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THE DISTRIBUTION APPROACH TO SPATIAL ECONOMICS

Thijs ten Raa*

1. INTRODUCTION

This article exposes a new approach to spatial economics which is general yet operational. This idea is straightforward. The approach is essentially a modification, or more precisely a spatialization, applicable to any standard, nonspatial model.

Standard models are usually comprised of structures and of activities, formally represented by elements of real vector spaces, such as scalars, vectors, or matrices. Basically, the structures and activities will now be represented by spatial distributions of those elements. In other words, the scalars, vectors, and matrices are spread out over geographical space. Formally, the real vector spaces are replaced by vector-valued distribution spaces.

So far the notion of spatialization will not have excited the revolutionary zeal of the theorist for (s)he is already acquainted with studies in which structures and activities have spatial components. Moreover, such refinements merely seem to increase computational complexity without affecting the mechanics of the model at hand. It is at this junction that our approach deviates and is truly new. We shall not chop up structures and activities in spatial bits and then proceed as usual. Instead we will consider spatialized structures and activities as single elements in a distribution space. The novelty is that the manipulations will apply directly to the spatialized structures and activities, as if they are ordinary numbers, vectors, or matrices. For this purpose we shall draw on the theory of distributions developed in 1945 by Schwartz (1957). Economists who want to make creative use of the present article must definitely read Schwartz' (1961) introduction to the theory, since in the present work the mathematics chapter "Theory of Distributions" is applied for the first time in economic science.

Our distribution approach to spatial economics will prove to be powerful. First, spatial structures of economic models become much more transparent. Second, solutions are brought within reach. For example, the spatial equilibrium analysis of urban density by Beckmann (1977) can now be applied to two-dimensional cities, a noteworthy extension of the fictitious railroad town to real life cities. Other open problems in spatial economics will be tackled by the distribution approach. The main purpose of this article is, however, the exposition of a new method for spatial economic analysis.

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*Assistant Professor of Spatial Economics, Erasmus University, Rotterdam, The Netherlands.

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Finally, let me note that the method is also useful when economic structures or activities are temporally distributed or even when they are distributed over space and time jointly, as in economic dynamics and econometric models with lags. Since these cases are more complicated (the distribution equations involve derivatives) they are less suitable for a basic exposition of the new approach and are left for future publications.

2. BENCH MARK: KEYNES SPATIALIZED

To fix ideas let us take the simplest model, a simple version of Keynes' model, and then see how it is spatialized. We consider

$$y = cy + x \tag{1}$$

where y is national income which is divided into consumption cy and other expenditures x including investment, government outlays, and net exports. Coefficient c is the propensity to consume.

Although it looks simplistic, the model and its spatialization below contain all the essential ingredients of the spatial economic models analyzed in this article. The Keynesian model serves as a benchmark for our approach to spatial economics.

The solution of the model is

$$y = (1 - c)^{-1}x \tag{2}$$

or

$$y = (1 + c + c^2 + c^3 + \ldots)x \tag{3}$$

This shows how a raise in exogenous expenditures x have a multiplier effect on national income y. The total effect equals the exogenous expenditure itself, x, plus the direct effect, cx, plus the indirect effects, $c^2x + c^3x + \ldots$. The total effect converges because of the economic fact of life that the propensity to consume lies between zero and one. Of every dollar earned, less than a dollar is spent on consumption. Formally

$$0 \leq c < 1 \tag{4}$$

Now let us spatialize. Exogenous expenditures and national income are reinterpreted as spatial distributions x and y. (A formal definition will be given in the next section.) In principle, c may remain a scalar; however, that would represent the very special case in which people spend their income only at the locations where they earn it. In general, the consumption part of one dollar earned at some location, say the origin, will be allocated according to some spatial pattern. This pattern is the spatial distribution of the propensity to consume. Maintaining notation, this distribution is denoted c. For simplicity we assume that this consumption pattern is the same for all people irrespective of their locations of income. Note that the special case of exclusive local consumption is recaptured by the distribution c which is concentrated in the origin.

How is the model affected? Let us begin with condition (4). Of course, still less than one dollar is spent on consumption out of every dollar earned. Now how much is spent on consumption (per dollar earned)? The distribution is c. The total amount is obtained by summing over space: $\int c$. Thus we get

(4′) $$0 \leq \int c < 1$$

Let us consider Equation (1) next. Consider income at point r: $y(r)$ (heuristically). It is built up of two parts: consumption expenditures at r and other expenditures at r: $x(r)$. How much are the consumption expenditures at r? Of each dollar earned at some point s, a fraction $c(r-s)$ is spent at point r. Hence point s contributes $c(r-s)y(s)$. Total consumption expenditures at point r are obtained by summation over all points s: $\int c(r-s)y(s)ds$. This expression is known as the convolution of c and y (at r) and denoted $(c*y)(r)$ (heuristically). In sum (dropping the arguments r) the income equation becomes

(1′) $$y = c*y + x$$

This completes the spatialization. Recapitulating, we reinterpreted all scalars as distributions and we replaced the direct product by the convolution.

