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SPATIAL INTERACTION ANALYSIS*

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This paper formulates equilibrium problems of spatial interaction theory without appealing to vector force fields at all, and determines equilibrium densities for non-Newtonian distance response functions. The first section reviews Amson's "plasma" model which is considered a general paradigm in the field. Here "species" are subject to local repulsion and global attraction effects. In equilibrium these effects are balanced.

Section 2 replaces Amson's mechanical equilibrium notion by a more utilitarian one. Section 3 analyzes equilibrium for general interaction functions, including exponential ones. In view of Smith's "cost-efficiency" principle of spatial interaction behavior, this extension fills a gap. In Section 4, an application of the general analysis contributes to the modality issue for equilibrium densities. Conclusions are summed up in the last section.

1. SPATIAL INTERACTION THEORY: DIGRESSION

Spatial interaction theory concerns the interplay between global attraction effects and local repulsion effects. Smith (1976, p. 97) identifies Amson's plasma model as a general paradigm that unifies a remarkable number of spatial theories (urban economics, traffic engineering, sociology of social and ethnic groups, to mention a few). A plasma is a system of gravitational species subject to self-attractions and to mutual attractions, and also to local pressure forces which depend on the local densities of the species [Amson (1972, p. 430)]. Definitions 2 and 3 of Amson (1972, p. 431–432) comprise a formal description. They are recapitulated here in reduced form and without subscripts, or, more precisely, for the single species-single pressure case. While reflecting my distaste for subscripts, this simplification does not infringe upon the generality of the argument.

Definition (Amson): An abstract city is a system (τ, k, d^+, s) where τ is a civic mass density on civic space R^2 (the Euclidian plane), k is a coercion strength coefficient, d^+ is a distance response function, and s is a state function which associates satisfaction pressure with local density.

Definition (Amson): The civic coercion experienced by the species at the point

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x in response to the species at the point y is the vector $c(x, y) = k\tau(x)\tau(y)d^{+}(\|y - x\|)[y - x/\|y - x\|]$.

In equilibrium civic coercion in response to all y (summed over the plane) is balanced by the negative gradient of the satisfaction pressure $\iint c(x, y)\, dy = -\text{grad}\, s[\tau(x)]$. Let me explain the definitions. $c(x, y)$ is an interaction vector for the body of species at x in response to the same at y, so that $\iint c(x, y)\, dy$ is, typically, a total attraction vector at x. A typical dissatisfaction pressure, $-s$, is total rent. In equilibrium, the attraction vector is equal to the rent gradient. Note that if we divide through by $\tau(x)$ we obtain the per capita attraction vector and satisfaction pressure. Thus, the equilibrium condition is that the per capita attraction vector is equal to the per capita rent gradient. Any individual is subject to global attraction and local pressure. The former is a positive force and the latter a negative one. In equilibrium they cancel out. Per capita rent density is typically an increasing function of civic density, such as $-s/\tau = K\tau^{\gamma-1}$ with rental constant $K > 0$ and rental exponent $\gamma > 1$. This agrees with Equation (7) of Amson (1972, p. 437–438) except for the minus sign. [Initially, Amson (1972, p. 433) correctly identifies housing rental as a dissatisfaction level, but gradually Amson (1972, pp. 433 and 437) begins to wrongly call it a satisfaction pressure.]

Following Amson (1972, 1973), this preliminary investigation focuses on $\gamma = 2$. This is the case in which the per capita dissatisfaction pressure is a linear function of civic density, a good first-order approximation. The distance response function serves as a kernel in the interaction term of the equilibrium equation. Amson (1972, p. 434) observes that most gravity models use an inverse power kernel $d^{+}(\|y - x\|) = \|y - x\|^{-a}$ ($a > 0$) having a pole wherever $y = x$, while others use an exponential decay kernel $d^{+}(\|y - x\|) = A \exp(-\beta\|y - x\|)$.

Although Amson is aware of the theoretical superiority of the exponential function in the sense of Wilson (1970, pp. 16–19), he assumes an inverse power form with exponent $a = 1$ in order to exploit Newtonian potential theory. This focus is natural in light of the state of affairs in attraction theory, including the foundation of Wilson's theory, at the time. However, Smith (1978) provided a strong behavioral foundation for the exponential interaction kernel, the so-called cost-efficiency principle. In view of this, it is now pressing to develop an equilibrium analysis that can handle non-Newtonian interaction kernels. This paper seeks to initiate such a line of research.

