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CONSISTENT SETS OF ESTIMATES FOR REGRESSIONS WITH CORRELATED OR UNCORRELATED MEASUREMENT ERRORS IN ARBITRARY SUBSETS OF ALL VARIABLES

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1. INTRODUCTION

OVER THE LAST DECADE the problem of measurement errors in the independent variables of a regression equation has attracted renewed interest among econometricians. In the fifties and sixties, the problem was considered to be more or less hopeless due to its inherent underidentification (e.g., Theil (1971)). Apart from instrumental variables, the most frequently cited textbook solution was Wald's method of grouping (Wald (1940)). Recent insight into the properties of the method of grouping can be interpreted as making this method worthless in most practical cases (Pakes (1982)). Since about 1970, new approaches to the problem have been explored, basically along three lines, viz. embedding the error-ridden equation into a set of multiple equations (e.g., Zellner (1970), Goldberger (1972)), into a set of simultaneous equations (e.g., Hsiao (1976), Geraci (1976)), and using the dynamics of the equation, if present (e.g., Maravall and Aigner (1977)). In view of the underidentification of the basic model, it is clear that all these methods invoke additional information of some kind. If this information takes the form of exact or stochastic knowledge of certain parameters in the model, the construction of consistent estimators is fairly straightforward (e.g. Fuller (1980), Kapteyn and Wansbeek (1984)). For an overview of the state of the art, see Aigner et al. (1984).

An approach somewhat orthogonal to the ones described above has been to take the model as it is and to use prior ideas about the size of the measurement errors to diagnose how serious the probem is. Examples are Blomqvist (1972), Hodges and Moore (1972), and Davies and Hutton (1975). Leamer (1983) starts from the opposite direction by asking how serious the measurement error problem has to be in order to render the data useless for inference, that is to say, when measurement error is large enough to make it impossible to put bounds on regression parameters. In an empirical example, he shows that even very small measurement errors in some explanatory variables would open up the possibility of perfectly collinear explanatory variables and hence make the data useless for statistical inference (at least without additional prior information).

The most systematic analysis of the information loss caused by measurement error is due to Klepper and Leamer (1984). They start out by invoking a minimal amount of prior information and then ask the question under what conditions it is still possible to make some inferences regarding the vector of unknown regression parameters β. In the special case where the measurement errors are assumed uncorrelated and the $k+1$ estimates of β, obtained by regressing each of the $k+1$ variables involved (i.e. the one dependent variable and the k independent variables) on the remaining k variables, are all in the same orthant, one can bound the ML estimates of β. In that case, the convex hull of the $k+1$ regressions contains all possible ML estimates and any point in the hull is a possible ML estimate. If the $k+1$ regressions are not all in the same orthant then the set of ML estimates is unbounded.

In that case Klepper and Leamer (1984) introduce extra prior information which allows them to bound the set of maximum likelihood estimates. The prior information comes in two forms. Firstly, a researcher is supposed to be able to specify a maximum value of R^2 if all exogenous variables were measured accurately. It is shown that if this maximum is low enough, one can again bound the set of ML estimates by a convex hull. Secondly, if

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the R^2 bound does not help in bounding the estimates, a researcher is assumed to be able to give upper and lower bounds for the measurement error variance. If the upper bound is tight enough, so that the true explanatory variables cannot be perfectly collinear, the set of maximum likelihood estimates is shown to be bounded by an ellipsoid. In the derivation of the ellipsoid, based on a result in Leamer (1982), it is assumed that all exogenous variables are measured with error. Obviously, this is restrictive.

Bekker, Kapteyn, and Wansbeek (1984) have generalized Klepper and Leamer's result to the case where the variance covariance matrix of the measurement errors may be singular, but they still assumed, as did Klepper and Leamer, that the endogenous variable is measured without error or that the measurement error in the endogenous variables is uncorrelated with the errors in the exogenous variables. In this paper we relax this assumption. Not only are there many cases where a nonzero correlation between errors in the endogenous variable and in the explanatory variables is likely (for instance when all variables in an equation are deflated by the same imperfect price index), but the importance of the generalization also lies in the possibility to extend the analysis to more complicated models than just the linear regression model. Section 2 presents this result.

