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A Foundation of Location Theory: Consumer Preferences and Demand

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Location models with a continuum of consumers have severe conceptual shortcomings. This article considers an alternative foundation for location economics. The commodity space of each of a finite number of consumers is the collection of measurable subsets of land. This space is so large that a solution to the consumer maximization problem does not always exist. The main result is the definition and use of a topology on land parcels to show that a solution will always exist for utilities continuous with respect to the topology. The underlying preferences are exposed. *Journal of Economic Literature* Classification Numbers: 022, 930, 021. © 1988 Academic Press, Inc.

I. INTRODUCTION

The canonical model of location theory has a continuum of consumers distributed over a geographical space such that each consumer values and owns a density of land (see, for example, [1, pp. 121–130] or [20]). Thus,

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utility is a function of a *density* of land, while location theorists take no exception to modelling consumers as a continuum. Hildenbrand [11] and related papers provided the axiomatic underpinnings for the use of such continuum models as mathematically convenient approximations to large but finite economies. It is demonstrated in this literature that if the number of consumers in a sequence of finite economies becomes large, for instance, by increasing proportionately the number of agents of various types, the core shrinks to the set of equilibria and the equilibria of the finite economies approximate those of the continuum economy. In the course of replicating the finite economies, the total endowment of the economy is increased in proportion to the number of agents so that it increases without bound. As a consequence, the equilibrium concept of the continuum economy equates mean supply to mean demand, so that in all of the finite economies as well as the continuum economy, agents are endowed with, consume, and derive utility from positive quantities of goods. The average amount of goods held by consumers is positive.

Location theory does not use this type of approach. If a land parcel is to be represented by a subset of a Euclidean space (say $\mathscr{R}^2$), then the σ-finiteness of the space implies that there is only a countable number of parcels of positive area in each partition of the space. As a consequence, a continuum of consumers must be endowed with and trade parcels of land of zero area (on average). Furthermore, any sequence of large but finite economies close to such a continuum economy has the property that average land area holdings must become close to zero, so that economies approximated by a continuum economy are pathological. Thus, the equilibria and comparative statics results of a continuum model are not necessarily close to those of a reasonable finite model. The densities of land in the continuum model cannot be interpreted as areas (unless the commodity space is not σ-finite), so consumers must have preferences over parcels of zero area (see [3] for proofs of these statements). An alternative interpretation is that the continuum of agents represents fractions of individual consumers rather than individuals themselves. This interpretation has severe limitations as well (see [6]). The time has come to develop a new theoretical basis for location theory.

It is not without reason that location theory has strayed from the use of classical modelling techniques such as the assumption of a finite number of agents with initial endowments and preferences over alternative commodity bundles. Perhaps the framework of location theory that is closest to established economic theory is the one in which land is modelled as just another commodity (see [18]), measured in acres and supplied in a quantity equal to the total area of land. While this approach conveniently casts land in the general equilibrium mold, it fails to differentiate parcels by qualities other than size, such as shape, and therefore is unable to address the issues of location theory. If, however, parcels are differentiated by spatial characteristics of location, shape, and so on, then there are infinitely many commodities present, in fact uncountably many. To make matters even more complicated, each separate parcel is indivisible in the sense that the acts of splitting or combining produce parcels with other characteristics, different commodities. In short, the properties of land imply that the assumptions concerning the commodity space, preferences, and production technologies used by standard general equilibrium models do not apply to land. Thus, models such as those of Debreu [8], Bewley [7], Mas-Colell [13], and Jones [12] cannot be used directly in this context; see [3, 21] for more detail.

The purpose of this paper is to analyze consumer behavior head-on when a commodity space that is a natural representation of land, a collection of subsets of the plane, is used. We shall explicitly address the complications that are due to the size of the commodity space and the indivisibility of the elements. The first approach is due to Berliant [2], who models land as the σ-algebra of measurable subsets of a two-dimensional set of finite measure that represents the world. In modelling the consumer, however, a key linearity assumption is made by postulating the existence of a marginal utility density such that the utility of a parcel is the integral of the density over the parcel. To state the assumption more intuitively, it is as if there is a fertility density over land and the value of a parcel to a consumer is the total fertility it carries. So while land is not a simple, homogeneous good such that only area matters, we can reduce its utility to underlying fertility units. Expressed in these units, land does become homogeneous, much as composite labor is reduced to simple labor in the Marxist theory of value. Thus, location and shape matter only through the "fertility" they embody, but are of no intrinsic value. The present paper generalizes [2] to the extent that the linearity assumption is dropped and preference for location or shape, such as cohesiveness, is admissible.

The main contribution of this article is the demonstration that a solution to the consumer's problem exists in the context of a large number of recombinable, indivisible commodities. The focus on the demand side is justified by noting that for many commodities, such as land, supply is rather inelastic. While location theory is the prime example, there are other applications. For example, it may be argued that all units of some good are differentiated. Each apple is slightly different from every other apple, and half an apple is not the same as two halves. If so, by splitting an apple two new commodities are produced. If the utility of the whole apple is not necessarily the same as the sum of the utilities of the two halves, then our framework can be imposed. Furthermore, the mathematical problem solved below is that of showing the existence of a solution to an infinite dimensional programming problem with a nonlinear objective function (i.e., a function that is not separable) and a linear constraint. As a consequence, the result might be of interest to those attempting to solve a programming problem with time, as the dimensionality of the space in which the subsets reside is arbitrary.