Keep in mind these two paradigms and the fact that the Dirac distribution or point mass at the origin, δ, is the unit distribution for, heuristically, $(x*\delta)(r) = \int x(r-s)\delta(s)ds = x(r-0) = x(r)$. Then the standard solutions (2) and (3) suggest the solution of the spatial model, namely

(2)′) $$y = [(\delta - c)^{*-1}]*x$$

or

(3′) $$y = (\delta + c + c^{*2} + c^{*3} + \ldots)*x$$

The * symbols are inserted to indicate that products and, hence, powers are in the sense of convolutions. (2′) and (3′) will be justified in the next section. Now let us discuss the economics of the solution. We see that a distributed increase of exogenous expenditures has a multiplier effect on national income, very much like before. The total effect now equals the exogenous distributed expenditure itself, x, plus the direct effect, the *convolution* $c*x$, plus the indirect effects, the further convolutions $(c^{*2})*x + (c^{*3})*x + \ldots$. In other words, an exogenous distributed expenditure spreads through the spatial economy in convolution multiplier fashion. The total effect converges in the sense of distributions precisely because of the spatial economic condition (4′). This will be proved in the next section.

Thus the theory of distributions enables us to handle the seemingly complex problem of spatialization without loss of operational results by reframing the economic variables and their algebraic relationships in a distribution space. A referee of this paper, Tony Smith, rightly noted that in measuring consumption fractions relative to the source of income, we assume spatial stationarity, and, in summing directly over all those sources, we assume spatial homogeneity. We may well analyze less regular spatial economies, but that would complicate the expressions and hamper the exposition of our new approach.

3. ANALYSIS: SCALAR DISTRIBUTIONS

Those who merely want to grasp the main thrust of this article should skip this technical section. The purpose of this section is an analysis of Equation (1′).

The classical approach is to confine oneself to locally summable coefficient c, known x, and unknown y. Then (1′) is an integral equation with kernel c. Through successive approximations a solution is arrived at under the premise that the surface between the horizontal axis and the graph of the kernel is contained in some rectangle of area less than one.

The classical approach lacks in two respects. First, c need not be locally summable. The elementary case of exclusive local consumption falls short of this condition, since it involves a concentrated distribution. Second, the economic condition that the propensity to consume is less than one, i.e., (4′), means that the surface between the horizontal axis and the graph of the kernel itself has area less than one without necessarily being contained in a rectangle. To overcome these shortcomings, we shall look at the equation afresh.

c, x, and y are scalar distributions over space in the sense of Schwartz, i.e., continuous linear functionals from the test space of infinitely differentiable functions on geographical space with compact supports to the reals. c is assumed to be nonnegative. By Schwartz (1957), c is a Radon measure and can thus be extended to the larger test space of continuous and bounded functions. (Infinite values of the measure will be ruled out by an economic assumption on c which is seen to imply boundedness.) The enlarged test space contains the constants. $\langle c,1\rangle$ is denoted by $\int c$. $\int$ properly generalizes the Lebesgue-Stieltjes integral. c is assumed to fulfill the economic condition $\int c < 1$.

Invoking the Dirac distribution, δ, Equation (1′) reads

$$(\delta - c)*y = x \tag{1''}$$

Consider the left-hand side operator. By convoluting through one sees that if $\delta + c + c^{*2} + c^{*3} + \ldots$ exists, then it is the inverse of $\delta - c$. Thus, if $\Sigma_0^\infty c^{*k}$ exists, then, convoluting through (1″), one obtains the solution for y, namely (3′). In fact, we shall show that $\Sigma_0^\infty c^{*k}$ converges and is continuous on the enlarged test space equipped with the sup-norm. [Then this holds a fortiori on the standard test space of Schwartz (1957).] For this purpose we first present:

Lemma 1: $\int c^{*k} = (\int c)^k \qquad (k = 0, 1, 2, \ldots)$.

Proof: $k = 0$ and $k = 1$ are trival. For $k = 2$, $\int c^{*2} = \langle c^{*2}, 1\rangle = \langle c*c, 1\rangle = \langle c,\langle c, 1\rangle\rangle = \langle c, \int c\rangle = \langle c, 1\rangle \int c = \int c \int c = (\int c)^2$. For $k > 2$ one proceeds in the same way. Q.E.D.

Corollary: $\int \Sigma_0^\infty c^{*k} = \Sigma_0^\infty (\int c)^k$.

Proof: $\int \Sigma_0^\infty c^{*k} = \langle \Sigma_0^\infty c^{*k}, 1\rangle = \Sigma_0^\infty \langle c^{*k}, 1\rangle = \Sigma_0^\infty \int c^{*k} = \Sigma_0^\infty (\int c)^k$ by Lemma 1. Q.E.D.

Now we obtain the inverse distribution.

Proposition 1: c is nonnegative and fulfills the economic condition $\int c < 1$. Then $\Sigma_0^\infty c^{*k}$ exists and is continuous (on the enlarged test space).

