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APPLIED DYNAMIC INPUT–OUTPUT WITH DISTRIBUTED ACTIVITIES*

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This paper implements dynamic input–output analysis with distributed activities. For this purpose, the input–output equation is formulated and solved in discrete time. Existing dynamic input–output models are shown to be instances. The fully distributed input–output model is applied to analyse the dynamic structure of the Polish economy. The effects of investment distributions are expressed by comparison with conventional, non-distributed input–output results.

1. Introduction

The analysis of distributed activities like capital construction is greatly facilitated by defining the flow and stock coefficients matrices of the economy, A and B, as *distributions* on the past in the sense of ten Raa (1986). For example, $A(-s)$ represents the direct unit requirements s time units prior to the delivery of output. In this paper, for comparison with other dynamic input–output studies, we want to allow for technical change. Then the whole input profile, A, depends on time, say of delivery, which we will denote by a subscript. Thus, $A_t(-s)$ are the direct unit requirements s time units prior to delivery time t, that is at time $t-s$. The stock coefficients distribution, B, is made time dependent in the same way.

Following Leontief (1970), $B_{t+1}(0)$ are the direct and immediate unit investment requirements at time t for new capacity usable the year after. Moreover, this capacity also requires investment quantities of $B_{t+1}(-s)$ s time units prior to t, at least in our model which mimics the distribution of investment over time. Note that we doggedly follow Leontief's (1970) convention of indexing technology not by final year of production of the particular capital goods, but rather the year in which they are first put to use.

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A_t and B_{t+1}, thus defined, constitute the general input–output model presented and solved in section 2. The model that comes closest is in Kigyóssy-Schmidt and Schwarz (1983) who consider distributed investment only and solve implicitly by presenting an algorithm. Sections 3–5 show how existing dynamic input–output models with explicit solutions fall out as special cases. Computational aspects are dealt with in section 6. Section 7 discusses choice of time unit issues such as the bias involved. Section 8 applies the model to analyze the dynamic structure of the Polish economy. Section 9 concludes with a call for further data collection.

2. The material balance equation and its solution

The material balance between output $x(t)$ and final demand $z(t)$ reads [ten Raa (1986)]

$$x(t)=\sum_{s=0}^{\infty} A_{t+s}(-s)x(t+s)+\sum_{s=0}^{\infty} B_{t+s+1}(-s)[x(t+s+1)-x(t+s)]+z(t). \tag{1}$$

Here we have chosen a sufficiently small unit of time such that the derivative $\dot{x}(t)$ may be approximated by $x(t+1)-x(t)$. A later section will dwell on this. To ease notation, define

$$G_t(0)=I-A_t(0)+B_{t+1}(0) \quad \text{and for } s=1,2,3,\ldots,$$

$$G_t(s)=-A_{t+s}(-s)+B_{t+s+1}(-s)-B_{t+s}(-s+1). \tag{2}$$

Then the material balance, (1), reduces to

$$\sum_{s=0}^{\infty} G_t(s)x(t+s)=z(t), \quad \text{or} \tag{3}$$

$$\begin{bmatrix} \ddots & & & & & \\ & G_0(0) & G_0(1) & G_0(2)\ldots & & \\ & & G_1(0) & G_1(1) & G_1(2)\ldots & \\ & 0 & & G_2(0) & G_2(1) & G_2(2)\ldots \\ & & & \ddots & & \end{bmatrix} \begin{bmatrix} \vdots \\ x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ z(0) \\ z(1) \\ z(2) \\ \vdots \end{bmatrix}. \tag{4}$$

The supermatrix that appears here is a generalization of the structural matrix of Leontief (1970) that underlies his dynamic inverse. The presence of production times greater than one and of investment lead times disrupts the familiar bidiagonal structure. However, since these lags are non-negative, the structure is still triangular. This enables us to find the inverse of the general structural matrix. [Singularity problems are taken care of in ten Raa (1986).] The inverse will be triangular and its diagonal entries will be the inverses of the respective ones on the diagonal of the super G-matrix, like in

$$\begin{bmatrix} \ddots\, G_0(0)^{-1} D_0(1) \ldots D_0(v) \ldots & & \\ \ddots & & \ddots \\ & & G_t(0)^{-1} D_t(1) \ldots D_t(v) \ldots \\ & & \ddots \end{bmatrix} \tag{5}$$

There remain to be determined the off-diagonal matrices $D_t(v), t=\ldots, 0,1,2,\ldots.$ and $v=1,2,\ldots.$

Proposition 1. $D_t(v)=\sum R_t(s_1)\ldots R_{t+s_1+\cdots+s_{l-1}}(s_l)G_{t+v}(0)^{-1}$, *where* $R_t(s)=-G_t(0)^{-1}G_t(s)$ *and the summation is over all* $(s_1,\ldots,s_l)$ *with each component in* $\{1,\ldots,v\}$ *and the sum of the components equal to* v.

Proof. See the appendix.

It follows that

$$x(t)=G_t(0)^{-1}z(t)+\sum_{v=1}^{\infty} D_t(v)z(t+v), \tag{6}$$

with $D_t(v)$ given by Proposition 1. This completes the formulation and solution of the general discrete input–output equation.

Before we turn to specializations of the model in the next sections, we address two interrelated issues raised by the referee: (i) the accounting for existing capacities, and (ii) the investment requirements solution to a mere increase of final demand.

Existing capacities are accounted for only in so far they are currently needed for plan fulfillment. For example, imagine an economy with final demand from time zero on that lasts for some extended period. Then, after an initial period of capital construction, the stock of capital is transferred from year to year. The investment term, the second one on the right-hand side of (1), is zero as there are no output changes in the intermediate period. Capital stock that is given in excess of minimum plan fulfillment requirements, due to an 'initial endowment' or, more historically, to overinvestment, is not accounted for explicitly. To remedy the model one proceeds as follows. To account for overinvestment, final demand is widened as to include excess capacity additions. Or should one desire a particular level of output and capital stock at some 'initial' point of time, then solution (6) implicitly determines all feasible future paths of final demand, z, and sustaining output paths, x. The selection of such a path of final demand, z, is a matter of plan. The restriction of investment requirements to those called for by mere increases of final demand is indeed an alternative way to deal with existing capacities. Luckily the model need not be respecified for this purpose. Since the model is linear in an abstract mathematical sense, the change in requirements associated with alternative values of final demand equals the requirements for the increase. In other words, the linearity insures that changes are governed by the same eq. (1) and solution (6). The unit requirements which will be evaluated in section 8 can thus be interpreted as deviations from an overall development of the economy that result from unit changes in final demand such as an export program increase of one unit. In particular, the negative values reported there are reductions in output levels facilitated by disinvestment of capital constructed for the program. The disinvested quantities are absorbed by the rest of the economy in its production of the established final demand quantities.

3. No technical change

In this case the coefficients matrix distributions, A and B, do not depend on time and their subscripts can be suppressed. Consequently, the subscripts of the structural matrix, (2), drop out too:

$$G_t(s) = G(s) \tag{7}$$

and the solution, (6), becomes

$$x(t) = G(0)^{-1}z(t) + \sum_{v=1}^{\infty} D(v)z(t+v), \quad \text{with} \tag{8}$$

$$D(v) = \sum R(s_1)\dots R(s_l)G(0)^{-1}, \tag{9}$$

where $R(s) = -G^{-1}(0)G(s)$ and the summation is over all $(s_1, \dots, s_l)$ with each component in $\{1, \dots, v\}$ and the sum of the components equal to v. This case will be relevant for our application to the Polish economy below.

4. Unitary lags

In Nikaidô's (1962) lagged model all production times are unity while fixed capital is absent. Formally, structural matrix (2) becomes

$$G_t(0)=I, \quad G_t(1)=-A_{t+1}(-1), \quad G_t(2)=G_t(3)=\cdots=0. \tag{10}$$

Hence, in Proposition 1, $R_t(s)=0$ for $s\geqq 2$, and, therefore, the summation in $D_t(v)$ is over $(1,\dots,1)$ with v components and over that only. Thus

$$D_t(v)=R_t(1)\dots R_{t+v-1}(1)G_{t+v}(0)^{-1}=[-G_t(1)]\dots[-G_{t+v-1}(1)]$$

$$=A_{t+1}(-1)\dots A_{t+v}(-1)$$

and

$$x(t)=z(t)+\sum_{v=1}^{\infty} A_{t+1}(-1)\dots A_{t+v}(-1)z(t+v).$$

So far the model is more general than Nikaidô's who assumes technical change away. Then A's subscripts may be dropped and one obtains

$$x(t)=\sum_{v=0}^{\infty} A(-1)^v z(t+v). \tag{12}$$

This agrees with the particular solution of Proposition 3 of Nikaidô (1962).

