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IDENTIFICATION IN THE LINEAR ERRORS IN VARIABLES MODEL

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1. INTRODUCTION

CONSIDER THE FOLLOWING multiple linear regression model with errors in variables:

(1.1) $y_j = \xi_j'\beta + \epsilon_j \qquad (j = 1, \ldots, n),$

(1.2) $x_j = \xi_j + v_j,$

where ξ_j, x_j, v_j, and β are k-vectors, y_j, ϵ_j are scalars. The ξ_j are unobservable variables: instead the x_j are observed. The measurement errors v_j are unobservable as well and we assume $v_j \sim N(0, \Omega)$ for all j. The ϵ_j are assumed to follow a $N(0,\sigma^2)$ distribution. The v_j and ϵ_j are mutually independent and independent of ξ_j. The ξ_j are considered as random drawings from some, as yet unspecified, multivariate distribution with zero mean. (In the usual terminology this means that we deal with the *structural* version of the model.)

It is fairly easy to show that if ξ_j is drawn from a multivariate normal distribution the parameter vector β is not identified. For the case $k = 1$ Reiersøl [4] has shown that normality of ξ_j is the *only* distributional assumption which spoils identification. Here we generalize his result to the case where k may be larger than one.

2. STATEMENT OF THE RESULT AND PROOF

PROPOSITION: *Under the assumptions above, the parameter vector β is identified if and only if there does not exist a linear combination of ξ_j which is normally distributed.*

PROOF: We first show that nonidentifiability of β implies the existence of a normally distributed linear combination of ξ_j. Let s be a scalar and t a k-vector. The characteristic function, $\varphi_{\epsilon_j,v_j}(s,t)$, of ϵ_j and v_j is

(2.1) $$\varphi_{\epsilon_j,v_j}(s,t) = \exp\{-\tfrac{1}{2}(\sigma^2 s^2 + t'\Omega t)\}.$$

Define

(2.2) $$\eta_j \equiv \xi_j'\beta.$$

The characteristic function of η_j and ξ_j is

(2.3) $$\varphi_{\eta,\xi}(s,t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\{i(s \cdot \eta_j + t'\xi_j)\}\, dF_{\eta,\xi}(\eta_j, \xi_j)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\{i(\beta s + t)'\xi_j\}\, dF_{\eta,\xi}(\eta_j, \xi_j) = \varphi_\xi(\beta s + t),$$

where $F_{\eta,\xi}$ is the joint distribution function of η_j and ξ_j. Assuming that β is not fully identified amounts to saying that there exist parameter sets $\{\beta, \sigma^2, \Omega\}$ and $\{\beta^*, \sigma^{*2}, \Omega^*\}$, with at least one element of β^* different from the corresponding element in β, generating

[1]The views expressed in this paper are those of the authors and do not necessarily reflect the policies of the Netherlands Central Bureau of Statistics. We thank Professor H. Schneeweiss for drawing our attention to Aufm Kampe [2].

the same distribution of the observable variables y_j, x_j. Consequently, the characteristic function of y_j, x_j should be the same for both sets of parameters:

$$(2.4) \qquad \exp\{-\tfrac{1}{2}(\sigma^2 s^2 + t'\Omega t)\}\varphi_\xi(\beta s + t) = \exp\{-\tfrac{1}{2}(\sigma^{*2}s^2 + t'\Omega^* t)\}\varphi_\xi^*(\beta^* s + t).$$

Notice that a separate characteristic function φ_ξ^* has been introduced since in general a different set of structural parameters will only give the same distribution of observables if the distribution of ξ_j is also different in both cases.

Equality (2.4) holds for all possible values of s and t. In particular, (2.4) holds if we let s and t vary in such a way that

$$(2.5) \qquad \beta^* s + t = 0.$$

For values of s and t satisfying (2.5), $\varphi_\xi^*(\beta^* s + t) = \varphi_\xi^*(0) = 1$, by the definition of a characteristic function. Thus (2.4) carries over into

$$(2.6) \qquad \varphi_\xi((\beta - \beta^*)s) = \exp\{-\tfrac{1}{2}[(\sigma^{*2} - \sigma^2)s^2 + s^2\beta^{*\prime}(\Omega^* - \Omega)\beta^*]\},$$

where t has been replaced by $-\beta^* s$ according to (2.5)

