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The evolution of mixed conjectures in the rent-extraction game*

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Abstract

This paper adopts an evolutionary perspective on the rent-extraction model with conjectural variations (CV). We analyze the global dynamics of the model with three CVs under the replicator equation. We find that the end points of the evolutionary dynamics include the pure-strategy consistent CVs. However, there are also mixed-strategy equilibria that occur. These are on the boundaries between the basins of attraction of the pure-strategy sinks. We develop a more general notion of consistency which applies to mixed-strategy equilibria. In a three conjecture example, we find that in contrast to the pure-strategy equilibria, the mixed-strategy equilibria are not ESS: under the replicator dynamics, there are three or four mixed equilibria that may either be totally unstable, or saddle-stable. There also exist heteroclinic orbits that link equilibria together.

JEL: D03, L15, H0.

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1 Introduction.

In this paper, we adopt a dynamic approach to analyze the evolution of beliefs underlying agents' behavior in the context of a rent-extraction game à la Tullock (1980). The idea is that the boundedly-rational agents employ decision rules, such as reaction functions, based on certain beliefs about other players' behavior. But how are these beliefs formed? Recently, some authors have adopted an evolutionary approach to explaining such beliefs using Maynard-Smith's notion of evolutionary stable strategies (ESS)[1]. ESS is however a *local* stability condition: it considers the effects on payoffs of a small deviation in the make-up of the population. In this paper we broaden the focus to consider the *global* dynamics of an explicit evolutionary process - the replicator equation. We apply this dynamic evolutionary approach to explain belief formation in the context of a rent-seeking game (Tullock 1967, 1980, 1987, Posner1975) where agents spend resources to dispute over rents or some prizes. Agents' beliefs about other players' behavior are particularly important in such models as it can directly impact the magnitude of the rent extracted by altering the success function. Importantly, rent-seeking models have many applications in economics and politics e.g. in elections where resources allocated to campaigning directly affect the candidate's probability of success and where the allocation itself is done based on the agent's belief about his opponent's behavior. Menezes and Quiggin (2010) have provided several different interpretations of such rent-extraction models and have argued that such models should be viewed as oligopsonistic markets for influence.

A decision rule in this context can be thought of as a reaction function (RF) which specifies the choice of action as a function of other agents' actions. Whilst there are various ways of parametrizing such decision rules, the one we adopt in this paper is the concept of *Conjectural Variations*. The notion of conjectures has maintained a long history in the Industrial Organization theory ever since the introduction of *Conjectural Variations Equilibria* by Bowley (1924) and Frisch (1933)[2]. Not only are conjectural variations (henceforth CV) models able to capture a range of behavioral outcomes - from competitive to cooperative, but also they have one parameter which has a simple economic interpretation. CV models have also been found quite useful in the empirical analysis of firm behavior in the sense that they provide a more general description of firms' behavior than the standard Nash equilibrium

[1] See, e.g. Dixon and Somma (2003), Müller and Norman (2005), Possajennikov (2009). See Jean-Marie and Tidball (2006) for a non-evolutionary approach to formation of conjectures in a dynamic context.

[2] See Giacoli (2005) for a detailed account of the role of conjectural variations in the history of oligopoly games.

(Slade 1995).

In this context, the concept of *consistent* conjectures was developed by a number of authors in the 1980s (see Bresnehan 1981, Boyer and Moreaux 1983, Klemperer and Meyer 1988) and has been widely applied in a variety circumstances such as public goods (Cornes and Sandler 1984[3], Itaya and Okamura 2003), strategic investment models (Dixon 1986), export subsidies (Tanaka 1991), natural resource extraction (Quérou and Tidball 2009). In Public Economics, Michaels (1989) applied this concept in the context of Tullock's rent-seeking game to show that the fraction of rents dissipated by seekers depends upon the type of CV assumed. In games with quadratic payoffs where the best-response functions with CVs are linear, the natural formulation for consistent conjectures is that the CV of one player equals the actual slope of the other player's RF. However, in games where the payoffs are not quadratic and therefore the RFs are non-linear (such as the ones in rent-seeking models with CVs), the notion of consistency can accordingly be adapted: consistency should imply that CVs are equal to the slopes of RFs at the *equilibrium* point.

Recently, the link between *consistency* and *evolutionary stability* has been made within the CV framework. One can think of economic agents' behavior being summarized by the CV term. One can imagine a population consisting of firms with different CVs which will earn different payoffs (on average) and a process of "natural selection" or social learning takes place (the CV is a *meme*). Firms with particular CVs do better than those with others: a process of imitation or adaption leads agents to switch from less successful CVs to more successful CVs. Dixon and Somma (2003) established that in a standard oligopoly setting with a quadratic payoff function[4], the consistent conjectures are the unique Nash equilibrium in a hypothetical "conjecture game": firms choose their CVs given the CVs of the other firms so as to maximize their payoffs in the output game[5]. This Nash equilibrium in the conjecture game was the *consistent conjecture*. This enabled the link to be made with evolutionary stable strategies (ESS). In the case where there is a strict-Nash equilibrium in the conjecture game, the resultant consistent conjecture will be ESS. Müller and Normann (2005) generalized this result to a wider class of oligopoly models[6]. Both Dixon and Somma (2003) and Müller and Normann (2005) were in the class of quadratic payoff models. Possajennikov (2009) showed that the link between ESS models and consistent conjectures

[3]See also Cornes and Sandler (1985) and Sugden (1985).

[4]Specifically, they consider a homogeneous good Cournot oligopoly with linear demand and quadratic costs.

[5]This idea was first developed in Dixon and Somma (1995).

[6]Specifically, differentiated oligopoly with linear demands.

extends to some non-quadratic payoff models, including the rent-seeking model (such as the one considered by Michaels (1989)).

However, all of the above studies were limited in that they focussed exclusively on *pure-strategy* equilibria and that they only studied *local stability* using the ESS condition. In contrast, the main contribution of this paper is to extend the focus to analyze the *global* evolutionary dynamics in the context of *mixed-strategies*. We do indeed find that in addition to the pure-strategy equilibria, mixed-strategy equilibria will exist in a finite version of the conjecture game where we restrict the set of permissible CVs to a finite set of three conjectures. We provide a bifurcation analysis and show that in addition to the three pure-strategy equilibria, there will exist at least three mixed-strategy equilibria (Proposition 2).

Further, we define a new concept of consistency that is applicable to the case of mixed-strategy equilibria. This is the notion of the probability that the conjectures will be consistent *ex post*. In the case of a pure-strategy equilibrium, the standard consistent conjectures are 100% consistent ex post. With mixed strategy equilibria, the conjectures will only be consistent a certain proportion of the time. We also define a notion of *ex ante* consistency, that the average conjecture equals the average slope. We find that in the rent-extraction model in which RFs are non-linear, the average conjecture in a mixed-equilibrium is not equal to the ex-post average slope of the reaction function. Hence, whilst the link between consistency and equilibrium in the conjecture game still exists, it is weaker in the case of mixed-strategies than for pure-strategy equilibria.

Our main results about the dynamics are as follows. Proposition 3 summarizes the local dynamics: the pure strategy-equilibria are sinks (both eigenvalues stable), whilst the strictly-mixed stationary points can either be saddle-path stable (one stable, one unstable eigenvalue) or unstable sources (both eigenvalues unstable). For the global dynamics, in Proposition 4 we find that there is a network of *heteroclinic* orbits[7] that connect equilibria. The heteroclinic orbits connecting these mixed-strategy stationary points with each other and the pure-strategy sinks constitute the boundaries of the basins of attraction for the pure-strategy sinks. There are two generic phase diagrams which describe the exact pattern of equilibria: in particular, if the most competitive conjecture is competitive enough we can have an internal mixed-equilibrium (with all three conjectures with strictly positive shares) which is a source. Otherwise, we have the more general case where there are three

[7] An *heterclinic orbit* is an equilibrium path that connects two (or more) stationary points. This contrasts to *homoclinic orbits* which have only one stationary point as an end-point.

stationary points involving only two conjectures with strictly positive probabilities: two of these stationary points are Nash equilibria (and saddle-path stable) with the third being a non-Nash equilibrium unstable source.

We can use the global dynamics as a guide to equilibrium selection. The most cooperative pure-strategy equilibrium is Pareto-dominant (from the point of view of the rent seekers) and involves the least rent dissipation and highest payoff. However, we do not find that in general the most cooperative conjecture has the biggest basin. Indeed, in the three conjecture case we might expect the intermediate conjecture to have the bigger basin. The reason is that in the rent-extraction model, the intermediate CV can do quite well against the two extremes, whilst the two extremes do badly against each other. Moderation can pay. This means that the intermediate conjecture can end up with a share of 1 even if it starts from a share of almost zero. In contrast, the two extreme conjectures require an initial base which is bounded well away from zero if they are to be selected. In contrast, the two extremes require an initial base which is bounded well away from zero if they are to be selected. Whilst we cannot in general rank the most cooperative and the intermediate conjecture, we can in general say that the most competitive equilibrium will have a smaller basin than the most cooperative. Indeed, in the extreme case of a "Bertrand" CV of -1, the basin of attraction shrinks to zero.

The notion of evolutionary dynamics (such as the replicator) is not unproblematic[8]: if one takes a literal view of the equations, they are based on random matching with the game played repeatedly in continuous time. However, one can think of this more as an *evolutionary metaphor*: over time, more successful strategies become more common. There are a variety of ways this can happen in social learning models. However, to explore the dynamics without recourse to simulating simple models we need to use a specific evolutionary process: the replicator equation is a robust framework that can stand for a wider class of payoff-monotone dynamics.

The organization of the paper is as follows. In section 2, we outline the basic rent-seeking model, which can also be thought of as a Cournot Oligopoly game, where we treat the conjectures as given. In section 3, we consider the underlying conjecture game and pure-strategy equilibria in the case where the strategy sets are a closed convex subset of the real line, and mixed-strategy equilibria where the strategy sets are a finite subset of the pure-strategy case. In section 4, we consider the relation between consistency and the equilibria in the conjecture game. In section 5, we analyze the evolutionary process of the

[8] We would like to thank Hans Haller for stressing this point to us.

model using the replicator equation. Section 6 concludes. All proofs are in the appendix.

2 The model.

We consider the following game where two firms X and Y choose actions $(x,\ y)$ independently with payoff functions given as follows:

$$\begin{aligned} U^X(x,y) &= \frac{x}{x+y} - x \\ U^Y(x,y) &= \frac{y}{x+y} - y \end{aligned}$$

This can be thought of as a simple rent-seeking game *à la* Tullock (1980) where players choose actions (e.g. effort or investment) to win a prize of fixed value (which is unity in the above formulation), where the first term in the payoff function denotes the probability of player i's winning the contest, $i = X, Y$, and the second term denotes constant unit cost of the action. Alternatively, this game can also be thought of as a homogeneous good Cournot duopoly[9] with unit elastic demand and constant unit cost where the market price is given by

$$P = \frac{1}{x+y}$$

so that total revenue equals 1, each firm receives a share of that revenue equal to its share of output[10], and the total cost of player i equals player i's output. For economically meaningful outcomes, we can restrict our attention to the strategy-space:

$$S = \{(x,y) : x \geq 0, y \geq 0 \text{ and } x + y \leq 1\}$$

The above payoff-function is strictly concave for $(x,y) \in S \cap (0,1)^2$. The corresponding iso-payoff sets for X are characterized by

$$\bar{U}^X = \{(x,y) : U^X(x,y) = \bar{U}\}$$

[9]It has been shown that a standard Tullock contest of the above type is strategically equivalent to a Cournot oligopoly game, and that the same strategic equivalence applies also with a more general success function in the original Tullock game (see Okuguchi 1995, Szidarovsky and Okuguchi 1997).