Although Klepper and Leamer (1984) assume throughout their paper that all measurement errors are uncorrelated, they do not exploit that information in the derivation of the ellipsoid. For any point in the ellipsoid we can find an Ω (the variance covariance matrix of the errors in the explanatory variables) that yields this point as an ML estimate, but such an Ω need not be diagonal. In Section 3 we investigate the consequences of the extra requirement that Ω is diagonal. In that case the ML estimates are bounded by a polyhedron, which need not be convex. Of course, the polyhedron lies within the ellipsoid. The convex hull of the polyhedron is determined by 2^l vertex points that all lie on the ellipsoid, where l is the number of nonzero measurement error variances. These points can be computed easily and then used to find, for all elements of β, intervals that bound the ML estimates. Generally, these intervals are tighter than the ones obtained from the ellipsoid.

Section 4 concludes by briefly discussing extensions to simultaneous equations models.

2. THE MODEL AND THE ELLIPSOID

Throughout we deal with the following model:

(2.1) $$\eta = \Xi\beta_0 + \varepsilon,$$

(2.2) $$y = \eta + u,$$

(2.3) $$X = \Xi + V;$$

equation (2.1) is the classical linear model, which relates the n-vector of dependent variables η to the $n\times k$-matrix of explanatory variables Ξ and the n-vector of disturbances ε. We assume that the distribution of ε is independent of Ξ and satisfies $E\varepsilon = 0$, $E\varepsilon\varepsilon' = \sigma_0^2 I$. The k-vector of parameters β_0 and σ_0^2 are unknown and have to be estimated.

Both η and Ξ are unobservable. Instead, y and X are observed and u and V therefore are the errors of measurement in y and X. We assume that u and V are uncorrelated with Ξ, η, and ε and that $Eu = 0$, $EV = 0$. Moreover, letting u_i be the ith element of u and v_i' the ith row of V, we assume that

$$E\begin{bmatrix} u_i \\ v_i \end{bmatrix}(u_i, v_i') \equiv \Phi \equiv \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Omega \end{bmatrix}$$

for all i and that (u_i, v_i') is stochastically independent of (u_j, v_j') for $i \neq j$.

Let Φ be known and define β and σ^2 by

(2.4) $$\beta \equiv (A-\Omega)^{-1}(Ab - \Phi_{21}),$$

(2.5) $$\sigma^2 \equiv \frac{1}{n}y'y - \Phi_{11} - \beta'(A-\Omega)\beta,$$

where $A \equiv X'X/n$, $b \equiv (X'X)^{-1}X'y$. Under normality of all variables involved, β and σ^2 are ML estimates. Under a variety of alternative assumptions, (β, σ^2) will still be a consistent estimate of (β_0, σ_0^2). Of course, if $\Phi = 0$, (β, σ^2) reduces to the OLS estimate (b, s^2), where $s^2 \equiv y'y/n - b'Ab$.

Although Φ will usually be unknown, it seems reasonable to assume that a researcher will be able to specify bounds for Φ, i.e.,

$$0 \leqslant \Phi \leqslant \Phi^* \equiv \begin{bmatrix} \Phi_{11}^* & \Phi_{12}^* \\ \Phi_{21}^* & \Omega^* \end{bmatrix}, \tag{2.6}$$

where Φ^* is specified by the researcher.[2] This bound on Φ will be used to derive bounds on the estimates β defined by (2.4). We assume that Φ^* is symmetric and that

$$0 \leqslant \Phi^* < B \equiv \frac{1}{n}\begin{bmatrix} y'y & y'X \\ X'y & X'X \end{bmatrix}, \tag{2.7}$$

thereby guaranteeing the existence of the estimate β and also the positiveness of the estimate σ^2 for any choice of Φ satisfying (2.6).[3] The latter can be shown easily by writing the positive definite matrix $(B-\Phi)^{-1}$ as

$$(B-\Phi)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & (A-\Omega)^{-1} \end{bmatrix} + \sigma^{-2}\binom{-1}{\beta}(-1, \beta'), \tag{2.8}$$

so that

$$\sigma^2 = \{e_1'(B-\Phi)^{-1}e_1\}^{-1} > 0, \tag{2.9}$$

where e_1 is the first unit vector. Furthermore, if we denote the estimate (β, σ^2) by (b^*, s^{*2}) if $\Phi = \Phi^*$, it is readily established that, as a consequence of the boundedness of Φ, also σ^2 is bounded:

$$s^2 \geqslant \sigma^2 \geqslant s^{*2} > 0. \tag{2.10}$$

We may now ask the question whether we can also delimit the set of estimates β given that Φ satisfies (2.6). The answer to that question is contained in Proposition 1. Define

$$F^* \equiv (A - \Omega^*)^{-1} - A^{-1}; \tag{2.11}$$

then we have the following proposition.

