times Q)^t((x, x'), \mathbb{G}) > 0$ for *arbitrary* pair $x, x' \in S$. Often this is just as easy, and much of the following discussion proceeds accordingly.

Example 2.3. Suppose we are studying a dynamic model of household wealth. Informally, the model is order reversing, if, for two households receiving idiosyncratic shocks from the same distribution, the wealth of the first household

[9] Sets of the form $A \times B$ with $A, B \in \mathscr{B}$ provide a semi-ring in the product σ-algebra $\mathscr{B} \otimes \mathscr{B}$ that also generates $\mathscr{B} \otimes \mathscr{B}$. Defining the probability measure $Q((x, x'), \cdot)$ on this semi-ring uniquely defines $Q((x, x'), \cdot)$ on all of $\mathscr{B} \otimes \mathscr{B}$. See, e.g., Dudley (2002, theorem 3.2.7).

is less than that of the second at some point in time with non-zero probability, regardless of initial wealth for each of the two households.

Example 2.4. Consider the stochastic kernel $Q(x, B) = \mathbb{P}\{\rho x + \xi_t \in B\}$ on $S = \mathbb{R}$ associated with the linear AR(1) model

$$X_{t+1} = \rho X_t + \xi_{t+1}, \qquad \{\xi_t\} \overset{\text{IID}}{\sim} N(0, 1). \tag{5}$$

This kernel is order reversing. To see this, fix $(x, x') \in \mathbb{R}^2$, and take two independent Markov-Q processes

$$X_{t+1} = \rho X_t + \xi_{t+1} \text{ with } X_0 = x, \qquad X'_{t+1} = \rho X'_t + \xi'_{t+1} \text{ with } X'_0 = x',$$

where $\{\xi_t\}$ and $\{\xi'_t\}$ are IID, standard normal, and independent of each other. We can see that $\mathbb{P}\{X_t \leq X'_t\} > 0$ is satisfied with $t = 1$, because

$$\mathbb{P}\{X_1 \leq X'_1\} = \mathbb{P}\{\rho x + \xi_1 \leq \rho x' + \xi'_1\} = \mathbb{P}\{\xi_1 - \xi'_1 \leq \rho(x' - x)\}.$$

Since $\xi_1 - \xi'_1$ is Gaussian, this probability is strictly positive.

Example 2.5. Order reversing is weaker than the monotone mixing condition (MMC) of Razin and Yahav (1979), Stokey et al. (1989) and Hopenhayn and Prescott (1992). To see this, let $S := \{x \in \mathbb{R}^n : a \leq x \leq b\}$, and let Q be a given kernel on S. In this setting, Q is said to satisfy the MMC whenever

$$\exists\, \bar{x} \in S \text{ and } k \in \mathbb{N} \text{ such that } Q^k(a, [\bar{x}, b]) > 0 \text{ and } Q^k(b, [a, \bar{x}]) > 0. \tag{6}$$

In view of remark 2.2, one implication of (6) is that

$$Q^k(x, [\bar{x}, b]) > 0 \text{ and } Q^k(x, [a, \bar{x}]) > 0 \text{ for all } x \in S. \tag{7}$$

Under the MMC, Q is order reversing. To see this, let $\bar{x}$ and k be as in (6), fix $x, x' \in S$ and let $\{X_t\}$ and $\{X'_t\}$ be independent, Markov-(Q, x) and Markov-(Q, x') respectively. Since $X_k \leq X'_k$ whenever $X_k \leq \bar{x} \leq X'_k$, we have

$$\mathbb{P}\{X_k \leq X'_k\} \geq \mathbb{P}\{X_k \leq \bar{x} \leq X'_k\} = \mathbb{P}\{X_k \leq \bar{x}\}\mathbb{P}\{\bar{x} \leq X'_k\}.$$

Both $\mathbb{P}\{\bar{x} \leq X'_k\}$ and $\mathbb{P}\{X_k \leq \bar{x}\}$ are strictly positive by (7). Hence Q is order reversing.

Example 2.6. Order reversing is also weaker than the "splitting condition" used by Dubins and Freedman (1966), Bhattacharya and Lee (1988) and Bhattacharya and Majumdar (2001). Their environment consists of a sequence of IID random maps $\{\gamma_t\}$ from S to itself. The maps generate $\{X_t\}$ via

$$X_t = \gamma_t(X_{t-1}) = \gamma_t \circ \cdots \circ \gamma_1(X_0).$$

The corresponding stochastic kernel is $Q(x, B) = \mathbb{P}\{\gamma_1(x) \in B\}$. The splitting condition requires existence of a $\bar{x} \in S$ and $k \in \mathbb{N}$ such that

(a) $\mathbb{P}\{\gamma_k \circ \cdots \circ \gamma_1(y) \leq \bar{x}, \ \forall y \in S\} > 0$; and

(b) $\mathbb{P}\{\gamma_k \circ \cdots \circ \gamma_1(y) \geq \bar{x}, \ \forall y \in S\} > 0$.

If the splitting condition holds, then Q is order reversing. To see this, let $\bar{x}$ and k be the constants in the splitting condition, fix $x, x' \in S$ and let $\{X_t\}$ and $\{X'_t\}$ be independent, Markov-(Q, x) and Markov-(Q, x') respectively. As in example 2.5, we have $\mathbb{P}\{X_k \leq X'_k\} \geq \mathbb{P}\{X_k \leq \bar{x}\}\mathbb{P}\{\bar{x} \leq X'_k\}$. Moreover, both terms in this product are positive. For example, $\mathbb{P}\{X_k \leq \bar{x}\} > 0$ because

$$\mathbb{P}\{X_k \leq \bar{x}\} = \mathbb{P}\{\gamma_k \circ \cdots \circ \gamma_1(x) \leq \bar{x}\} \geq \mathbb{P}\{\gamma_k \circ \cdots \circ \gamma_1(y) \leq \bar{x}, \ \forall y \in S\},$$

and the final term is strictly positive by the splitting condition.

Remark 2.5. In a separate technical note, Kamihigashi and Stachurski (2010) use an order mixing condition to establish a certain convergence result that is needed for one component of the proof of our main theorem. This order mixing condition is considerably stricter than order reversing.

3 Main Results

Our main theorem generalizes the well-known stability result of Hopenhayn and Prescott (1992). It provides conditions both necessary and sufficient for global stability of increasing and order reversing Markov processes:

Theorem 3.1. *Let Q be a stochastic kernel that is both increasing and order reversing. Then Q is globally stable if and only if*

1. *Q is bounded in probability, and*
2. *Q has either a deficient distribution or an excessive distribution.*

The kernel Q is called *Feller* if $Qh \in cbS$ whenever $h \in cbS$. If Q is Feller, then condition 2 can be omitted. Since this result is likely to be useful in many applications, we state it as a second theorem.

Theorem 3.2. *Let Q be increasing, order reversing, and Feller. Then Q is globally stable if and only if Q is bounded in probability.*

To illustrate theorem 3.1, we show how it generalizes the stability results of Razin and Yahav (1979), Stokey et al. (1989, theorem 12.12) and Hopenhayn and Prescott (1992, theorem 2). To begin, let $a, b \in \mathbb{R}^n$ with $a \leq b$, and let $S := \{x \in \mathbb{R}^n : a \leq x \leq b\}$. Recall that Q satisfies the MMC whenever (6) holds. Generalizing the earlier results of Razin and Yahav (1979) and Stokey et al. (1989), theorem 2 of Hopenhayn and Prescott (1992, p. 1397) states that if the state space S is of this form and Q is an increasing kernel satisfing the MMC, then Q is globally stable.[10]

Theorem 3.1 further generalizes this result. To see this, suppose that $S = \{x \in \mathbb{R}^n : a \leq x \leq b\}$, Q is increasing and the MMC holds. We now check the conditions of theorem 3.1. First, Q is order reversing, as shown in example 2.5. Second, Q is bounded in probability, because $\{Q^t(x, \cdot)\}$ is always tight. Indeed, $Q^t(x, \cdot)$ is supported on S by definition, and S itself is compact. Finally, Q has a deficient measure, since S has least element a (see remark 2.3).