[10]Henceforth, we will refer to x and y as 'outputs'.

and have slopes given by

$$\left.\frac{dy}{dx}\right|_{\bar{U}^x} = \frac{y - x^2 - 2xy - y^2}{x}$$

For $\bar{U} \in (0,1)$, the iso-payoff curve intersects the $x-$ axis at $\left(1-\bar{U},0\right)$. However, all iso-payoff sets with $\bar{U} \in (0,1)$ originate from $(0,0)$. Appendix A provides further details about the shape of the iso-payoff functions. The payoff function is undefined for $x = y = 0$. However, in order to convert the joint profit supremum into a maximum, we adopt the definition $U^X(0,0) = U^Y(0,0) = 0.5$. In the event of neither player doing anything, the prize is split.

2.1 Conjectural variation (CV) output game.

Each firm has a *conjecture* about the response of the other firm to variations in its own output. $\phi_x = \partial y/\partial x$ and $\phi_y = \partial x/\partial y$ denote such conjectures held by firms X and Y respectively where $\phi_i \in [-1,+1]$, $i = x, y$. This gives the reaction functions (RFs) defined by the following first-order conditions:

$$\begin{aligned}
\frac{1}{(x+y)} - \frac{x}{(x+y)^2}(1+\phi_x) - 1 &= 0 \\
\frac{1}{(x+y)} - \frac{y}{(x+y)^2}(1+\phi_y) - 1 &= 0
\end{aligned}$$

From above, we get the reaction functions in the following form:

$$x = R(y,\phi_x) = -\frac{1}{2}\phi_x - y + \frac{1}{2}\sqrt[+]{\phi_x^2 + 4\phi_x y + 4y} \tag{1}$$

$$y = R(x,\phi_y) = -\frac{1}{2}\phi_y - x + \frac{1}{2}\sqrt[+]{\phi_y^2 + 4\phi_y x + 4x} \tag{2}$$

For $\left\{(\phi_x,\phi_y) \in [-1,1]^2 \text{ and } 1-\phi_y\phi_x > 0\right\}$, the equilibrium values of output are given by:

$$x\left(\phi_x,\phi_y\right) = \frac{(1+\phi_y)(1-\phi_y\phi_x)}{(2+\phi_y+\phi_x)^2} \tag{3}$$

$$y\left(\phi_x,\phi_y\right) = \frac{(1+\phi_x)(1-\phi_y\phi_x)}{\left(2+\phi_y+\phi_x\right)^2} \tag{4}$$

In the cases where $\phi_y\phi_x = 1$, we set $x(1,1) = 0$ and $x(-1,-1) = \frac{1}{2}$ and likewise for

y, these being the limiting values[11]. In case of *symmetric* conjectures ($\phi_x = \phi_y = \phi$), equilibrium outputs will be given by

$$x(\phi,\phi) = y(\phi,\phi) = \frac{1-\phi}{4} \tag{5}$$

We can consider the following special cases:

(i) Cournot-Nash conjectures: $\phi_x = \phi_y = 0$

(1) and (2) then yield

$$\begin{aligned} x &= -y + \sqrt{y} \\ y &= -x + \sqrt{x} \end{aligned}$$

so that Cournot-Equilibrium values are

$$\begin{aligned} x^c &= y^c = \frac{1}{4} \text{ and} \\ U^X &= U^Y = U|_{\text{Cournot}} = \frac{1}{4} \end{aligned}$$

(ii) Bertrand-Nash conjectures: $\phi_x = \phi_y = -1$

(1) and (2) then yield

$$\begin{aligned} x &= 1 - y \\ y &= 1 - x \end{aligned}$$

which has the set of solutions $x + y = 1$, with the symmetric solution being at $x = y = \frac{1}{2}$ with corresponding equilibrium payoffs $U|_{\text{Bertrand}} = 0$.

(iii) Fully collusive conjectures: $\phi_x = \phi_y = 1$

In this case, (3) and (4) imply $x = y = 0$.This is the joint profit maximum.

[11] Alternatively, one can restrict the strategy set to $[-1+\varepsilon, 1-\varepsilon]$ for some arbitrarily small $\varepsilon > 0$.

3 The Conjecture Game.

In order to analyze the evolutionary properties of conjectures, following Dixon and Somma (2003), we consider a further stage of the game where firms are choosing their conjectures.[12] We will first analyze this hypothetical "conjecture game" in terms of pure-strategies, where the strategy sets are intervals on the real line $[-1, 1]$. We will then consider the case of finite strategy sets in order to analyze the possible existence of mixed-strategy equilibria where more than one strategy is played with a positive probability.

3.1 Pure-strategy equilibria.

Given the equilibrium outputs as a function of the conjectures, we can think of a *reduced* form game of the equilibrium given conjectures with each firm choosing its conjecture. The payoffs for the conjecture game, after simplification, are:

$$U^X(\phi_x, \phi_y) = \frac{(1+\phi_x)(1+\phi_y)^2}{(2+\phi_y+\phi_x)^2} \tag{6}$$

$$U^Y(\phi_x, \phi_y) = \frac{(1+\phi_y)(1+\phi_x)^2}{(2+\phi_y+\phi_x)^2} \tag{7}$$

Firms' equilibrium choice of conjectures will then be obtained from the following first-order conditions:

$$\frac{dU^X(\phi_x, \phi_y)}{d\phi_x} = -(1+\phi_y)\frac{(\phi_y+\phi_y\phi_x-\phi_y-\phi_y^2)}{(2+\phi_y+\phi_x)^3} = 0$$

$$\frac{dU^Y(\phi_x, \phi_y)}{d\phi_y} = -(1+\phi_x)\frac{(\phi_x+\phi_y\phi_x-\phi_x-\phi_x^2)}{(2+\phi_y+\phi_x)^3} = 0$$

This yields the following reaction functions in the conjecture game:

$$R^X(\phi_y) = \phi_y$$

$$R^Y(\phi_x) = \phi_x$$

[12] The entire game can equivalently be considered as a two-stage game where, as if, firms choose their conjectures in the first stage, and then given their choice of conjectures in the first stage, they choose outputs in the second stage.

That is, the best-response of firm is to choose the same conjecture as the other firm[13]. Thus, we have the following proposition (stated without proof):

Proposition 1 *Pure strategy Nash equilibrium conjectures are symmetric.*

Thus, there is a continuum of "strict" Nash equilibria, each parameterized by the symmetric conjecture $\phi \in [-1, 1]$ with equilibrium output levels given by (5) and symmetric payoffs given by:

$$U(\phi) = \frac{(1+\phi)(2\phi + \phi^2 + 1)}{4(1+\phi)^2} = \frac{1+\phi}{4} \tag{8}$$

There is also a "Bertrand" Nash equilibrium which is not strict: if one firm sets $\phi = -1$, then the other firm earns zero profits whatever conjecture it has. Clearly, the equilibria are Pareto-ranked: the higher the conjecture, the higher the profits, with the limiting profit being half the joint profit maximum $U(1) = \frac{1}{2}$ and the minimum being the Bertrand case $U(-1) = 0$. The structure of the conjecture game is similar to a coordination game, except that the "off-diagonal" elements vary with the conjectures.

3.2 Mixed-strategy Equilibria.

Mixed-strategy equilibria will also exist if we take a finite subset of conjectures. In this section, we provide an example, prior to a more general analysis when we model the evolutionary dynamics. Consider a finite subset of conjectures Φ taken from $[-1, +1]$, with $\#\Phi = n$, so that $\Phi = \{\phi_i\}_{i=1}^n$ This then gives us an $n \times n$ payoff matrix A :

$$A(n) = \left[\pi_{ij} = U^I(\phi_i, \phi_j\right]_{i=1...n}^{j=1...n}, I = X, Y$$

The row i gives the payoff to the firm playing each strategy j (conjecture) against i, while the column j gives the payoff of playing strategy j against each of the strategies i.

The mixed-strategy for player X is the n vector of probabilities $z_x \in \Delta^{n-1}$, and likewise for player Y. The expected payoff for player X is then:

$$EU_x(z_x, z_y) = z_x' A z_y$$

[13] The second order conditions are satisfied, since

$$\frac{d^2U^X(\phi_x, \phi_y)}{d\phi_x^2} = -\frac{2(1+\phi_y)}{(2+\phi_y+\phi_x)^4}(1 + 3\phi_y + 2\phi_y^2 - \phi_x(1-\phi_y))$$

which is strictly negative when $\phi_y = \phi_x$.

or

$$EU_x = \sum_{i=1}^{n}\sum_{j=1}^{n} z_{ix} z_{jy} \pi_{ij}$$

That is, each element π_{ij} in the payoff matrix A is weighted with the probability of X playing i and Y playing j. A (symmetric) mixed-strategy equilibrium $\mathbf{z}^*$ occurs when for all $\mathbf{z} \in \Delta^{n-1}$,

$$EU_x(\mathbf{z}^*, \mathbf{z}^*) \geq EU_x(\mathbf{z}, \mathbf{z}^*)$$

In particular, all mixed-strategy equilibria have the property that if $z_i^* > 0$, then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} z_i^* z_j^* \pi_{ij} = EU_x(\mathbf{z}^*, \mathbf{z}^*)$$

That is, all strategies that are played with a strictly positive probability earn the same expected profit in equilibrium.

If we consider the 3×3 payoff matrix generated by conjectures $\Phi = \left\{0, \frac{1}{2}, 1\right\}$, we have:

$$A = \begin{pmatrix} 0.25 & 0.36 & 0.4444 \\ 0.24 & 0.375 & 0.4898 \\ 0.2222 & 0.3673 & 0.5 \end{pmatrix} \tag{9}$$

In addition to the 3 pure strategy equilibria, there are also 2 mixed equilibria. Adapting the notation slightly, so that $z(\phi)$ is the probability that conjecture ϕ is played, the 2 mixed equilibria are given by:

- $z^*(1) = 0.4302$, $z^*(\frac{1}{2}) = 1 - z^*(1)$, $z^*(0) = 0$.
- $z^*(1) = 0$, $z^*(\frac{1}{2}) = 0.4$, $z^*(0) = 0.6$.

There is the following profile in which $z^*(\frac{1}{2}) = 0$, and the two conjectures $(1, 0)$ earn equal payoffs:

$$z^*(1) = \frac{1}{3},\ z(\frac{1}{2}) = 0,\ z^*(0) = \frac{2}{3}$$

This is not an equilibrium, because the expected payoff from playing 0.5 exceeds the payoffs of the other two. Note that in this example, both mixed-equilibria involve only pairs of strategies being played with strictly positive probabilities, there being no equilibrium with all three strategies being played.