The consumer's problem we address is that of maximizing a (nonlinear) utility functional subject to a budget constraint by appropriate choice of an element of the σ-algebra of measurable subsets of a Euclidean compactum. As usual, we would like to have a continuous utility function and a compact budget set. Hence the σ-algebra must be endowed with a topology. Given a utility function and a budget set of financially feasible commodity bundles, the topology ought to be strong enough to make the former continuous, but weak enough to make the latter compact. Moreover, the topology should be strong enough so that the budget constraint is preserved under limits.

So far, the problem of finding a suitable topology looks similar to the one Bewley [7] faced in generalizing equilibrium analysis to economies with infinitely many commodities. The Mackey topology did the job there, and it is in fact the strongest topology consistent with the desired duality, making it a sharp solution. But we face an additional complication, namely that our commodities do not constitute a linear space (as they are indivisible). While it might seem natural to ensure compactness along with the continuity of a utility and dual elements (prices) by embedding the σ-algebra in a large linear space endowed with a "sharp" topology, too many elements might be introduced into the commodity space in the process. Such an added element could arise as the limit of a sequence of elements of the σ-algebra of increasing utility. In this case, the limit would not be a commodity bundle. For example, imagine a preference for world-wide presentation such as that once possessed by the queen of the British empire. For simplicity, assume a uniform rent (or cost) density and a budget that can cover half the world of the unit interval (the world is still flat). Let

$$\left[0, \frac{1}{n}\right] \cup \left[\frac{2}{n}, \frac{3}{n}\right] \cup \left[\frac{4}{5}, \frac{5}{n}\right] \cup \cdots \cup \left[\frac{n-2}{n}, \frac{n-1}{n}\right] \qquad (n = 1, 2, \ldots)$$

represent a sequence of commodities of increasing utility. If the indicator functions of these sets are embedded in L^∞ with the weak* topology, then the limit is $\frac{1}{2} 1_{[0,1]}$, half the indicator function on the interval. This, indeed is a natural limit that is not in the commodity space.

A topology on a space of parcels that does ensure that limits are elements of the space is the one induced by the Hausdorff metric on closed subsets of a compactum (see [11, p. 17]), which we describe next. Let d be a metric on a compact space M, and let E be a closed subset of M. Define the ε-neighborhood of E, $B_\varepsilon(E) = \{x \in M \mid \exists y \in E$ such that $d(x, y) \leqslant \varepsilon\}$. The Hausdorff metric on closed subsets, say E and F, of M is given by $H(E, F) = \inf\{\varepsilon \in (0, \infty] \mid E \subset B_\varepsilon(F)$ and $F \subset B_\varepsilon(E)\}$. To see how this metric measures distance, let E be a closed ball in a convex, compact subset of $\mathbb{R}^n$ and let F be this ball along with one point not in the ball. We employ the Euclidean metric on $\mathscr{R}^n$. The Hausdorff distance between E and F is simply the Euclidean distance between the added point and the point in the ball closest to the added point. Naturally, this distance tends to zero as the added point gets closer to the ball. If E is a ball in $\mathbb{R}^n$ and F is just a point, then the Hausdorff distance between E and F is the Euclidean distance between F and the point in E *furthest* from the point in F. This distance is positive whenever E and F differ.

The Hausdorff metric is potentially useful in location theory since it differentiates parcels by shape as well as location, and is compact. However, four problems associated with the application of this topology to our problem can be identified. First, it does not preserve the budget under limits. For instance, the Hausdorff limit of the sequence specified above is the entire unit interval, which is twice as expensive as any element of the sequence. Second, the topology does not generalize the special case of the linear utility functions of Berliant [2]. For example, the sequence specified above demonstrates that the integral of any positive density function is discontinuous in the topology. Third, the topology is restricted to closed sets. Strictly speaking, this does not hinder the demonstration of the existence of a solution to the consumer's problem. However, equilibrium analysis would be difficult as closed sets cannot fill the compactum without overlap. Fourth, the Hausdorff topology is not extendable to a linear space in any obvious way, so price analysis along standard duality lines is difficult.

The topology we propose for location theory is a modification of the Hausdorff metric topology that overcomes the first three problems. Its construction is motivated by an observation on the first problem. The remark on Lemma 1 of the Appendix shows that the value of the limit of a sequence of parcels converging in the Hausdorff topology is at least as large as the limit of the values of parcels in the sequence. In other words, linear valuations may be discontinuous, but they are upper semicontinuous. Even though the Hausdorff topology is not stronger than the weak* topology, it is stronger in an upper semicontinuous sense. At first sight, it is disappointing that the value discontinuity is in the wrong direction: upward rather than downward. However, it is possible to turn this around by comparing (open) sets through the Hausdorff distance of their *complements*. This alteration eliminates the first problem with the Hausdorff topology. A value discontinuity is now downward and in agreement with the budget constraint. In such a case, the limit set could be small. In particular, the sequence consisting of complements of elements of the sequence specified above converges, in the new topology, to the empty set. To correct for this and to resolve the second and third problems with the Hausdorff topology, the latter manifesting itself by the confinement of the Hausdorff metric on complements (or "outer Hausdorff" topology) to open sets, we augment as follows. Given a space of measurable sets, the outer Hausdorff metric is applied to the interiors of sets. In conjunction, all sets are subjected to a topology generated by a marginal utility density as in [2]. Thus, two measurable sets are close in our topology if and only if the complements of their interiors are close in the Hausdorff metric and the whole sets are close in aggregate marginal utility density. The linear utilities of Berliant [2] can be seen to be continuous with respect to this topology (see Section III for more detail).

