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Supportability and Anonymous Equity*

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W. W. Sharkey and L. G. Telser (*J. Econom. Theory* **18** (1978), 23–37) feel that invulnerability of a natural monopoly to the threat of competitive entry is well reflected in the concept of supportability. G. R. Faulhaber and S. B. Levinson (*Amer. Econom. Rev.* **71** (1981), 1083–1091) point out that supportability is necessary for the achievability of anonymous equity, i.e., absence of consumer subsidies in public enterprise pricing. This paper reconciles supportability with market clearance and shows that supportability is sufficient for the achievability of anonymous equity. *Journal of Economic Literature* Classification Numbers: 022, 611, 614.

INTRODUCTION

A cost function is *supportable at an output vector* x^0 if prices exist which cover total costs and render supply of any part of the output $x \leqslant x^0$ unprofitable. (Precise definitions will be presented in the next section.) A cost function is *supportable* if it is supportable at any output vector. The supporting prices clearly have to do with invulnerability to the threat of entry. But do they call forth the output under consideration: x^0; are supporting prices market clearing? Sharkey and Telser kill this complication by assuming that demand is completely inelastic. This paper will relax the complete inelasticity assumption drastically. If there are threshold quantities of demand for all goods, then supportability and market clearance *can* be reconciled.

The problem also arises in another related context; Willig [6] and Faulhaber and Levinson [2] define the absence of consumers subsidies in public enterprise pricing as anonymous equity. Prices are *anonymously equitable* if they cover total costs of the quantities called forth by demand

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and any part of demand generates revenues no greater than its stand-alone costs. Faulhaber and Levinson observe that supportability of the cost function at the quantities demanded is a necessary condition for anonymous equity to be achievable. In fact, the preceding definitions imply that anonymous equity *is* market clearing supportability at the quantities demanded. Hence the proclaimed result is equivalent to the following. If there are threshold quantities of demand for all goods, then supportability is a *sufficient* condition for anonymous equity to be achievable.

In the recent theories of natural monopoly and public enterprise pricing two elements are of importance. One is invulnerability to the threat of entry or, depending on the setting, lack of subsidization. The other is market clearance. These elements seem quite different if not independent. The result of this paper indicates though that the second element, market clearance, is hidden in the first one, invulnerability or subsidy-freeness. Solution concepts which include market clearance, such as anonymous equity, are only superficially tighter than those which do not, such as supportability.

PRELIMINARIES

First some notation and definitions are copied from Sharkey and Telser [5]. $\mathbb{R}^n_+$ will represent the nonnegative orthant of n-dimensional Euclidean space. For any two members x and y the expression $x \geqslant y$ is to be interpreted $x_i \geqslant y_i$ for $i = 1,..., n$ and $x > y$ will be written if $x \geqslant y$ and $x \neq y$. Furthermore, the expression $x \gg y$ is to be interpreted $x_i > y_i$ for $i = 1,..., n$. The inner product of x and y will be written $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. Any nondecreasing function on $\mathbb{R}^n_+$ will be considered a *cost function*. A cost function c is *subhomogeneous* if $c(\lambda x) \leqslant \lambda c(x)$ for all $x \geqslant 0$ and $\lambda \geqslant 1$. A cost function c is *supportable at* x^0 if $p(x^0) = \{p \in \mathbb{R}^n_+ \mid \langle p, x^0\rangle = c(x^0)$ while $0 \leqslant x \leqslant x^0$ implies $\langle p, x\rangle \leqslant c(x)\}$ is nonempty. c is *supportable* if it is supportable at any $x^0 > 0$. The latter property will be briefly referred to by the notion of supportability. Observing axiomatic value theory, a *demand correspondence* q will be an upper hemicontinuous convex-valued correspondence from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$. (Upper hemicontinuity means that $x^m \to x$, $y^m \in q(x^m)$ implies $y^{m_k} \to y$, $y \in q(x)$ for some subsequence. Upper semicontinuity means that $x^m \to x$, $y^m \in q(x^m)$, $y^m \to y$ implies $y \in q(x)$. Note that upper hemicontinuity is sufficient for upper semicontinuity.) By definition, *threshold* quantities ε for all goods fulfill $0 \ll \varepsilon \leqslant q$. Following Faulhaber and Levinson [2], *anonymous equity* is said to be *achievable* if there are $p^0 \in \mathbb{R}^n_+$ and $x^0 \in \mathbb{R}^n_+$ with $p^0 \in p(x^0)$ and $x^0 \in q(p^0)$. In other words, anonymous equity is achievable if and only if there is market clearance and supportability at the quantities demanded.