PROPOSITION 1: *The set of solutions β satisfying* (2.4), *with Φ satisfying* (2.6), *is given by:*

$$(\beta - \tfrac{1}{2}(b+b^*))'F^{*-}(\beta - \tfrac{1}{2}(b+b^*)) \leqslant \tfrac{1}{4}(s^2 - s^{*2}), \tag{2.12}$$

$$F^*F^{*-}(\beta - \tfrac{1}{2}(b+b^*)) = \beta - \tfrac{1}{2}(b+b^*), \tag{2.13}$$

where F^{*-} is an arbitrary g-inverse of F^*. This bound is minimal, i.e., for any β satisfying (2.12) *and* (2.13) *there exists a Φ such that* (2.4) *and* (2.6) *hold true.*

[2] The notation $C \leqslant D$ means that $D - C$ is a positive semidefinite matrix; $C < D$ means $D - C$ is positive definite.

[3] Note that Φ^* has to be strictly less than B. Among other things, this excludes the possibility that the true explanatory variables in Ξ are perfectly collinear. If Ξ could have less than full column rank, no bounds for β exist.

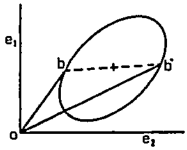

FIGURE 1.—The ellipsoid when (2.16) holds.

The proof follows from results obtained in Bekker (1986).

Equation (2.12) describes a cyclinder and (2.13) presents a projection of the cylinder onto the space spanned by F^*. Thus (2.12) and (2.13) describe an ellipsoid in the space spanned by F^*. It is easy to show that

(2.14) $$s^2 - s^{*2} = (b^* - b)'F^{*-}(b^* - b) + \Phi_{11}^* - \Phi_{12}^*\Omega^{*-}\Phi_{21}^*.$$

The nonnegative definiteness of Φ^* implies that

(2.15) $$\Phi_{11}^* \geqslant \Phi_{12}^*\Omega^{*-}\Phi_{21}^*.$$

If (2.15) holds as an equality, i.e.,

(2.16) $$\Phi_{11}^* = \Phi_{12}^*\Omega^{*-}\Phi_{21}^*,$$

then (2.12) and (2.14) imply that both b and b^* lie on the surface of the ellipsoid and the center of the ellipsoid is located at the midpoint of the segment joining b and b^*. See Figure 1.

If (2.16) holds, the measurement error u_i in y is linearly dependent upon the measurement errors v_i in the exogenous variables, in the sense that the mean square of their difference is zero. To see this, define $\lambda \equiv \Omega^{*-}\Phi_{21}^*$, so that (2.16) is equivalent to

(2.17) $$(-1, \lambda')\Phi^* = 0.$$

This implies, in conjunction with (2.6),

(2.18) $$0 \leqslant (-1, \lambda')\Phi\binom{-1}{\lambda} \leqslant (-1, \lambda')\Phi^*\binom{-1}{\lambda} = 0,$$

so that $(-1, \lambda')\Phi = 0$, which is equivalent to $E(u_i - \lambda' v_i)^2 = 0$. That is, the measurement error in y is a fixed linear combination of the measurement errors in X with probability one. One particular case in which this holds is where $\Phi_{21} = 0$ and $\Phi_{11} = 0$, i.e., no measurement errors in y.

If we let Φ_{11}^* increase, keeping all other elements of Φ^* constant, so that (2.15) becomes a strict inequality, $s^2 - s^{*2}$ increases according to (2.14). As b, b^*, and F^* do not depend on Φ_{11}^*, this means that the ellipsoid expands. In that case b and b^* are no longer on the surface on the ellipsoid, but the midpoint of the line joining b and b^* is still the center of the ellipsoid. See Figure 2.