4 Consistency of conjectures.

There are several definitions of consistency of conjectures available[14]. However, we use the one in the sense of Bresnahan (1981), that in the output game each firm's conjecture about the slope of the other firm's reaction function is correct at the equilibrium outputs. Unlike the quadratic payoff framework considered by Dixon and Somma (2003) and Müller and Normann (2005), the CV reaction functions are not linear in this model, so that consistency-correctness at equilibrium outputs does not imply correctness elsewhere. This has important implications for the evolutionary stability of equilibria as we shall see.

From (1) and (2), the slopes of the reaction functions are:

$$\frac{dR(y,\phi_x)}{dy} = -1 + \frac{1+\phi_x}{\sqrt{\phi_x^2 + 4\phi_x y + 4y}} \tag{10}$$

$$\frac{dR(x,\phi_y)}{dx} = -1 + \frac{1+\phi_y}{\sqrt{\phi_y^2 + 4\phi_y x + 4x}} \tag{11}$$

Now, we can set the outputs (x, y) at their equilibrium values given $\left(\phi_x, \phi_y\right)$ using (3), (4), and then consider whether or not the conjectures are consistent.

4.1 Pure-strategy Equilibria and consistency.

From Proposition 1, we can focus attention only on the symmetric conjectures: $\phi_x = \phi_y = \phi$. Equations (10) and (11) then simplify as:

$$\frac{dR(y,\phi)}{dy} = -1 + \frac{1+\phi}{\sqrt{\phi^2 + 4\phi y + 4y}} \tag{12}$$

$$\frac{dR(x,\phi)}{dx} = -1 + \frac{1+\phi}{\sqrt{\phi^2 + 4\phi x + 4x}} \tag{13}$$

Evaluating the above slopes at the equilibrium values of output given by (5) and simplifying, we find:

$$\frac{dR(y,\phi)}{dy} = \phi \tag{14}$$

$$\frac{dR(x,\phi)}{dx} = \phi \tag{15}$$

[14] See, e.g. Hahn 1977, 1978; Perry 1982; Kamien and Schwartz 1983; Boyer and Moreaux 1983.

Hence, all pure-strategy (symmetric) Nash equilibrium conjectures are consistent.[15] This is true for any $\phi \in [-1, 1]$ so that:

Observation 1 The set of consistent conjectures equilibria is equivalent to the set of pure-strategy Nash equilibria in the conjecture game.

Further, we also observe that,

Observation 2 Unlike Bresnahan (1981), Cournot conjectures are consistent in this model.

To see that, note for $\phi_x = 0$,the slope of firm x's RF:

$$\frac{dR(y,0)}{dy} = -1 + \frac{1}{2\sqrt{y}}$$

which when evaluated at Cournot output level $y = 1/4$, yields $\frac{dR(y,0)}{dy} = 0$. Likewise for $\phi_y = 0$.

However, if the conjectures are asymmetric i.e. $\phi_x \neq \phi_y$ (as is the case in mixed-strategy) then that will involve inconsistent conjectures (in the above sense) by one or both of the firms.[16].

4.2 Mixed-strategy equilibria and consistency.

Is there any sense in which a mixed-strategy equilibrium in the conjecture game is consistent? We need to modify the notion of consistency in this case. Now, there are two possible ways of defining consistency in the context of mixed strategies.

Definition 1: The Probability of consistency. *In equilibrium, there is a probability that both players will choose the same conjectures.*

[15] A similar result is also be found in Michaels (1989) who showed that there can be multiple equilibria in the standard symmetric form of the game where any CV can be consistent. Michaels however does not consider a conjecture stage of the game as we do in this paper.

[16] To see that, note that the slopes of the reaction functions given by $(10, 11)$ when evaluated at equilibrium output values given by $(3), (4)$ yields (since $\phi_y \neq \phi_x$):

$$\frac{dR(y,\phi_x)}{dy} = \frac{\phi_x + \phi_y + 2\phi_x\phi_y}{\phi_x^2 - \phi_x\phi_y + 2(1+\phi_x)} \neq \phi_y$$

and similarly for the other firm.

If both players choose the same conjecture, their conjectures are "consistent" in the resultant game *ex post*. If they choose different conjectures, they will not be consistent. Hence, we can define the probability of *ex post* consistency:

$$PC(z^*) = \sum_{i=1}^{n} (z_i^*)^2$$

For example, in the two mixed Nash equilibria identified in the 3×3 example, we have:

$$\begin{aligned} PC(0.6, 0.4, 0) &= 0.36 + 0.16 = 0.52 \\ PC(0.4302,\ 0.5698, 0) &= (0.430\,2)^2 + (\ 0.569\,8)^2 = 0.509\,8 \end{aligned}$$

In the case of pure-strategy equilibria, of course $PC(1) = 1$: the conjecture is correct in equilibrium. However, when we have strictly mixed-strategies, the conjectures will only be correct a certain proportion of the time: in the three mixed equilibria in our game they are correct $51 - 52\%$ of the time.

Definition 2: The average conjecture equals the average slope. *The average conjecture equals the average slope of the reaction function encountered.*

We have the result that in a symmetric equilibrium, the slopes of the RFs are equal to the conjecture. Now, *if the RFs were linear*, this would immediately imply that the average slope must equal the average conjecture. Hence with linear RFs, any mixed-strategy equilibria must satisfy Definition 2. However, here the matter is more complicated due to the non-linearity in the RFs. If we look at the *ex post* game, when the firms have different conjectures, the outputs will be different and we will have asymmetric outcomes. In the case of linear RFs, asymmetry in the outcome does not affect the slope of the reaction function, since this is constant. With non-linear reaction functions such as the ones we have in the rent-extraction game, the slopes of the reaction functions vary as we move away from the 45^0 line in output space. We know the equilibrium outputs given the conjectures from (3), (4); we know the *slopes* of the reaction functions given outputs and conjectures from (10), (11). Hence we can define the *average* slope and the *average* conjecture.

Let us take the simple example of the mixed-strategy equilibrium where $z^*(1) = 0.4302$, $z^*(\frac{1}{2}) = 0.5698$, $z^*(0) = 0$. Here the average conjecture $\bar{\phi} = 0.4302 + (0.5)0.5698 = 0.715\,1$. For 49% of the time there will be an asymmetric equilibrium where one firm will have a zero conjecture and the other a conjecture of 1. If $\phi_x = 0.5$, $\phi_y = 1$, then we have the following slopes at the resulting outputs

$$\frac{dy}{dx} = 0.555\,6 > \phi_x = 0.5. \quad (16)$$
$$\frac{dy}{dx} = 0.909\,1 < \phi_y = 1 \quad (17)$$

We can see that the slopes in the asymmetric equilibria are not so different from the conjectures in this example (only about a 10% difference). However, note that the average slope in the asymmetric case is 0.732 4 which is almost equal to the average conjecture in the asymmetric case 0.75. Y's slope is a little greater than X's conjecture ϕ_x: X's slope is a little less than Y's conjecture, but the two effects partially cancel each other out, so that the average slope is only a little less than the average conjecture.

Denote the average slope of the reaction functions by $\bar{R}$ for the two symmetric cases (when both conjectures equal 1 and 0.5, in turn) and the asymmetric case (when one conjecture equals 1, and the other equals 0.5). Then, in this example we have $\bar{R} = 0.7065$,[17] which does not equal the average conjecture $\bar{\phi} = 0.7151$. Hence, the average conjecture does not equal the average slope of the reaction function in this case. However, the deviation is not so great: that is because the slopes in the asymmetric case are (on average) not so much different from the conjectures see (16) and (17).

Hence we can conclude that,

Observation 3. Being an equilibrium in the conjecture game need not imply consistency of conjectures.

The above result differs from Dixon and Somma (2003) where it was derived for a simple game in which there were no mixed equilibria and the RFs were linear. However, in the context of our rent-extraction game, there are many pure-strategy equilibria and also many mixed equilibria. The possibility of mixed-equilibria does imply a weaker degree of consistency of conjectures: this is the ex post proportion of outcomes in which the conjectures turn out to be consistent. With pure-strategy equilibria, this is 100%. With strictly mixed

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$$\begin{aligned}\bar{R} &= \Pr\left[\phi_x = \phi_y = 1\right] 1 + \Pr\left[\phi_x \neq \phi_y\right]\left[\frac{1.5}{2}\right] + \Pr\left[\phi_x = \phi_y = 0.5\right] 0.5 \\ &= (0.4302)^2 + (1 - (0.4302)^2 - (1 - 0.4302)^2)\left(\frac{0.555\,6 + 0.909\,1}{2}\right) + (1 - 0.4302)^2 0.5\end{aligned}$$

equilibria, the conjectures may not be correct all of the time, and may not be correct on average.

5 Global Evolutionary Dynamics.

In this section, we analyze the global dynamics of the model using the replicator equation. Previous authors have focussed only on the local stability of consistent conjectures using Maynard Smith's notion of an evolutionary stable strategy (ESS). Analyzing global dynamics is important as it will enable us to understand how the "population" behaves from any given starting point, rather than assuming a small deviation from a proposed equilibrium. Furthermore, this is particularly important in our context because of the large number of equilibria and the possibility of the dynamics providing a criterion for equilibrium selection.

For the purpose of analyzing the dynamics and for notational simplicity, we will reparamterize the model by defining the conjectures as: $\varphi_x \equiv 1 + \phi_x \in [0, 2]$ and $\varphi_y \equiv 1 + \phi_y \in [0, 2]$. Then, (6) and (7) can be written as:

$$
\begin{aligned}
U^X(\varphi_x, \varphi_y) &= \frac{\varphi_x \varphi_y^2}{(\varphi_y + \varphi_x)^2} \\
U^Y(\varphi_x, \varphi_y) &= \frac{\varphi_y \varphi_x^2}{(\varphi_y + \varphi_x)^2}
\end{aligned}
$$

To construct the matrix we take our finite set of n conjectures $\Phi = \{\phi_i\}_{i=1}^n$. This then gives an $n \times n$ payoff matrix $\mathbf{A}$:

$$\underset{n\times n}{\mathbf{A}} = \left[\pi_{ij} = U^I(\varphi_i, \varphi_j)\right]_{i=1\ldots n}^{j=1\ldots n}, I = X, Y.$$

The row i gives the payoff to the firm playing each strategy i (conjecture) against j. Column j gives us the payoff of playing strategy j against each of the strategies i. Matrix A is not symmetric - see (9). Let

$$r_{ij} = \frac{\varphi_i \varphi_j}{\left(\varphi_i + \varphi_j\right)^2}$$

For $\varphi_i \varphi_j \neq 0$, note that[18] $\pi_{ij}/\varphi_j = \pi_{ji}/\varphi_i = r_{ij}$. Further, note that, for any pair ij, r_{ij} is

[18] The asymmetry of A arises becuase when $\varphi_i \neq \varphi_j$, $\pi_{ij} \neq \pi_{ji}$. Also note that from (8) $\pi_{ii} = \varphi_i/4$.

smaller than 1/4. Hence, for notational convenience, we further define:

$$m_{ij} \equiv \frac{1}{4} - r_{ij} = \left(\frac{\varphi_i - \varphi_j}{2(\varphi_i + \varphi_j)} \right)^2 \geq 0, \text{ for all } j = i, \ldots, 3, \; i = 1, 2.$$

It then follows immediately that $m_{ij} = m_{ji}$ and $m_{ii} = 0$.