To see that the conditions of theorem 3.1 are strictly weaker than those of Hopenhayn and Prescott (1992), consider the AR(1) model (5) with $\rho \in [0, 1)$.

[10]Hopenhayn and Prescott's result also holds when S is a compact metric space with closed partial order $\leq$, provided that S contains both a lower bound a and an upper bound b. Our results are likewise valid in this setting. In fact, theorem 3.1 continues to hold if $\leq$ is any closed partial order, S is separable and completely metrizable, order intervals of S are compact, and compact subsets of S are order bounded.

Here the Gaussian shocks force us to choose the state space $S = \mathbb{R}$, rather than an order interval of $\mathbb{R}^n$, and Hopenhayn and Prescott's theorem cannot be applied. On the other hand, all the conditions of theorem 3.1 are satisfied.[11] (Of course the AR(1) model is a trivial example. Nontrivial applications are presented in section 5.)

It is worth noting that in many cases, Hopenhayn and Prescott's theorem can be used if the distribution of the shocks is truncated, or, more generally, if the support of the shocks was chosen to be bounded. At first pass, the major implication of our results appears to be that restricting the support of the shock in this way is unnecessary. However, our results are, in a sense, more significant than this. For example, when we consider the nonlinear autoregression discussed in section 5.1, we will see that globally stability cannot be proved using earlier results unless far more structure is imposed. The reason is that, structural aspects of the model being given, large shocks are often stabilizing due to the mixing they imply.

4 Verifying the Conditions

Theorem 3.1 requires that Q is increasing, order reversing, bounded in probability, and possesses an excessive or deficient measure. A sufficient condition for Q to be increasing was given in example 2.2. In this section, we present a number of sufficient conditions for the remaining properties. The most significant of these is proposition 4.3, which provides sufficient conditions for order reversing.

Throughout the following discussion, we use the simple AR(1) model for illustrative purposes. Significant applications are deferred to section 5.

[11] That the model is order reversing was shown in example 2.5. Monotonicity follows from example 2.2. Boundedness in probability is shown in example 4.1 below. For existence of a μ with $\mu \preceq \mu Q$, we can take the stationary distribution $\mu = N(0, (1-\rho^2)^{-1})$.

4.1 Boundedness in Probability

Boundedness in probability is a standard condition in the Markov process literature. In this section, we review some well-known techniques for checking boundedness in probability, and introduce a new one based on order-theoretic ideas.

Let Q be a stochastic kernel on $S = \mathbb{R}$, and let $\{X_t\}$ be Markov-(Q, x). Then Q is bounded in probability when $\sup_t \mathbb{E}\,|X_t| < \infty$ for any initial x. The same statement is valid if we replace $|X_t|$ with X_t^2. Intuitively, boundedness of these moments means that the process does not diverge.

To go beyond the case of $S = \mathbb{R}$, recall the notion of a coercive function: $V \colon S \to \mathbb{R}_+$ is called *coercive* if the sublevel set $L_a := \{x \in S : V(x) \leq a\}$ is precompact for all $a > 0$.[12] It is known that Q is bounded in probability whenever there exists a coercive function V with

$$\sup_t \int V(y) Q^t(x, dy) < \infty, \qquad \forall\, x \in S. \tag{8}$$

(See, e.g., Meyn and Tweedie, 2009. The function V may depend on x, in which case the condition is also necessary.) Condition (8) can sometimes be verified via a "drift" condition. For example, let Q_F be the kernel (2). Then (8) will be satisfied if there exist positive constants α and β with $\alpha < 1$ and

$$\mathbb{E}\, V[F(x, \xi_t)] \leq \alpha V(x) + \beta, \qquad \forall\, x \in S. \tag{9}$$

Example 4.1. Consider the AR(1) process (5) with $S = \mathbb{R}$. Here (9) is satisfied for $V(x) := |x|$ whenever $|\rho| < 1$. Indeed, by the triangle inequality, $\mathbb{E}\,|\rho x + \xi_t| \leq |\rho| \cdot |x| + \mathbb{E}\,|\xi_t|$. This corresponds to (9) with $\alpha := |\rho|$ and $\beta := \mathbb{E}\,|\xi_t|$.

Examples of how to construct coercive functions satisfying (8) are given in section 5. Further examples can be found in Stachurski (2002), Nishimura and

[12]For example, if $V(x) = \|x\|$ and $S = \mathbb{R}^n$, then V is coercive, because L_a is the closed ball $\bar{B}(0, a) = \{x \in \mathbb{R}^n : x \leq a\}$. However, if $S = \mathbb{R}^n_{++}$, then $V(x) = \|x\|$ is not coercive, because $L_a = \mathbb{R}^n_{++} \cap \bar{B}(0, a)$, which is not compact. In essence, V is coercive if $V(x_n) \to \infty$ whenever x_n "diverges" towards the "edges" of the state space.

Stachurski (2005), Kamihigashi (2007), and Kristensen (2008) and Meyn and Tweedie (2009).

We now introduce a new result that can be used to check boundedness in probability, and also relates to our techniques for checking existence of deficient and excessive measures discussed in section 4.2 below. To begin, let Q and Q' be any two stochastic kernels on S. If Q' dominates Q pointwise on $\mathscr{P}$, in the sense that $\mu Q \preceq \mu Q'$ for all $\mu \in \mathscr{P}$, then we write $Q \preceq Q'$. Equivalently, $Q \preceq Q'$ if $Qh \leq Q'h$ pointwise on S whenever $h \in ibS$.[13]

Example 4.2. Let F and Q_F be as in example 2.1. Consider a second process

$$X_{t+1} = G(X_t, \xi_{t+1}), \qquad \{\xi_t\} \overset{\text{IID}}{\sim} \phi,$$

where $G \colon S \times Z \to S$ is measurable. Let Q_G be the corresponding stochastic kernel. If $G(x,z) \leq F(x,z)$ for all $(x,z) \in S \times Z$, then $Q_G \preceq Q_F$.

Proposition 4.1. *Let Q_ℓ, Q, Q_u be stochastic kernels on S. If $Q_\ell \preceq Q \preceq Q_u$ and both Q_ℓ and Q_u are bounded in probability, then Q is bounded in probability.*

4.2 Existence of Excessive and Deficient Measures

Condition 2 of theorem 3.1 requires existence either an excessive or a deficient distribution. In some cases this is easy to verify. For example, if S has a least element or a greatest element then the condition always holds (remark 2.3). However, there are many settings where S has neither ($S = \mathbb{R}^n$ and $S = \mathbb{R}^n_{++}$ are obvious examples). In this case, one can work more carefully with the definition of the model to construct excessive and deficient distributions. One example is Zhang (2007), who constructs such measures for the stochastic optimal growth model.