The intuitive explanation for this is that if Φ_{11}^* increases, we do not only allow more measurement error in y (which is indistinguishable from errors in the equation anyway) but also more covariance between the errors in y and X. Thus, the bound on Φ becomes less tight and the ellipsoid expands.

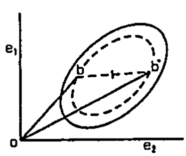

FIGURE 2.—The ellipsoid when (2.15) is a strict inequality.

If the number of regressors exceeds two, it will be hard in practice to represent the ellipsoid given by (2.12) and (2.13) in a transparant way. For that reason it is useful to derive bounds for linear functions of β. Let ψ be a known vector; then bounds for $\psi'\beta$ are implied by the following proposition.

PROPOSITION 2: *The maximum and minimum of $\psi'\beta$, with ψ fixed and β satisfying* (2.12) *and* (2.13), *are given by*

$$\psi'\hat{\beta}=\tfrac{1}{2}\psi'(b+b^*)\pm\tfrac{1}{2}((s^2-s^{*2})\psi'F^*\psi)^{1/2}. \tag{2.19}$$

The proof follows from results in Bekker (1986).

3. UNCORRELATED MEASUREMENT ERRORS

In this section we assume that, in addition to the bounds on Φ as given in (2.6), a researcher is also willing to assume that Φ^* and Φ are *diagonal.* That is, measurement errors in different variables are uncorrelated.

The first thing to notice is that in this case the measurement error in the regressand is completely indistinguishable from the error in the equation. Therefore it is of no consequence for the set of estimates β. Since Φ is diagonal, $\phi_{21}=0$ and the estimator β is simply given by

$$\beta=(A-\Omega)^{-1}Ab, \tag{3.1}$$

where Ω is diagonal and bounded by

$$0\leqslant\Omega\leqslant\Omega^*<A. \tag{3.2}$$

Clearly, the set of estimates is unchanged if we choose $\Phi_{11}^*=\Phi_{11}=0$. Consequently the ellipsoid (2.12)-(2.13) only depends on Ω^*. We will refer to (2.12)-(2.13), with $\Phi_{21}^*=0$ and $\Phi_{11}^*=0$, as "the ellipsoid spawned by Ω^*". This ellipsoid is still a bound for the set of estimators β, but it is no longer a minimal bound if Ω and Ω^* are restricted to be diagonal.

In order to derive a more satisfactory bound we define the following points:

$$\beta_\delta\equiv(A-\Omega_\delta^*)^{-1}Ab, \tag{3.3}$$

where $\Omega_\delta^*=\Omega^*\Delta=\Delta\Omega^*=\Delta\Omega^*\Delta$, with $\Delta=\text{diag}(\delta)$ and δ a vector with ones and zeros as elements. If Ω^* has l nonzero diagonal elements then there are 2^l different matrices Ω_δ^*, which all satisfy (3.2). Clearly the 2^l solutions β_δ are bounded by the ellipsoid spawned by Ω^*. We shall refer to the β_δ as "generated by Ω^*".

PROPOSITION 3: *All β_δ lie on the surface of the ellipsoid spawned by Ω^*.*

PROOF: See the Appendix.

Having established that all β_δ lie on the surface of the ellipsoid spawned by Ω^*, we next show that β lies in the convex hull of the 2^l points β_δ that are generated by Ω^*.

PROPOSITION 4: *If Ω and Ω^* are diagonal and satisfy* (3.2), *then the set of estimates β satisfying* (3.1) *is contained in the convex hull of the 2^l points β_δ generated by Ω^*.*

PROOF: See the Appendix.

Thus, the diagonality of Ω further reduces the region where β may lie when measurement error is present. In practical applications, the most obvious use of this result is to compute all 2^l points β_δ and to derive the interval in which each coefficient lies. These intervals will in general be smaller than the ones obtained from Proposition 2 by choosing for ψ

the k unit vectors successively. It should be noted that the convex polyhedron need not be a minimal bound. In fact, it can be shown by a counterexample (available upon request) that the set of estimates β may not be convex. However, this does not affect the intervals for the separate coefficients. Proposition 4 is similar to a result given by Chamberlain and Leamer (1976) (employing a result by Leamer and Chamberlain (1976)) that bounds the posterior mean by 2^k regressions if the prior covariance matrix is diagonal. In terms of the present framework, their proof assumes that Ω is an unbounded diagonal negative definite matrix. However, their result may be amended to allow for bounded diagonal positive definite matrices Ω. Still, for their proof it must be assumed that Ω is nonsingular (so $l=k$, among other things).