The Hausdorff and outer Hausdorff topologies are not comparable. However, some partial comparison is possible, namely with respect to the subtraction or addition of single points. The Hausdorff topology records the addition of an isolated point, but not the subtraction of an interior point. The outer Hausdorff topology does the opposite. Since subtraction of interior points seems more important, the outer Hausdorff topology looks finer, at least to consumers who care more about the absence of holes than the presence of isolated points. Thus, employment of the outer Hausdorff metric is not only convenient mathematically, but also intuitively attractive.

The topology just described is a good device for evaluating commodities in terms of location and shape as well as size. For instance, two parcels are close substitutes when their interiors are close in the outer Hausdorff topology and their boundaries have substantial overlap. The program we envisage is a reconstruction of location theory along the lines of classical general equilibrium theory with a finite number of agents trading parcels that enter utility and production functions *directly*. This contrasts with the prevailing strand of thought, the so called new urban economics, in which densities of land feed densities of utility and production. As Mas-Colell [14] points out, the main difficulty with a program that deals with infinitely many commodities is the determination of the existence of a (non-trivial) solution to the consumer's problem or of a Pareto optimum. In fact, he assumes this away and is then able to demonstrate the existence of an equilibrium. While we admit that his context of Banach lattices is incomparable to our σ-algebra of parcels, this paper represents an attempt to fill a gap by deriving demand. A next step is equilibrium analysis. An existence theorem for the case of linear utilities can be found in [4]. Further existence results will not be easy in view of the possible nonextendability of the topology to linear spaces. Yet such an equilibrium analysis will be an important undertaking as it paves the way for an explicit treatment of the spatial externalities that underlie phenomena such as agglomeration and spatial interaction. The techniques of Shafer and Sonnenschein [19] could be useful.

In the next section we will be more specific by presenting the formal framework of the analysis. Section III contains examples of utilities continuous with respect to the topology and examines the relationship of our work to classical urban economics. Since the main result is the derivation of demand from utilities and prices, this analysis is undertaken straightaway in Section IV, leaving the study of the relation between preferences and utilities to Section V. Section VI concludes the article.

II. The Model

Let m be Lebesgue measure on $\mathscr{R}^n$, let L be a compact subset of $\mathscr{R}^n$, and let $\mathscr{B}$ be the σ-algebra of measurable subsets of L. Elements of $\mathscr{B}$ that are the same almost surely are *not* considered to be equivalent. If the framework is interpreted in location theoretic terms, L is land, a subset of $\mathscr{R}^2$, and $\mathscr{B}$ is the consumption set of each agent. Land can be heterogeneous and anything can be embedded in it, so it is a differentiated commodity that can be divided and recombined in an infinity of varieties. Combination with a null set may create a nonequivalent parcel by virtue of, for example, the new set having a larger connected area. Furthermore, there is only one instance of each potential parcel of land, so that there is a discrete choice as to whether to purchase it or not; there is an indivisibility associated with this commodity.

Let H be the Hausdorff metric on nonempty, closed sets in $\mathscr{B}$ (see [11, p. 16]).

For A, $B \in \mathscr{B}$, define the difference set $A \backslash B = \{x \in A \mid x \notin B\}$ and for $\varepsilon > 0$, the ε-ball $B_\varepsilon(A) = \{y \in L \mid \exists x \in A \text{ with } \|x - y\| \leqslant \varepsilon\}$. $A \subset B$ is defined to be $x \in A$ implies $x \in B$. If $B \in \mathscr{B}$, $\mathring{B}$ is the interior of B in the relative topology on L. Let $B^c = L \backslash B$. ∂B is the boundary of B in the relative topology on L, the set of points in L each of whose neighborhoods (relative to L) contains members of both B and B^c.

A topology on $\mathscr{B}$ is now defined. All further references to continuity which do not specify a topology implicitly employ the one given below. It is based on the Hausdorff metric on exteriors.

Fix $h \in L^1$. A basis for the topology is given by the collection of sets of the form

$$\{L\}, \left\{C \in \mathscr{B} \mid H((\mathring{C})^c, (\mathring{B})^c) < \delta, \left| \int_C h(x)\, dm(x) - \int_B h(x)\, dm(x) \right| < \delta \right\}$$

for $B \in \mathscr{B}$, $\delta > 0$. Note that, as not all points can be separated, this is not a

Hausdorff space, let alone a metric space. (For example, B and C may have common interior and remainders with equal h-values but different locations.) L is isolated in this topology.

As argued in the Introduction, additive utility functions are continuous in this topology. The topology is a generalization designed to capture spatial features such as location and shape. For example, a utility function that reflects preference for cohesiveness is $U(B) = \sup[\int_B h(x)\, dm(x) + \sum_{i=1}^{\infty} 2^{-i} \cdot V(B_i)]$ with the supremum taken over all ordered partitions $(B_1, B_2, \ldots)$ of B and $V(B) = \sup_{x \in B}\{\varepsilon \mid B_\varepsilon(x) \subset B\}$. This utility function is also continuous in our topology, although some effort is required to prove this fact.