Proof: $\Sigma_0^\infty c^{*k}$ is nonnegative since c is. Consequently it is of order zero

[Schwartz (1957)] and we can estimate for any test function ϕ: $\langle \Sigma_0^\infty c^{*k}, \phi \rangle \leq \langle \Sigma_0^\infty c^{*k}, \| \phi \|_\infty \rangle = \langle \Sigma_0^\infty c^{*k}, 1 \rangle \| \phi \|_\infty = \int \Sigma_0^\infty c^{*k} \| \phi \|_\infty = \Sigma_0^\infty (\int c)^k \| \phi \|_\infty$ by the corollary to Lemma 1. The coefficient on the right-hand side is finite by the assumption on c.

Q.E.D.

Next we will examine properties of the inverse operator. The unknown y is as regular as the known vector distribution x in terms of integrability and differentiability.

Proposition 2: c is as in Proposition 1. Then $\Sigma_0^\infty c^{*k}$ preserves the combination of nonnegativity and p-integrability, $1 \leq p \leq \infty$. (That is, if $x \geq 0$ and $\| x \|_p \leq \infty$, then the same holds for y.)

Proof: By nonnegativity of c, $\Sigma_0^\infty c^{*k}$ preserves nonnegativity. Moreover, choose $L^1_{loc,+} \ni f_m \uparrow \Sigma_0^\infty c^{*k}$ and define $y_m = f_m * x$. Then $y_m \uparrow y$ and $\| y_m \|_p \leq \| f_m \|_1 \| x \|_p = \langle f_m, 1 \rangle \| x \|_p \leq \langle \Sigma_0^\infty c^{*k}, 1 \rangle \| x \|_p = \int \Sigma_0^\infty c^{*k} \| x \|_p = \Sigma_0^\infty (\int c)^k \| x \|_p$ by the corollary to Lemma 1. By the assumption on c and the principle of monotone convergence of integration theory, y_m converges in the p-norm. In fact, $\| y_m \|_p \uparrow \| y \|_p$ for our $y_m \uparrow y$. Taking the limit in our inequality, $\| y \|_p \leq \Sigma_0^\infty (\int c)^k \| x \|_p$. Q.E.D.

Lemma 2: c is as in Proposition 1. Then $\Sigma_0^\infty c^{*k}$ preserves the combination of boundedness and uniform continuity.

Proof: A bounded and uniform continuous x belongs to the (enlarged) test space. Thus it can be convoluted with $\Sigma_0^\infty c^{*k}$ by direct application of this distribution on $x(r-.)$ which is defined by $s \to x(r-s)$. Now $\| y(r+h) - y(r) \| = \| \langle \Sigma_0^\infty c^{*k}, x(r+h-.) \rangle - \langle \Sigma_0^\infty c^{*k}, x(r-.) \rangle \| = \| \langle \Sigma_0^\infty c^{*k}, x(r+h-.) - x(r-.) \rangle \| \leq \langle \Sigma_0^\infty c^{*k}, \| x(r+h-.) - x(r-.) \|_\infty \rangle = \int \Sigma_0^\infty c^{*k} \| x(.+h) - x(.) \|_\infty = \Sigma_0^\infty (\int c)^k \| x(.+h) - x(.) \|_\infty$ by the corollary to Lemma 2. Using the assumption on c, we see by unlimited variation of h that y is bounded when x is, and, by taking h sufficiently small, that y is uniformly continuous when x is. Q.E.D.

Proposition 3: c is as in Proposition 1. Then $\Sigma_0^\infty c^{*k}$ preserves p times bounded and continuous differentiability.

Proof: Since $y^{(p)} = (\Sigma_0^\infty c^{*k} * x)^{(p)} = \Sigma_0^\infty c^{*k} * x^{(p)}$, it suffices to show that y is bounded and continuous when x is. Define $\Sigma_m = \Sigma_0^\infty c^{*k}. \Delta_m$ by $\langle \Sigma_0^\infty c^{*k} . \Delta_m, \phi \rangle = \langle \Sigma_0^\infty c^{*k}, 0.\Delta_m \rangle$ where $\Delta_m(r) = 1$ on $\| r \| \leq m$, $m + 1 - \| r \|$ on $m \leq \| r \| \leq m+1$ and 0 elsewhere. By Proposition 1, $\Sigma_0^\infty c^{*k} - \Sigma_m$ vanishes from above and, in particular, is of order zero. Define $y_m = \Sigma_m * x$. Then $y_m(r) = \langle \Sigma_m, x(r-.) \rangle = \langle \Sigma_0^\infty c^{*k}, (x . \Delta_m)(r-.) \rangle$. But x is continuous, hence $x . \Delta_m$ is uniformly continuous and bounded and so is y_m by Lemma 2. Thus if we can show $y_m \to y$ uniformly, then y is continuous. This is easy to demonstrate: $0 \leq y(r) - y_m(r) = \langle \Sigma_0^\infty c^{*k}, x(r-.) \rangle - \langle \Sigma_m, x(r-.) \rangle = \langle \Sigma_0^\infty c^{*k} - \Sigma_m, x(r-.) \rangle \leq \langle \Sigma_0^\infty c^{*k} - \Sigma_m, \| x(r-.) \|_\infty \rangle = \langle \Sigma_0^\infty c^{*k} - \Sigma_m, 1 \rangle \| x \|_\infty \downarrow 0$ when x is bounded. Then y is bounded too by Proposition 2. Q.E.D.