4. CONCLUSION

It is very simple to apply Propositions 2 and 4 to empirical problems, and the analysis could easily be incorporated in regression packages. Since the propositions cover a wide range of cases, the researcher has considerable freedom to express his prior ideas about Ω as precisely or as vaguely as he wants. The result of the analysis will then summarize succinctly the sensitivity of estimation outcomes to assumptions about the quality of the data used.

It appears that the framework developed in this paper will allow for extensions to more complicated models. Consider for example the jth structural equation in a linear simultaneous equations system:

$$y_j = Y_j\alpha_0 + \Xi_j\gamma_0 + \varepsilon_j, \tag{5.1}$$

where Y_j and Ξ_j are matrices of endogenous and exogenous variables, respectively, included as explanatory variables in this equation; y_j is the vector of endogenous variables to be explained by this equation and ε_j is a vector of errors. Let Ξ be the matrix of all exogenous variables in the system. Then 2SLS amounts to GLS applied to

$$\Xi' y_j = \Xi' Y_j\alpha_0 + \Xi'\Xi_j\gamma_0 + \Xi'\varepsilon_j. \tag{5.2}$$

If Ξ is measured with error, this model becomes similar to (2.1)–(2.3). Since Ξ occurs on both sides of the equation, the measurement errors in the left and right hand side variables will in general be correlated. For the special case where $\gamma_0=0$, it is easy to show that Proposition 1 can be applied directly to derive an ellipsoid for a consistent estimate of α_0, defined analogously to β (cf. (2.4)). (Bekker, Kapteyn, and Wansbeek (1984) have derived the same ellipsoid without reference to Proposition 1, assuming that all exogenous variables are measured with error.) Proposition 1 is not applicable when $\gamma_0 \neq 0$. For that more general case further research is needed.

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APPENDIX

PROOF OF PROPOSITION 3

Clearly $\Omega_\delta^* = \Omega_\delta^*\Omega^{*-}\Omega_\delta^*$ and $\Omega_\delta^* = \Omega^*\Omega^{*-}\Omega_\delta^*$. If we define

$$F_\delta^* \equiv (A-\Omega_\delta^*)^{-1} - A^{-1},$$

then

(A.1) $$F_\delta^*(A-\Omega^*)\Omega^{*-}AF_\delta^* = (A-\Omega_\delta^*)^{-1}\Omega_\delta^* A^{-1}(A-\Omega^*)\Omega^{*-}AA^{-1}\Omega_\delta^*(A-\Omega_\delta^*)^{-1} = F_\delta^*.$$

So $(A-\Omega^*)\Omega^{*-}A$ is a g-inverse of F_δ^* for every δ; in particular it is a g-inverse of F^*. As

(A.2) $$F_\delta^* = F^*(A-\Omega^*)\Omega^{*-}\Omega_\delta^*(A-\Omega_\delta^*)^{-1},$$

it follows that

(A.3) $$F_\delta^* F^{*-} F_\delta^* = F_\delta^*$$

for any g-inverse F^{*-}. As $2\beta_\delta - b - b^* = (2F_\delta^* - F^*)Ab$, and using (2.14) with $\Phi_{11}^* = 0$ and $\Phi_{21}^* = 0$, it follows that (2.12) becomes an equality if we substitute β_δ for β. *Q.E.D.*

Proof of Proposition 4

Before we present the proof we need some auxiliary results.

Lemma 1: *Let A be a positive definite matrix, k a vector, and μ a scalar, $0 \leq \mu \leq 1$. Then*

(A.4) $$(A+\mu kk')^{-1} = \lambda A^{-1} + (1-\lambda)(A+kk')^{-1},$$

where

(A.5) $$0 \leq \lambda = \frac{1-\mu}{1+\mu k' A^{-1} k} \leq 1.$$

Proof: Straightforward.