If q is single-valued, then the definition of anonymous equity becomes, substituting the second condition into the first one: there is a $p^0 \in \mathbb{R}^n_+$ with $\langle p^0, q(p^0)\rangle = c(q(p^0))$ while $0 \leqslant x \leqslant q(p^0)$ implies $\langle p^0, x\rangle \leqslant c(x)$. In this case the closely related concept of *sustainability* is defined as follows: there is a $p^0 \in \mathbb{R}^n_+$ with $\langle p^0, q(p^0)\rangle = c(q(p^0))$, while $0 \leqslant p \leqslant p^0$, $0 \leqslant x \leqslant q(p)$ implies $\langle p, x\rangle \leqslant c(x)$ (see [2] and references therein). Reference [2, Proposition 7] neatly organizes the solution concepts we have come across. Supportability at the quantities demanded is necessary for anonymous equity to be achievable which on its turn is necessary for sustainability. In fact, from supportability at the quantities demanded one obtains anonymous equity by the additional requirement of market clearance. And from anonymous equity one obtains sustainability by requiring that the condition of subsidy freeness or invulnerability to the threat of entry is also met for lower prices. The first tightening up is but superficial, as the result of this paper indicates. The second restriction, in going from anonymous equity to sustainability, is an open issue. This, however, is not so much a matter of market clearance and beyond the scope of the paper.

Analysis

Our strategy of proving that under supportability anonymous equity is achievable consists of finding a fixed point of $(p, x) \mapsto p(x) \times q(p)$. For this purpose we want the constituent mappings to be upper semicontinuous convex-valued correspondences. q is like that by definition but for $p(\cdot)$ we have to prove that it is in fact an upper semicontinuous convex-valued correspondence. The assumed supportability guarantees the existence of $p(\cdot)$ as a correspondence. Further, it is convex-valued by definition. Thus the point is to prove upper semicontinuity. The difficulty is that only supportability is assumed whereas we need some kind of regularity. Lemma 1 will close the gap by proving that for cost functions an aspect of supportability (namely, subhomogeneity) implies continuity. Then Lemma 2 will finish off by showing that $p(\cdot)$ is as desired. In this way, invoking some straightforward compactness considerations and Kakutani's fixed point theorem, we shall find a fixed point of the described mapping which clearly proves that anonymous equity is achievable given supportability.

LEMMA 1. *A subhomogeneous cost function is continuous from below. It is continuous on its domain's interior.*

Proof. Let the cost function be c and let $\mathbb{R}^n_+ \ni x^m \to x$. Define λ_m (μ_m) as the maximum (minimum) of the existing x_i/x_i^m and unity. Then $\lambda_m \downarrow 1$ and, for $x \gg 0$, $\mu_m \uparrow 1$. For large m, $\lambda_m x_i^m \geqslant x_i \geqslant \mu_m x_i^m$. This follows from λ_m's (μ_m's) maximizing (minimizing) property for $x_i > 0$ ($x_i^m > 0$) and is otherwise obvious. Thus for large m, $\lambda_m x^m \geqslant x \geqslant \mu_m x^m$. It follows that by nondecreasingness of cost function c and subhomogeneity, $\lambda_m^{-1} c(x) \leqslant \lambda_m^{-1} c(\lambda_m x^m) \leqslant c(x^m) \leqslant \mu_m^{-1} c(\mu_m x^m) \leqslant \mu_m^{-1} c(x)$. It follows that $c(x) \leqslant \lim c(x^m) \leqslant c(x)$ where the second inequality is true for $x \gg 0$. Q.E.D.