The price space corresponding to the commodity space $\mathscr{B}$ is somewhat problematic, as duality theory does not supply a natural space. It is desirable to have no arbitrage in equilibrium, for otherwise traders would always wish to change their demands. In the context of the model, no arbitrage means that traders cannot put parcels together or take them apart and make a profit. Hence prices should be additive as a function of parcels. If traders are not to make a profit by putting together or taking apart an infinity of parcels, then prices should be countable additive. If, furthermore, a parcel of zero area is to have a zero price, then the Radon–Nikodym theorem yields a price space that is the set of all integrable functions on L; that is, if $B \in \mathscr{B}$ the price of B is $\int_B p(x)\, dm(x)$ for an integrable p. The zero area–zero price condition is expected to be fulfilled at equilibrium for continuous utility as null additions to closed sets are not recorded by our topology. (*Proof*: Let A be closed and B null. Compare $A \cup B$ to A. By Lemma 2 of the Appendix, $(A \cup B)^0 \subset (A^0 \cup \varnothing \cup \partial A) = A$, hence the two sets have equal interior. By Lemma 3, both sets have null boundary, hence equal h-value.)

III. Examples

It is natural to ask what utility functions are continuous with respect to this topology and to wonder about whether our subsequent derivation of demand gives any hints as to the form of demand derived from such utilities. To see that linear utilities expressible as integrals are continuous with respect to such a topology, pick a utility $u(B) = \int_B f(x)\, dm(x)$, where f is in L^1. Note that one can always choose $h \equiv f$ for the topology, which makes this utility continuous with respect to it. Clearly, with more than one consumer and different densities f, it might not be possible to define one topology (with a given h) for all consumers. The trick in this case is to define a separate topology for each of a finite number of consumers (with different h terms) and take the coarsest topology consistent with each individual topology. However, this strategy is more closely related to the demonstration that an equilibrium exists rather than the study of demand, so it is deferred to future work. Suffice it to comment that linear utilities can be made continuous with respect to the topology by choice of h; demand has been characterized in [2]. With a little more work, it is possible to verify that the example of Section II also is continuous with respect to the topology.

Of more interest is a set of examples that provides a link between our theory and classical location theory. In the latter literature, utilities are generally specified as $u(x, q, t)$, where $x \in \mathscr{R}$ is a quantity of composite good or numeraire, $q \in \mathscr{R}$ is the quantity of land consumed, and $t \in L$ is the location of the land or consumer (see [22]). For simplicity, we suppress x, leaving the inclusion of other goods to the Conclusion.

The variable q has a natural analog in our model, $\int h(x)\, dm(x)$ (especially if $h \equiv 1$). The variable t presents more of a problem. There are many analogs of this variable, the most natural of which seem to be continuous with respect to our topology.

To be specific, let $s\colon \mathscr{B} \to C^0(L)$, where $C^0(L)$ is the set of continuous functions on L with the C^0 topology, be a continuous map such that $s(B)(t) = 0$ for all $t \notin \mathring{B}$. An example is the map $s(B)(t) = r(B, t)$ where $r(B, t) = \sup\{r \geqslant 0 \mid y \in \mathring{B}$ for all y with $\|t - y\| \leqslant r\}$. Further, let $g\colon C^0(L) \to L$ be a nonempty, upper hemi-continuous correspondence such for all t, $t' \in g(S(B))$, $u(q, t) = u(q, t')$ for all q. An example is $g(f) = \operatorname{argmax}_{t \in L} f(t)$. It is clear why g is defined to be a correspondence and not a function, for otherwise discontinuities in many topologies would arise in this particular example. We call a map g a *locator map*. In fact, $g(s(B))$ is the center of the largest ball contained in B. Defining $W(B) = u(\int_B h(x)\, dm(x),\ g(s(B)))$, using the uniform continuity theorem [15, p. 180], W is continuous in our topology. A more specific example is $W(B) = v(\int_B h(x)\, dm(x),\ \sup_{t \in L} r(B, t))$ for some continuous v.

Given that demand exists, it is clearly possible to construct first order conditions for specific forms of g, s and h. For example, if $h \equiv 1$ while g and s take the special forms above, a necessary condition is

$$u_1 \frac{1}{p_1} = u_2 \frac{1}{p_2} \qquad \text{at demand } B^*,$$

where subscripts represent derivatives,

$$p_1 = \liminf_{\substack{B \subset (B^*)^c \\ m(B) \to 0}} \frac{\int_B p(x)\, dm(x)}{m(B)} \qquad \text{and} \qquad p_2 = \lim_{\varepsilon \to 0} \frac{\int_{B_\varepsilon(\mathring{B}^*)} p(x)\, dm(x)}{m(B_\varepsilon(\mathring{B}^*))}.$$

p_1 is the "essential infimum" of p on $(B^*)^c$, analogous to the standard "essential supremum." It is shown below that demand is nonempty, so these conditions are not vacuous. Such conditions are of use when calculating equilibria for specific economies where the coherence of land parcels matters to agents.