Let $\mathbf{z} = (z_1, \ldots, z_n) \in \Delta^{n-1}$, where z_i is the probability that conjecture i will be played . Then, the expected payoff of strategy i is

$$u_i(\mathbf{z}) = (\mathbf{Az})_i = \sum_{j=1}^{n} \pi_{ij} z_j, \; i = 1, \ldots, n,$$

and the $n \times 1$-vector of expected payoffs for all strategies $\mathbf{u}$ is

$$\mathbf{u} = \mathbf{Az}.$$

The mean across all strategies is then:

$$\bar{u}(\mathbf{z}) = \mathbf{z}^\top \mathbf{Az} = \sum_{i=1}^{n} \sum_{j=1}^{n} z_i z_j \pi_{ij}$$

The replicator dynamics is given by the n-dimensional ordinary differential equation system [19]

$$\dot{z}_i = F_i(\mathbf{z}) \equiv z_i \left(u_i(\mathbf{z}) - \bar{u}(\mathbf{z}) \right), \; i = 1, \ldots, n \tag{18}$$

where $\mathbf{z} \in \Delta^{n-1}$.

5.1 The 3×3 case

Here, we consider the case in which there are three conjectures, $n = 3$, where $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$, with $0 \leq \varphi_1 < \varphi_2 < \varphi_3 \leq 2$. In this case the probability profiles $\mathbf{z} = (z_1, z_2, z_3) \in \Delta^2$. The three profiles located at the vertices of the simplex Δ^2, $\mathbf{z} \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ or $\mathbf{e}_3 = (0, 0, 1)$, correspond to pure strategy equilibria. All the other profiles correspond to mixed strategies. The three boundary profiles located at one of the three edges of the simplex, excluding the vertices, $\mathbf{z} \in \{\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}\}$, where $\mathbf{e}_{12} = \{\mathbf{z} \in \Delta^2 : z_3 = 0\}$, $\mathbf{e}_{13} = \{\mathbf{z} \in \Delta^2 : z_2 = 0\}$ and $\mathbf{e}_{23} = \{\mathbf{z} \in \Delta^2 : z_1 = 0\}$, correspond to mixed strategies.

[19]See Sandholm (2010) for a recent account of the properties of this type of evolutionary dynamics.

We call them boundary mixed strategies in order to distinguish from the interior mixed strategies, which are located at the interior of the simplex, $\mathbf{z} \in \mathrm{Int}(\Delta^2)$.

Stationary states for system (18) are the probability profiles $\mathbf{z}^* \in \Delta^2$ such that $F_i(\mathbf{z}^*) = 0$ for all $i = 1, 2, 3$. However, not all steady-states of the replicator dynamics represent Nash equilibria: it is a property of the replicator dynamics that once extinct, a strategy can never return, so that $z_i = 0$ implies $\dot{z}_i = 0$ (since payoffs are bounded).

Now, for every strategy i there are two types of steady states that correspond to Nash equilibria: either $z_i > 0$ and $u_i(\mathbf{z}) = \bar{u}(\mathbf{z})$ or $z_i = 0$ and $u_i(\mathbf{z}) \leq \bar{u}(\mathbf{z})$. However, a steady-state with $z_i = 0$ and $u_i(\mathbf{z}) > \bar{u}(\mathbf{z})$ will not be a Nash-equilibrium (since the expected payoff for i can be increased by choosing $z_i > 0$).

Define

$$\Gamma = \Gamma(\varphi_1, \varphi_2, \varphi_3) \equiv m_{13} - (m_{12} + m_{23}) \tag{19}$$

Function Γ defined over $(\varphi_1, \varphi_2, \varphi_3)$, determines the number of stationary states and whether or not they are Nash equilibria as given by the following proposition[20]:

[20]Observe that $\varphi_1 = 0$ (i.e. the Bertrand case) implies $\Gamma < 0$.

Proposition 2: Stationary Profiles.

(a) Let $\Gamma > 0$. Then there are six stationary probability profiles, of which five are Nash equilibria :

(i) the three pure strategy profiles, $\mathbf{z}^* = \mathbf{e}_1$, $\mathbf{z}^* = \mathbf{e}_2$ and $\mathbf{z}^* = \mathbf{e}_3$ are all Nash equilibria;

(ii) three boundary mixed-strategy profiles belonging to the three edges are stationary states $\mathbf{z}^* = \mathbf{e}_{12}^*$, $\mathbf{z}^* = \mathbf{e}_{13}^*$, $\mathbf{z}^* = \mathbf{e}_{23}^*$, where

$$\mathbf{e}_{12}^* \equiv \left(\frac{\varphi_2}{\varphi_1 + \varphi_2}, \frac{\varphi_1}{\varphi_1 + \varphi_2}, 0 \right) \in \mathbf{e}_{12},$$

$$\mathbf{e}_{13}^* \equiv \left(\frac{\varphi_3}{\varphi_1 + \varphi_3}, 0, \frac{\varphi_1}{\varphi_1 + \varphi_3} \right) \in \mathbf{e}_{13}$$

and

$$\mathbf{e}_{23}^* \equiv \left(0, \frac{\varphi_3}{\varphi_2 + \varphi_3}, \frac{\varphi_2}{\varphi_2 + \varphi_3} \right) \in \mathbf{e}_{23},$$

of which $\mathbf{e}_{12}^*$ and $\mathbf{e}_{23}^*$ are Nash equilibria and $\mathbf{e}_{13}^*$ is not a Nash equilibrium.

(b) *Let $\Gamma < 0$. If $\varphi_1 > 0$ then there are seven stationary population profiles: the six described in part (a), which are all Nash equilibria, and the interior mixed profile equilibrium:*

$$\mathbf{z}^* = \hat{\mathbf{z}} \equiv \left(\frac{\varphi_2\varphi_3 d_{23}}{D}, \frac{\varphi_1\varphi_3 d_{13}}{D}, \frac{\varphi_1\varphi_2 d_{12}}{D} \right) \in \mathrm{Int}(\Delta^2) \tag{20}$$

where $d_{12} \equiv m_{12}(m_{12} - m_{13} - m_{23}) < 0$, $d_{13} \equiv m_{13}(m_{13} - m_{12} - m_{23}) < 0$, $d_{23} \equiv m_{23}(m_{23} - m_{12} - m_{13}) < 0$ and $D \equiv \varphi_1\varphi_2 d_{12} + \varphi_1\varphi_3 d_{13} + \varphi_2\varphi_3 d_{23} < 0$. If $\varphi_1 = 0$ then equilibria $\mathbf{e}_{12}^*$, $\mathbf{e}_{13}^*$ and $\hat{\mathbf{z}}$ merge with $\mathbf{e}_1$ and there are four Nash equilibria.

(c) *When $\Gamma = 0$, there are six stationary profiles as described in (a) and they are all Nash equilibria.*

Figure 1: Bifurcation diagram

Clearly, the precise value of Γ is crucial in determining whether we have 1, 3 or 4 mixed equilibria. We can take (19), and assume the three strategies are equally spaced, by setting $\varphi_2 = (\varphi_1 + \varphi_3)/2$, and plot a bifurcation diagram in the space of conjectures (φ_1, φ_3) in Figure 1. There are two bifurcation loci $\{(\varphi_1, \varphi_3) : \varphi_1 = 0\}$ and $\{(\varphi_1, \varphi_3) : \Gamma(\varphi_1, \varphi_3) = 0\}$. The last set divides the conjecture space into two: there is a small area where φ_1 is less than 0.066, for which $\Gamma < 0$. Most of the parameter space results in $\Gamma > 0$. This means that in the 3×3 example the vast majority of combinations of conjectures will yield only two boundary mixed equilibria with a third mixed non-Nash boundary stationary point. In this sense, the interior mixed equilibrium is a rarity, and requires one firm to have a very competitive conjecture ($\phi_1 < -0.94, \varphi_1 < 0.066$). We can now see that the example in section 3.2 where $\phi_1 = 0$ and $\phi_3 = 1$ is firmly in the region where $\Gamma > 0$, so that there are only three stationary points on the edges.

Figure 2: Phase diagrams

We can think about the strategy profiles in terms of the unit-simplexes, depicted in Figure 2 [21] for the cases not corresponding to bifurcations. The pure-strategy equilibria are on the vertices: the most competitive is in the bottom right corner ($z_1 = 1$), the least competitive at the top ($z_3 = 1$). All those equilibria are sinks.

When $\Gamma > 0$ we have the generic simplex as depicted in Figure 2(a). There are three partially mixed stationary states: one on each of the edges between the three vertices. There are two stationary profiles $\mathbf{e}_{12}^*$ and $\mathbf{e}_{23}^*$ that involve conjecture φ_2 with each of the other two conjectures: these are both Nash equilibria and are saddle-points with the stable manifold belonging to the interior of the simplex. Note that $\mathbf{e}_{12}^*$ is closer to $\mathbf{e}_1$ than $\mathbf{e}_2$: this follows because to equate the payoffs, the more competitive conjecture needs a higher probability of meeting itself. Likewise, $\mathbf{e}_{23}^*$ is closer to $\mathbf{e}_2$ than $\mathbf{e}_3$. There is a third stationary state that is not a Nash-equilibrium, which is a mixed profile with $z_3 = 0$, and is a source.

When $\Gamma < 0$ we have the simplex as depicted in Figure 2(b). In this case, there are two differences: first, the stationary mixed profile with $z_2 = 0$ becomes a saddle-point stable Nash-equilibrium, and, second, an additional interior mixed stationary state emerges, which is also a Nash-equilibrium but is a source. Again the stable manifold associated to boundary equilibria for $z_2 = 0$ belongs to the interior of the simplex. When $\Gamma \downarrow 0$, the mixed equilibria gets closer to the interior mixed equilibrium in $\mathbf{e}_{13}$, and when $\Gamma = 0$ the two merge. In this case, the boundary mixed equilibrium is a Nash-equilibrium. This property however does not show up when $\Gamma \uparrow 0$. This corresponds to a local bifurcation point.

The next proposition formally assert that the local dynamics at the stationary points displayed at the two phase diagrams hold generically:

Proposition 3: local dynamics.

[21] We have used Dynamo by Sandholm, Dokumaci and Franchetti (2010), to draw the phase diagrams.

(a) The pure strategy Nash equilibria, $\mathbf{z}^* = \mathbf{e}_1$, $\mathbf{z}^* = \mathbf{e}_2$ *and* $\mathbf{z}^* = \mathbf{e}_3$, *are always sinks, and the two boundary mixed Nash equilibria* $\mathbf{z}^* = \mathbf{e}_{12}^*$ *and* $\mathbf{z}^* = \mathbf{e}_{23}^*$ *are always saddle points.*
(b) Let $\Gamma > 0$*: then the boundary non-Nash stationary state* $\mathbf{z}^* = \mathbf{e}_{13}^*$ *is a source.*
(c) Let $\Gamma < 0$.*. If* $\varphi_1 > 0$*: then the boundary mixed Nash equilibrium* $\mathbf{z}^* = \mathbf{e}_{13}^*$ *is a saddle point and the interior mixed Nash equilibrium* $\mathbf{z}^* = \hat{\mathbf{z}}$ *is a source.* If $\varphi_1 = 0$, *then* $\mathbf{e}_{12}^*$, $\mathbf{e}_{13}^*$ *and* $\hat{\mathbf{z}}$ *merge with* $\mathbf{e}_1$, *which is a fold bifurcation.*
(d) If $\Gamma = 0$, *then there is a local fold bifurcation at equilibrium point* $\mathbf{z}^* = \mathbf{e}_{13}^* = \hat{\mathbf{z}}$.