4. FIRST CASE STUDY: EXPENDITURE DIFFUSION

The expenditure diffusion model of Paelinck (1982, p.5) is designed to assess the interregional effects of regional expenditure programs. Expenditures are transmitted through people who spend their income not only locally, but also at adjacent regions. Consequently, demand in some region depends also on income in other regions. Total consumption in region ρ is specificed by Paelinck as

$$\alpha y_\rho + \alpha^* \sum y_\sigma + \alpha^{**} \sum y_\tau \tag{5}$$

where we use strictly Greek symbols for regions, reserving Arabic letters for point locations. y_ρ is income in region ρ, obtained by integrating the income density $y(r)$ over all points r in ρ

$$y_\rho = \int_{\|r-\rho\|=0} y(r)dr \tag{6}$$

where $\| \, . \, \|$ is the contiguity distance of Paelinck (1982, p. 44). Σy_σ is income in the regions σ contiguous to ρ

$$\sum y_\sigma = \int_{\|s-\rho\|=1} y(s)ds \tag{7}$$

Σy_τ is income in further regions, contiguous to σ

$$\sum y_\tau = \int_{\|t-\rho\|=2} y(t)dt \tag{8}$$

where α is the propensity to consume locally and α^* is the propensity to consume in some directly contiguous region. Therefore, the propensity to consume in the directly contiguous regions equals α^* times the number of such regions, i.e., $4\alpha^*$ for a grid structure of regions. The propensity to consume in the second-degree contiguous regions then equals $8\alpha^{**}$. If the propensity to consume decreases with contiguity, then $\alpha > 4\alpha^* > 8\alpha^{**}$. Note that the propensity to consume adds up to $\alpha + 4\alpha^* + 8\alpha^{**}$. This figure must be less than one. (In the case of decreasing propensity, combination of the inequalities yields $8\alpha^* < \alpha + 4\alpha^* < 1$ or $\alpha^* < 0.125$, and $24\alpha^{**} < \alpha + 4\alpha^* + 8\alpha^{**} < 1$ or $\alpha^{**} < 0.042$.)

The income balance of region ρ reads income y_ρ equals consumption (5) plus exogenous expenditures x_ρ:

$$y_\rho = \alpha y_\rho + \alpha^* \sum y_\sigma + \alpha^{**} \sum y_\tau + x_\rho \tag{9}$$

The model clearly resembles "Keynes spatialized." In fact, we shall reveal the distribution structure of the present model and show that (9) can be obtained from (1′) through the propensity to consume density, $c(r)$, which equals $\alpha / \int_{\|s-r\|=0} ds$, $\alpha^* / \int_{\|s-r\|=0} ds$, $\alpha^{**} / \int_{\|s-r\|=0} ds$ or 0 for $\| r \| = 0, 1, 2$ or $3, 4, \ldots$, respectively. For then (1′) becomes, evaluated at point r

$$y(r) = \int c(r-s)y(s)ds + x(r) = \left(\int_{\|s-r\|=0} + \int_{\|s-r\|=1} + \int_{\|s-r\|=2}\right)$$

$$[c(r-s)y(s)ds] + x(r) = \left(\alpha\int_{\|s-r\|=0} + \alpha^*\int_{\|s-r\|=1} + \alpha^{**}\int_{\|s-r\|=2}\right)$$

$$[y(s)ds]\Big/\int_{\|s-r\|=0} ds + x(r) = \left(\alpha y_\rho + \alpha^* \sum y_\sigma + \alpha^{**} \sum y_\tau\right)\Big/\int_{\|s-\rho\|=0}$$

$$ds + x(r) \text{ for } \rho \ni r$$

by (6), (7), and (8). Integrating over all r in ρ, (9) emerges. It should be mentioned that Paelinck's contiguity distance is only a pseudo-distance between points, for distinct points may be at zero distance from each other, namely, when they belong to a common region. This complication may be overcome by taking a Minkowski distance instead. In fact, when the regional classification becomes finer and finer, the contiguity pseudo-distance tends to the sup distance. However, I would prefer a more realistic Minkowski distance. For these metric matters I refer to my earlier paper [ten Raa (1983)].

Having laid down the distribution structure of the model, we are now prepared to address Paelinck's problem of assessing interregional expenditure effects. The consequent income distribution is given by (3′), as has been justified by Proposition 1. The effects of a unit expenditure impulse are found by substituting $x = \delta$, the Dirac distribution. Thus these effects are distributed like $\Sigma_0^\infty c^{*k}$.

Paelinck (1982, p. 9) hypothesizes that "the effect of an impulse will probably decrease; however the effects may undulate across the area, and the peak of a wave quite far . . . may well get a larger amplitude than one nearer. . . ." We shall argue, however, that the model does not generate such a wave pattern. For this purpose we may make Paelinck's assumptions that the direct consumption effect (c) is symmetric with respect to directions, diminishing, and limited to some finite area. Then c can be interpreted as a distribution on the one-dimensional space of distances, with bounded support and nonpositive derivative. (This is done by projection; purists would introduce a new symbol for this c. The derivative is in the sense of distributions.) We shall derive that the total effects ($\Sigma_0^\infty c^{*k}$) are damped and even diminishing, which clearly settles the issue.