[13] For example, suppose that the latter condition holds. Pick any $\mu \in \mathscr{P}$. Fixing $h \in ibS$, we have $Qh \leq Q'h$. Integrating with respect to μ gives $\langle \mu, Qh \rangle \leq \langle \mu, Q'h \rangle$, or, equivalently, $\langle \mu Q, h \rangle \leq \langle \mu Q', h \rangle$. Since h was an arbitrary element of ibS, we have shown that $\mu Q \preceq \mu Q$. The proof of the converse is also straightforward.

However, identifying excessive and deficient measures may be nontrivial. For this reason, we now provide a sufficient condition which is relatively straightforward to check in applications.

Proposition 4.2. *Let Q be a stochastic kernel on S. If there exists another kernel Q_u such that Q_u is Feller, bounded in probability and $Q \preceq Q_u$, then Q has an excessive distribution. Likewise, if Q_ℓ is Feller, bounded in probability and $Q_\ell \preceq Q$, then Q has a deficient distribution.*

Examples of how to use this result are provided in the applications. In addition, we note that propositions 4.1 and 4.2 can be combined with theorem 3.1 to obtain the following stability result:

Theorem 4.1. *Suppose that Q is increasing and order reversing. If there exist kernels $Q_\ell \preceq Q \preceq Q_u$ such that Q_ℓ and Q_u are bounded in probability and at least one of them is Feller, then Q is globally stable.*

4.3 Order Reversing

In this section we give sufficient conditions for order reversing. To state them, we introduce two new definitions: We call kernel Q on S *upward reaching* if, given any x and c in S, there exists a $t \in \mathbb{N}$ such that $Q^t(x, \{y \in S : c \leq y\}) > 0$; and *downward reaching* if, given any x and c in S, there exists a $t \in \mathbb{N}$ such that $Q^t(x, \{y \in S : y \leq c\}) > 0$.

Example 4.3. The AR(1) process in (5) is both upward and downward reaching. For example, fix x, c in $S = \mathbb{R}$, and take $t = 1$. We have

$$Q(x, \{y \in S : y \leq c\}) = \mathbb{P}\{\rho x + \xi_1 \leq c\} = \mathbb{P}\{\xi_1 \leq c - \rho x\},$$

which is positive because $\xi_t \sim N(0, 1)$. Hence Q is downward reaching.

We can now present the main result of this section.

Proposition 4.3. *Suppose that Q is increasing and bounded in probability. If Q is either upward or downward reaching, then Q is order reversing.*

Using proposition 4.3, we can also provide more specialized results for the model in example 2.1. To simplify the exposition, we assume without loss of generality that Z is the support of ϕ.[14] Also, observe that each finite path of shock realizations $\{z_i\}_{i=1}^t \subset Z$ and initial condition $X_0 = x \in S$ determines a path $\{x_i\}_{i=0}^t$ for the state variable up until time t via (1). Let $F^t(x, z_1, \ldots, z_t)$ denote the value of x_t determined in this way.[15]

Proposition 4.4. *Suppose that $x \mapsto F(x,z)$ is increasing for each $z \in Z$, F is continuous on $S \times Z$, and Q_F is bounded in probability. If any one of*

1. $\forall\, x, c \in S, \;\; \exists\, \{z_i\}_{i=1}^k \subset Z$ *such that* $F^k(x, z_1, \ldots, z_k) < c$
2. $\forall\, x, c \in S, \;\; \exists\, \{z_i\}_{i=1}^k \subset Z$ *such that* $F^k(x, z_1, \ldots, z_k) > c$
3. $\forall\, x, x' \in S, \;\; \exists\, \{z_i\}_{i=1}^k$ *and* $\{z'_i\}_{i=1}^k$ *with* $F^k(x, z_1, \ldots, z_k) < F^k(x', z'_1, \ldots, z'_k)$

holds, then Q_F is globally stable.

The interpretation of the strict inequality for vectors in conditions 1–3 is that $(x_i)_{i=1}^n < (y_i)_{i=1}^n$ if $x_i < y_i$ for all i.

Example 4.4. Consider the AR(1) model $F(x,z) = \rho x + z$ with $0 \leq \rho < 1$. Clearly the model is increasing and continuous in the sense of proposition 4.4. We showed in example 4.1 that boundedness in probability holds. Thus, to show the model is order reversing, it remains to verify one of conditions 1–3 in proposition 4.4. Taking condition 1, fix $x, c \in \mathbb{R}$. We need to choose a shock sequence that drives the process below c when it starts at x. This can be done in one step, by choosing z_1 such that $\rho x + z_1 < c$.

5 Applications

We now turn to more substantial applications of the results described above.

[14] That is, $\phi(Z) = 1$, and $\phi(G) > 0$ whenever $G \subset Z$ is nonempty and open. Z can always be re-defined so that this assumption is valid.

[15] Formally, $F^1 := F$ and $F^{t+1}(x, z_1, \ldots, z_{t+1}) := F(F^t(x, z_1, \ldots, z_t), z_{t+1})$ for all $t \in \mathbb{N}$.

5.1 Nonlinear Autoregression

Our first application is a non-specific additive shock model, which helps to illustrate the generality our results. The dynamics are given by

$$X_{t+1} = F(X_t, \xi_{t+1}) = f(X_t) + \xi_{t+1}, \quad \{\xi_t\} \overset{\text{IID}}{\sim} \phi, \quad \mathbb{E}\,\|\xi_t\| < \infty, \tag{10}$$

where $S = \mathbb{R}^n$, and $f \colon \mathbb{R}^n \to \mathbb{R}^n$. We assume that

1. The function f is increasing.
2. $\mathbb{P}\{\xi_t \leq z\}$ is non-zero for all $z \in \mathbb{R}^n$.
3. $\exists\, \alpha \in [0, 1)$ and $L \geq 0$ such that $\|f(x)\| \leq \alpha\|x\| + L$ for all $x \in \mathbb{R}^n$.

The last assumption is a growth condition on f. Global stability cannot hold without some restriction along these lines. The second assumption can be replaced by: $\mathbb{P}\{\xi_t \geq z\}$ is non-zero for all $z \in \mathbb{R}^n$.

Let Q_F be the stochastic kernel associated with (10) via (2). For this model, the MMC does not apply, Q_F is not irreducible, the splitting condition fails, the model is not an expected contraction, the standard Harris recurrence conditions are not satisfied, and the process is not Feller.[16] Indeed, to the best of our knowledge, global stability of Q_F—or even existence of a stationary distribution—cannot be established using any result in the existing literature.

On the other hand, a straightforward proof of global stability can be constructed via theorem 3.1 without additional assumptions. To begin, note that $F(x, z) := f(x) + z$ is increasing in x for each z because f is increasing, and hence, by example 2.2, Q_F is increasing. Second, Q_F is bounded in probability, as can be shown by taking $V(x) := \|x\|$ in (9).[17] Third, Q_F is order reversing by proposition 4.3. Indeed, if we fix $x, c \in S = \mathbb{R}^n$ and let E be the event

[16] For a discussion of irreducibility, see Meyn and Tweedie (2009, ch. 4). On the splitting condition, see, e.g., Bhattacharya and Lee (1988), or Bhattacharya and Majumdar (2001). For expected contractions, see, e.g., Santos and Peralta-Alva (2005, p. 1952). For more on Harris recurrence, see Harris (1956). A beautiful modern interpretation of Harris' method is given in Hairer and Mattingly (2008).

[17] By the triangle inequality, $\mathbb{E}\,\|f(x) + \xi_1\| \leq \|f(x)\| + \mathbb{E}\,\|\xi_1\| \leq \alpha\|x\| + L + \mathbb{E}\,\|\xi_1\|$.