Without loss of generality, we assume that the first l diagonal elements of Ω^*, $\omega_1^*, \omega_2^*, \ldots, \omega_l^*$, are nonzero ($l \leq k$) and the remaining $k-l$ elements are zero. Let us index the 2^l vectors δ by a subscript j, with $j = 1, \ldots, 2^l$. A typical element of δ_j is δ_{ij}, $i = 1, \ldots, k$. We order the δ_j in such a way that, for $j \leq 2^m$ and $0 \leq m \leq l-1$, $\delta_{j+2^m} = \delta_j - e_{m+1}$, with e_{m+1} the $(m+1)$th unit vector. An example of such an ordering, for $k=4$ and $l=3$, is

$$\begin{bmatrix}1\\1\\1\\0\end{bmatrix} \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$
$$\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4 \quad \delta_5 \quad \delta_6 \quad \delta_7 \quad \delta_8$$

Define $K_j \equiv A - \sum_{i=1}^{l} \delta_{ij}\omega_i^* e_i e_i'$ (this would be denoted as $A - \Omega_\delta^*$ in Section 3, with $\delta = \delta_j$). Then we have $K_{j+2^m} = K_j + \omega_{m+1}^* e_{m+1} e_{m+1}'$.

Lemma 2: *Let μ_i, $i = 1, \ldots, 2^m$, be scalars satisfying $\mu_i \geq 0$, $\sum_i \mu_i = 1$; then there exist scalars λ_j, $j = 1, \ldots, 2^m$, satisfying $\lambda_j \geq 0$, $\sum_j \lambda_j = 1$, such that*

(A.6) $$\left\{\sum_{i=1}^{2^m} \mu_i K_i\right\}^{-1} = \sum_{j=1}^{2^m} \lambda_j (K_j)^{-1}, \quad \textit{for all} \quad 0 \leq m \leq l.$$

Proof: The proof is by induction. Assume (A.6) holds for $m \leq l-1$; then we show that it also holds for $m+1$.

(A.7) $$\sum_{i=1}^{2^{m+1}} \mu_i K_i = \sum_{i=1}^{2^m} \mu_i K_i + \sum_{i=1}^{2^m} \mu_{i+2^m} K_{i+2^m} = \sum_{i=1}^{2^m} (\mu_i + \mu_{i+2^m}) K_i + \left\{\sum_{i=1}^{2^m} \mu_{i+2^m}\right\} \omega_{m+1}^* e_{m+1} e_{m+1}'.$$

Lemma 1 implies

(A.8) $$\left\{\sum_{i=1}^{2^{m+1}} \mu_i K_i\right\}^{-1} = \lambda \left\{\sum_{i=1}^{2^m} (\mu_i + \mu_{i+2^m}) K_i\right\}^{-1} + (1-\lambda)\left\{\sum_{i=1}^{2^m} (\mu_i + \mu_{i+2^m})(K_i + \omega_{m+1}^* e_{m+1} e_{m+1}')\right\}^{-1}$$
$$= \lambda \left\{\sum_{i=1}^{2^m} (\mu_i + \mu_{i+2^m}) K_i\right\}^{-1} + (1-\lambda)\left\{\sum_{i=1}^{2^m} (\mu_i + \mu_{i+2^m}) K_{i+2^m}\right\}^{-1},$$

with $0 \leq \lambda \leq 1$. Assuming that the proposition holds for m, (A.8) implies that it holds also for $m+1$. Furthermore, (A.6) holds if $m=0$. *Q.E.D.*

We can now prove Proposition 4. Consider $K \equiv A - \Omega$. Given that $0 \leq \Omega \leq \Omega^*$ and that Ω and Ω^* are diagonal we can write K as

$$K = \sum_{j=1}^{2^l} \mu_j K_j, \tag{A.9}$$

where $\mu_j \geq 0$, $\sum_j \mu_j = 1$. According to (3.1) and Lemma 2 we have

$$\beta = (A-\Omega)^{-1}Ab = K^{-1}Ab = \left\{ \sum_{j=1}^{2^l} \mu_j K_j \right\}^{-1} Ab = \sum_{j=1}^{2^l} \lambda_j K_j^{-1} Ab = \sum_{j=1}^{2^l} \lambda_j \beta_{\delta_j}, \tag{A.10}$$

with $\lambda_j \geq 0$, $\sum_j \lambda_j = 1$. *Q.E.D.*

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