In fact, the damped behavior is due to the boundedness of c's support alone, irrespective of c's diminishing behavior. For by appropriate choice of a distance unit, supp $c \subset [0, 1]$ and therefore supp $c^{*k} \subset [0, k]$. To investigate $\Sigma_0^\infty c^{*k}$ at r we must apply it to a test function with support in a small neighborhood of r. The terms c^{*k} with $k \leq r$, the floor of r, have no impact. Consequently, the total effect at r is like $\Sigma_r^\infty c^{*k}$. But this vanishes as r tends to infinity since $\Sigma_0^\infty c^{*k}$ converges by Proposition 1. Thus the total effects are damped. Next we shall demonstrate that the transition from the direct effect to the total effects generates no wave pattern. The crux is:

Proposition 4: b and c are nonnegative distributions over n-dimensional Euclidian space, depend on distance only, and in weakly decreasing fashion. Then the same holds for $b*c$.

Proof: By method of descent, i.e., induction on n. Since nonnegativity and exclusive distance dependence hold trivially, it remains to be shown that $b*c$ is weakly decreasing in distance. By the technique in the proof of Proposition 2 we may confine ourselves to locally summable b and c. For $n = 1$, $r \in R$ and $d/d\|r\|$ $(b*c)\,(r)$

$$= sgn(r)\frac{d}{dr}(b*c)(r)$$
$$= sgn(r)\left[\int_0^{\infty} b(r-s)\dot{c}(s)ds + \int_{-\infty}^{0} b(r-s)\dot{c}(s)ds\right]$$
$$= sgn(r)\left[\int_0^{\infty} b(r-s)\dot{c}(s)ds + \int_0^{\infty} b(r+s)\dot{c}(-s)ds\right]$$
$$= sgn(r)\int_0^{\infty} [b(r-s) - b(r+s)]\dot{c}(s)ds$$

by symmetry of $c(.)$ (which depends on distance only). By weak decreasingness of b, using $s \geq 0, r \geq 0 \Longrightarrow \| r - s \| \leq \| r + s \| \Longrightarrow b(r-s) \geq b(r+s)$ and $r \leq 0 \Longrightarrow \| r - s \| \geq \| r + s \| \Longrightarrow b(r-s) \leq b(r+s)$; in short, $sgn(r)[b(r-s) - b(r+s)] \geq 0$. But, by weak decreasingness of c, $\dot{c}(s) \leq 0$ on $[0, \infty)$. Substitution of these inequalities in the expression for $d/d\| r \| (b*c)(r)$ yields that $b*c$ is weakly decreasing in distance for $n = 1$. Now suppose the proposition is true for dimension n. Then we shall prove it for dimension $n + 1$. By exclusive distance dependence, it suffices to consider $b*c$ at points in R^{n+1} with last component zero, i.e. $(r, 0)$ with $r \in R^n$. By Fubini's theorem

$$(b*c)(r,0) = \int_{R^{n+1}} b[(r,0) - (s, s_{n+1})]c(s, s_{n+1})d(s, s_{n+1})$$
$$= \int_{-\infty}^{\infty}\left[\int_{R^n} b(r - s, -s_{n+1})c(s, s_{n+1})ds\right]ds_{n+1}$$
$$= \int_{-\infty}^{\infty}\left[\int_{R^n} b(r - s, s_{n+1})c(s, s_{n+1})ds\right]ds_{n+1}$$

by symmetry of $b(r,.)$ (b depending on distance only). For all s_{n+1}, define $b_{s_{n+1}}: R^n \rightarrow R_+$ by $b_{s_{n+1}}(r) = b(r, s_{n+1})$, and $c_{s_{n+1}}$ similarly. Then our expression becomes

$$(b*c)(r,0) = \int_{-\infty}^{\infty}\left[\int_{R^n} b_{s_{n+1}}(r-s)c_{s_{n+1}}(s)ds\right]ds_{n+1}$$

Here, by the induction hypothesis [.] is weakly decreasing in $\| r \|$, for all s_{n+1}. Hence, $(b*c)(r,0) = \int_{-\infty}^{\infty} [.]ds_{n+1}$ is weakly decreasing in $\| r \|$ or $\| (r, 0) \|$. Q.E.D.

Corollary: c is as in Proposition 4. Then the same holds for $\Sigma_0^{\infty} c^{*k}$.

Proof: By induction on k, c^{*k} is as in Proposition 4. Summation over k yields the result. Q.E.D.

That is, the total effects are diminishing. I am grateful to referee Tony Smith for exposing a shortcoming in my original argument.