$\{\xi_1 \leq c - f(x)\}$, then, by assumption, $\mathbb{P}(E) > 0$. Moreover, if E occurs, then $f(x) + \xi_1 \leq c$. Hence Q_F is downward reaching, and order reversing holds.

To complete the proof of global stability via theorem 3.1, then, the only difficulty is to show existence of an deficient or excessive measure. For this purpose, we use proposition 4.2. To apply the lemma, we need to find a dominating process that possesses at least one stationary distribution. For the dominating process, consider

$$X_{t+1} = G(X_t, \xi_{t+1}) = g(X_t) + \xi_{t+1}, \qquad g(x) := \alpha \|x\| \mathbf{1} + L\mathbf{1}. \tag{11}$$

Here $\mathbf{1}$ is the unit vector in $\mathbb{R}^n$, and α and L are as above. The model (11) is easily shown to be Feller and bounded in probability.[18] Moreover, $F \leq G$, because if $f_i(x)$ is the i-th component of $f(x)$, then $f_i(x) \leq |f_i(x)| \leq \|f(x)\| \leq \alpha\|x\| + L$.

$$\therefore \quad f(x) \leq \alpha\|x\|\mathbf{1} + L\mathbf{1} = g(x).$$

$$\therefore \quad F(x,z) = f(x) + z \leq g(x) + z = G(x,z).$$

We conclude that the conditions of proposition 4.2 are satisfied, and the proof of global stability is done.

NOTE: Clean up above to reflect new results.

5.2 Optimal Growth

NOTE: Clean up

Variations on the stochastic optimal growth model form the foundations of many economic studies, and the existence of ergodic, non-trivial stochastic equilibria is of fundamental importance when comparing predictions with data. We begin by looking at the most elementary case, where consumption is chosen to maximize $\mathbb{E} \sum_{t=0}^{\infty} \delta^t u(c_t)$ subject to $k_{t+1} + c_t \leq \xi_t f(k_t)$. All variables are nonnegative and $\{\xi_t\} \stackrel{\text{IID}}{\sim} \phi$. For now, we assume that u is bounded with

[18] Since g is continuous, the model is Feller. In addition, (9) is valid for the coercive function $V(x) := \|x\|_\infty := \max_{i=1}^n |x_i|$, and hence boundedness in probability also holds.

$u' > 0$, $u'' < 0$, and $u'(0) = \infty$; while $f(0) = 0$, $f' > 0$, $f'' < 0$, $f'(0) = \infty$ and $f'(\infty) = 0$. (Extensions are discussed below.)

To study the dynamics of the optimal process, we take $y_t = \xi_t f(k_t)$ as the state variable, and consider the income process $y_{t+1} = \xi_t f(y_t - \sigma(y_t))$, where $\sigma(\cdot)$ is the optimal consumption policy. Let Q be the corresponding stochastic kernel. For the state space we take $S = \mathbb{R}_{++}$. (Zero is deliberately excluded so that any stationary distribution on S is automatically non-trivial.) It is well-known that optimal savings $y \mapsto y - \sigma(y)$ is increasing and continuous, and hence Q is increasing and Feller on S (cf., e.g., Stokey et al., 1989, p. 393).

Brock and Mirman (1972) were the first to prove global stability of this process, for the case where ξ_t has support $[a, b]$, with $0 < a < b$. The same result can be obtained from theorem 3.1. Indeed, Hopenhayn and Prescott (1992) show that Q satisfies the conditions of their stability result, which, as discussed in section 3, is a special case of theorem 3.1.[19]

The assumption that ξ_t has bounded support can be removed.[20] Instead, one can assume that ξ_t has sufficiently small tails. In particular, suppose now that $\mathbb{E}\,\xi_t < \infty$ and $\mathbb{E}\,(1/\xi_t) < \infty$. (These restrictions bound the right and left-hand tails respectively.) Boundedness in probability is known to hold (see, e.g., Nishimura and Stachurski (2005) or Kamihigashi, 2007), so, in view of theorem 3.2, order reversing is sufficient for global stability. To verify order reversing, it suffices to show that Q is either upward or downward reaching (proposition 4.3). Suppose that $\mathbb{P}\{\xi_t \leq z\} > 0$ for all $z \in S$. If we fix any $x, c \in S$, then $\mathbb{P}\{\xi_t f(x - \sigma(x)) \leq c\} = \mathbb{P}\{\xi_t \leq c / f(x - \sigma(x))\} > 0$. Thus, Q is downward reaching,[21] and hence globally stable.[22]

The stability results for optimal growth models presented in this section can be extended in many ways. For example, in the previous result, we can

[19] See Olson and Roy (2006), Kamihigashi (2007) or Chatterjee and Shukayev (2008) for additional discussion of the case where ξ_t has bounded support.

[20] Stability without bounded shocks was first shown by Stachurski (2002), using stricter conditions than those considered here.

[21] Since $S = \mathbb{R}_{++}$ and optimal consumption is interior, we have $f(x - \sigma(x)) > 0$.

[22] A closely related result was proved by Zhang (2007). His result is also a special case of theorem 3.1.

remove the assumption that f is concave.[23] Without concavity of f, optimal consumption may be discontinuous, and Q is no longer Feller. However, monotonicity of Q still holds, as does boundedness in probability (Nishimura and Stachurski, 2005). Existence of a excessive measure is not difficult to establish.[24] Moreover, the order reversing proof in the previous paragraph goes through unchanged. This proves global stability under weaker conditions than those used in Nishimura and Stachurski (2005).

Remark 5.1. The preceding results feel counterintuitive. To prove stability we used order reversing, and to prove order reversing we relied on nonzero probability of arbitrarily bad productivity shocks. These shocks are stabilizing, rather than destabilizing, because, as a result of the Inada conditions we imposed, the fundamental structure of the economy acts against divergence. Large shocks do not destabilize, they simply promote mixing.

5.3 An Open Economy with Borrowing Constraints

Next we consider an overlapping generations model of wealth distribution, which is a variation of the small open economy studied by Matsuyama (2004). Agents live for two periods, consuming only when old. Each household consists of one old and one young agent (child). There is a unit mass of such households indexed by $i \in [0, 1]$. In each period t, the old agent of household i provides financial support b_t^i to her child. The child has the option to become an entrepreneur, investing one unit of the consumption good in a "project,"

[23]In models of renewable resource exploitation, f is biologically determined, and typically non-concave. For motivation and further discussion, see, for example, Dechert and Nishimura (1983), Majumdar, Mitra and Nyarko (1989), or Mitra and Roy (2006).

[24]To do so we can use proposition 4.2. Since $f'(\infty) = 0$, we can choose positive constants α, β with $\alpha \mathbb{E}\, \xi_t < 1$ and $f(x) \leq \alpha x + \beta$ (Nishimura and Stachurski, 2005, proposition 4.3). Now take $G(x, z) := z(\alpha x + \beta)$, so that $F(x, z) := z f(x - \sigma(x)) \leq z f(x) \leq G(x, z)$. Letting Q_F and Q_G be the corresponding kernels, the last inequality implies $Q_F \preceq Q_G$. In view of proposition 4.2, it remains only to show that Q_G is Feller and bounded in probability. Since $G(\cdot, z)$ is continuous, Q_G is Feller. Using $\alpha \mathbb{E}\, \xi_t < 1$, condition (9) can be established for $V(x) = x + 1/x$, which is coercive on $S = \mathbb{R}_{++}$. Boundedness in probability then follows.

and receiving stochastic output of $\theta + \eta_{t+1}^i$ in period $t+1$. Let $k_{t+1}^i \in \{0, 1\}$ be young agent i's investment in the project. If the remainder $b_t^i - k_{t+1}^i$ between current income and investment on the project is positive, then she invests this quantity at the world risk-free rate R. If it is negative then she borrows $k_{t+1}^i - b_t^i$ at the same risk-free rate. Independent of her investment choices choice, she receives an endowment of e_{t+1}^i units of the consumption good when old.