Phase diagrams in Figure 2 displays not only local dynamics but also global dynamics, for the two generic cases. It shows there is a heteroclinic network which is joining all the stationary points of the replicator dynamics. Heteroclinic orbits exist in the intersection of the stable manifold associated to one equilibrium point to the unstable manifold associated to another equilibrium point. Therefore, there are heteroclinic orbits linking sinks to saddle points, in the interior of the simplex, and saddle points to sinks, in the boundaries of the simplex. This implies that the heteroclinic orbits in the interior of the simplex separate the basins of attractions of the three pure strategy Nash equilibria

$$\mathcal{B}_i \equiv \left\{ \mathbf{y} \in \Delta^2 : \lim_{t\to\infty} \mathbf{z}(t,\mathbf{y}) = \mathbf{e}_i \right\} \ , \ i = 1, 2, 3.$$

Proposition 4: global dynamics.
(a) Let $\Gamma > 0$. *Then there is a heteroclinic network composed of 8 heteroclinic orbits: six heteroclinic orbits join the boundary mixed equilibria to the pure strategy equilibria, and two heteroclinic orbits join the steady state on edge* $\mathbf{e}_{13}$ *to the boundary mixed equilibria on the edges* $\mathbf{e}_{12}$ *and* $\mathbf{e}_{23}$. *These two heteroclinics separate the boundaries for the basins of attraction* $\mathcal{B}_1$, $\mathcal{B}_2$ *and* $\mathcal{B}_3$ *associated to the three pure strategies equilibria* $\mathbf{e}_1$, $\mathbf{e}_2$ *and* $\mathbf{e}_3$.
(b) Let $\Gamma < 0$. *If* $\varphi_1 > 0$, *then there is a heteroclinic network composed of 9 heteroclinic orbits, six heteroclinic orbits join the boundary mixed equilibria to the pure strategy equilibria, and three heteroclinic orbits join the interior mixed equilibrium,* $\hat{\mathbf{z}}$ *to the boundary mixed equilibrium on the edges* $\mathbf{e}_{13}$, $\mathbf{e}_{12}$ *and* $\mathbf{e}_{23}$. *These three heteroclinics separate the basins of attraction* $\mathcal{B}_1$, $\mathcal{B}_2$ *and* $\mathcal{B}_3$ *associated to the three pure strategy equilibria* $\mathbf{e}_1$, $\mathbf{e}_2$ *and* $\mathbf{e}_3$. *If* $\varphi_1 = 0$, *then there is a heteroclinic network composed of 5 heteroclinic orbits, three heteroclinic orbits joining* $\mathbf{e}_1$ *to* $\mathbf{e}_2$, $\mathbf{e}_3$ *and* $\mathbf{e}_{23}$, *and two heteroclinic orbits joining* $\mathbf{e}_{23}$ *to* $\mathbf{e}_2$ *and* $\mathbf{e}_3$. *The heteroclinic orbit between* $\mathbf{e}_1$ *and* $\mathbf{e}_{23}$ *separates the basins of attraction* $\mathcal{B}_2$ *and* $\mathcal{B}_3$. *Basin* $\mathcal{B}_1$ is empty.

From the above proposition, we can see that:

(a) The three pure-strategy equilibria are asymptotically stable and are ESS (so long as $\varphi_1 > 0$ to rule out the Bertrand case).

(b) None of the non-pure strategy fixed-points are asymptotically stable or ESS.

(c) The non-pure strategy fixed points are either unstable sources or saddle-stable with a stable manifold of dimension 1. We can see that the non-pure-strategy stationary states are on the borders of the basins of attraction of the three pure-strategy equilibrium conjectures. The boundaries of the basins are heteroclinic orbits which connect the "mixed" stationary states with each other and with the pure strategy equilibria. Hence, there is a sense in which the non-pure strategy stationary points are "fragile": the replicator dynamics on the two dimensional simplex results in a stable manifold of at most one dimension. This means that these stationary states are not locally stable, since a small deviation will almost always lead away to one of the three pure-strategy sinks. Whilst they are fragile in this sense, they are also *essential* to the model, as with their heteroclinic orbits they define the boundaries between the basins of attraction of the pure-strategy sinks.

5.2 Equilibrium Selection.

Clearly, the evolutionary dynamics imply that the initial position determines which equilibrium comes about in the long-run. However, what can we say about the size of the basins of attraction? In particular, what determines the size of the basins of attraction? Does the Pareto dominant equilibrium have a larger basin of attraction? If we consider each point in the unit simplex to be equally likely, we can interpret the size of the basin as the *probability* of the corresponding equilibrium. In the general case of $\Gamma > 0$, we are able to approximate each basin under the assumption that the heteroclinic orbits are all linear, so that the three basins can be broken down into triangles using Proposition 4[22]. Let us call $P(\mathbf{e}_i)$ the (approximate) probability of asymptotic convergence to pure strategy $\mathbf{e}_i$. Our approximations

[22] We show in the proof of Proposition 4 that there is not an analytic first integral for the replicator dynamic system and therefore the separatices of the basins of attraction cannot be determined analytically. However, we also prove that they will be close to the straight lines connecting equilibria, which allows us to approximate of the dimension of basins of attraction for the pure strategies.

are:

$$
\begin{aligned}
P(\mathbf{e}_1) &= \frac{\varphi_1^2}{(\varphi_1+\varphi_2)(\varphi_1+\varphi_3)}, \\
P(\mathbf{e}_2) &= \frac{\varphi_2}{\varphi_1+\varphi_3}\left(\frac{\varphi_1}{\varphi_1+\varphi_2}+\frac{\varphi_3}{\varphi_2+\varphi_3}\right), \\
P(\mathbf{e}_3) &= \frac{\varphi_3^2}{(\varphi_1+\varphi_3)(\varphi_2+\varphi_3)}.
\end{aligned}
$$

Since $\varphi_3 > \varphi_1$we can see that the basin of attraction of the Pareto dominant equilibrium $\mathbf{e}_3$ is larger than that of the most competitive equilibrium $\mathbf{e}_1 : P(\mathbf{e}_3)/P(\mathbf{e}_1) = \varphi_3^2/\varphi_1^2 > 1 \Rightarrow P(\mathbf{e}_3) > P(\mathbf{e}_1)$. However, the relative size of $P(\mathbf{e}_2)$ is more complicated to understand. To take the simplest case, if $\varphi_1 = 0$ (Bertrand), then the exact probabilities are

$$
\begin{aligned}
P(\mathbf{e}_3) &= \frac{\varphi_3}{(\varphi_2+\varphi_3)} \\
P(\mathbf{e}_2) &= \frac{\varphi_2}{(\varphi_2+\varphi_3)}
\end{aligned}
$$

and so we have the unambiguous ranking $P(\mathbf{e}_3) > P(\mathbf{e}_2)$. In general, however, it is more than possible to have $P(\mathbf{e}_3) < P(\mathbf{e}_2)$. In particular, as $\varphi_1 \to \varphi_3$, then $P(\mathbf{e}_1)$ and $P(\mathbf{e}_3)$ both tend to 1/4 whilst $P(\mathbf{e}_2)$ tends to 1/2. If we take another example with $\varphi_1 = 1$ (Cournot) and $\varphi_3 = 2$ (Joint profit maximization), then $P(\mathbf{e}_2) > P(\mathbf{e}_3)$ for $\varphi_2 > 1.155$.

Hence we cannot claim that the Pareto dominant equilibrium will have the largest basin. The reason for this is from the payoff function of the rent-extraction game: whilst the most competitive CV will do worst, the middle conjecture does better than the most cooperative against the most competitive and does better against the most cooperative than the most competitive. This was why (for $\Gamma > 0$) the stationary point with only the most and least cooperative conjectures is not a Nash equilibrium and is an unstable source. The result is that the basin of attraction for the intermediate conjecture φ_2 is often (although certainly not always) larger than the most cooperative conjecture φ_3.

There is also a key difference between the intermediate conjecture and the most cooperative. If we look at Figure 2a, we can see that for $\Gamma > 0$, if the population starts close to $\mathbf{e}_{23}$ but in $\mathcal{B}_2$, there are equilibrium paths that start with an almost zero share for φ_2 but tend asymptotically to $\mathbf{e}_2$ where the share is 1. The cooperative conjecture φ_3 however requires a minimum share to start off with. The lowest starting share for φ_3 occurs on the boundary of $\mathcal{B}_3$ at stationary point $\mathbf{e}_{13}$: its initial share must be just above that at $\mathbf{e}_{13}$ for it to be able to get to $\mathbf{e}_1$. So long as $\varphi_1 > 0$, this is bounded away from zero.

6 Conclusion.

In this paper, we have taken the rent-extraction model with conjectural variations and applied a social learning model to it in the form of the evolutionary replicator dynamics. CVs become more (less) common as their average payoffs are above (below) average. The end-points of this evolutionary process can be both pure-strategy equilibria and mixed-strategy equilibria. However, the mixed-equilibria are either unstable, or have limited saddle-path stability and hence are not ESS. The pure-strategy equilibria have large basins of attraction, and their boundaries are separated by heteroclinic orbits that connect the mixed-equilibria. Whilst all the pure-strategy equilibrium conjectures are consistent conjectures, the standard definition of consistency does not apply to mixed equilibria. We therefore develop two generalizations of the standard consistency condition to apply to the case of mixed-equilibria and show that, whilst the link between consistency and equilibrium in the conjecture game still exists, it is weaker in the case of mixed-strategies than it is for pure-strategy equilibria.

In our analysis of the rent-extraction game, we do not find a tendency for all of the rent to be extracted in the evolutionary long-run. The rent is only fully dissipated when there are competitive (Bertrand) conjectures, which is not ESS and will have no basin of attraction. The Pareto-optimum of zero-rent dissipation is not only possible, but also has a significant basin of attraction which in certain cases may be the biggest. However, in the three conjecture case we have analyzed, the intermediate conjecture may well have the larger basin of attraction.

There are very many shortcomings to using simple evolutionary dynamics: they certainly are not a literal real-time representation of how agents behave. However, the long-run dynamics give us a guide as to what social institutions and individual strategies might emerge over time. In the case of the rent-extraction model they have given us an insight into what types of behavior and associated beliefs will succeed in earning above average payoffs, and in so doing become more common.

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7 Appendix.

7.1 Proof of Proposition 2.

Proof The ODE (18) has three steady states in the vertices of the simplex, three steady states in the edges of the simplex and we prove there is one additional steady state which is in the interior of the simplex only if $\Gamma < 0$.