5. SECOND CASE STUDY: URBAN DENSITY

The urban density model of Beckmann (1977) is a spatial equilibrium model designed to explain observed patterns of density distributions of economic activities in cities. To reveal basic density patterns Beckmann assumes away all incidental nonspatial causes. All households are equal as regards a utility function, income, consumption of nonspatial goods, and attractiveness for other households. Then in equilibrium households must be equally satisfied in all locations. All households' utility levels are the same, u_0. Utility consists of two terms. One represents the net utility of housing, corrected for the disutility of housing density. The other represents the net utility of interaction, taking into account transportation economies of density. Housing density trades off these two utilities: that is the crux of the equilibrium model.

The net utility of housing is written as an indirect utility term of housing density at point r, $m(r)$, and assuming linearity it becomes $\alpha - \beta m(r)$. The net utility of interactions is also written as such an indirect utility term and is based on the entropy function: $\int e^{-\|x-r\|} m(x)\,dx$. Thus the equation of spatial equilibrium is

$$u_0 = \alpha - \beta m(r) + \int e^{-\|x-r\|} m(x)\,dx$$

or

$$m(r) = \frac{\alpha - u_0}{\beta} + \frac{1}{\beta} \int e^{-\|x-r\|} m(x)\,dx \tag{10}$$

This is Equation (2) of Beckmann (1977, p. 126). His careful theorizing has been insulted by my hasty derivation.

Observe that when the city extends over the whole real line or plane, circle or ball, then the solution consists of a uniform housing density. Therefore, the model is not so rich that it can explain location and size of the city. To be fair, such an objective would require a nonhomogeneity assumption on geographical space such as favorable conditions around the origin, possibly reflected through space-dependent parameters in the utility function. Otherwise, any solution would be arbitrary in that it could be translated or rotated. Beckmann circumvents all this by fixing the location and size of the city on $\Omega = [-R, R]$. Thus the integrations are performed over Ω.

Beckmann's equation of equilibrium requires integrability. Densities concentrated at single points such as the Empire State Building are ruled out a priori. This condition is relaxed by going to the distribution of housing, m. Then Equation (10) becomes

$$m = \frac{1}{\beta} e^{-\|\cdot\|} * m + \frac{\alpha - u_0}{\beta} \tag{10'}$$

To solve for the housing distribution, let us check the economic assumption of Proposition 1. By nonnegativity it is necessary and sufficient that $\int (1/\beta) e^{-\|\cdot\|} < 1$ or $\beta > \int e^{-\|\cdot\|}$. For $\Omega = [-R, R]$ this yields $\beta > 2(1 - e^{-R})$ which is precisely the condition of Beckmann (1977, p. 127). The condition on β enables us to solve (10') through the existence and regularity propositions. By Proposition 1, $m = \Sigma_0^{\infty} (1/\beta e^{-\|\cdot\|})^{*k} * \alpha - u_0/\beta$ and m is as regular as $(\alpha - u_0)/\beta$ in the sense of Propositions 2 and 3, that is ∞-integrable on Ω and ∞ times bounded and continuous differentiable on Ω. Moreover, $m = \Sigma_0^{\infty} * \alpha - u_0/\beta$ is unique since Σ_0^{∞} is unique, for if T were another inverse of $\delta - 1/\beta\, e^{-\|\cdot\|}$, then $T = T * \delta = T * (\delta -$

$1e/\beta^{-\|\cdot\|})*\Sigma_0^\infty = \delta*\Sigma_0^\infty = \Sigma_0^\infty$. This observation is added to cover the further result of Beckman (1977, p. 127).

Our distribution analysis of housing equilibrium is very powerful, for it does not hinge on $\Omega = [-R, R]$. In fact, the space may be of any dimension. Beckmann (1977, p. 129) reports that, unfortunately, his analysis of the two-dimensional city is impeded by some difficult partial differential equations except in some singular cases. We have surmounted this problem by direct inversion of the interaction distribution. This approach extends the theory of urban density to two-dimensional cities.

6. ANALYSIS: VECTOR DISTRIBUTIONS

To pave the way for the third and last case study we must extend the analysis to vector-valued distributions. As before, followers of just the main thrust of the article should skip this material. Nothwithstanding, the extension is straightforward.

The roles of coefficient c, known x and unknown y, are now assumed by an $(n \times n)$-dimensional nonnegative coefficients matrix distribution $\mathbf{A}$ and known and unknown n-dimensional vector distributions $\mathbf{f}$ and $\mathbf{q}$, respectively. The equation to be analyzed, parallel to (1′), becomes

$$\mathbf{q} = \mathbf{A}*\mathbf{q} + \mathbf{f}$$

The convolution product is defined by $(\mathbf{A}*\mathbf{q})_i = \Sigma_{j=1}^n a_{ij}*q_j$ where $*$ stands for familiar scalar convolution.