5. Instantaneous production

In Leontief's (1970) dynamic model there are neither producion times nor investment lead times. Formally, structural matrix (2) becomes

$$G_t(0)=I-A_t(0)+B_{t+1}(0),$$

$$G_t(1)=-B_{t+1}(0),$$

$$G_t(2)=G_t(3)=\cdots=0. \tag{13}$$

As in section 4, the solution matrix of Proposition 1 can be specialized as follows:

$$D_t(v)=R_t(1)\dots R_{t+v-1}(1)G_{t+v}(0)^{-1}$$

$$=R_t(1)\dots R_{t+v-1}(1)[I-A_{t+v-1}(0)+B_{t+v-1}(0)]^{-1},$$

with

$$R_t(1) = -G_t(0)^{-1}G_t(1) = [I - A_t(0) + B_{t+1}(0)]^{-1}B_{t+1}(0).$$

This agrees with the typical element of Leontief's (1970) dynamic inverse. Recalling our general solution, (6), we now obtain

$$\begin{aligned} x(t) &= G_t(0)^{-1}z(t) + \sum_{v=1}^{\infty} D_t(v)z(t+v) \\ &= [I - A_t(0) + B_{t+1}(0)]^{-1}z(t) + \sum_{v=1}^{\infty} [I - A_t(0) \\ &\quad + B_{t+1}(0)]^{-1}(0)\dots[I - A_{t+v-1}(0) + B_{t+v}(0)]^{-1}B_{t+v}(0) \\ &\quad \times [I - A_{t+v}(0) + B_{t+v+1}(0)]^{-1}z(t+v). \end{aligned} \tag{14}$$

Note that when $z(t) = 0$ for $t \neq$ some s, the solution reduces to

$$\begin{aligned} x(t) &= [I - A_t(0) + B_{t+1}(0)]^{-1}z(t) + [I - A_t(0) \\ &\quad + B_{t+1}(0)]^{-1}B_{t+1}(0)\dots[I - A_{s-1}(0) + B_s(0)]^{-1}B_s(0) \\ &\quad \times [I - A_s(0) + B_{s+1}(0)]^{-1}z(s) \end{aligned} \tag{15}$$

with the second term eliminated when $t \geqq s$. When technical change is absent, the solution is further simplified to

$$\begin{aligned} x(t) &= [I - A(0) + B(0)]^{-1}z(t) \\ &\quad + \{[I - A(0) + B(0)]^{-1}B(0)\}^{s-t}[I - A(0) + B(0)]^{-1}z(s) \end{aligned} \tag{16}$$

with the second term eliminated when $t \geqq s$.

6. Computational aspects

In general, the greatest lag, be it production time or investment lead time, is important. Recall that the lagged technical coefficients are collected in matrix $G_t(s)$ of (2), where s denotes the lag. Thus the general lag can be represented by

$$\sigma = \sup_{s,t} \{s \mid G_t(s) \neq 0\}. \tag{17}$$

The implication for the computational build up of a typical element of the general solution, that is $D_t(v)$ of Proposition 1, is as follows:

Proposition 2. *$D_t(v)$ of Proposition 1 consists of one term if $\sigma=1$. Otherwise, writing $v=s\sigma+k$ ($s=0,1,\ldots$ and $k=1,\ldots,\sigma$), $D_t(v)=D_t(s\sigma+k)$ consists of $2^{k-s-1}\sum_{j=0}^{s}(-1)^j[((s-j)\sigma_j+k)+((s-j)\sigma_{j-1}+k-1)]2^{(s-j)(\sigma+1)}$ terms. Each term in $D_t(v)$ is the product of at most $n\times n$-matrices where n is the number of vectors.*

Proof. See the appendix.

The unknown itself, output $x(t)$, is an infinite expansion of $D_t(v)$'s, as given by (6). The convergence of the series can be analyzed in the same manner as of the dynamic inverse of Leontief (1970). Furthermore, eigensolutions of the homogeneous equation may arise just like in Nikaidô (1962). But basically our equation is a discretization of the distribution equation in ten Raa (1986) which has been analyzed in detail. This discretization error involved will be discussed now.

7. Time unit

The time unit was chosen sufficiently small to facilitate discretization of the material balance equation. Since this choice is clearly not unique, it is desirable to know the impact of variation of the time unit.

The model of instantaneous production of section 5, that is Leontief's (1970) dynamic inverse, reveals the essence of the problem as it contains a single lag – representing investment requirements. Thus we consider the discrete input–output equation,

$$x_\tau(t)=A(0)x_\tau(t)+B_\tau(0)[x_\tau(t+\tau)-x_\tau(t)]+z(t), \tag{18}$$

where the subscript, τ, as well as same τ in the capacity term refer to the unit of time in the following sense. To study variation of the time unit, we must measure the unit with some objective yardstick which is fixed once and for all. Say this measurement is in years, then τ is the chosen time unit expressed in years and x_τ is the implied output rate, measured against the objective yardstick, that is quantity per year, for comparison with other x_τ's under variation of τ. As is well known, the choice of time unit affects the capital matrix, unlike the flow matrix. B's subscript refers to this, not to technical change which is neglected in this section as it constitutes an independent problem.

The benchmark for our comparisons is the limiting case of the continuous input–output equation,

$$x(t)=A(0)x(t)+B(0)\dot{x}(t)+z(t). \tag{19}$$

The first equation, (18), is a direct discretization of (19), provided that

$$B_\tau(0)=B(0)/\tau. \tag{20}$$

This is true indeed, as the dimensional argument of Leontief (1970) demonstrates.

As Wassily Leontief told me, the choice of the time unit is an aggregation issue. All output produced during a period of length τ by a sector is lumped together. This observation suggests a framework for the determination of the impact of the choice of the time unit, τ. We define the *bias distribution* as $x_\tau - x$. The total bias, or shortly *bias*, is defined as the bias distribution summed over time, interpreting x_τ as a step function. Note that we allow negative and positive parts of the bias distribution to cancel out. A zero (total) bias need not imply that the bias distribution vanishes everywhere on the time axis. It merely implies that the *total requirements* x_τ summed over time, are equal in the discrete and the continuous cases.

By linearity and absence of technical change, it suffices to determine the bias distribution associated with $z(t)=0$ for $t\neq$ some s and $z_j(s)=\tau^{-1}\delta_{i,j}$, where $j=1,\dots,n$ and $\delta_{i,j}$ is the Kronecker symbol, for $i=1,\dots,n$, the number of sectors. (In the continuous case τ^{-1} is replaced by its limit, δ_s, the Dirac distribution concentrated at s.) In other words, we consider the n unit final demand vectors at time s, all having one quantity equal to one and the other quantities zero. By linearity these final demand vectors can be handled simultaneously through formal substitution of the matrix $\tau^{-1}I$ for $z(s)$ in the eqs. (18) and (19). In the first, discrete, case we thus obtain a *matrix* of output vectors $x_\tau(t)$, say $X_\tau(t)$. Similarly, the second, continuous case will produce a *matrix* $X(t)$ of output vectors $x(t)$. The issue is to determine $X_\tau - X$ which summarizes the n elementary bias distributions, and its sum over time which summarizes the biases. The first proposition presents explicit expressions for X_τ and X. The second proposition concerns the bias itself.

Proposition 3. Assume that $B(0)$ *is invertible. Then*

$$X_\tau(t)=\{I+B(0)^{-1}[I-A(0)]\tau\}^{(t-s-\tau)/\tau}B(0)^{-1}$$

and $X(t)=\exp\{B(0)^{-1}[I-A(0)](t-s)\}B(0)^{-1}$, *both for* $t\leqq s$ *and both zero otherwise.*

Proof. See the appendix.

Remark. The assumption is inessential. If necessary, one can use the generalized inverse and decomposition device of ten Raa (1986). Moreover, ultimately we want the bias. The result will extend to singular $B(0)$ by perturbation of such a matrix and a limiting argument.

Proposition 4. The choice of the time unit is asymptotically pointwise unbiased in the sense that the bias distributions, $X_\tau - X$, tend to zero for vanishing τ. The total requirements, summed over time, of both X_τ and X equal $[I-A(0)]^{-1}$ so that, a fortiori, *the biases are zero for all sectors and time units.*

Apparently, the time unit and even the capital matrix may be chosen arbitrarily as far as the total requirements, summed over time, are concerned. The choice merely affects the time scheduling of the required output levels, not their total amounts. Thus, we expect our study to be relevant for the timing of production which, of course, is not surprising in the context of a theory of distributed input–output. When the time unit is large, the economy need not produce capital shortly before the delivery of the final goods, but can even disinvest already; investment must be at an early stage. On the other hand, an economy with a small time unit can, relatively speaking, postpone investment.

It does not follow that the choice of the time unit and the capital matrix is immaterial for output matters other than timing. The capability to postpone investment required for a given bill of final goods is a positive one, not from a subjective time preference point of view, but in an objective sense. The capability to postpone adds growth potential. Thus, while the time unit is immaterial for the total requirements of a *given* bill of final goods, it does affect the class of admissible bills of final goods, that is the potence of the economy. Choice of a large time unit is biased in that it reduces the maximum growth rate.