Another example of interest is given by the utility $\bar{U}(B) \equiv \int_B r(B, t)\, dm(t) = \int_{\mathring{B}} r(B, t)\, dm(t) = \int_L r(B, t)\, dm(t)$. It is easy to see that if $\{B_n\}_{n=1}^{\infty}$ converges to B in the topology, then $\{r(B_n, t)\}_{n=1}^{\infty}$ converges to $r(B, t)$ for each t. By Lebesgue's dominated convergence theorem, $\bar{U}(B_n)$ converges to $\bar{U}(B)$, so $\bar{U}$ is continuous in the topology. Each of these examples values shape by measuring the cohesiveness of a set, unlike the traditional model. The example W also records location, and utilities from the traditional model can be used in place of v.

IV. Derivation of Demand

Consumers determine demand for land along with demand for all other goods. However, the complicating pecularities, noncompactness of the consumption set and a nontrivial topology thereon, are specific to land as an indivisible commodity in the sense of the Introduction. Therefore, the other goods may be ignored, at least for the time being. After this section's analysis of a pure land market, the linkage with other goods will be discussed in the Conclusion. Thus, consider a consumer with a utility function U defined on the commodity space $\mathscr{B}$ and with a budget y. In partial equilibrium analysis y would represent income not spent on other goods, while in general equilibrium analysis it would be the value of the initial endowment or parcel. The utility function is taken to be continuous in the topology of the preceding section. This means that parcels are close substitutes when their interiors are near in the outer Hausdorff metric and they are of approximately the same measure (in terms of h).

The consumer's problem is

$$\max_{B \in \mathscr{B}} U(B) \tag{1}$$

subject to

$$\int_B p(x)\, dm(x) \leqslant y. \tag{2}$$

The main result is:

Theorem 1. *If U is continuous with respect to the topology defined in Section II, then there exists a solution to* (1) *subject to* (2).

Proof. Utility is defined up to a monotonic transformation and, therefore, may be assumed to be bounded. Moreover, since the empty set fulfills the budget constraint, there is a supremum value in the consumer's problem, (1) subject to (2). Thus let $\{B_k\}_{k=1}^{\infty} \subset \mathscr{B}$ be a sequence with $U(B_k)$ tending to the supremum and $\int_{B_k} p(x)\, dm(x) \leqslant y$ for all k. If $B_k = L$ infinitely often, then L solves the problem. Otherwise we may assume that $\{(\mathring{B}_k)^c\}_{k=1}^{\infty}$ converges in the Hausdorff topology since the latter is compact, as L is [11, p. 17].

Letting $G = \{g \in L^{\infty} \mid 0 \leqslant g \leqslant 1\}$ and embedding $\{1_{B_k}\}_{k=1}^{\infty}$, indicators of the sets B_k, in L^{∞}, $1_{B_k} \in G$ for all k. Note that G is weak*-closed. Since G lies in the closed unit ball of L^{∞}, the Banach–Alaoglu theorem (see [17, p. 66]) implies that G is weak*-compact. Hence, we may take $\{1_{B_k}\}_{k=1}^{\infty}$ to be convergent in the weak* topology on L^{∞}.

The limit of the sequence $\{(\mathring{B}_k)^c\}_{k=1}^{\infty}$, which is convergent in the Hausdorff topology, is closed and thus can be denoted B^c with B open relative to L.

Let

$$v = \lim_{k \to \infty} \int_L 1_{B_k}(x)(1 - 1_B(x))\, h(x)\, dm(x) = \lim_{k \to \infty} \int_{B_k \setminus B} h(x)\, dm(x)$$

and

$$w = \lim_{k \to \infty} \int_L 1_{B_k}(x)(1 - 1_B(x))\, p(x)\, dm(x) = \lim_{k \to \infty} \int_{B_k \setminus B} p(x)\, dm(x).$$

Both v and w exist since 1_{B_k} is weak*-convergent and hence converges on $(1 - 1_B)\, h$, $(1 - 1_B)\, p \in L^1$. Let $\mathscr{C}$ be $\mathscr{B}$ restricted to subsets of B^c. By Lyapunov's theorem, the image of $\mathscr{C}$ under the vector measure $\binom{\int h\, dm}{\int p\, dm}$ is closed. Also,

$$\lim_{k \to \infty} \begin{pmatrix} \int_{B_k \setminus B} h(x)\, dm(x) \\ \int_{B_k \setminus B} p(x)\, dm(x) \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix}.$$

Hence there exists $C \in \mathscr{C}$ with

$$\begin{pmatrix} \int_C h(x)\, dm(x) \\ \int_C p(x)\, dm(x) \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix}.$$

Also, $C \cap B = \varnothing$. Let $A = B \cup C$.

We want to eliminate an annoying asymmetry between $A = B \cup C$ and $B_k = \mathring{B}_k \cup (B_k \backslash \mathring{B}_k)$. In a sense, namely of the "outer" Hausdorff topology, B is the limit of $\mathring{B}_k$. But B need not be the interior of solution A. In principle, C may have added interior to B. To correct for this, we subtract some points from C, namely $S \cup \{[B \cup (C \backslash S)]^0 \backslash B\}$, where S is a countable dense subset of B^c which indeed, as a subset of $\mathscr{R}^n$ and therefore second countable, is separable.