Suppressing the i superscript to simplify notation, her wealth at the beginning of period $t+1$ is

$$w_{t+1} = (\theta + \eta_{t+1})k_{t+1} - R(k_{t+1} - b_t) + e_{t+1}. \tag{12}$$

We assume that

$$e_{t+1} = \rho e_t + \epsilon_{t+1}, \quad 0 < \rho < 1, \tag{13}$$

and that the idiosyncratic shocks $\{\eta_t\}$ and $\{\epsilon_t\}$ are IID, nonnegative, and $\mathbb{P}\{\epsilon_t > \alpha\} > 0$ for any $\alpha \geq 0$. We also assume that

$$R < \theta, \quad \gamma R < 1. \tag{14}$$

The first inequality in (14) implies that becoming an entrepreneur is always profitable, even ex post, and every agent would choose to do so absent additional constraint. Due to market imperfection, however, each agent may borrow only up to a fraction $\lambda \in (0, 1)$ of $\theta + \rho e_t$, the minimum possible value of her old-age income. That is,

$$R(k_{t+1} - b_t) \leq \lambda(\theta + \rho e_t). \tag{15}$$

Note that his constraint rules out default even in the worst case.

Letting c denote consumption, young agents maximize $\mathbb{E}_t[c_{t+1}^{1-\gamma} b_{t+1}^{\gamma}]$ subject to (12), (15), and

$$c_{t+1} + b_{t+1} = w_{t+1}. \tag{16}$$

Since maximization of $c_{t+1}^{1-\gamma} b_{t+1}^{\gamma}$ subject to (16) implies that $b_{t+1} = \gamma w_{t+1}$, old agents give a fixed fraction γ of their wealth to their child. Since becoming an entrepreneur is always profitable, young agents do so whenever feasible, implying

$$k_{t+1} = \kappa(b_t, e_t) := \begin{cases} 1 & \text{if } R(1 - b_t) \leq \lambda(\theta + \rho e_t), \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Recalling $b_{t+1} = \gamma w_{t+1}$ and (12), we can now write

$$b_{t+1} = \sigma(b_t, e_t, \eta_{t+1}, \epsilon_{t+1}) \tag{18}$$

where σ is defined by

$$\sigma(b, e, \eta, \epsilon) := \gamma[(\theta + \eta - R)\kappa(b, e) + Rb + \rho e + \epsilon] \tag{19}$$

The right-hand side of (12) is increasing in k_{t+1} by (14), and κ easily seen to be increasing. As a result, $\sigma(b, e, \eta, \epsilon)$ is increasing in (b, e, η, ϵ). The system of equations (18) and (13) determines a (discontinuous) Markov process on $S := [0, \infty) \times [0, \infty)$ with state vector $X_t := (b_t, e_t)$.[25] The corresponding kernel Q is increasing (see example 2.2).

We now show that Q is globally stable. Let $m_\eta := \mathbb{E}\, \eta_t$ and $m_\epsilon := \mathbb{E}\, \epsilon_t$. To see that Q is bounded in probability, note from (13) that

$$\mathbb{E}\, e_t \leq m_\epsilon / (1 - \rho) + \rho^t e_0 \leq \bar{e} := m_\epsilon / (1 - \rho) + e_0 \tag{20}$$

for all t. In addition, it follows from (18) and (19) that

$$\mathbb{E}\, b_{t+1} \leq \gamma[\theta + m_\eta - R + R\mathbb{E}\, b_t + \bar{e}].$$

Using $\gamma R < 1$, we obtain the bound

$$\mathbb{E}\, b_t \leq \gamma[\theta + m_\eta - R + \bar{e}] / (1 - \gamma R) + b_0 \tag{21}$$

for all t. Together, (20) and (21) imply that Q is bounded in probability.[26] Since $\mathbb{P}\{\epsilon_t > \alpha\} > 0$ for any $\alpha \geq 0$, it is easy to see that Q is upward reaching,

[25] We do not exclude $(0, 0)$ from the state space since it is not an absorbing state.

[26] The function $V(b, e) = b + e$ is coercive on S, and equations (20) and (21) imply that $\sup_t \mathbb{E}\, [V(b_t, e_t)] \leq \sup_t \mathbb{E}\, [b_t] + \sup_t \mathbb{E}\, [e_t] < \infty$, which gives (8).

and thus order reversing by proposition 4.3. In view of these results and theorem 3.1, Q will be globally stable whenever it has a deficient or excessive measure. Since $(0,0)$ is a least element for S, remark 2.3 implies that Q has a deficient measure, and we conclude that Q is globally stable.

We have shown that given any initial distribution μ_0 of (b_0, e_0), the distribution μ_t of (b_t, e_t) converges to a unique stationary distribution μ^*. Recalling Remark ???, we can easily examine the effect of a change in a parameter. For example, suppose that the initial distribution is μ^*, and that the credit constraint (15) is relaxed by increasing $\lambda \leq 1$. This change shifts κ and thus σ upward. Hence the distribution of (b_t, e_t) (in fact only the distribution of b_t since e_t is exogenous) keeps shifting upward over time and converges to the new stationary distribution, which stochastically dominates the initial stationary distribution μ^*.

6 Conclusion

This paper considered global stability of stochastic economies and time series models, based on a new mixing condition called order reversing. Our main theorems (theorem 3.1 and theorem 3.2) generalizes earlier results based on monotone mixing due to Razin and Yahav (1979), Stokey et al. (1989), and Hopenhayn and Prescott (1992), significantly extending the range of possible applications, and shedding light on the interactions between shocks, structural dynamics and stochastic stability.

Other new results contained in the paper are propositions 4.1–4.4 and theorem 4.1. These results provide additional stability conditions and aid in verification of order reversing and other conditions of our main stability theorem. Their usefulness is illustrated in the applications discussed in section 5.

7 Proofs

State general assumptions for state and order!

Before proving theorem 3.1, we need some additional results and notation. To begin, let Φ be any stochastic kernel on $D \subset \mathbb{R}^q$, let $x \in D$ and let D-valued stochastic process $\{X_t\}$ be Markov-(Φ, x). The joint distribution of $\{X_t\}$ over the sequence space D^∞ will be denoted by $\mathbf{P}_x$. For example, $\mathbf{P}_x\{X_t \in B\} = \Phi^t(x, B)$ for any $B \subset D$, and $\mathbf{P}_x \cup_{t=0}^\infty \{X_t \in B\}$ is the probability that the process ever enters B. Evidently $\mathbf{P}_x$ depends on Φ as well as x, but this dependence is suppressed in the notation.

We say that Borel set $B \subset D$ is

- *Strongly accessible* for Φ if $\mathbf{P}_x \cup_{t=0}^\infty \{X_t \in B\} = 1$ for all $x \in D$, and
- *uniformly accessible* for Φ if, for all compact $C \subset D$, there exists an $n \in \mathbb{N}$ and $\delta > 0$ with $\inf_{x \in C} \Phi^n(x, B) \geq \delta$.

The following lemma is fundamental to our results, although the proofs is delayed to maintain continuity.