Proposition 5. The maximum growth rate, say g_τ, is inversely related to the time unit, τ.

Proof. See the appendix.

This section is completed with some more philosophical musings on time. While the last proposition suggests that discretization yields a downward bias of the estimated maximum growth rate, it may also be that the true world is discrete, and neat, continuous modelling will produce an upward bias. So far the proposition bears on model selection. But there is more to it, as can be seen by considering two economies, one relatively continuous, with a small time unit, the other relatively discrete, with a large unit. Then the first economy is superior in that it can sustain higher growth rates. Thus, with a view to enhancing the potence of an economy, it makes sense to smooth investment. The improvement rests on the possibility of fine tuning the productive capacity to the instantaneous output requirements. This should not be confused with the betterment of performance which can be strived for by direct reduction of delays. The mere metering with a finer time unit has a more modest impact than the speeding up of production which severely affects the structure of the economy as we shall see in the next section.

8. The dynamic structure of the Polish economy

The fully distributed input–output model (1), and its solution, (6) or (8), are now evaluated for the case of Poland, using input–output data of 1969. The purpose is to determine the direct and indirect requirements of the 1969 Polish final demand components. The requirements will be distributed over time, extending into the preceding years. The distributions will be presented in the form of plots.

Abstracting from technical change, central plans can be rationally drawn by superposition of the requirements distributions, weighted by final demand target values. The 'total mass' of the distributions must be consistent with resource availability, while their time patterns dictate the scheduling of production in agreement with technical lags and interindustry links. In this first exercise the economy is divided in fifteen sectors only: (1) Fuel and Energy, (2) Metallurgy, (3) Machinery and Electrical Equipment, (4) Chemicals, (5) Stone, Clay and Glass Products, (6) Wood and Paper Products, (7) Textiles, Leather and Clothing, (8) Food Products, (9) Unspecified Manufactured Products, (10) Construction, (11) Agriculture, (12) Forestry, (13) Transport and Communication, (14) Trade, (15) Other Material Services. Following Czerwiński, Jurek, Panek and Sledziński (1980) the flow matrix, A, is instantaneous, while the stock matrix, B, is distributed over four years. $A(0)$, $B(0)$, $B(-1)$, $B(-2)$, and $B(-3)$, as defined in the introduction of the paper, are taken from Central Statistical Office (1971) and Czerwiński, Guzik, Jurek, Panek, Runka and Sledziński (1982), respectively. To make this article self contained, the matrices are reproduced in table 1. The (i, j)th entry of $A(0)$ is the amount of flow i needed per unit of j. The (i, j)th entry of say $B(-2)$ is the quantity of i to be invested in sector j, 2 years prior to a unit of capacity expansion. Since the model is linear, it suffices to consider the fifteen unit final demand vectors, $z(0)$, with one component unity and the others zero. For each final demand vector we have computed the sectoral total requirements distributions, summed up by vectors $x(t)$, $t=0, -1, -2, \dots$, not by direct evaluation of solution (6), but by going through its derivation, that is the proof of Proposition 1. Since there are 15 sectors, the output consists of $15\times 15=275$ unit requirement plots. To save space, we have selected ten plots, representing the requirements of the two typical investment sectors, (3) (Machinery and Electrical Equipment) and (10) (Construction), for the five typical final demand sectors, (3), (7), (8), (11) and (14). Sectors were classified on the basis of 1969 investment/- and consumption/output ratios. Fig. 1

displays the investment sector requirements for final demand sector (3), fig. 2 for sector (7), fig. 3 for sector (8), fig. 4 for sector (11), and fig. 5 for final demand sector (14). Throughout the paper, final demand excludes investment which is endogenized in our model.

Let us explain one plot in detail. Fig. 1 shows the output requirements of sector (10) (Construction) for the final delivery of one unit of good 3 (Machinery and Electrical Equipment) in zlotys per zlotys. (The continuous graph is relevant, the dashed one will be explained below.) Thus, if it is decided to increase exports of Machinery and Electrical Equipment at some future year, say 1990, the fig. 1 presents the required change in Construction. The level of construction must be adjusted practically six years in advance, that is 1984, to observe investment lead times and interindustry balances. Negative adjustment values, notably in the year prior to final delivery, represent output reduction quantities which are compensated by Construction stock releases in the Machinery and Electrical Equipment or its supply sectors. This familiar disinvestment emerges in the context of interrupted demand, here as well as in Leontief (1970). We have found additional disinvestment three years before final delivery. This is a consequence of the temporal distribution of capital construction. In the absence of future demand, initial capital layers are released at intermediate stages of production.

Disinvestments make a cyclical pattern of direct and indirect requirements, even though the investment coefficients themselves are smoothly distributed. This is caused by the interindustry interplay of the investment distributions. Especially Construction [sector (10)] undergoes wild cycles in the fulfillment of final demand components. These business cycles are purely technical, independent of investors behaviour. Their existence restrains the power of the price system to clear markets on the basis of current supply and demand conditions and calls for some conscious timing of sectoral activities in the course of plan fulfillment.

While figs. 1–5 display the requirements of Machinery and Electrical Equipment (sector 3) for various separate sectors of final demand, they can be combined to obtain the total Machinery and Electrical Equipment requirements for the 1969 final demand vector. Each plot is blown up by a factor equal to the receiving final demand component and then they are added. The results are displayed in figs. 6–13 for all productive sectors and, in particular, in fig. 7 for sector (3).

The noted fluctuations at the sector-to-sector level do not wash out when the requirement distributions are aggregated by final demand sectors. To fulfill aggregated final demand of 1969, Construction (sector 10) started around 1960 in the absence of productive capacities left over from before 1969 final demand fulfillment, as depicted in fig. 10. The cyclical path demonstrates that the price system problems with market clearance do not

Table 1

Flow coefficients matrix $A(0)$ and stock matrix distribution B.

$A(0)$	0.2470	0.1249	0.0224	0.0895	0.1051	0.0289	0.0129	0.0145	0.0192	0.0273	0.0139	0.0118	0.1137	0.0177	0.0678
	0.0062	0.6023	0.1887	0.0187	0.0400	0.0066	0.0012	0.0024	0.0100	0.0584	0.0011	0.0026	0.0100	0.0021	0.0349
	0.0590	0.0491	0.2933	0.0359	0.0666	0.0126	0.0147	0.0148	0.0393	0.1231	0.0159	0.0345	0.0894	0.0215	0.1579
	0.0171	0.0122	0.0433	0.2681	0.0300	0.0932	0.0904	0.0072	0.0288	0.0269	0.0374	0.0122	0.0213	0.0088	0.0380
	0.0119	0.0239	0.0095	0.0156	0.1189	0.0075	0.0007	0.0079	0.0036	0.1290	0.0011	0.0113	0.0071	0.0059	0.0355
	0.0177	0.0028	0.0154	0.0234	0.0380	0.2133	0.0087	0.0016	0.0300	0.0103	0.0007	0.0262	0.0082	0.0206	0.0366
	0.0112	0.0045	0.0109	0.0431	0.0182	0.0367	0.3604	0.0049	0.0242	0.0115	0.0328	0.0064	0.0118	0.0247	0.0137
	0.0005	0.0004	0.0009	0.0277	0.0011	0.0022	0.0160	0.1882	0.2973	0.0003	0.0143	0.0084	0.0017	0.0144	0.0002
	0.0008	0.0012	0.0048	0.0052	0.0033	0.0310	0.0080	0.0069	0.1707	0.0033	0.0283	0.0012	0.0029	0.0094	0.0248
	0.0259	0.0092	0.0036	0.0040	0.0128	0.0038	0.0021	0.0034	0.0038	0.0257	0.0062	0.0247	0.0064	0.0134	0.0217
	0.0001	0.0001	0.0001	0.0057	0.0007	0.0018	0.0651	0.3634	0.1355	0.0013	0.4482	0.0108	0.0026	0.0163	0.0119
	0.0001	0.0009	0.0003	0.0175	0.0019	0.1817	0.0005	0.0009	0.0007	0.0037	0.0002	0.0200	0.0006	0.0001	0.0002
	0.0631	0.0182	0.0177	0.0273	0.0790	0.0300	0.0097	0.0228	0.0249	0.0902	0.0007	0.1115	0.0481	0.1223	0.0164
	0.0049	0.0169	0.0117	0.0109	0.0114	0.0138	0.0050	0.0649	0.0199	0.0173	0.0151	0.0044	0.0179	0.0071	0.0135
	0.0015	0.0009	0.0021	0.0013	0.0035	0.0010	0.0008	0.0016	0.0021	0.0007	0.0248	0.0032	0.0056	0.0039	0.0111
$B(0)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.2542	0.1763	0.1537	0.4777	0.4219	0.4365	0.2359	0.1650	0.0995	0.4024	0.4474	0.6972	0.1892	0.2089	0.8083
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0005	0.0003	0.0006	0.0014	0.0022	0.0016	0.0006	0.0010	0.0004	0.0005	0.0114	0.0138	0.0112	0.0020	0.0044
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.2469	0.1353	0.0456	0.1228	0.2228	0.0949	0.0291	0.0398	0.0248	0.0075	0.3117	0.4359	0.0000	0.0980	0.9537
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0079	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0006	0.0007	0.0004	0.0026	0.0016	0.0012	0.0005	0.0004	0.0003	0.0009	0.0021	0.0022	0.0018	0.0021	0.0018
	0.0050	0.0035	0.0030	0.0086	0.0079	0.0092	0.0046	0.0030	0.0019	0.0089	0.0091	0.0140	0.0233	0.0037	0.0097
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$B(-1)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.1963	0.1651	0.0998	0.0989	0.2993	0.3282	0.0740	0.1034	0.0571	0.0451	0.2132	0.4969	0.6475	0.0673	0.6961
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0004	0.0003	0.0004	0.0003	0.0015	0.0012	0.0002	0.0006	0.0002	0.0001	0.0054	0.0098	0.0061	0.0006	0.0037