Basically, S destroys the interior of C, and any interior that may still be created in uniting B and $C \backslash S$ is also subtracted away. Nevertheless, the points we subtract form only a null set: They belong either to S, which is countable, or to $[B \cup (C \backslash S)]^0 \backslash B$. But the latter, by Lemma 2, is contained in ∂B, which is null by Lemma 3. In short, we define $C' = C \backslash \{S \cup \{B \cup (C \backslash S)]^0 \backslash B\}\}$ and $A' = B \cup C'$. Then $\mathring{A}' = B$ and $C' = C$ a.s. Also, as $C' \subset C \subset B^c$, $C' = A' \backslash B$.

Now

$$\left| \int_{B_k} h(x)\, dm(x) - \int_{A'} h(x)\, dm(x) \right|$$

$$= \left| \int_{B_k \backslash B} h(x)\, dm(x) + \int_{B_k \cap B} h(x)\, dm(x) - \int_B h(x)\, dm(x) - \int_{C'} h(x)\, dm(x) \right|$$

$$\leqslant \left| \int_{B_k \backslash B} h(x)\, dm(x) - \int_{C'} h(x)\, dm(x) \right| + \left| \int_{B_k \cap B} h(x)\, dm(x) - \int_B h(x)\, dm(x) \right|$$

$$= \left| \int_{B_k \backslash B} h(x)\, dm(x) - \int_C h(x)\, dm(x) \right| + \left| \int_{B_k \cap B} h(x)\, dm(x) - \int_B h(x)\, dm(x) \right|.$$

The first term tends to zero by choice of C, while the second term tends to zero by Lemma 1 and Lebesgue's dominated convergence theorem. Hence

$$\lim_{k \to \infty} \int_{B_k} h(x)\, dm(x) = \int_{A'} h(x)\, dm(x).$$

By a similar argument,

$$\lim_{k\to\infty}\int_{B_k} p(x)\,dm(x)=\int_{A'} p(x)\,dm(x).$$

Since $\int_{B_k} p(x)\,dm(x)\leqslant y$ for all k, $\int_{A'} p(x)\,dm(x)\leqslant y$.

Since $\mathring{B}_k\to B$ in the outer Hausdorff topology and U is a continuous function, $U(A')$ attains the supremum value of U over the budget set.

Q.E.D.

Implicitly, it has been proved that the budget set is sequentially compact in the given topology.

V. Underpinning by Preferences

The topology described in Section II can be given a metric structure under certain conditions. For $A, B\in\mathscr{B}$, define a distance (or pseudometric)

$$d(A,B)=H((\mathring{A})^c,(\mathring{B})^c)+\left|\int_A h(x)\,dm(x)-\int_B h(x)\,dm(x)\right|.$$

If an equivalence class of $\mathscr{B}$ is defined to be a collection of elements of $\mathscr{B}$, each pair of which has distance zero, then d defines a metric on such equivalence classes of $\mathscr{B}$. Let $\tilde{\mathscr{B}}$ be the set of equivalence classes of elements of $\mathscr{B}$.

Note that Theorem 1 yields as a solution a particular element of $\mathscr{B}$, not of $\tilde{\mathscr{B}}$. The reason is that not all elements of an equivalence class of $\mathscr{B}$ have the same value, so some could violate the budget constraint. However, it is convenient to use this metric space to define preferences, since all elements of an equivalence class of $\tilde{\mathscr{B}}$ must have the same utility if the utility is to be continuous with respect to the topology defined in Section II.

In this section conditions on preferences, objects more primitive than utilities, that imply the existence of continuous utilities are investigated.

A preference order $\gtrsim$ over $\tilde{\mathscr{B}}$ is called *continuous* if sets of the form $\{B\in\tilde{\mathscr{B}}\mid B\gtrsim C\}$ and $\{B\in\tilde{\mathscr{B}}\mid C\gtrsim B\}$ are closed in the topology of Section II for each $C\in\tilde{\mathscr{B}}$.

THEOREM 2. *If a preference order $\gtrsim$ over $\tilde{\mathscr{B}}$ is continuous, then there exists a continuous utility representation U of $\gtrsim$.*

Proof. First we show that $\tilde{\mathscr{B}}$ is separable. The proof in [5, p. 360] shows that there is a countable set $\mathscr{E}\subset\mathscr{B}$ of open elements such that if A is compact and C is open, $A\subset C$, then there exists $E\in\mathscr{E}$ with $A\subset E\subset C$. We have added L and $\varnothing$ to $\mathscr{E}$.

For each $E \in \mathscr{E}$, Halmos [10, Theorem 40.5] gives us that $\{B \in \mathscr{B} \mid B \subset E^c\}$ with the L^1 topology on indicators of sets is separable; call its countable dense subsets $\mathscr{F}_E$.

In constructing a countable subset of $\tilde{\mathscr{B}}$ that is dense in our metric, it is natural to take equivalence classes of unions of $\mathscr{E}$ and respective $\mathscr{F}_E$ members, $E \cup F$. The interior of the latter is relevant in evaluating the outer Hausdorff component of the metric, but, unfortunately, need not be simply E, since F may have added interior. To correct for this, we subtract some points from F, namely $S \cup \{[E \cup (F \backslash S)]^0 \backslash E\}$, where S is a countable dense subset of E^c which indeed, as a subset of $\mathscr{R}^n$ and therefore second countable, is separable. Basically, S destroys the interior of F, and any interior that may still be created in uniting E and $F \backslash S$ is also subtracted away. In short, for each $E \in \mathscr{E}$ and $F \in \mathscr{F}_E$ we define $F' = F \backslash \{S \cup \{[E \cup (F \backslash S)]^0 \backslash E\}\}$ and $\mathscr{F}'_E$ the collection of such F'. Then for each $E \in \mathscr{E}$, $F' \in \mathscr{F}'_E$, $(E \cup F')^0 = E$ and $F' = F$ a.s. for some $F \in \mathscr{F}_E$.