Lemma 7.1. *Let B be a Borel subset of D. If Φ is bounded in probability and B is uniformly accessible, then B is strongly accessible.*

Now we return to the specific setting of theorem 3.1, where S is a subset of $\mathbb{R}^n$ such that order intervals are compact and compact sets are order bounded. Let Q be a given kernel on S, and let $Q \times Q$ be the product kernel (4). For given pair $(x, x') \in S \times S$, let $\{X_t\}$ and $\{X'_t\}$ be Markov-(Q, x) and Markov-(Q, x') respectively, and also independent of each other. As discussed in section 2.1, the bivariate process $\{(X_t, X'_t)\}$, that takes values in $S \times S$, is Markov-$(Q \times Q, (x, x'))$. Its joint distribution over the sequence space $(S \times S)^\infty$ is denoted by $\mathbf{P}_{x,x'}$. In this notation, Q is order reversing if

$$\forall\, x, x' \in S \text{ with } x \geq x', \;\; \exists\, k \geq 0 \text{ such that } \mathbf{P}_{x,x'}\{X_k \leq X'_k\} > 0.$$

In addition, Q is called *order mixing* (Kamihigashi and Stachurski, 2010) if

$$\forall\, x, x' \in S, \quad \mathbf{P}_{x,x'} \cup_{t=0}^\infty \{X_t \leq X'_t\} = 1.$$

Put differently, Q is order mixing if $\mathbb{G} := \{(y, y') \in S \times S : y \leq y\}$ is strongly accessible for the product kernel $Q \times Q$. Order mixing is clearly stronger than order reversing, and significantly more difficult to check in applications. However, we will show below that if Q is increasing and bounded in probability, then order reversing and order mixing are equivalent.

Lemma 7.2. *If Q is bounded in probability, then so is the product kernel $Q \times Q$.*

Lemma 7.3. *If Q is increasing and bounded in probability, then $\{\mu Q^t\}$ is tight for all $\mu \in \mathscr{P}$.*

Lemma 7.4. *If Q is increasing and order reversing, then $\mathbb{G}$ is uniformly accessible for $Q \times Q$.*

Proofs are given at the end of this section.

Let us now turn to the proof of theorem 3.1. The proof proceeds as follows: First we show that under the conditions of the theorem, Q is order mixing. Using order mixing, we then go on to prove existence of a stationary distribution, and global stability.

Regarding the first step, to show that Q is order mixing we need to prove that $\mathbb{G}$ is strongly accessible for $Q \times Q$ under the conditions of theorem 3.1. Since Q is bounded in probability, $Q \times Q$ is also bounded in probability (lemma 7.2), and hence, by of lemma 7.1, it suffices to show that $\mathbb{G}$ is uniformly accessible for $Q \times Q$. This follows from lemma 7.4.

This is an important result in its own right, and we state it as a theorem:

Theorem 7.1. *If Q is increasing and bounded in probability, then Q is order mixing if and only if Q is order reversing.*

We now prove global stability, making use of order mixing. In the sequel, we define *icbS* to be the bounded, increasing and continuous functions from S to $\mathbb{R}$ (i.e., $icbS = ibS \cap cbS$). We will make use of the following results:[27]

Lemma 7.5. *Let $\mu, \mu', \mu_n \in \mathscr{P}$.*

1. *$\mu \preceq \mu'$ iff $\langle \mu, h \rangle \leq \langle \mu', h \rangle$ for all $h \in icbS$,*
2. *$\mu = \mu'$ iff $\langle \mu, h \rangle = \langle \mu', h \rangle$ for all $h \in icbS$, and*
3. *$\mu_n \to \mu$ iff $\langle \mu_n, h \rangle \to \langle \mu, h \rangle$ for all $h \in icbS$.*

Proof of theorem 3.1. We begin by showing that if Q is globally stable, then conditions 1–2 of the theorem hold. Regarding condition 1, fix $x \in S$. Global stability implies that $\{\mu Q^t\}$ is convergent for each $\mu \in \mathscr{P}$, and hence $\{Q^t(x, \cdot)\} = \{\delta_x Q^t\}$ is convergent. Since convergent sequences are tight (Dudley, 2002, proposition 9.3.4) and $x \in S$ was

[27]See Torres (1990) for proofs.

arbitrary, we conclude that Q is bounded in probability, and condition 1 is satisfied. Condition 2 is trivial (take $\mu = \mu^*$ to generate a constant sequence).

Next we show that if Q is increasing, order reversing and conditions 1–2 hold, then Q has at least one stationary distribution. By theorem 7.1, Q is order mixing, and Kamihigashi and Stachurski (2010, theorem 3.1) implies that, for any ν and ν' in $\mathscr{P}$ we have

$$\lim_{t\to\infty} |\langle \nu Q^t, h\rangle - \langle \nu' Q^t, h\rangle| = 0, \qquad \forall\, h \in ibS. \tag{22}$$

Now let $\{\mu Q^t\}$ be a tight and monotone sequence, existence of which is guaranteed by conditions 1–2 and lemma 7.3. We suppose without loss of generality that $\{\mu Q^t\}$ is increasing, since the other case changes nothing in what follows except the direction of the inequalities.

By Prohorov's theorem, tightness implies existence of a a subsequence of $\{\mu Q^t\}$ converging to some $\psi^* \in \mathscr{P}$. By monotonicity, the entire sequence converges: $\mu Q^t \to \psi^*$. But then $\mu Q^t \preceq \psi^*$ for all $t \geq 0$, because for any $h \in icbS$ and $t \geq 0$ we have

$$\langle \mu Q^t, h\rangle \leq \sup_{t\geq 0} \langle \mu Q^t, h\rangle = \lim_{t\to\infty} \langle \mu Q^t, h\rangle = \langle \psi^*, h\rangle.$$

From part 1 of lemma 7.5 we conclude that $\mu Q^t \preceq \psi^*$.

Next, we claim that $\psi^* \preceq \psi^* Q$. To see this, pick any $h \in icbS$. Since $\mu Q^t \preceq \psi^*$ for all t, and since $Qh \in ibS$,

$$\langle \mu Q^t, Qh\rangle \leq \langle \psi^*, Qh\rangle = \langle \psi^* Q, h\rangle.$$

Using the fact that $h \in cbS$ we can take the limit to obtain

$$\langle \psi^*, h\rangle = \lim_{t\to\infty} \langle \mu Q^{t+1}, h\rangle = \lim_{t\to\infty} \langle \mu Q^t, Qh\rangle \leq \langle \psi^* Q, h\rangle.$$

Hence $\langle \psi^*, h\rangle \leq \langle \psi^* Q, h\rangle$ for all $h \in icbS$, and $\psi^* \preceq \psi^* Q$ as claimed. Iterating on this inequality we obtain $\psi^* \preceq \psi^* Q^t$ for all t.

To summarize our results so far, we have

$$\mu Q^t \preceq \psi^* \preceq \psi^* Q \preceq \psi^* Q^t.$$

for all $t \geq 0$. Fixing $h \in icbS$, this implies that

$$\langle \mu Q^t, h\rangle \leq \langle \psi^*, h\rangle \leq \langle \psi^* Q, h\rangle \leq \langle \psi^* Q^t, h\rangle.$$

Applying (22), we obtain $\langle \psi^*, h \rangle = \langle \psi^* Q, h \rangle$ for all $h \in icbS$. By lemma 7.5, this implies that $\psi^* = \psi^* Q$, and ψ^* is stationary for Q.