First, all the steady states in the vertices are all Nash equilibria, because for $\mathbf{z}^* = \mathbf{e}_1$, we have $u_1(\mathbf{e}_1) - \bar{u}(\mathbf{e}_1) = 0$, $u_2(\mathbf{e}_1) - \bar{u}(\mathbf{e}_1) = -\varphi_1 m_{12} < 0$, and $u_3(\mathbf{e}_1) - \bar{u}(\mathbf{e}_1) = -\varphi_2 m_{13} < 0$; for $\mathbf{z}^* = \mathbf{e}_2$, we have $u_1(\mathbf{e}_2) - \bar{u}(\mathbf{e}_2) = -\varphi_2 m_{12} < 0$, $u_2(\mathbf{e}_2) - \bar{u}(\mathbf{e}_2) = 0$, and $u_3(\mathbf{e}_2) - \bar{u}(\mathbf{e}_2) = -\varphi_2 m_{23} < 0$ and, for $\mathbf{z}^* = \mathbf{e}_3$, we have $u_1(\mathbf{e}_3) - \bar{u}(\mathbf{e}_3) = -\varphi_3 m_{13} < 0$, $u_2(\mathbf{e}_3) - \bar{u}(\mathbf{e}_3) = -\varphi_3 m_{23} < 0$, and $u_3(\mathbf{e}_3) - \bar{u}(\mathbf{e}_3) = 0$.

Second, for the steady states in the edges of the simplex we have: steady state in edge $\mathbf{e}_{12}$, joining vertices $\mathbf{e}_1$ and $\mathbf{e}_2$, $\mathbf{z}^* = \mathbf{e}_{12}^* = (\varphi_2/(\varphi_1+\varphi_2), \varphi_1/(\varphi_1+\varphi_2), 0)$ is a Nash equilibrium because $u_1(\mathbf{e}_{12}^*) - \bar{u}(\mathbf{e}_{12}^*) = u_2(\mathbf{e}_{12}^*) - \bar{u}(\mathbf{e}_{12}^*) = 0$ and

$$u_3(\mathbf{e}_{12}^*) - \bar{u}(\mathbf{e}_{12}^*) = -\frac{\varphi_1\varphi_2\,(m_{13}+m_{23}-m_{12})}{\varphi_1+\varphi_2} < 0$$

as

$$m_{13} - m_{12} = \varphi_1 \frac{(\varphi_3-\varphi_2)(\varphi_2\varphi_3-\varphi_1^2)}{(\varphi_1+\varphi_3)^2(\varphi_2+\varphi_3)^2} > 0;$$

the steady state in edge $\mathbf{e}_{13}$, joining vertices $\mathbf{e}_1$ and $\mathbf{e}_3$, $\mathbf{z}^* = \mathbf{e}_{13}^* = (\varphi_3/(\varphi_1+\varphi_3), 0, \varphi_1/(\varphi_1+\varphi_3), 0)$ verifies $u_1(\mathbf{e}_{13}^*) - \bar{u}(\mathbf{e}_{13}^*) = u_3(\mathbf{e}_{13}^*) - \bar{u}(\mathbf{e}_{13}^*) = 0$ and

$$u_2(\mathbf{e}_{13}^*) - \bar{u}(\mathbf{e}_{13}^*) = \frac{\varphi_1\varphi_3\,(m_{13}-m_{12}-m_{23})}{\varphi_1+\varphi_3}$$

as $m_{12} > 0$ and

$$m_{13} - m_{23} = \varphi_3 \frac{(\varphi_2-\varphi_1)(\varphi_3^2-\varphi_1\varphi_2)}{(\varphi_1+\varphi_3)^2(\varphi_2+\varphi_3)^2} > 0.$$

As $\mathrm{sign}(u_2(\mathbf{e}_{13}^*) - \bar{u}(\mathbf{e}_{13}^*)) = \mathrm{sign}(\Gamma)$ (see equation (19)) then $\mathbf{e}_{13}^*$ is a Nash equilibrium if and only if $\Gamma < 0$; and, finally, steady state in edge $\mathbf{e}_{23}$, joining vertices $\mathbf{e}_2$ and $\mathbf{e}_3$, $\mathbf{z}^* = \mathbf{e}_{23}^* = (0, \varphi_3/(\varphi_2+\varphi_3), \varphi_2/(\varphi_2+\varphi_3), 0)$ is also a Nash equilibrium because it verifies $u_2(\mathbf{e}_{23}^*) - \bar{u}(\mathbf{e}_{23}^*) = u_3(\mathbf{e}_{23}^*) - \bar{u}(\mathbf{e}_{23}^*) = 0$ and

$$u_1(\mathbf{e}_{23}^*) - \bar{u}(\mathbf{e}_{23}^*) = \frac{\varphi_2\varphi_3\,(m_{23}-m_{12}-m_{13})}{\varphi_2+\varphi_3} < 0$$

as $m_{13} - m_{23} > 0$.

The steady state outside the boundary of Δ^2, $\mathbf{z}^* = \hat{\mathbf{z}}$, is formally given in equation (20) . If it belongs to the interior of Δ^2 it is always a Nash equilibrium because it verifies $u_1(\hat{\mathbf{z}}) = u_2(\hat{\mathbf{z}}) = u_3(\hat{\mathbf{z}}) = \bar{u}(\hat{\mathbf{z}})$. Then we have to check if it belongs to the simplex: first observe that $\hat{z}_1 + \hat{z}_2 + \hat{z}_3 = 1$; second, from the previous conditions over the quantities m_{12}, m_{13} and m_{23} we readily see that $d_{12} < 0$ and $d_{23} < 0$. Then $\mathbf{z}$ is positive and its components are less than one if $d_{13} < 0$ as well, which implies $D < 0$. But $d_{13} = m_{13}\Gamma < 0$ if and only if $\Gamma < 0$.

Hence $\Gamma < 0$ is a necessary and sufficient condition both for the steady state $\mathbf{e}_{13}^*$ to be a Nash equilibrium and also for the steady state to be inside the simplex. If $\Gamma > 0$ $\mathbf{e}_{13}^*$ is not a Nash equilibrium, and there is no interior steady-state. If $\Gamma = 0$ it is easy to see that the interior mixed-strategy equilibrium and the boundary equilibrium coalesce, which means that the boundary equilibrium will be a Nash equilibrium, because in this case $d_{12} = d_{23} = -2m_{12}m_{23}$ and $d_{13} = 0$. □

7.2 Proof of Proposition 3.

proof We obtain equivalent results if we study the local dynamics of the 3-dimensional ODE, (18), or if we study its 2-dimensional projection of the dynamic system (18) into the space (z_1, z_3) by the relation $z_2 = 1 - z_1 - z_3$,

$$\dot{z}_1 = z_1 \left[(1 - z_1 - z_3) \left(-\varphi_2 m_{12} + m_{12}(\varphi_1 + \varphi_2) z_1 + m_{23}(\varphi_2 + \varphi_3) z_3 \right) + \right.$$
$$\left. + m_{13} z_3 ((\varphi_1 + \varphi_3) z_1 - \varphi_3) \right] \quad (21)$$

$$\dot{z}_3 = z_3 \left[(1 - z_1 - z_3) \left(-\varphi_2 m_{23} + m_{12}(\varphi_1 + \varphi_2) z_1 + m_{23}(\varphi_2 + \varphi_3) z_3 \right) + \right.$$
$$\left. + m_{13} z_1 ((\varphi_1 + \varphi_3) z_3 - \varphi_1) \right]. \quad (22)$$

The Jacobian of this system , evaluated at steady state $(z_1^*, 1 - z_1^* - z_3^*, z_3^*)$, $\mathbf{J}(z_1^*, z_3^*)$, has spectrum

$$\lambda(z_1^*, z_3^*) \equiv \{ \lambda : \det \left[\mathbf{J}(z_1^*, z_3^*) - \lambda \mathbf{I} \right] = 0 \} = \{ \lambda_1(z_1^*, z_3^*), \lambda_2(z_1^*, z_3^*) \}$$

First, the spectra of the reduced Jacobian for the pure strategies are: $\lambda(\mathbf{e}_1) = \{-\varphi_1 m_{13}, -\varphi_1 m_{12}\}$,

$\lambda(\mathbf{e}_2) = \{-\varphi_2 m_{12}, -\varphi_2 m_{23}\}$, and $\lambda(\mathbf{e}_3) = \{-\varphi_3 m_{13}, -\varphi_3 m_{23}\}$. In all the three cases all the eigenvalues are negative and, if φ_2 is equally spaced, we have the relationship

$$\lambda_1(\mathbf{e}_i) < \lambda_2(\mathbf{e}_i) < 0, i = 1, 2, 3.$$

Note further that whilst $\lambda(\mathbf{e}_2)$ and $\lambda(\mathbf{e}_3)$ are strictly negative, $\lambda(\mathbf{e}_1)$ is zero if and only if $\varphi_1 = 0$. This gives rise to the fold bifurcation at $\mathbf{e}_1$when $\varphi_1 = 0$.

Second, the spectra of the reduced Jacobian for the steady states in the edges are:

$$\lambda(\mathbf{e}_{12}^*) = \left\{-\frac{\varphi_1\varphi_2(m_{13}+m_{23}-m_{12})}{\varphi_1+\varphi_2}, \frac{\varphi_1\varphi_2 m_{12}}{\varphi_1+\varphi_2}\right\},$$

$$\lambda(\mathbf{e}_{13}^*) = \left\{\frac{\varphi_1\varphi_3\Gamma}{\varphi_1+\varphi_3}, \frac{\varphi_1\varphi_3 m_{13}}{\varphi_1+\varphi_3}\right\}$$

and

$$\lambda(\mathbf{e}_{23}^*) = \left\{-\frac{\varphi_2\varphi_3(m_{12}+m_{13}-m_{23})}{\varphi_1+\varphi_2}, \frac{\varphi_2\varphi_3 m_{23}}{\varphi_2+\varphi_3}\right\},$$

so that , $z^* = \mathbf{e}_{12}^*$ and $\mathbf{z}^* = \mathbf{e}_{23}^*$ are saddle points because $\lambda_1(\mathbf{e}_{12}^*) < 0 < \lambda_2(\mathbf{e}_{12}^*)$ and $\lambda_1(\mathbf{e}_{23}^*) < 0 < \lambda_2(\mathbf{e}_{23}^*)$ and $\mathbf{z}^* = \mathbf{e}_{13}^*$ is also a saddle point if $\Gamma < 0$ and $\varphi_1 > 0$, and is a source if $\Gamma > 0$. In the case of $\Gamma < 0$ and $\varphi_1 = 0$, we have $\lambda(\mathbf{e}_{12}^*) = \lambda(\mathbf{e}_{13}^*) = \{\ 0, 0\}$ and from Proposition 2 (b) these equilibria merge which gives rise to the fold bifurcation.

Third, recall that steady state $\hat{\mathbf{z}}$ will only belong to the simplex Δ^2 if $\Gamma < 0$. The trace and the determinant of the Jacobian evaluated at $\hat{\mathbf{z}}$ are both positive, under that condition:

$$\mathrm{tr}(\mathbf{J}(\hat{\mathbf{z}})) = -2\frac{4^2\varphi_1\varphi_2\varphi_3}{D} m_{12}m_{13}m_{23} > 0$$

$$\det(\mathbf{J}(\hat{\mathbf{z}})) = -\left(\frac{4^2\varphi_1\varphi_2\varphi_3}{D}\right)^2 m_{12}m_{13}m_{23}(m_{12}+m_{13}-m_{23})(m_{13}+m_{23}-m_{12})\Gamma > 0.$$

If we assume further that φ_2 is the average of the other two strategies, then the discriminant of the Jacobian is also positive, and, as the two eigenvalues of the Jacobian are real and positive, the interior mixed equilibrium is a source. In the case where $\varphi_1 = 0$ from Proposition 2(b) this equilibrium merges with $\mathbf{e}_1$, at a fold bifurcation point.