Now $\mathscr{G} = \{E \cup F' \in \tilde{\mathscr{B}} \mid E \in \mathscr{E},\ F' \in \mathscr{F}'_E\}$ is countable as it is of the same cardinality as of a countable product of countable sets [16, Corollary 2.14]. We show that it is dense in $\tilde{\mathscr{B}}$. Let $B \in \tilde{\mathscr{B}}$ and fix $\delta > 0$. If $B = L$, put $E = L$. If $\mathring{B} = \varnothing$, put $E = \varnothing$.

Otherwise, let $B' = \{x \in \mathring{B} \mid \inf_{y \notin \mathring{B}} \|x - y\| \geq \delta/3k\}$, where $k \geq 1$ is chosen so large that $\int_{\mathring{B} \backslash B'} |h(x)|\, dm(x) \leq \delta/3$; then $\mathring{B} \backslash B'$ is nonempty, is open, and has positive Lebesgue measure. Clearly, $B' \subset L$ is closed, hence compact. By definition of $\mathscr{E}$, the latter contains an E with $B' \subset E \subset \mathring{B}$. Also, $H[(\mathring{B})^c, E^c] < \delta/3k$. (This fact is established easily: On the one hand $(\mathring{B})^c \subset E^c$. On the other, if $x \in E^c$, then $x \in B'^c$, hence $x \notin \mathring{B}$ or $\inf_{y \notin B} \|x - y\| < \delta/3k$, hence $\inf_{y \notin \mathring{B}} \|x - y\| < \delta/3k$, so $\|x - y\| < \delta/3k$ for some $y \in (\mathring{B})^c$.)

$B \backslash \mathring{B}$ is situated in E^c. Since $\mathscr{F}'_E$ is dense in $\{B \in \mathscr{B} \mid B \subset E^c\}$ in the L^1 topology of indicators of sets, it contains, by Lebesgue's dominated convergence theorem, a member $F' \subset E^c$ with $\int |1_{F'}(x) - 1_{B \backslash \mathring{B}}(x)|\, |h(x)|\, dm(x) < \delta/3$. Defining $B^* = E \cup F' \in \mathscr{G}$ and using $E \cap F' = \varnothing$, $H[(\mathring{B})^c, (\mathring{B}^*)^c] = H[(\mathring{B})^c, E^c] < \delta/3k \leq \delta/3$,

$$\left| \int_B h(x)\, dm(x) - \int_{B^*} h(x)\, dm(x) \right|$$

$$= \left| \int_{\mathring{B}} h(x)\, dm(x) + \int_{B \backslash \mathring{B}} h(x)\, dm(x) - \int_E h(x)\, dm(x) - \int_{F'} h(x)\, dm(x) \right|$$

$$\leq \left| \int_{\mathring{B}} h(x)\, dm(x) - \int_E h(x)\, dm(x) \right|$$

$$+\left|\int_{B\setminus\hat{B}} h(x)\, dm(x) - \int_{F'} h(x)\, dm(x)\right|$$

$$\leqslant \int_{\hat{B}\setminus E} |h(x)|\, dm(x) + \int |1_{F'}(x) - 1_{B\setminus\hat{B}}(x)|\, |h(x)|\, dm(x)$$

$$< \int_{\hat{B}\setminus B'} |h(x)|\, dm(x) + \delta/3 \leqslant \frac{2\delta}{3}.$$

In sum, the distance between $B \in \tilde{\mathscr{B}}$ and $B^* \in \mathscr{G}$ is less than δ. Hence $\mathscr{G}$ is dense in $\tilde{\mathscr{B}}$. Since the former is countable, the latter is separable. By Rudin [16, p. 39], $\tilde{\mathscr{B}}$ is second countable, so $\gtrsim$ has a continuous utility representation according to Debreu [9, Proposition 3]. Q.E.D.

An alternative, but less direct, technique of proof would be to show that $\tilde{\mathscr{B}}$ is compact and metrizable, hence second countable.

VI. Conclusion

In this article, it has been shown that a continuum of consumers is not necessary to the development of location theory. A useful topology has been proposed as a tool for the analysis of an economy with a finite number of consumers, and a solution to the consumer problem has been shown to exist when this topology is employed. Further questions related to the existence of equilibria and their welfare properties have yet to be tackled. Standard commodities can be added to the model with land in a relatively straightforward fashion. If $U: \mathscr{B} \times \mathscr{R}^l_+ \to \mathscr{R}$ is continuous in the product topology on $\mathscr{B} \times \mathscr{R}^l_+$, then there exists a solution to the maximization problem

$$\max_{\substack{B \in \mathscr{B} \\ Z \in \mathscr{R}^l_+}} U(B, Z) \tag{3}$$

subject to

$$\int_{B \in \mathscr{B}} p(x)\, dm(x) + q \cdot Z \leqslant y, \tag{4}$$

where $p \in L^1$, $p \geqslant 0$, and $q \in \mathscr{R}^l$, $q^i > 0$ for each i (see the corollary of the Appendix).