LEMMA 2. *For a supportable cost function, $p(\cdot)$ is an upper semicontinuous convex-valued correspondence. It is bounded on $\{x^0 \in \mathbb{R}_+^n \mid x^0 \geqslant \varepsilon\}$ for $\varepsilon \gg 0$.*

Proof. By supportability, $p(\cdot)$ is a correspondence. The convex valuedness is obvious. To prove upper semicontinuity, let $\mathbb{R}_+^n \ni x^m \to x^0 > 0$ and $p(x^m) \ni p^m \to p^0$. Then

$$p^0 \in \mathbb{R}_+^n. \tag{1}$$

By [5, Proposition 2], the cost function, say c, is subhomogeneous, and, by Lemma 1, c is continuous from below. Consequently, $\lim c(x^m) \geqslant c(x^0)$, and, using $p^m \in p(x^m)$,

$$\langle p^0, x^0 \rangle - c(x^0) \geqslant \lim [\langle p^m, x^m \rangle - c(x^m)] \geqslant 0. \tag{2}$$

For $0 \leqslant x \leqslant x^0$ there are $y^m \leqslant x^m$ with $y^m \uparrow x$. Using $p^m \in p(x^m)$ and the nondecreasingness of cost function c, $\langle p^m, y^m \rangle \leqslant c(y^m) \leqslant c(x)$. It follows that

$$\langle p^0, x \rangle \leqslant c(x). \tag{3}$$

By (1), (2), and (3), $p^0 \in p(x^0)$ which proves the upper semicontinuity. Finally, on $x^0 \geqslant \varepsilon \gg 0$, $p^0 \in p(x^0)$ implies $\langle p^0, \varepsilon \rangle \leqslant c(\varepsilon)$ which implies that p^0 is bounded independently of x^0 and therefore $p(\cdot)$ is bounded. Q.E.D.

Now we present our main result.

PROPOSITION. *If there are threshold quantities of demand for all goods, then supportability is a sufficient condition for anonymous equity to be achievable.*

Proof. By Lemma 2, a convex compactum $\mathscr{S}$ contains $p(\{x \in \mathbb{R}_+^n \mid x \geqslant \varepsilon\})$. By the upper hemicontinuity of q, a convex compactum $\mathscr{E}$ contains $q(\mathscr{S})$. By the demand assumption $\mathscr{E}$ can be situated in $\{x \in \mathbb{R}_+^n \mid x \geqslant \varepsilon\}$. Now take $p \in \mathscr{S}$ and $x \in \mathscr{E}$. It follows that $p(x) \subset p(\mathscr{E}) \subset p(\{x \in \mathbb{R}_+^n \mid x \geqslant \varepsilon\}) \subset \mathscr{S}$ and $q(p) \subset q(\mathscr{S}) \subset \mathscr{E}$. Consequently, by Lemma 2 and Kakutani's fixed point theorem, $\mathscr{S} \times \mathscr{E} \ni (p, x) \mapsto p(x) \times q(p) \subset \mathscr{S} \times \mathscr{E}$ has a fixed point (p^0, x^0) with $p^0 \in p(x^0) \subset \mathbb{R}_+^n$ and $x^0 \in q(p^0) \subset \mathbb{R}_+^n$. Q.E.D.

Our demand assumption merely rules out a boundary complication which is independent and has been analyzed in [1]. An alternative set of demand assumptions which yields the same result consists of weak gross substitutability and normality as defined in [4] along with the condition that every output ray (consisting of bundles with fixed proportions) contains a profitable point. This can be proved in the same way as [3, Theorem 3]. Both the threshold and the profitability assumptions prevent the solution from degenerating into the trivial one that reflects unwillingness to pay the costs of the goods.

Discussion

An anonymous referee made an interesting comment. (S)he interpreted that if the cost function is supportable then any output is stable in a certain sense provided that demand is completely inelastic, but otherwise there is at least one stable output by our proposition. But (s)he wondered if there is any way to *identify* the anonymously equitable prices and quantities. Here I would like to make two remarks. One is on the nature of the problem. The issue is to find, simultaneously, $p^0 \in p(x^0)$ and $x^0 \in q(p^0)$. These relationships are, respectively, a supply schedule and a demand schedule. The problem is essentially to find the intersection of these schedules. The appropriate tools are the approximation techniques of equilibrium analysis. The other remark is a brief reference to [2, Proposition 9]. This proposition provides conditions under which the Ramsey optimum is anonymously equitable.

Conclusion

The entry concept of supportability and the equilibrium concept of market clearance can be reconciled. For anonymous equity to be achievable, supportability is not only necessary as pointed out by Faulhaber and Levinson [2] but also sufficient provided that there are threshold quantities of demand for all goods.

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