Rewriting $\varphi_\xi((\beta - \beta^*)s)$, we have that

$$(2.7) \qquad \varphi_\xi((\beta - \beta^*)s) \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\{is(\beta - \beta^*)'\xi_j\}\, dF_\xi(\xi_j)$$

The right hand side of (2.7) arises as the characteristic function of ξ_j, where $(\beta - \beta^*)s$ is its argument. Alternatively we can also interpret it as the characteristic function of the scalar variable $z \equiv (\beta - \beta^*)'\xi_j$ with s as its argument, say $\varphi_z(s)$. Write $a^2 \equiv (\sigma^{*2} - \sigma^2) + \beta^{*\prime}(\Omega^* - \Omega)\beta^*$; then (2.6) carries over into

$$(2.8) \qquad \varphi_z(s) = \exp\{-\tfrac{1}{2}a^2 s^2\},$$

which is the characteristic function of a normally distributed variable. Thus nonidentifiability of β implies the existence of a linear combination of the latent variables (i.e., $z = (\beta - \beta^*)'\xi_j$) which follows a normal distribution (with variance a^2).

To prove the second part of the proposition we assume that there exists a k-vector d of constants, not all zero, such that $d'\xi_j$ follows a normal distribution. Define $\beta^* \equiv \beta - d$. Then $\nu_j \equiv y_j - \beta^{*\prime}\xi_j$ follows a normal distribution with mean zero and variance σ^{*2}, say, because

$$(2.9) \qquad \nu_j = y_j - \beta'\xi_j + d'\xi_j = \epsilon_j + d'\xi_j,$$

which is the sum of two independently distributed normal variables. Moreover ν_j and v_j are independent. Thus

$$(2.10) \qquad f(\nu_j, v_j) \propto \exp\{-\tfrac{1}{2}[\nu_j^2/\sigma^{*2} + v_j'\Omega^{-1}v_j]\}.$$

Obviously, there also holds

$$(2.11) \qquad f(\epsilon_j, v_j) \propto \exp\{-\tfrac{1}{2}[\epsilon_j^2/\sigma^2 + v_j'\Omega^{-1}v_j]\}.$$

On the basis of (2.10) we have

$$(2.12) \qquad f(y_j, x_j) \propto \exp\{-\tfrac{1}{2}[(y_j - \beta^{*\prime}\xi_j)^2/\sigma^{*2} + v_j'\Omega^{-1}v_j]\},$$

whereas (2.11) implies

$$(2.13) \qquad f(y_j, x_j) \propto \exp\left\{-\tfrac{1}{2}\left[(y_j - \beta'\xi_j)^2/\sigma^2 + v_j'\Omega^{-1}v_j\right]\right\}.$$

One observes that the true β cannot be distinguished from β^* since they imply the same density for y_j and x_j. The existence of a linear combination of the ξ_j which is normally distributed thus implies nonidentifiability of β.

3. DISCUSSION

Our proof generalized Reiersøl's. For $k = 1$, it reduces to his proof. For the case $k \geq 1$ and the ξ_j mutually uncorrelated, Willassen [6] employs Cramér's decomposition theorem to show that none of the ξ_j should be normally distributed to guarantee identifiability of β. This is obviously a specialization of our result. Aufm Kampe [2] has shown that nonidentifiability of β implies the existence of a normally distributed linear combination of ξ_j. This result is also stated (without proof) by Wolfowitz [7]. Rao [3] has proven a theorem implying that an element of β is unidentifiable if the corresponding ξ_j is normally distributed. This is also a specialization of the proposition.

The proposition clearly rests on the assumed normality of ϵ_j and v_j. If these random variables follow a different distribution, a normally distributed ξ_j need not spoil identifiability.

The proposition also has implications for the functional model where the ξ_j are considered to be fixed unknown constants. As observed by Aigner et al. [1] it follows from a result by Wald [5] that in the functional model there will exist a consistent estimator of β if and only if β is identified in the structural model under any distributional assumption regarding the ξ_j. Under our normality assumptions regarding v_j and ϵ_j, the proposition implies that normality is the worst possible assumption for the ξ_j. Thus the extraneous information that will be required to identify β in the structural model with normally distributed ξ_j is identical to that which is needed to guarantee the existence of a consistent estimator of β in the functional model.

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