	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.9274	0.5108	0.1680	0.4518	0.8278	0.3636	0.1091	0.1523	0.0885	0.1840	0.9802	0.9826	0.7613	0.4583	0.8213
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0005	0.0006	0.0002	0.0005	0.0011	0.0009	0.0002	0.0003	0.0002	0.0000	0.0010	0.0016	0.0010	0.0007	0.0016
	0.0038	0.0032	0.0020	0.0018	0.0056	0.0069	0.0015	0.0019	0.0011	0.0011	0.0044	0.0100	0.0127	0.0012	0.0084
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$B(-2)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.1516	0.1546	0.0422	0.0204	0.2123	0.0000	0.0231	0.0000	0.0000	0.0000	0.0000	0.3543	0.3527	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0003	0.0003	0.0003	0.0001	0.0011	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0070	0.0033	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.2760	0.1537	0.0493	0.1325	0.2512	0.1110	0.0326	0.0465	0.0251	0.0000	0.0000	0.5205	0.3136	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0004	0.0006	0.0001	0.0001	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011	0.0005	0.0000	0.0000
	0.0030	0.0030	0.0013	0.0004	0.0040	0.0000	0.0004	0.0000	0.0000	0.0000	0.0000	0.0071	0.0069	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$B(-3)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.1171	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0002	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0049	0.0027	0.0008	0.0020	0.0045	0.0018	0.0005	0.0007	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0003	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0023	0.0000	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

cancel out at the economy level. Their persistence underscores the need for some planning, although two problems must be acknowledged. The referee rightly noted that planning is difficult in the sense that business cycles in final demand have to be forecasted and that it may be a recipe worse than the cure. Planning in the centrally planned economies such as Poland until now managed to create a strong investment cycle: investment boom in the starting years and investment stagnation at the end of the five years period. Intelligent use of the distributed dynamic inverse of this article may muffle the cycle, but, due to the first complication, not to the extent of complete elimination.

The investment–disinvestment accelerator effects are dramatic in sectors (3), (10) and (13), typical capital industries, but also present in sectors (1), (2), (5) and (12), which are typical intermediate goods producers. (See figs. 7, 10 and 12, and figs. 6, 8 and 11 respectively.) Total requirement distributions tell a better story about the role of sectors in the economy than the direct input–output coefficients. On the other end of the spectrum we have the typical final consumption sectors (8) (Food Products), (7) (Textiles, Leather and Clothing), and (11) (Agriculture) which can postpone production practically till the year of final delivery, even when the direct and indirect requirements are taken into account. (See figs. 9 and 11.)

When final demand of 1969 is appropriately embedded in a sequence of similar vectors for the surrounding years, the cycles of the magnitude in the discussed 1969 requirements figures wash out in summing up to the overall development of the economy. The *increases* of final demand produce cycles of smaller order. These minor cycles, as argued in section 2, have the same shape as the ones discussed so far. Thus, the irregularities in the sectoral pattern of the final demand increases generate oscillations about the overall development of the economy. The referee has correctly conjectured that even these oscillations could very well be smoothed out by the structure of the irregularities of the final demand increases since the latter include inventory investments caused by delays in the fulfillment of investment plans. It should be mentioned though that this smoothing out is no virtue. Efficient planning of sectoral outputs entails variations that precisely match the exogenous increases of final demand. Smoothing, especially through filling final demand with inventory investment, is a form of overinvestment that keeps the economy beyond its true production possibilities.

While analysis of the distributed structure of the economy is appealing both theoretically and empirically, it remains to be seen if the whole exercise is worthwhile on pure pragmatic grounds. In other words, do the results deviate significantly from classical input–output analysis, without distributed activities? For comparison we have performed all the computations for the non-distributed case too with $B(-1)$, $B(-2)$, and $B(-3)$ suppressed and $B(0)$ equal to the standard capital matrix of Czerwiński, Jurek, Panek and

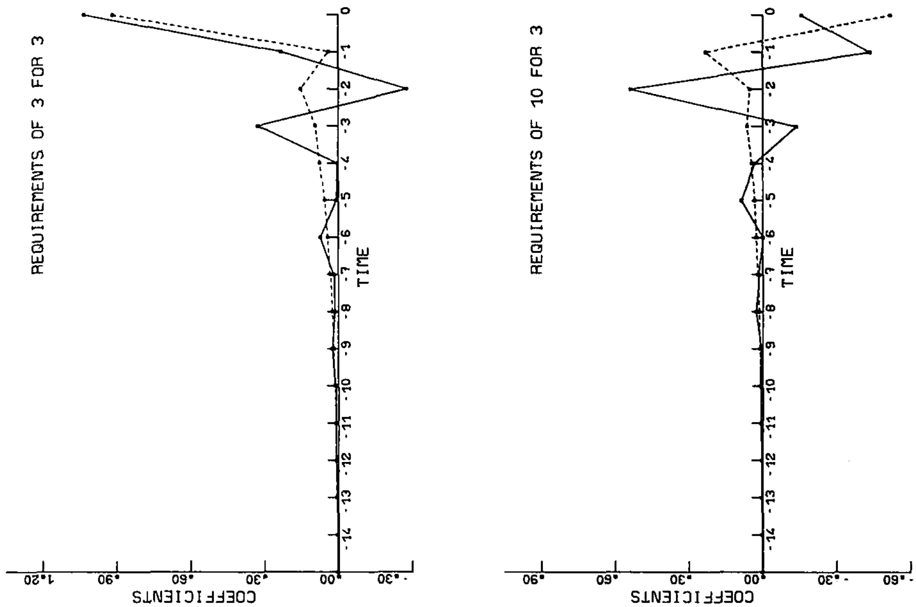
REQUIREMENTS OF 3 FOR 3
TIME
COEFFICIENTS
REQUIREMENTS OF 10 FOR 3
TIME
COEFFICIENTS

Fig. 1

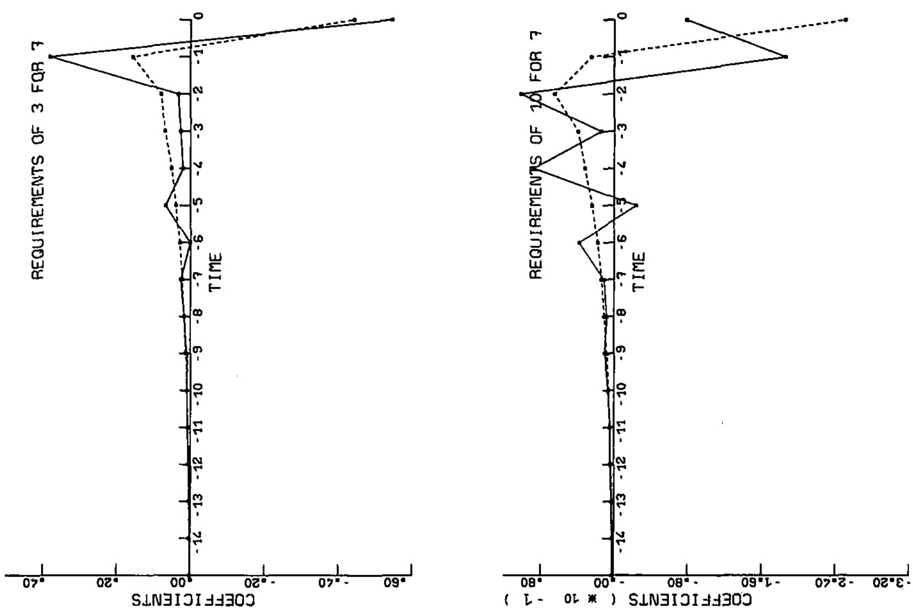
REQUIREMENTS OF 3 FOR 7
TIME
COEFFICIENTS
REQUIREMENTS OF 10 FOR 7
TIME
COEFFICIENTS (* 10 -1)