□

7.3 Proof of Proposition 4.

proof Zeeman (1980) presents a complete classification of the phase portraits of the replicator dynamics (RD) for the 3×3 case. They include the phase portraits in figure **??**. These phase portraits suggest there is a heteroclinic network which is not an heteroclinic cycle as in some RD games (e.g., the rock - scissor -paper RD game). The heteroclinic network consists of six heteroclinic orbits joining equilibria on the edges, $\mathbf{e}_{12}$, $\mathbf{e}_{13}$ and $\mathbf{e}_{23}$, to equilibria on the vertices of the simplex, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, and two interior heteroclinic orbits joining steady state $\mathbf{e}_{13}^*$ to steady states $\mathbf{e}_{12}^*$, and $\mathbf{e}_{23}^*$, respectively. Those two heteroclinic orbits separate the boundaries of the basins of attractions in the interior of Δ^2. Next we prove that the phase diagram in figure 2, for case $\Gamma > 0$, is generic. The proof for case $\Gamma < 0$ is similar.

Heteroclinic orbits lay along invariants of type $\{(z_1, z_2, z_3) : F(z_1, z_2, z_3) = \text{constant}\}$. The best way to prove that their layout as in figure 2 is generic, is to determine a first integral of the RD system (18) explicitly. If we transform the 3-dimensional RD system (18) into a 2-dimensional Lotka-Volterra (LV) system, using a well known transformation (see Hofbauer and Sigmund 1998, p.77), and if we draw upon the relevant literature on the determination of the first integrals of the LV equation, e.g. Llibre and Valls (2007), we find that there is not an analytic first integral for the associated LV equation.

Therefore we resort to a heuristic proof by using equations (21)-(22).

The orbits along the edges of the simplex lay along invariants $\{(z_1, z_3) : z_1 = 0\}$, $\{(z_1, z_3) : 1 - z_1 - z_3 = 0\}$, and $\{(z_1, z_3) : z_3 = 0\}$. In the first case the dynamics is given by $\dot{z}_1 = 0$ and $\dot{z}_3 = z_3(1 - z_3)(z_3 - z_3(\mathbf{e}_{23})m_{23}(\varphi_2 + \varphi_3)$, which means that $z_1(t) = 0$, for any $t \geq 0$ and if $0 < z_3(0) < z_3(\mathbf{e}_{23}^*)$ $(1 > z_3(0) > z_3(\mathbf{e}_{23}^*))$ then $z_3(t)$ will converge asymptotically to vertex $\mathbf{e}_2$ ($\mathbf{e}_3$). In the second case the dynamics is given by $\dot{z}_3 = -\dot{z}_1$ and $\dot{z}_1 = z_1(1 - z_1)(z_1 - z_1(\mathbf{e}_{13})m_{13}(\varphi_1 + \varphi_3)$, which means that $z_3(t) = 1 - z_1(t)$, for any $t \geq 0$, and if $0 < z_1(0) < z_1(\mathbf{e}_{13}^*)$ $(1 > z_1(0) > z_1(\mathbf{e}_{13}^*))$ then the trajectory $z_1(t)$ will converge asymptotically to vertex $\mathbf{e}_1$ ($\mathbf{e}_3$). In the last case, the dynamics is given by $\dot{z}_3 = 0$ and $\dot{z}_1 = z_1(1 - z_1)(z_1 - z_1(\mathbf{e}_{12})m_{12}(\varphi_1 + \varphi_2)$, which means that $z_3(t) = 0$, for $t \geq 0$, and if $0 < z_1(0) < z_1(\mathbf{e}_{12}^*)$ $(1 > z_1(0) > z_1(\mathbf{e}_{12}^*))$ then the trajectory $z_1(t)$ will converge asymptotically to vertex $\mathbf{e}_1$ ($\mathbf{e}_2$).

Next, we prove that, if $\Gamma > 0$ there are two heteroclinic orbits inside a closed trapping area $\mathbb{T}$ which is bounded by equilibrium points $\mathbf{e}_2$, $\mathbf{e}_{12}^*$, $\mathbf{e}_{13}^*$, and $\mathbf{e}_{23}^*$:

$$\mathbb{T} = \left\{ (z_1, z_3) : \ z_1 \geq 0, \ z_3 \geq 0, \ \frac{-\varphi_2 + (\varphi_1 + \varphi_3)z_1}{\varphi_3 - \varphi_2} \leq z_3 \leq \frac{\varphi_2(1 - z_1) + \varphi_1}{\varphi_2 + \varphi_3} \right\}.$$

As we already saw, all the points belonging to segments of the edges $\mathbf{e}_2$-$\mathbf{e}_{12}$, and $\mathbf{e}_2$-$\mathbf{e}_{23}$, converge to the pure strategy steady state $\mathbf{e}_2$. By continuity, given any initial point close to the those edges, the replicator dynamics will also imply asymptotic convergence to $\mathbf{e}_2$. However, all the dynamics starting close to the straight line $\mathbf{e}_{12} - \mathbf{e}_{13}$, passing through points $\mathbf{e}^*_{13}$ and $\mathbf{e}^*_{12}$, will exit $\mathbb{T}$ and converge to vertex $\mathbf{e}_1$. Similarly, all the dynamics starting close to the straight line $\mathbf{e}_{13} - \mathbf{e}_{23}$, passing through points $\mathbf{e}^*_{13}$ and $\mathbf{e}^*_{23}$, will exit $\mathbb{T}$ and converge to vertex $\mathbf{e}_3$. This means that there are two separatrices belonging to the interior of $\mathbb{T}$: the first is in the intersection of the stable manifold associated to the saddle point $\mathbf{e}^*_{12}$ with the unstable manifold associated with the source $\mathbf{e}^*_{13}$, $W^s(\mathbf{e}^*_{12}) \cap W^u(\mathbf{e}^*_{13})$; and the second is in the intersection of the stable manifold associated to the saddle point $\mathbf{e}^*_{23}$ with the unstable manifold associated with the source $\mathbf{e}^*_{13}$, $W^s(\mathbf{e}^*_{23}) \cap W^u(\mathbf{e}^*_{13})$.

Those separatrices partition $\mathbb{T}$ in three subsets, where there will be asymptotic convergence towards one and only one of the three vertices of the simplex. The subset associated to $\mathbf{e}_2$ is the basin os attraction of $\mathbf{e}_2$ and the other subsets of $\mathbb{T}$ belong to the basins of attraction of $\mathbf{e}_1$ or $\mathbf{e}_3$. The separatrices are invariants and contain all the heteroclinic orbits converging asymptotically to either $\mathbf{e}^*_{12}$ or $\mathbf{e}^*_{23}$.

To prove this formally, observe that the formal expression of line $\mathbf{e}_{13} - \mathbf{e}_{12}$ is

$$z_3 = -\frac{\varphi_2}{\varphi_3 - \varphi_2} + \frac{\varphi_1 + \varphi_3}{\varphi_3 - \varphi_2} z_1 : \ (z_1, z_3) \in \mathbb{T}$$

which is positively sloped. Evaluating equations (21)-(22) along that line we get

$$\begin{aligned}\dot{z}_1 = -z_1 \frac{(\varphi_1 + \varphi_2)(\varphi_1 + \varphi_3)}{(\varphi_3 - \varphi_2)^2} \ & (z_1 - z_1(\mathbf{e}^*_{13}))\,(z_1 - z_1(\mathbf{e}^*_{12})) \times \\ & \times (\varphi_2(m_{13} + m_{23} - m_{12}) + \varphi_3(m_{12} + m_{23} - m_{13})) > 0\end{aligned}$$

$$\begin{aligned}\dot{z}_3 = -\frac{(\varphi_1 + \varphi_2)^2(\varphi_1 + \varphi_3)}{(\varphi_3 - \varphi_2)^3} \ & (z_1 - z_1(\mathbf{e}^*_{13}))\,(z_1 - z_1(\mathbf{e}^*_{12})) \times \\ \times \left[\ (\varphi_2(m_{13} + m_{23} - m_{12}) + \varphi_3(m_{12} + m_{23} - m_{13}))\, z_1 - \frac{2\varphi_2\varphi_3 m_{23}}{\varphi_1 + \varphi_2} \ \right] < 0. &\end{aligned}$$

Then the vector field is negatively sloped along line $\mathbf{e}_{13} - \mathbf{e}_{12}$ and, locally, z_1 is increasing and z_3 is decreasing towards $\mathbf{e}_1$. Therefore, the global dynamics involves exit from trapping area $\mathbb{T}$.

The formal expression of line $\mathbf{e}_{13} - \mathbf{e}_{23}$ is

$$z_3 = \frac{\varphi_2}{\varphi_2 + \varphi_3} - \frac{\varphi_2 - \varphi_1}{\varphi_2 + \varphi_3} z_1 : \ (z_1, z_3) \in \mathbb{T}$$

which is negatively sloped. Evaluating equations (21)-(22) along that line we get

$$\dot{z}_1 \quad = \quad -z_1 \left(z_1 - z_1(\mathbf{e}_{13}^*)\right) \frac{(\varphi_1 + \varphi_3)}{(\varphi_2 + \varphi_3)} \quad \left(\varphi_2 (m_{13} - m_{23} + m_{12})(z_1 - 1) - \varphi_1 \Gamma z_1\right) \quad < \quad 0$$

and

$$\dot{z}_3 = -z_1 \left(z_1 - z_1(\mathbf{e}_{13}^*)\right) \frac{(\varphi_1 + \varphi_3)}{(\varphi_2 + \varphi_3)^2} \ \left(\varphi_2 (1 - z_1) + \varphi_1 z_1\right) \times$$
$$\times \left(\varphi_2 (m_{13} - m_{23} + m_{12}) + \varphi_1 (m_{12} + m_{23} - m_{13})\right) > 0$$

which implies that the slope of the vector field along the line $\mathbf{e}_{13} - \mathbf{e}_{23}$ is also negative. But the slope of the vector field is steeper than the slope of line $\mathbf{e}_{13} - \mathbf{e}_{23}$, because

$$\left.\frac{dz_3}{dz_1}\right|_{(\dot{z}_1, \dot{z}_3)} - \left.\frac{dz_3}{dz_1}\right|_{e_{13} - e_{12}} = \frac{2\varphi_1 \varphi_2 m_{12}}{(\varphi_2 + \varphi_3)\left(\varphi_2 (m_{13} + m_{12} - m_{23})(z_1 - 1) + -\varphi_1 \Gamma z_1\right)} < 0.$$

Then, locally, z_1 is decreasing and z_3 is increasing towards $\mathbf{e}_3$. Therefore, the global dynamics also involves exit from trapping area $\mathbb{T}$.