The crux is the generalization of the economic condition $\int c < 1$. $\int \mathbf{A}$ is now defined by $(\int \mathbf{A})_{ij} = \int a_{ij}$. Note that $\int \mathbf{A}$ is an ordinary $n \times n$ matrix. What about the bound? Recall that it was used for the convergence of the inverse distribution $\Sigma_0^\infty c^{*k}$ through $\int \Sigma_0^\infty c^{*k} = \Sigma_0^\infty (\int c)^k$ (the corollary to Lemma 1). The latter equality is seen to hold for $\mathbf{A}$ by considering it component by component. Thus we now want convergence of $\Sigma_0^\infty (\int \mathbf{A})^k$. By nonnegativity, the necessary and sufficient conditions therefore are those of Hawkins and Simon (1949). In short, $\int \mathbf{A}$ is assumed to fulfill the Hawkins-Simon conditions. Indeed, these conditions properly generalize the economic condition $\int c < 1$.

Precisely as in the scalar case, one sees that if $\Sigma_0^\infty \mathbf{A}^{*k}$ exists, then it is the inverse distribution and one obtains the solution of Equation (12), in the following section,

$$\mathbf{q} = \sum_0^\infty \mathbf{A}^{*k}*\mathbf{f} \tag{11}$$

Here the following observations are pertinent. Lemma 1 and its corollary hold for $\mathbf{A}$. Propositions 1, 2, and 3—with the economic condition generalized by the Hawkins-Simon condition—also apply to $\mathbf{A}$. The proof of this extension proceeds in straightforward and uncomplicated component-by-component fashion and is therefore omitted.

7. THIRD CASE STUDY: WORLD MODEL

The United Nations world model of Leontief, Carter, and Petri (1977) is a multiregional input-output model, specified as follows: $\mathbf{q}$, $\mathbf{f}$, $\mathbf{m}$, $\mathbf{e}$, $\mathbf{A}_0$ and μ denote, respectively, a supply and a final demand vector, an import and an export vector, an input-output coefficients matrix, and an import coefficients vector. All these entities are functions of the regions r ($r = 1, \ldots, 15$, the number of regions in the world model). Furthermore, $\theta(r, s)$ denotes region r's export shares vector in market region s. For region r the equations are

$$\mathbf{q}(r) = \mathbf{A}_0(r)[\mathbf{q}(r) - \mathbf{m}(r)] + \mathbf{f}(r) + \mathbf{e}(r), \mathbf{m}(r) = \hat{\mu}(r)\mathbf{q}(r) \text{ and } \mathbf{e}(r) = \sum_s \hat{\theta}(r, s)\mathbf{m}(s) \tag{12}$$

where

$$\mathbf{0} \leq \mu(r) \leq \mathbf{i} = (1 \ldots 1), \sum_r \mu(r) < 15\mathbf{i}, \theta(r, s) \geq 0 \text{ and } \sum_r \theta(r, s) = \mathbf{i} \tag{12a}$$

Here we subscribe to the analytically satisfactory model as opposed to the computationally convenient one of Leontief, Carter, and Petri (1977, p. 22). (In fact, the analytically satisfactory model turns out to be more tractable!) The strict inequality excludes from consideration the banal possibility that some good is completely imported everywhere, i.e., produced nowhere. (In fact, for indecomposable technologies $\mathbf{A}_0(r)$, this would be the case of all supply and final demand equal to zero.) It should be noted that Leontief, Carter, and Petri (1977) assume that export shares are the same for all markets s: $\theta(r, s) = \theta(r)$. However, we shall maintain the refined picture of trade, $\theta(r, s)$. (In Paelinck's terminology, we use full information input-output, whereas Leontief, Carter, and Petri (1979) work with limited information input-output.)

To highlight the basic structure of the model, consider the case in which technology and import structure are uniform and export patterns are also basically the same in that only the relative location of a market s matters: $\mathbf{A}_0(r) = \mathbf{A}_0$, $\mu(r) = \mu$ and $\theta(r, s) = \theta(\|r - s\|)$. Here $\|r - s\|$ is a symbol for the distance between regions r and s, e.g., in the contiguity sense of Paelinck (1982). By substitution and simplification, (12) and (12a) reduce to

$$\mathbf{q}(r) = \mathbf{A}_0\widehat{[\mathbf{i} - \mu]}\mathbf{q}(r) + \sum_s \hat{\theta}(\|r - s\|)\mu\mathbf{q}(s) + \mathbf{f}(r) \tag{13}$$

where

$$\mathbf{0} \leq \mu < \mathbf{i}, \theta(\|r - s\|) \geq \mathbf{0} \text{ and } \sum_r \theta(\|r - s\|) = \mathbf{i} \tag{13a}$$

A region r is an element of space. Space can remain an index set as in the world model or can now be structured, e.g., into the Euclidian plane. Both interpretations are consistent with the subsequent argument.