Fig. 2

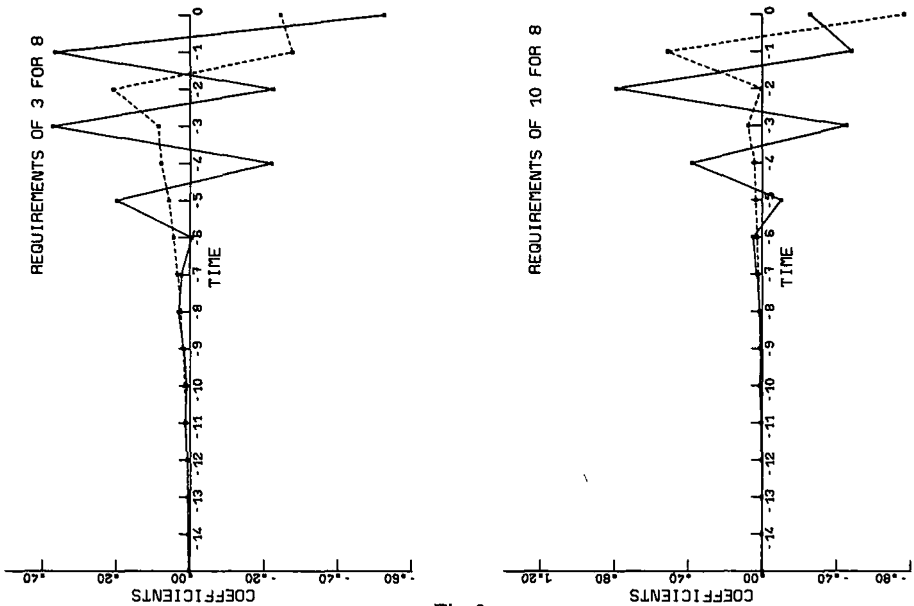
REQUIREMENTS OF 3 FOR 8
REQUIREMENTS OF 10 FOR 8
TIME
COEFFICIENTS

Fig. 3

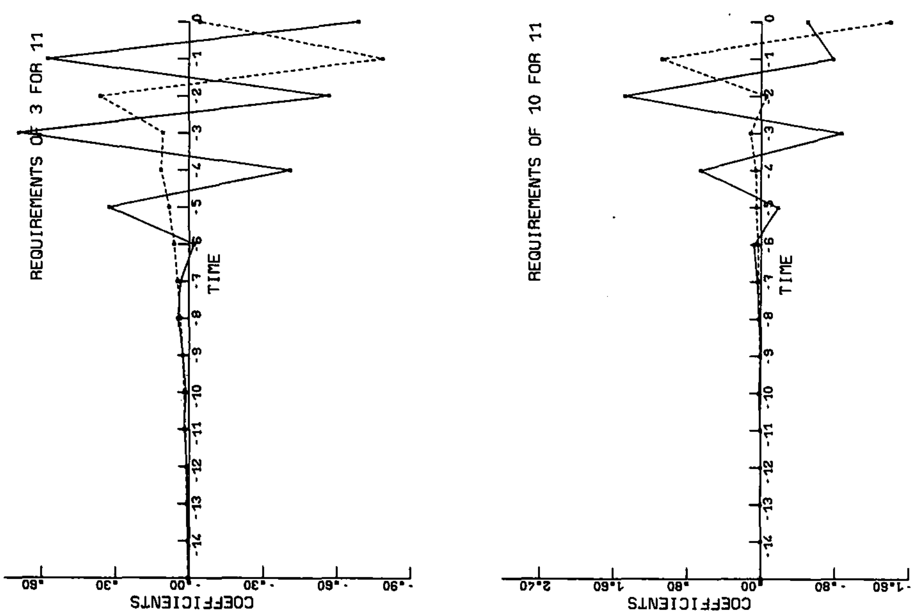
REQUIREMENTS OF 3 FOR 11
REQUIREMENTS OF 10 FOR 11
TIME
COEFFICIENTS

Fig. 4

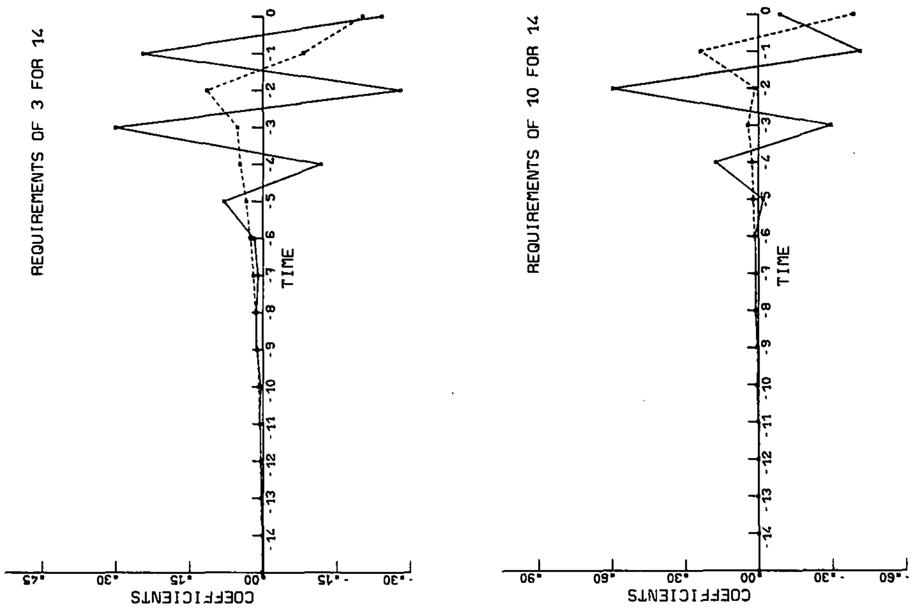
REQUIREMENTS OF 3 FOR 14
TIME
COEFFICIENTS
REQUIREMENTS OF 10 FOR 14
TIME
COEFFICIENTS

Fig. 5

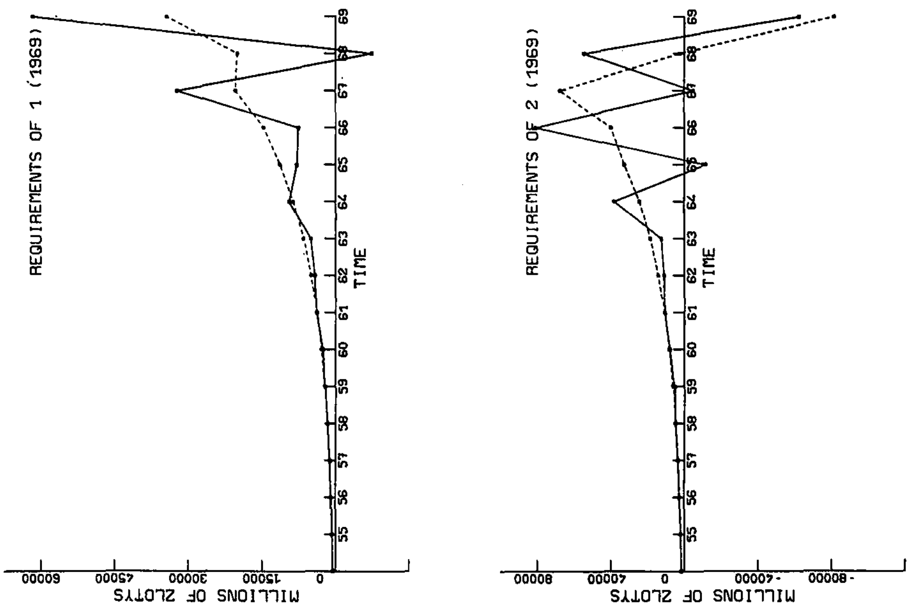
REQUIREMENTS OF 1 (1969)
TIME
MILLIONS OF ZLOTYS
REQUIREMENTS OF 2 (1969)
TIME
MILLIONS OF ZLOTYS

Fig. 6

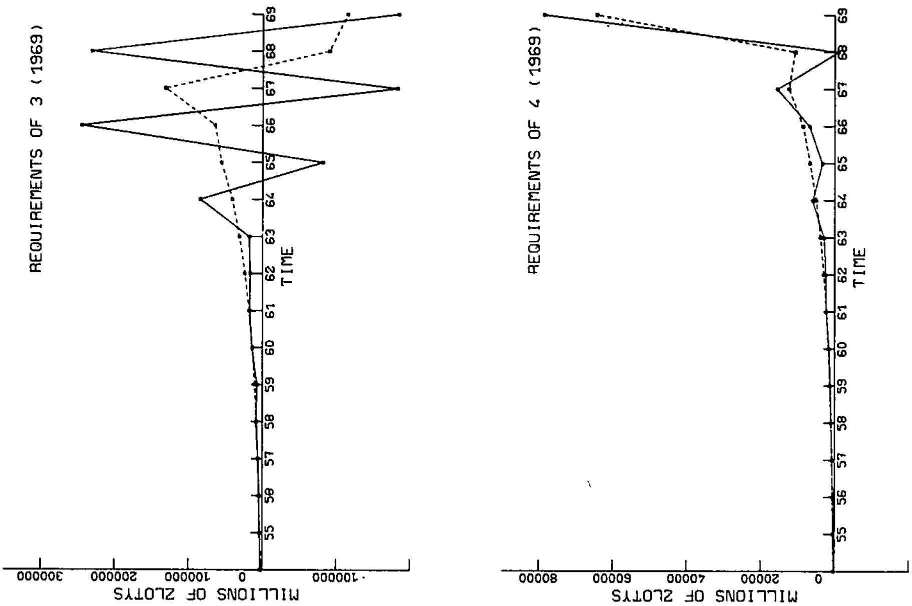
REQUIREMENTS OF 3 (1969)
REQUIREMENTS OF 4 (1969)
TIME
MILLIONS OF ZLOTYS
55 56 57 58 59 60 61 62 63 64 65 66 67 68 69
300000 200000 100000 0 -100000
80000 60000 40000 20000 0