It remains to show that Q is globally stable, and ψ^* is unique. Fixing $\mu \in \mathscr{P}$ and applying (22) again, we have

$$|\langle \mu Q^t, h \rangle - \langle \psi^*, h \rangle| \to 0, \quad \forall\, h \in ibS. \tag{23}$$

Since $icbS \subset ibS$, this implies that $\mu Q^t \to \psi^*$ (see lemma 7.5). Finally, uniqueness is also immediate, because if μ is also stationary, then by (23) we have $\langle \mu, h \rangle = \langle \psi^*, h \rangle$ for all $h \in icbS$. By lemma 7.5, we then have $\psi = \psi^*$. □

Proof of theorem 3.2. Under the conditions of the theorem, Q is order mixing, as proved in theorem 7.1. In addition, boundedness in probability and the Feller property guarantee the existence of a stationary distribution by the Krylov-Bogolubov theorem (see, e.g., Stachurski, 2009). Global stability then follows from Kamihigashi and Stachurski (2010, theorem 3.1). □

Proof of proposition 4.1. Pick any $x \in S$ and fix $\epsilon > 0$. Let K be a compact set such that $Q_\ell^t(x, K) \geq 1 - \epsilon$ for all t. By our assumptions on S, there exists an $x_\ell \in S$ with $Q_\ell^t(x, \{y \in S : y \geq x_\ell\}) \geq 1 - \epsilon$ for all t. By similar reasoning, there exists $x_u \in S$ such that $Q_u^t(x, \{y \in S : y \leq x_u\}) \geq 1 - \epsilon$ for all t. Since $Q_\ell \preceq Q \preceq Q_u$, it then follows that $Q^t(x, \{y \in S : y \geq x_\ell\}) \geq 1 - \epsilon$ and $Q^t(x, \{y \in S : y \leq x_u\}) \geq 1 - \epsilon$ for all t. Combining these bounds, we obtain $Q^t(x, [x_\ell, x_u]) \geq 1 - 2\epsilon$ for all t. Since the order interval $[x_\ell, x_u]$ is compact, it follows $\{Q^t(x, \cdot)\}$ is tight. As $x \in S$ was arbitrary, we have shown that Q is bounded in probability. □

Proof of proposition 4.2. If Y is a random variable, then let $\mathscr{L}Y$ be its distribution.[28] Let $\mu \in \mathscr{P}$. Consider the first case, where $F \leq G$. We claim that if μ is stationary for Q_G, then $\mu Q \preceq \mu$. To see this, let X and ξ be independent random variables taking values in S and Z respectively, with $\mathscr{L}X = \mu$ and $\mathscr{L}\xi = \phi$. It is immediate from the definitions that $\mathscr{L}F(X, \xi) = \mu Q_F$ and $\mathscr{L}G(X, \xi) = \mu Q_G = \mu$. Since $G \leq F$ we have $G(X, \xi) \leq F(X, \xi)$ pointwise on Ω. We conclude that $\mu = \mathscr{L}G(X, \xi) \preceq \mathscr{L}F(X, \xi) = \mu Q_F$, as was to be shown.

The proof of the second case is similar. □

[28]Formally, $\mathscr{L}Y$ is the image measure of Y under $\mathbb{P}$.

Now we turn to the proof of proposition 4.3.

Proof of proposition 4.3. Suppose first that Q is upward reaching. Pick any $(x, x') \in S \times S$. Let $\{X_t\}$ and $\{X'_t\}$ be indepedent, Markov-(Q, x) and Markov-(Q, x') respectively. We need to prove existence of a $k \in \mathbb{N}$ such that $\mathbb{P}\{X_k \leq X'_k\} > 0$.

Since Q is bounded in probability, there exists a compact $C \subset S$ with $\mathbb{P}\{X_t \in C\} > 0$ for all $t \geq 0$. By assumption, we can take an order interval $[a, b]$ of S with $C \subset [a, b]$. For this a, b we have

$$\mathbb{P}\{a \leq X_t \leq b\} > 0 \text{ for all } t \geq 0.$$

As Q is upward reaching, there is a $k \in \mathbb{N}$ such that $\mathbb{P}\{b \leq X'_k\} > 0$. Using independence, we now have

$$\mathbb{P}\{X_k \leq X'_k\} \geq \mathbb{P}\{X_k \leq b \leq X'_k\} = \mathbb{P}\{X_k \leq b\}\mathbb{P}\{b \leq X'_k\} > 0,$$

as was to be shown. The proof for the downward reaching case is similar. □

Proof of proposition 4.4. Let $\{\xi_t\}_{t=1}^{\infty}$ and $\{\xi'_t\}_{t=1}^{\infty}$ be IID from ϕ and independent of each other. Consider first condition 3. We claim that Q_F is order reversing. Fix $x, x' \in S$. Let $\{z_t\}_{t=1}^{k}$ and $\{z'_t\}_{t=1}^{k}$ be as in the statement of the corollary. Define the constant

$$\gamma := \mathbb{P}\{F^k(x, \xi_1, \ldots, \xi_k) < F^k(x', \xi'_1, \ldots, \xi'_k)\}.$$

We need only show that $\gamma > 0$. By hypothesis, $F^k(x, z_1, \ldots, z_k) < F^k(x', z'_1, \ldots, z'_k)$. By continuity of F, there exist open neighborhoods N_t of z_t and N'_t of z'_t such that

$$\tilde{z}_t \in N_t \text{ and } \tilde{z}'_t \in N'_t \text{ for } t \in \{1, \ldots, k\} \implies F^k(x, \tilde{z}_1, \ldots, \tilde{z}_k) < F^k(x', \tilde{z}'_1, \ldots, \tilde{z}'_k).$$

This leads to the estimate

$$\gamma \geq \mathbb{P} \cap_{t=1}^{n} \{\xi_t \in N_t \text{ and } \xi'_t \in N'_t\} = \prod_{t=1}^{n} \phi(N_t)\phi(N'_t).$$

Since Z is the support of ϕ, this last term is positive, and $\gamma > 0$.

The proof of the corollary will be complete if conditions 1–2 of the corollary imply that Q_F is upward and downward reaching respectively (see proposition 4.3). The arguments are very similar to the proof just completed and hence we omit them. □

Finally, we complete the proof of all remaining lemmas stated in this section.

Proof of lemma 7.1. Let B be a uniformly accessible subset of D. To prove the lemma, it suffices to show that $\mathbf{P}_x \cup_t \{X_t \in B\} = 1$ whenever $\{\Phi^t(x, \cdot)\}$ is tight. To this end, fix $x \in D$, and assume that $\{\Phi^t(x, \cdot)\}$ is tight. Let $\tau := \inf\{t \geq 0 : X_t \in B\}$. Evidently we have $\cup_{t=0}^{\infty}\{X_t \in B\} = \{\tau < \infty\}$. Thus, we need to show that $\mathbf{P}_x\{\tau < \infty\} = 1$.