We redefine export shares as a nonnegative vector distribution θ across space. Of a one unit impulse of imports at the origin, region r supplies, heuristically, $\theta(r)$. Since summing over r we must recapture the unit of imports, we assume $\int\theta = \mathbf{i}$. Then the middle term on the right-hand side of (13) becomes, heuristically, $\hat{\theta}*\hat{\mu}\mathbf{q}(r)$. Dropping the arguments r we capture the basics of the world model in a nutshell

(14) $$\mathbf{q} = \mathbf{A}^*\mathbf{q} + \mathbf{f}$$

with

(14a) $$\mathbf{A} = \mathbf{A}_0\widehat{[\mathbf{i} - \boldsymbol{\mu}]}\delta + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\mu}}, \mathbf{0} \leq \boldsymbol{\mu} < \mathbf{i}, \boldsymbol{\theta} \geq \mathbf{0}, \text{ and } \int\boldsymbol{\theta} = \mathbf{i}$$

Note that $\mathbf{A} \geq 0$ and $\int\mathbf{A} = \mathbf{A}_0\widehat{[\mathbf{i} - \boldsymbol{\mu}]} + \hat{\mathbf{i}}\hat{\boldsymbol{\mu}}$. As always, $\mathbf{A}_0$ is assumed to fulfill the Hawkins-Simon conditions. It follows that $\int\mathbf{A}$ is a convex combination of a matrix ($\mathbf{A}_0$) with spectral radius less than one and a matrix ($\hat{\mathbf{i}}$) with spectral radius one where the weight of the latter ($\hat{\boldsymbol{\mu}}$) is strictly less than $\hat{\mathbf{i}}$. Intuitively, then, the spectral radius of $\int\mathbf{A}$ itself must also be less than one. This fact can be established rigorously for $\mathbf{A}_0$ indecomposable: then $\int\mathbf{A}$ is a convex combination of two nonnegative and indecomposable matrices and, consequently, it is nonnegative and indecomposable. By Froebenius' theorem, $\rho(\int\mathbf{A}) = \lambda$ with $(\int\mathbf{A})\mathbf{x} = \lambda\mathbf{x}$ for some positive $\mathbf{x}$. Substituting $\int\mathbf{A} = \mathbf{A}_0\widehat{[\mathbf{i} - \boldsymbol{\mu}]} + \hat{\mathbf{i}}\hat{\boldsymbol{\mu}} = \mathbf{I} - (\mathbf{I} - \mathbf{A}_0)\hat{\mathbf{i}} - \boldsymbol{\mu}$ we obtain $\mathbf{x} - (\mathbf{I} - \mathbf{A}_0)\widehat{[\mathbf{i} - \boldsymbol{\mu}]}\mathbf{x} = \lambda\mathbf{x}$, $(1 - \lambda)\mathbf{x} = (\mathbf{I} - \mathbf{A}_0)\widehat{[\mathbf{i} - \boldsymbol{\mu}]}\mathbf{x}$ or $(1 - \lambda)\ \widehat{[\mathbf{i} - \boldsymbol{\mu}]}^{-1}$ $(\mathbf{I} - \mathbf{A}_0)^{-1}\,\mathbf{x} = \mathbf{x}$. Since $\mathbf{x}$ is positive, $1 - \lambda$ is nonzero and we may divide through $(\mathbf{I} - \mathbf{A}_0)^{-1}\widehat{[\mathbf{i} - \boldsymbol{\mu}]}^{-1}\mathbf{x} = (1 - \lambda)^{-1}\mathbf{x}$. Since $(\mathbf{I} - \mathbf{A}_0)^{-1} = \Sigma_0^\infty \mathbf{A}_0^k \geq \mathbf{I}$, $\widehat{[\mathbf{i} - \boldsymbol{\mu}]} > \mathbf{0}$ and $\mathbf{x} > \mathbf{0}$, the left-hand side is positive. Thus the right-hand side is positive and since $\mathbf{x} > \mathbf{0}$, $(1 - \lambda)^{-1} > 0$. Consequently, $1 - \lambda > 0$ or $\rho(\int\mathbf{A}) < 1$. A more general proof, without an appeal to indecomposability, would be more closely along the intuitive line which led to the statement on $\mathbf{A}$ or complete by a limiting argument. The result means that $\int\mathbf{A}$ fulfills the Hawkins-Simon conditions.

We thus have proved that the Hawkins-Simon conditions on a local scale ($\mathbf{A}_0$) carry over to the global operator ($\mathbf{A}$); this enables us to apply the existence and regularity propositions as extended for the vector case. By Proposition 1, the solution $\mathbf{q}$ of (14) is given by (11), and $\mathbf{q}$ is as regular as $\mathbf{f}$ in the sense of Propositions 2 and 3.

8. CONCLUSION

A new method for spatial economic analysis consists of four steps. First, standard, nonspatial models are taken as points of departure. Second, structures and activities are no longer considered point scalars or vectors, but distributions over space. Third, the ordinary product is replaced by the convolution product. Fourth, the consequent spatialized models are subjected to the Schwartz calculus of distributions.

The approach offers a unifying framework for spatial economic models as varied as the spatial Keynesian model, the expenditure diffusion model of Paelinck (1982), the spatial equilibrium model of urban density of Beckmann (1977), and the United Nations world model of Leontief, Carter, and Petri (1977).

The application of the theory of distributions of Schwartz (1957) seems promising for economic science. Our analysis of the various spatial economic models features the following results:

1. uncovering of the distribution structure; the transmission of mathematical properties from the exogenous to the endogenous variables;
2. description of the spatial pattern of expenditure diffusion;
3. determination of two-dimensional urban density; and

4. a concise account and analysis of the United Nations world model and some extensions.

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