Fig. 7

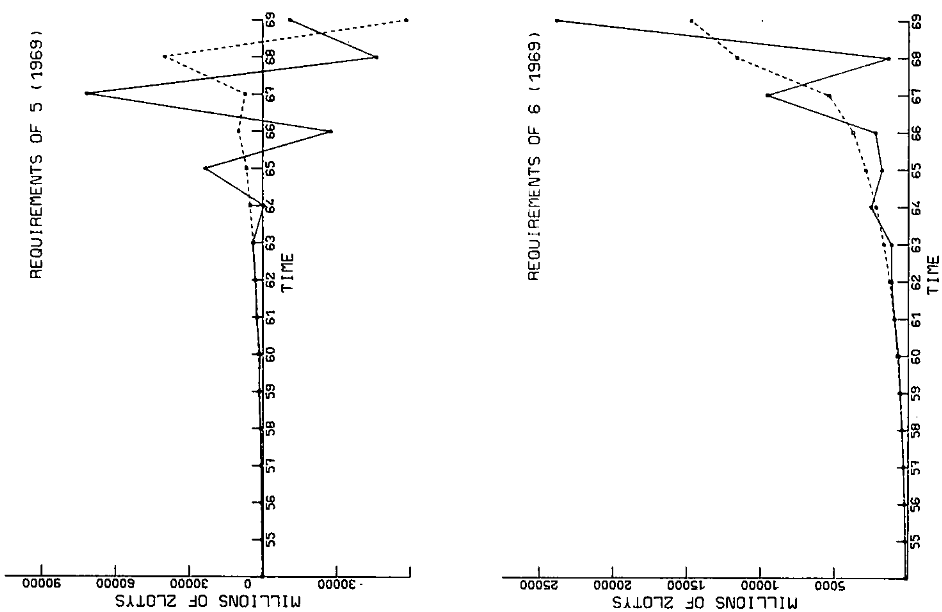
REQUIREMENTS OF 5 (1969)
REQUIREMENTS OF 6 (1969)
TIME
MILLIONS OF ZLOTYS
55 56 57 58 59 60 61 62 63 64 65 66 67 68 69
90000 60000 30000 0 -30000
25000 20000 15000 10000 5000

Fig. 8

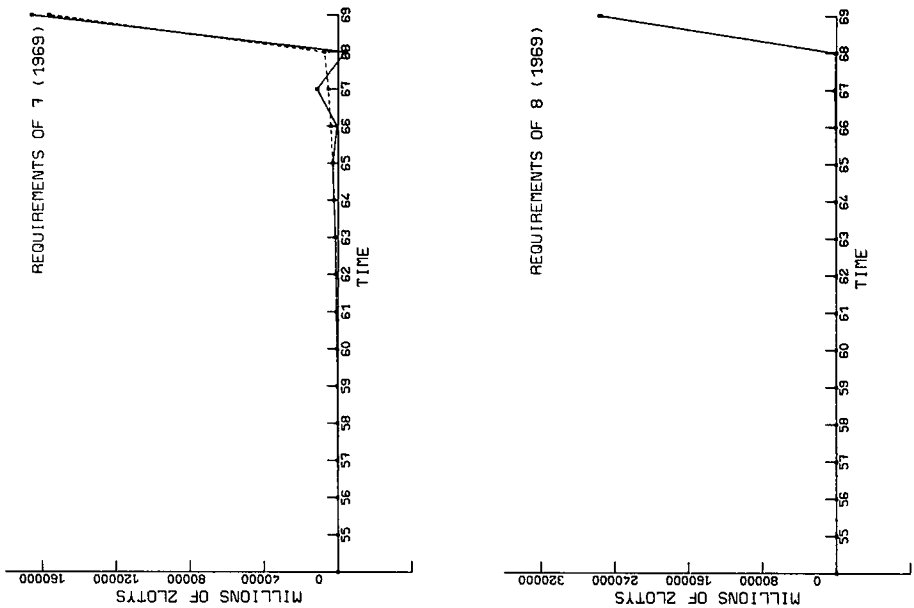
REQUIREMENTS OF 7 (1969)
TIME
MILLIONS OF ZLOTYS
REQUIREMENTS OF 8 (1969)
TIME
MILLIONS OF ZLOTYS

Fig. 9

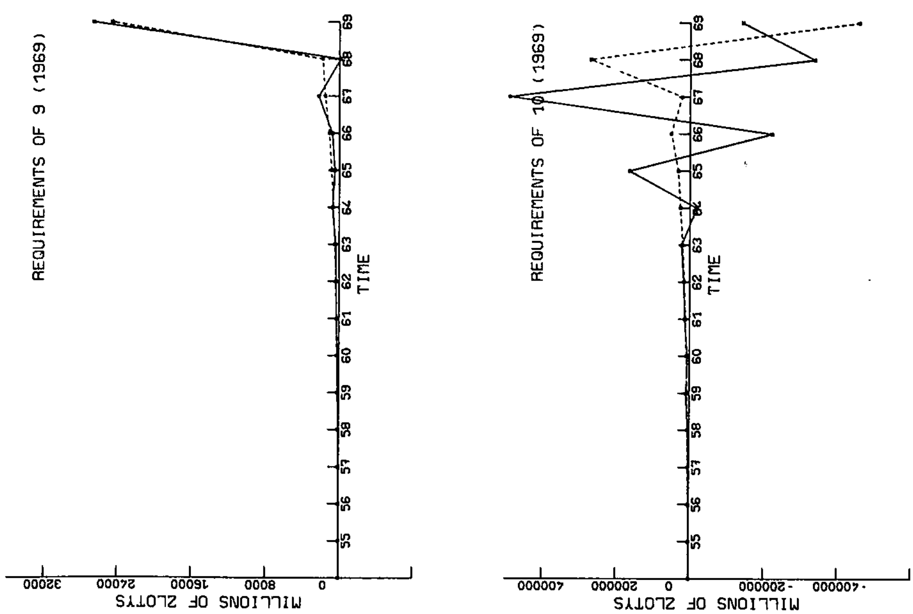
REQUIREMENTS OF 9 (1969)
TIME
MILLIONS OF ZLOTYS
REQUIREMENTS OF 10 (1969)
TIME
MILLIONS OF ZLOTYS

Fig. 10

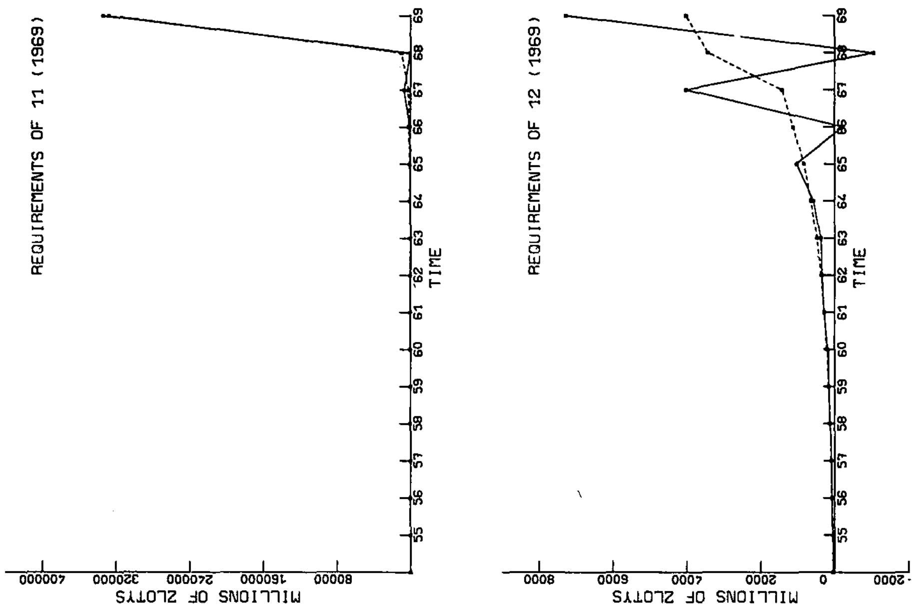
REQUIREMENTS OF 11 (1969)
TIME
MILLIONS OF ZLOTYS
REQUIREMENTS OF 12 (1969)
TIME
MILLIONS OF ZLOTYS