At last, we prove that the separatrices lay inside the trapping area $\mathbb{T}$. First, recall that the stable eigenspaces, $E^s(\mathbf{e}_{12}^*)$ and $E^s(\mathbf{e}_{23}^*)$, are tangent to the stable manifolds associated to the two boundary saddle points, $\mathbf{e}_{12}^*$ and $\mathbf{e}_{23}^*$. This means that the heteroclinic trajectories are asymptotically tangent to the stable eigenspaces. The stable eigenspace associated to $\mathbf{e}_{12}^*$ has slope

$$\left.\frac{dz_3}{dz_1}\right|_{E^s(e_{12}^*)} = \frac{(\varphi_1 + \varphi_2)(m_{13} + m_{23})}{(m_{13} - m_{23})\varphi_3 - (m_{13} + m_{23})\varphi_2}$$

which is positive if $(\varphi_1^2 + \varphi_3^2)\varphi_2 - 2\varphi_1 \varphi_3^2 > 0$, and is negative or vertical otherwise. In the second case, the separatrix is clearly inside $\mathbb{T}$. However, the separatrix is also inside $\mathbb{T}$ when it is positively sloped, because it is steeper than line $\mathbf{e}_{13} - \mathbf{e}_{12}$, as, in this case,

$$\left.\frac{dz_3}{dz_1}\right|_{E^s(e_{12}^*)} - \left.\frac{dz_3}{dz_1}\right|_{e_{13} - e_{12}} = \frac{\varphi_3 (\varphi_1 + \varphi_2)(\varphi_1 + \varphi_3)^2}{(\varphi_1^2 + \varphi_3^2)\varphi_2 - 2\varphi_1 \varphi_3^2} > 0$$

The stable eigenspace associated to $\mathbf{e}_{23}^*$ is also negatively sloped, because

$$\left.\frac{dz_3}{dz_1}\right|_{E^s(e_{23}^*)} = -\frac{(m_{12}-m_{13})\varphi_1 + (m_{12}+m_{13})\varphi_2}{(\varphi_2+\varphi_3)(m_{12}+m_{13})} < 0.$$

Again, it is inside $\mathbb{T}$ because it is steeper than line $\mathbf{e}_{13} - \mathbf{e}_{23}$ as

$$\left.\frac{dz_3}{dz_1}\right|_{E^s(e_{23}^*)} - \left.\frac{dz_3}{dz_1}\right|_{e_{13}-e_{23}} = -2\frac{\varphi_1 m_{12}}{(\varphi_1+\varphi_3)(m_{12}+m_{13})} > 0.$$

In the case of $\Gamma < 0$, the proof is similar in the case of $\varphi_1 > 0$. If $\varphi_1 = 0$, we have the additional factor of the merging of equilibria (Proposition 2) and resultant fold bifurcation and disappearance of $\mathcal{B}_1$. □

7.4 A. Iso-payoff sets.

The iso-payoff sets for X characterized by

$$\bar{U}^X = \{(x,y) : U^X(x,y) = \bar{U}\}$$

have the slope given by

$$\left.\frac{dy}{dx}\right|_{\bar{U}^x} = \frac{y - x^2 - 2xy - y^2}{x}$$

The payoffs are undefined at $(0,0)$ .For $\bar{U} \in (0,1)$, the iso-payoff curve intersects the $x-$ axis at $\left(1-\bar{U},0\right)$. However, all iso-payoff sets with $\bar{U} \in (0,1)$ originate from $(0,0)$ with slope defined by:

$$\begin{aligned} \frac{x}{x+y} &= \bar{U} \\ \lim_{x\to 0} \left.\frac{dy}{dx}\right|_{\bar{U}^x} &= \frac{y}{x} = \frac{1-\bar{U}}{\bar{U}} \end{aligned}$$

For $\bar{U}$ close to 0 the slope is very high; for $\bar{U}$ close to 0 the slope is very low; whilst the slope is zero at an intermediate level. The locus of points at which the iso-payoff functions have zero slope is defined by:

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{\bar{U}^x} &= 0 \\ y - x^2 - 2xy - y^2 &= 0 \\ x &= -y + \sqrt{y} \end{aligned}$$

This is depicted in Figure 1 for firm X's payoff, along with the lines $x+y=1$ and $x=y$:

Fig A1 here.

In the region to the left of the zero-slope locus, the slopes of the iso-payoff functions are strictly positive; whereas to the right they are strictly negative. The slopes are all strictly decreasing along each iso-payoff locus for $\bar{U} \in (1,0)$. The $\bar{U}=0$ locus is the union of the line segments $\{x+y=1, x \geq 0, y \geq 0\}$ with $\{x=0, y \in (0,1]\}$. The corresponding iso-payoffs for firm Y are simple the reflection of those of firm X in the 45^o line.

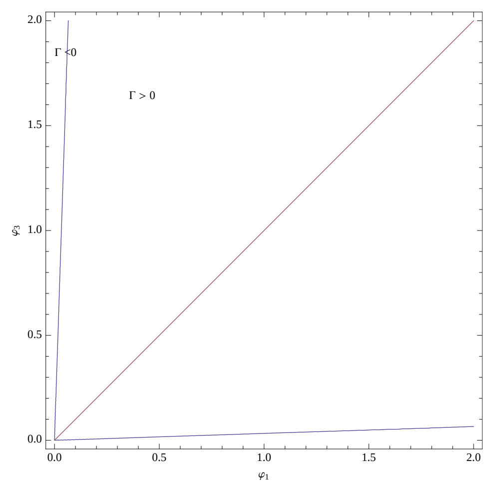

Figure 1: Bifurcation diagram in the space (φ_1, φ_3), for equally spaced conjectures $\varphi_2 = (\varphi_1 + \varphi_3)/2$.

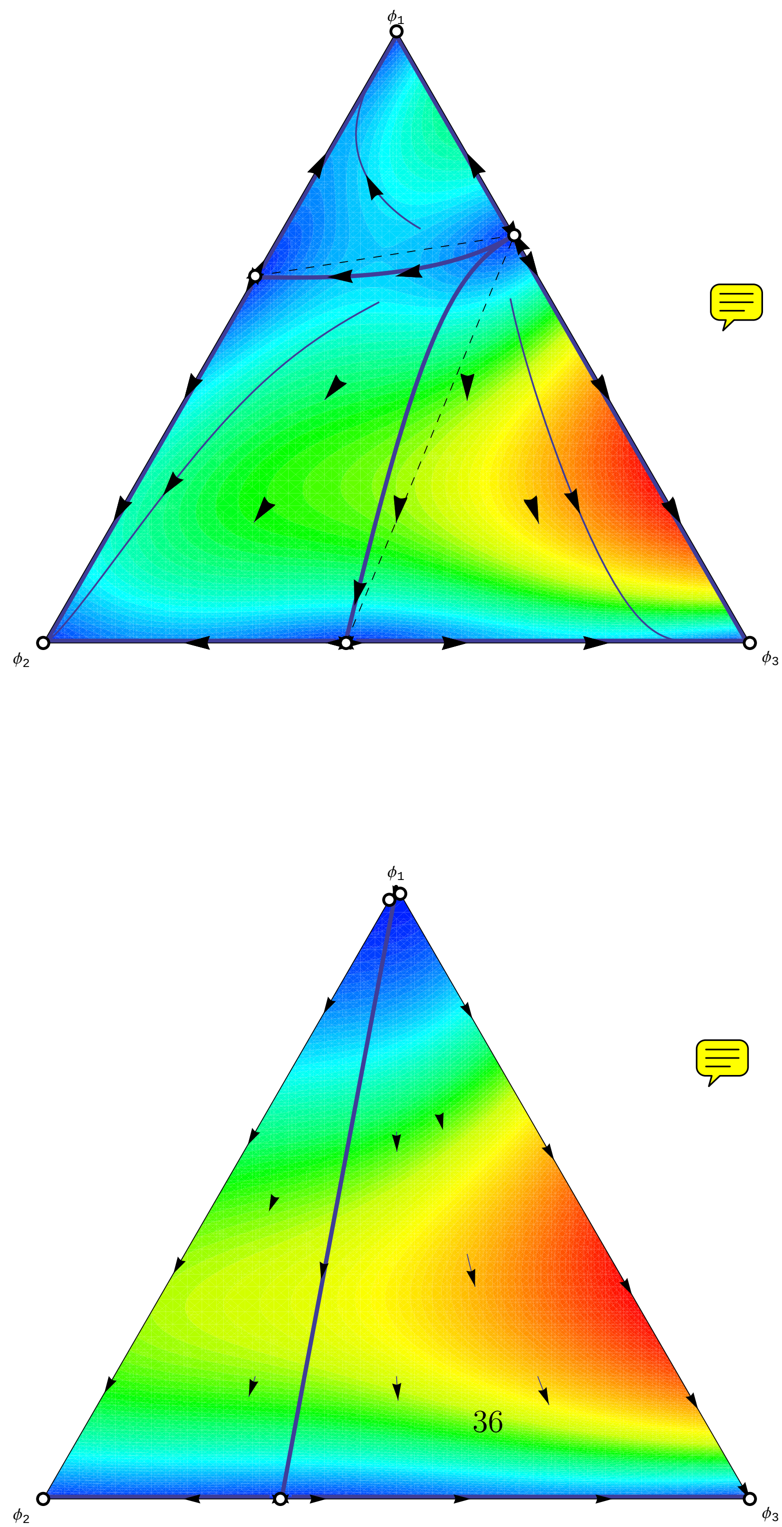

Figure 2: Phase diagrams over the simplex for equally spaced conjectures:

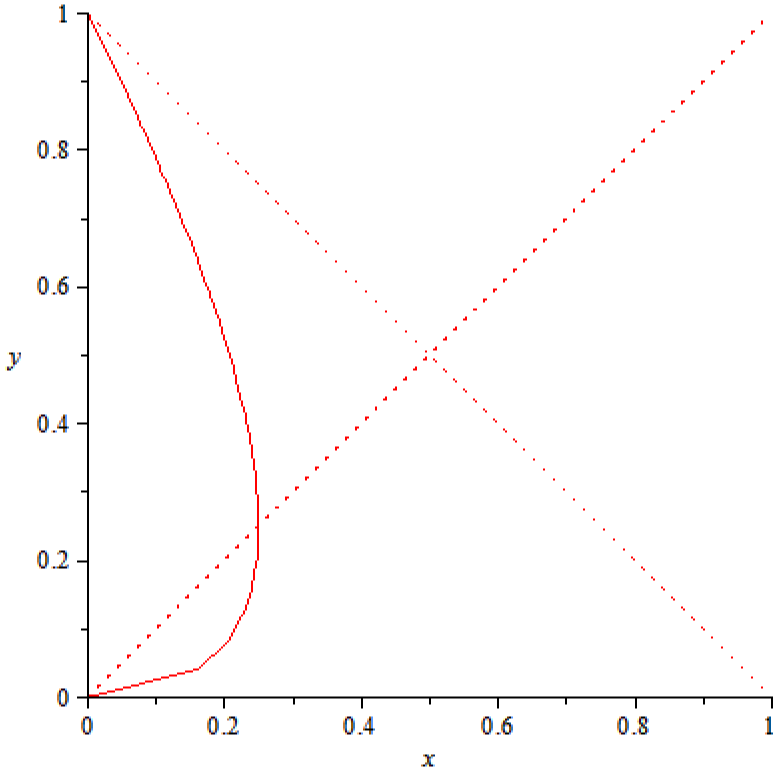

Figure A1: The locus of zero-sloped points on Firm Xs iso-payoff functions.