APPENDIX

LEMMA 1. *Let* $\{B_k\}_{k=1}^{\infty} \subset \mathscr{B}$ *be a sequence of open sets with nonempty complements such that* $B_k^c \to B^c$ *in the Hausdorff topology and let* $p \in L^1$. *Then* $\int_{B_k \cap B} p(x)\, dm(x) \to \int_B p(x)\, dm(x)$.

Proof. Let $x \in B$. Since B must be open, $B_\varepsilon(x) \subset B$ for some ε positive. Hence $x \notin B_\varepsilon(B^c)$. Since $B_k^c \to B^c$ in the Hausdorff topology, $B_\varepsilon(B^c) \supset B_k^c$ for k large enough, say $k \geqslant N$. It follows that $x \notin B_k^c$ and, therefore, $x \in B_k \cap B$, $k \geqslant N$. Hence $1_{B_k \cap B} \to 1_B$, pointwise. Consequently, using Lebesgue's dominated convergence theorem,

$$\lim_{k \to \infty} \int_{B_k \cap B} p(x)\, dm(x) = \int_L \lim_{k \to \infty} 1_{B_k \cap B}(x)\, p(x)\, dm(x) = \int_B p(x)\, dm(x).$$

Q.E.D.

Remark. Suppose $p \geqslant 0$ a.s. and $\int_{B_k} p(x)\, dm(x) \leqslant z$ for all k. Then $\int_{B_k \cap B} p(x)\, dm(x) \leqslant z$ and by Lemma 1, $\int_B p(x)\, dm(x) \leqslant z$. In other words, the imposition of Hausdorff convergence on *complements* insures preservation of the budget constraint in the limit.

LEMMA 2. *Let* A *and* B *be disjunct sets. Then* $(A \cup B)^0 \subset (\mathring{A} \cup \mathring{B} \cup \partial A)$.

Proof. Let $x \in (A \cup B)^0$. Then $x \in A \cup B$. If $x \in A \cup \mathring{B}$, we are done. Otherwise $x \in \partial B \backslash A$. Sufficiently small neighborhoods contain non-B-members which, by choice of x, must belong to A. Thus, while x does not belong to A, it must be in ∂A. Q.E.D.

LEMMA 3. *An open set in* $\mathscr{R}^n$ *has null boundary.*

Proof. Since $\mathscr{R}^n$ is separable metric, it has a countable base [16, p. 39]. Hence an open set, A, is the countable union of balls, $B_1, B_2, \ldots$. Therefore

$$\partial A \subset \bigcup_{k=1}^{\infty} \partial B_k \quad \text{and} \quad m(\partial A) \leqslant m\left(\bigcup_{k=1}^{\infty} \partial B_k\right) \leqslant \sum_{k=1}^{\infty} m(\partial B_k) = 0.$$

Q.E.D.

COROLLARY. *If* $\mathscr{B}$ *is given the topology defined in Section* II, $U: \mathscr{B} \times \mathscr{R}_+^l \to \mathscr{R}$ *is continuous in the product topology on* $\mathscr{B} \times \mathscr{R}_+^l$, $p \in L^1$ $(p \geqslant 0)$, $q \in \mathscr{R}^l$, $q^i > 0$ *for all* $i = 1, 2, \ldots, l$, *then there exists a solution to* (3) *subject to* (4).

Proof. $(\emptyset, 0) \in \mathscr{B} \times \mathscr{R}^l_+$ is in the budget set, so since the ordinal utility function can be chosen to be bounded,

$$C \equiv \sup \left\{ U(B, Z) \middle| (B, Z) \in \mathscr{B} \times \mathscr{R}^l_+, \int_B p(x)\, dm(x) + q \cdot Z \leqslant y \right\}$$

is well defined. Choose a sequence

$$(B_k, Z_k)_{k=1}^{\infty}, (B_k, Z_k) \in \mathscr{B} \times \mathscr{R}^l_+, \int_{B_k} p(x)\, dm(x) + q \cdot Z_k \leqslant y \qquad \text{for all } k,$$

with $\lim_{k \to \infty} U(B_k, Z_k) = C$. The projection of the budget set onto its second component is compact in $\mathscr{R}^l_+$ since $q^i > 0$ for all i, so without loss of generality $Z_k \to Z^* \in \mathscr{R}^l_+$.

As in the proof of Theorem 1, by passing to a subsequence we can take $\{B_k\}_{k=1}^{\infty}$ to be convergent to some B^* in our topology with

$$\lim_{k \to \infty} \int_{B_k} p(x)\, dm(x) = \int_{B^*} p(x)\, dm(x).$$

Hence

$$U(B^*, Z^*) = \lim_{k \to \infty} U(B_k, Z_k) = C.$$

Also,

$$y \geqslant \int_{B_k} p(x)\, dm(x) + q \cdot Z_k \qquad \text{for all } k,$$

so

$$y \geqslant \lim_{k \to \infty} \int_{B_k} p(x)\, dm(x) + \lim_{k \to \infty} q \cdot Z_k = \int_{B^*} p(x)\, dm(x) + q \cdot Z^*. \qquad \text{Q.E.D.}$$

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