Fig. 11

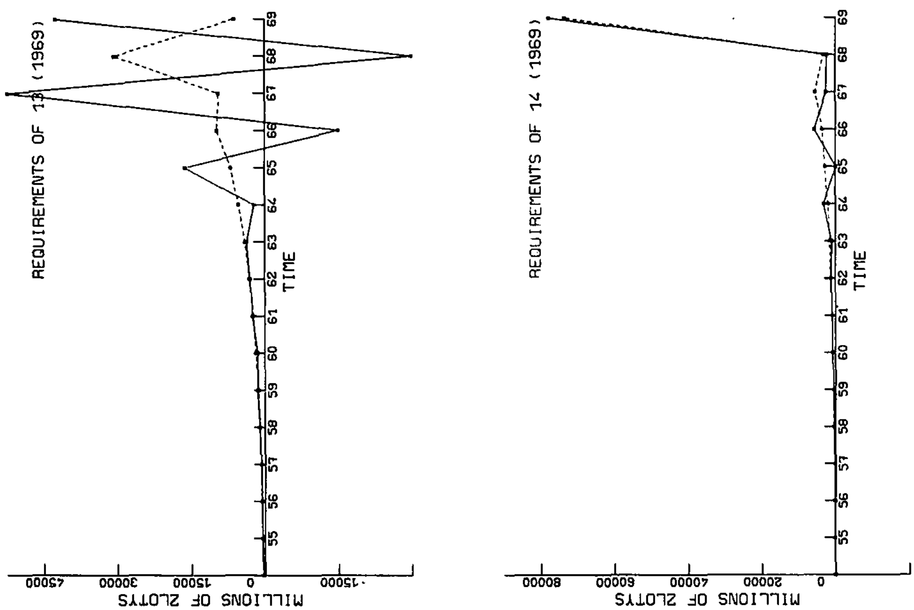
REQUIREMENTS OF 13 (1969)
TIME
MILLIONS OF ZLOTYS
REQUIREMENTS OF 14 (1969)
TIME
MILLIONS OF ZLOTYS

Fig. 12

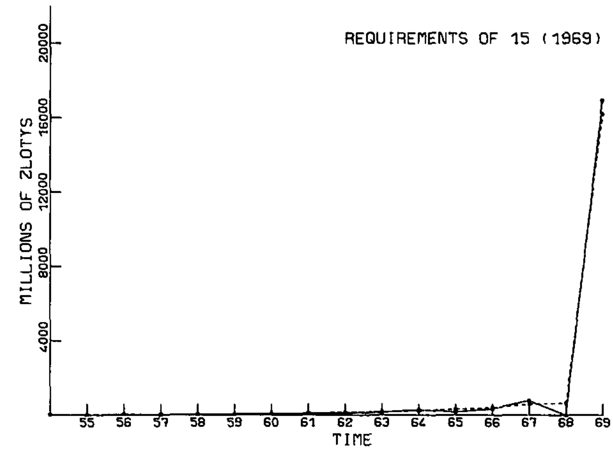

Fig. 13

Sledziński (1980). It should be mentioned that the nondistributed matrix is not obtained by simple aggregation of the distribution ones, but is a little bigger. If one neglects lags, then inputs are divided by current rather than future outputs or rates of change. But since current rates are relatively small, at least for a growing economy, the nondistributed coefficients must be relatively big. The classical input–output results are indicated by the dashed graphs in figs. 1–13. Comparison shows that investment distributions are responsible for great differences which, however, can be qualified. Careful inspection reveals that the requirements based on distribution activities (the continuous graphs) fluctuate around the classical requirements (the dashed graphs). The differences fluctuate quite evenly in the sense that over- and undershootings are balanced. This finding confirms Proposition 4 by which the total requirements, summed over time, are independent of the investment distribution B. [It is easy to see that Proposition 4 extends to stock coefficients, B, with finite total mass by the partial differentiation and Neumann series arguments of ten Raa (1986).]

An indirect test of distributed versus classical input–output is possible by looking at the typical final consumption sectors, (8), (7), and (11). Since their total requirements do not extend into the past (see fig. 3), future final demand does not add to the 1969 requirements which consequently must closely agree with actual output of those sectors in 1969. This is true of our results, but less so of the classical, nondistributed ones. For example, 1969 Food Products (sector 8) and Textiles, Leather and Clothing (sector 7) produced 256,986 and 168,639 millions zlotys of output, respectively, which is closely approximated by our results (257,500 and 166,260), but less so by classical input–output (255,600 and 156,900).

Table 2

Investment matrix $\bar{B}$.

Supplier sector	Recipient sector: Fuel & energy	Metallurgy	Machinery & electrical equipment	Chemicals	Stone, clay and glass products	Wood and paper products	Textiles, leather and clothing	Food products	Unspecified manufactured products	Construction	Agriculture	Forestry	Transport and communication	Trade	Other material services
Fuel and energy	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Metallurgy	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Machinery and electrical equipment	0.7877	0.5289	0.3963	0.6108	0.9855	0.7820	0.3421	0.2740	0.1641	0.4511	0.6714	1.5710	2.2991	0.2811	1.6165
Chemicals	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Stone, clay and glass products	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Wood and paper products	0.015	0.0009	0.0016	0.0018	0.0051	0.0028	0.0009	0.0016	0.0007	0.0006	0.0171	0.0310	0.0217	0.0027	0.0087
Textiles, leather and clothing	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Food products	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Unspecified manufactured products	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Construction	1.5688	0.8594	0.2899	0.7802	1.4131	0.6029	0.1846	0.2526	0.1573	0.2078	1.3411	1.9796	1.1872	0.5890	1.9074
Agriculture	0	0	0	0	0	0	0	0	0	0	0.0118	0	0	0	0
Forestry	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Transport and communication	0.0019	0.0021	0.0010	0.0033	0.0037	0.0021	0.0008	0.0007	0.0005	0.0010	0.0032	0.0050	0.0036	0.0029	0.0037
Trade	0.0155	0.0104	0.0078	0.0110	0.0185	0.0164	0.0067	0.0050	0.0032	0.0088	0.0137	0.0315	0.0450	0.0050	0.0194
Other material services	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The other sectors, capital industries and intermediate goods producers, show more significant absolute requirements differences when investment lead times are taken into account. Thus, distributed input–output merits attention and is especially useful for proper time scheduling of production.

9. Conclusion

The theory of dynamic input–output analysis with distributed activities lends itself to discretization and then generalizes existing dynamic input–output models by facilitating stock and flow input profiles without pre-empting empirical applications. Direct and indirect requirements for the Polish final demand vector of 1969 based on investment distributions fluctuate dramatically around classical results that ignore the time structure of production. The generalized inverse that summarizes the requirements is a central planning tool for proper time scheduling of production.

The results are sensitive with respect to the distributions of economic activities. Direct information on production and investment lead times at the level of input–output data collection is called for, especially since present research suggests that statistical inference is a poor substitute due to the presence of multicollinearities.

Appendix

Proof of Proposition 1. Multiply the tth row ($t \geqq 0$) of the original matrix with the $(t+v)$th column ($v>0$) of the inverse. This yields

$$G_t(0)D_t(v)+\cdots+G_t(v-1)D_{t+v-1}(1)+G_t(v)G_{t+v}(0)^{-1}=0$$

or

$$D_t(v)=\sum_{s=1}^{v-1} R_t(s)D_{t+s}(v-s)-G_t(0)^{-1}G_t(v)G_{t+v}(0)^{-1}.$$

This is a diophantine equation for $D_t(v)$ with different weights for the predecessors; the solution is presented in the statement of Proposition 1 and will now be derived by induction on v.

For $v=1$ the summation index consists of 1 and the proposition reads

$$D_t(1)=R_t(1)G_{t+1}(0)^{-1}$$

which is true by the established expression for $D_t(v)$ and the definition of $R_t(1)$.

Now suppose the proposition is valid for $1,\ldots,v-1$. Then the established expression for $D_t(v)$ becomes

$$D_t(v)=\sum_{s=1}^{v-1} R_t(s)\sum R_{t+s}(s_1)\dots R_{t+s+s_1+\dots+s_{l-1}}(s_l)G_{t+v}(0)^{-1}$$
$$-G_t(0)G_t(v)G_{t+v}(0)^{-1},$$

where the $\sum$-summation is over all $(s_1,\dots,s_l)$ with each component in $\{1,\dots,v-s\}$ and their sum equal to $v-s$. We have to prove that this expression coincides with that in the statement of Proposition 1 and shall do so by showing that any term in one expression shows up in the other. Since in each expression the summation indices – $(s,s_1,\dots,s_l)$ with $s_1,\dots,s_l$ in $\{1,\dots,v-s\}$ summing up to $v-s$ and s in $\{1,\dots,v-1\}$ plus the element 0 (representing the separate term $-G_t(0)^{-1}G_t(v)G_{t+v}(0)^{-1}$) over here and $(s_1,\dots,s_l)$ with $s_1,\dots,s_l$ in $\{1,\dots,v\}$ summing up to v in the statement of the proposition – assume different values, double counting of terms is avoided.