Fix $\epsilon > 0$. Since $\{\Phi^t(x, \cdot)\}$ is tight, there exists a compact set C such that

$$\inf_t \mathbf{P}_x\{X_t \in C\} = \inf_t \Phi^t(x, C) \geq 1 - \epsilon.$$

Since B is uniformly attracting, there exists an $n \in \mathbb{N}$ and $\delta > 0$ such that $\inf_{y \in C} \Phi^n(y, B) \geq \delta$. For $t \in \mathbb{N}$, define $p_t := \mathbf{P}_x\{\tau \leq tn\}$. We wish to obtain a relationship between p_t and p_{t+1}. To this end, note that

$$\begin{aligned} \mathbb{1}\{\tau \leq (t+1)n\} &= \mathbb{1}\{\tau \leq tn\} + \mathbb{1}\{\tau > tn\}\mathbb{1}\{\tau \leq (t+1)n\} \\ &\geq \mathbb{1}\{\tau \leq tn\} + \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{(t+1)n} \in B\} \\ &\geq \mathbb{1}\{\tau \leq tn\} + \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbb{1}\{X_{(t+1)n} \in B\}. \end{aligned}$$

Taking expectations yields

$$p_{t+1} \geq p_t + \mathbf{E}_x \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbb{1}\{X_{(t+1)n} \in B\}.$$

We estimate the last expectation as follows:

$$\begin{aligned} &\mathbf{E}_x \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbb{1}\{X_{(t+1)n} \in B\} \\ &\qquad = \mathbf{E}_x[\mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbf{E}_x[\mathbb{1}\{X_{(t+1)n} \in B\}|\mathscr{F}_{tn}]] \\ &\qquad = \mathbf{E}_x[\mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\Phi^n(X_{tn}, B)] \\ &\qquad \geq \mathbf{E}_x \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\delta \\ &\qquad = \mathbf{E}_x(1 - \mathbb{1}\{\tau \leq tn\})\mathbb{1}\{X_{tn} \in C\}\delta \\ &\qquad = \mathbf{E}_x \mathbb{1}\{X_{tn} \in C\}\delta - \mathbf{E}_x \mathbb{1}\{\tau \leq tn\}\mathbb{1}\{X_{tn} \in C\}\delta \\ &\qquad \geq (1 - \epsilon)\delta - \mathbf{E}_x \mathbb{1}\{\tau \leq tn\}\delta \\ &\qquad = (1 - \epsilon)\delta - p_t\delta. \end{aligned}$$

$$\therefore \quad p_{t+1} \geq p_t + (1 - \epsilon)\delta - p_t\delta = (1 - \delta)p_t + (1 - \epsilon)\delta.$$

The unique, globally stable fixed point of $q_{t+1} = (1 - \delta)q_t + (1 - \epsilon)\delta$ is $1 - \epsilon$, so

$$1 - \epsilon \leq \lim_{t \to \infty} p_t = \mathbf{P}_x\{\tau < \infty\} \leq 1$$

Since ϵ was arbitrary, we obtain $\mathbf{P}_x\{\tau < \infty\} = 1$. $\square$

Proof of lemma 7.2. Fix $x, x' \in S$ and $\epsilon > 0$. Since Q is bounded in probability, we can choose compact sets C and C' such that

$$Q^t(x, C) \geq (1-\epsilon)^{1/2} \quad \text{and} \quad Q^t(x', C') \geq (1-\epsilon)^{1/2} \quad \text{for all } t.$$

$$\therefore \quad (Q \times Q)^t((x, x'), C \times C') = Q^t(x, C) Q^t(x', C') \geq 1 - \epsilon \quad \text{for all } t.$$

Since $C \times C'$ is compact in the product space, $Q \times Q$ is bounded in probability. □

Proof of lemma 7.3. Fix $\mu \in \mathscr{P}$ and $\epsilon > 0$. Since individual elements of $\mathscr{P}$ are tight (Dudley, 2002, theorem 11.5.1), we can choose a compact set $C_\mu \subset S$ with $\mu(C_\mu) \geq 1 - \epsilon$. By assumption, we can take an order interval $[a, b]$ of S with $C_\mu \subset [a, b]$. For this a, b, we have

$$\mu([a,b]^c) = \mu(S \setminus [a,b]) \leq \epsilon. \tag{24}$$

By hypothesis, $\{Q^t(x, \cdot)\}$ is tight for all $x \in S$, so we choose compact subsets C_a and C_b of S with $Q^t(a, C_a) \geq 1 - \epsilon$ and $Q^t(b, C_b) \geq 1 - \epsilon$ for all t. Since $C_a \cup C_b$ is also compact, we can take an order interval $[\alpha, \beta]$ of S with $C_a \cup C_b \subset [\alpha, \beta] \subset S$. We then have $Q^t(a, [\alpha, \beta]) \geq 1 - \epsilon$ and $Q^t(b, [\alpha, \beta]) \geq 1 - \epsilon$ for all t. Letting $I_\alpha := \{x \in S : x \geq \alpha\}$ and $D_\beta := \{x \in S : x \leq \beta\}$, this leads to

$$\therefore \quad Q^t(a, I_\alpha) \geq 1 - \epsilon \quad \text{and} \quad Q^t(b, D_\beta) \geq 1 - \epsilon \quad \text{for all } t. \tag{25}$$

In view of remark 2.2 and (25), we have

$$a \leq x \implies Q^t(x, I_\alpha) \geq Q^t(a, I_\alpha) \geq 1 - \epsilon,$$

and, by a similar argument,

$$x \leq b \implies Q^t(x, D_\beta) \geq Q^t(b, D_\beta) \geq 1 - \epsilon.$$

Since $[\alpha, \beta] := \{x \in S : \alpha \leq x \leq \beta\} = I_\alpha \cap D_\beta$, we have

$$Q^t(x, [\alpha, \beta]^c) = Q^t(x, D_\beta^c \cup I_\alpha^c) \leq 2 - Q^t(x, D_\beta) - Q^t(x, I_\alpha).$$

This leads to the estimate

$$a \leq x \leq b \implies Q^t(x, [\alpha, \beta]^c) \leq 2\epsilon. \tag{26}$$

Combining (24) and (26), we now have

$$\begin{aligned}
\mu Q^t([\alpha,\beta]^c) &= \int Q^t(x,[\alpha,\beta]^c)\mu(dx) \\
&= \int_{[a,b]} Q^t(x,[\alpha,\beta]^c)\mu(dx) + \int_{[a,b]^c} Q^t(x,[\alpha,\beta]^c)\mu(dx) \\
&\le \int_{[a,b]} 2\epsilon\,\mu(dx) + \mu([\alpha,\beta]^c) \;\le\; 3\epsilon.
\end{aligned}$$

Since $[\alpha,\beta]$ is compact and t is arbitrary, we conclude that $\{\mu Q^t\}$ is tight. □

Proof of lemma 7.4. Let C be any compact subset of $S \times S$. We need to prove existence of an $n \in \mathbb{N}$ and $\delta > 0$ such that $(Q \times Q)^n((x,x'),\mathbb{G}) \ge \delta$ whenever $(x,x') \in C$. To do so, we introduce the function

$$\psi_n(x,x') := (Q \times Q)^n((x,x'),\mathbb{G}) = \mathbf{P}_{x,x'}\{X_n \le X'_n\}.$$

Intuitively, since Q is increasing, the event $\{X_n \le X'_n\}$ becomes less likely as x rises and x' falls, and hence $\psi_n(x,x')$ is decreasing in x and increasing in x' for each n. A routine argument confirm this is the case.

Since $C \subset S \times S$ is compact, we can take an order interval $[a,b]$ of S with $C \subset [a,b] \times [a,b]$.[29] Moreover, since Q is order reversing, we can take $n \in \mathbb{N}$ such that $\delta := \psi_n(b,a) = \mathbf{P}_{b,a}\{X_n \le X'_n\} > 0$. Observe that

$$(x,x') \in C \implies (x,x') \in [a,b]\times[a,b] \implies x \le b \text{ and } x' \ge a.$$

$$\therefore \quad (x,x') \in C \implies (Q\times Q)^n((x,x'),\mathbb{G}) = \psi_n(x,x') \ge \psi_n(b,a) = \delta.$$

In other words, $\mathbb{G}$ is uniformly accessible for $Q \times Q$. □

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