First take the derived expression. The separate term, $-G_t(0)^{-1}G_t(v)G_{t+v}(0)^{-1}$, shows up in the statement of the proposition when $l=1$ by definition of $R_t(v)$. Now pick any other term:

$$R_t(s)R_{t+s}(s_1)\dots R_{t+s+s_1+\dots+s_{l-1}}(s_l)G_{t+v}(0)^{-1}$$

with

$$s_1,\dots,s_l \quad \text{in } \{1,\dots,v-s\}$$

summing up to $v-s$ and s in $\{1,\dots,v-1\}$ summing up to v. Consequently the term shows up in the statement of the proposition.

Next take the expression in the statement of Proposition 1. When $l=1$, then $s_l=v$ and the term equals the separate term $-G_t(0)^{-1}G_t(v)G_{t+v}(0)^{-1}$. Otherwise $l\geqq 2$ and the term equals $R_t(s_1)\dots R_{t+s_1+\dots+s_{l-1}}(s_l)G_{t+v}(0)^{-1}$ with $s_1,\dots,s_l\geqq 1$ summing up to v. Consequently $s_2,\dots,s_l$ are in $\{1,\dots,v-s_1\}$ and sum up to $v-s_1$ while s_1 is in $\{1,\dots,v-1\}$. Consequently the term shows up in the derived expression. □

Proof of Proposition 2. Recall from the proof of Proposition 1 that

$$D_t(v)=\sum_{s=1}^{v-1} R_t(s)D_{t+s}(v-s)-G_t(0)^{-1}G_t(v)G_{t+v}(0)^{-1}$$

$$=\sum_{s=1}^{v} R_t(s)D_{t+s}(v-s)$$

with $D_t(0)=G_t(0)^{-1}$. In fact, since

$$R_t(s)=-G_t(0)^{-1}G_t(s),\, D_t(v)=\sum_{s=1}^{\sigma} R_t(s)D_{t+s}(v-s).$$

Here the number of terms is independent of t and can be denoted $\#_v^{(\sigma)}$. By the last equation, $\#_v^{(\sigma)}=\sum_{s=1}^{\sigma}\#_{v-s}^{(\sigma)}$. For $v<0$, $\#_v^{(\sigma)}=0$, and $\#_0^{(\sigma)}=1$.

It follows that $\#_v^{(\sigma)}$ are the generalized Fibonacci numbers of dimensions σ. If $\sigma=1$, the solution is obviously $\#_v^{(1)}=1(v\geqq 1)$. Otherwise, for $\sigma\geqq 2$, the solution is due to Bernstein (1971, p. 141): $\#_{s\sigma+k}^{(\sigma)}=2^{k-s-1}\sum_{j=0}^{s}(-1)^j[((s-j)\sigma_j+k)+((s-j)\sigma_{j-1}+k-1)]^{2(s-j)(\sigma+1)}$ $(s=0,1,\dots$ and $k=1,\dots,\sigma)$. By Proposition 1 each term in $\dot{D}_t(v)$ is the product of at most v $n\times n$-matrices where n is the number of sectors.

Proof of Proposition 3. Eq. (15) implies $X_\tau(t)$ values for $\tau=1$. This motivates the change of variables $t'=t/\tau$ and $X'_\tau(t')=X_\tau(\tau t')$. Then $X'_\tau(t')=X_\tau(t)$ and by (18) and (20),

$$\begin{aligned}[I-A(0)X'_\tau(t')&=[I-A(0)]X_\tau(t)\\ &=B_\tau(0)[X_\tau(t+\tau)-X_\tau(t)]+\tau^{-1}I\delta_{s,t}\\ &=\tau^{-1}B(0)[X'_\tau(t'+1)-X'_\tau(t')]+\tau^{-1}I\delta_{s/\tau,t'}.\end{aligned}$$

This is Leontief's (1970) dynamic equation with capital matrix $\tau^{-1}B(0)$. Substituting this for $B(0)$ in (16), we obtain

$$\begin{aligned}X'_\tau(t')=\{[I-A(0)+\tau^{-1}B(0)]^{-1}\tau^{-1}B(0)\}^{s/\tau-t'}[I-A(0)\\ +\tau^{-1}B(0)]^{-1}\tau^{-1},\end{aligned}$$

for $t\leqq s$ and zero otherwise.

It follows that

$$\begin{aligned}X_\tau(t)&=\{[I-A(0)+\tau^{-1}B(0)]^{-1}\tau^{-1}B(0)\}^{s/\tau-t/\tau}\\ &\quad\times[I-A(0)+\tau^{-1}B(0)]^{-1}\tau^{-1}\\ &=\{[I-A(0)+\tau^{-1}B(0)]^{-1}\tau^{-1}B(0)\}^{(s-t)/\tau+1}B(0)^{-1}\\ &=\{B(0)^{-1}\tau[I-A(0)+\tau^{-1}B(0)]\}^{-(s-t)/\tau-1}B(0)^{-1}\\ &=\{I+B(0)^{-1}[I-A(0)\tau\}^{(t-s-\tau)/\tau}B(0)^{-1},\end{aligned}$$

for $t\leqq s$ and zero otherwise.

By Remark 3 of section 8 of ten Raa (1986),

$$X(t)=\check{H}\exp\{B(0)^{-1}[I-A(0)]t\} * B(0)^{-1}\delta_s$$
$$=\check{H}(t-s)\exp\{B(0)^{-1}[I-A(0)](t-s)\}B(0)^{-1}$$
$$=\exp\{B(0)^{-1}[I-A(0)](t-s)\}B(0)^{-1},$$

for $t \leqq s$ and zero otherwise.

Proof of Proposition 4. $X_\tau - X$ tends to zero for vanishing τ if, roughly speaking, $\log X_\tau$ tends to $\log X$ or, more precisely and using Proposition 3, $\frac{t-s-\tau}{\tau}\log\{I+B(0)^{-1}[I-A(0)]\tau\}$ tends to $B(0)^{-1}[I-A(0)](t-s)$, which is clearly true. Here log $\{\cdot\}$ is defined for τ sufficiently small by the usual Taylor series.

By Proposition 3, the total requirements, X_τ, sum up over time

$$\sum_{t/\tau=-\infty}^{s/\tau} X_\tau(t)=\sum_{t/\tau=-\infty}^{s/\tau}\{I+B(0)^{-1}[I-A(0)]\tau\}^{(t-s)/\tau-1}$$
$$\times\{I+B(0)^{-1}[I-A(0)]\tau-I\}[I-A(0)]^{-1}$$
$$=\left\{\sum_{t/\tau=-\infty}^{s/\tau}\{I+B(0)^{-1}[I-A(0)]\tau\}^{(t-s)/\tau}\right.$$
$$\left.-\sum_{t/\tau=-\infty}^{s/\tau}\{I+B(0)^{-1}[I-A(0)]\tau\}^{(t-s)/\tau-1}\right\}[I-A(0)]^{-1}$$
$$=\{I+B(0)^{-1}[I-A(0)]\tau\}^{(s-s)/\tau}[I-A(0)]^{-1}=[I-A(0)]^{-1}.$$

By Proposition 3, the total requirements, X, sum up over time to

$$\int_{-\infty}^{s} X(t)\,\mathrm{d}t=\int_{-\infty}^{s} e^{B(0)^{-1}[I-A(0)](t-s)}B(0)^{-1}\,\mathrm{d}t=e^{B(0)^{-1}[I-A(0)](t-s)}$$
$$\times[I-A(0)]^{-1}\Big|_{-\infty}^{s}$$
$$=e^0[I-A(0)]^{-1}-0=[I-A(0)]^{-1}.$$

Proof of Proposition 5. Maximum growth is obtained when all surplus is invested. Then the input–output equations become homogeneous and the determination of the growth rates is an eigenvalue problem. See Bródy (1965, 1974). Thus consider $x_\tau(t)=e^{g_\tau t}x_\tau(0)$ for the discrete input–output equation and $x(t)=e^{gt}x(0)$ for the continuous one. Substitution in homogeneized (18) yields using (20), $[I-A(0)]x_\tau(0)=\tau^{-1}B(0)(e^{g_\tau\tau}-1)x_\tau(0)$ and in homogeneized (19), $[I-A(0)]x(0)=B(0)gx(0)$. Consequently $\tau^{-1}(e^{g_\tau\tau}-1)=g$.

By a Taylor expansion we see that $g_\tau \uparrow g$ for $\tau \downarrow 0$. This proves that the maximum growth rate, g_τ, is inversely related to the time unit, τ.

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