2. EQUILIBRIUM

Essentially, Amson transforms coercion strength and distance response, as well as pressure, into forces—namely, the attraction vectors and the satisfaction gradient, which subsequently are required to cancel out in a vectorial sense. This is an extremely dubious notion of spatial interaction equilibrium. As Amson (1972, p. 434) himself notes,

> for example, it seems very unlikely that a citizen, say, exposed to two simultaneous coercions to relocate in two different directions, will in fact relocate in a direction determined by an application of the "parallelogram law of resultant forces" from mechanics. From probabilis-

tic considerations alone, such a citizen is much more likely to accept relocation in the direction of the stronger of the two competing coercions. But however difficult this feature of the vector additivity of coercions is to accommodate within a general study of urban systems, it presents no serious obstacle in the restricted study of models of urban systems possessing circular symmetry. All that is then required is a postulate of a "principle of symmetry": (1) all civic coercions that arise through the circularly symmetric distribution of civic matter act in a radial direction (being *calculated* as if the vector law of addition were applicable to coercions), and (2) the resultant of any two *co-radial* coercions is their vector sum.

Unfortunately, the "principle of symmetry," part (2) in particular, is still dubious, as I shall discuss first. Then, more significantly, I shall show how an appropriate revision of the spatial interaction equilibrium definition avoids vectorial force addition entirely.

Consider a civic point mass in the origin and a civic line mass uniformly distributed over the unit circle. Consider the resultant coercion at a point in between, say $(0, \frac{1}{2})$. By symmetry considerations, the unit circle may be represented by a single point on the axis through $(0, \frac{1}{2})$, and, by appropriate choice of mass, this point may be located in either the origin, or $(0, \frac{1}{2})$ itself, or $(0, 1)$. (These cases correspond to $a < 1$, $a = 1$, or $a > 1$ when the distance response function has inverse power form; see the previous section.) The latter case is the most interesting, as it holds for exponential decay interaction kernels. The mass in $(0, 1)$ depends on the parameters in the interaction kernel and the original circular distributed mass, but is set equal to unity by choice of that original mass.

Now we have an abstract city meeting Amson's restriction of circular symmetry, represented by two unit point masses, one in the origin and one in $(0, 1)$. Species at the point under consideration, $(0, \frac{1}{2})$, experience two equal civic coercions in opposite directions. By Amson's definition, abstracting from satisfaction pressure, they are in equilibrium. Now this may be true for physical particles, but, in my opinion, not for civic species. When a civic species feels attracted by two points which lie in opposite directions, it does not remain undecided, but flips a coin and will move either to the left or to the right attractor. While Amson postulates that such species will remain at rest, I claim it will divide up and move to the left and right. An example will illustrate the issue at hand.

During the last decade much public effort has been made in the creation of a port in the Dutch province of Groningen. Proponents such as Professor Pen of the University of Groningen are now puzzled that the finished port attracts no investment, arguing that the location of the port is ideal: just half-way between the industrial centers of Rotterdam and Hamburg. Indeed, Amson's equilibrium theory would predict equilibrium civic species at the Groningen port, balanced by the Rotterdam and Hamburg coercions. Nonetheless, the port is now a good fishing pond at best. Why? Because firms do not care about attraction vectors, let alone their cancelling out. They are interested in attraction levels, and those levels are higher in Rotterdam or Hamburg. Therefore, firms would rather invest in Rotterdam and Hamburg themselves. Moral: Drop the transformation into attraction vectors and pressure gradients, and stick with coercion and pressure levels themselves. Then the per capita levels of attraction and satisfaction pressure should not cancel but add to a constant across space in equilibrium, in order not to induce individual species to relocate to a place with a higher perceived utility level.

To see the formalities of the revision, recall Amson's equilibrium equation $\iint k\tau(x)\tau(y)d^+(\|y - x\|)\ [y - x/\|y - x\|]ds + \text{grad}\ s[\tau(x)] = 0$. We drop the vector operators $y - x/\|y - x\|$ and grad, and after division through by $\tau(x)$ we let them add to u_0: $\iint k\tau(y)d^+(\|y - x\|)dy + s[\tau(x)]/\tau(x) = u_0$. The latter revision seems appropriate for spatial interaction modeling, since it retains attraction and satisfaction pressure levels, now considered utility components, without analogy to the mechanical law of resultant forces.

Invoking Amson's functional pressure form as discussed in Section 1, $-s/\tau = K_\tau^{2-1}$, the equilibrium equation becomes, dividing through by the rental constant K, $\iint (k/K)\ d^+\ (\|y - x\|)\tau(y)\ dy - \tau(x) = u_0/K$. In sum, this equation is essentially Amson's "plasma balance," stripped of vectorial forces, featuring his functional pressure form but not his Newtonian distance response function. Instead, the model remains general with respect to the interaction kernel.

3. ANALYSIS

The spatial interaction equilibrium equation just derived happens to be another instance of the fundamental Equation (1′) of my recent paper [ten Raa 1984)]. Therefore, we can solve for the equilibrium density τ by inverting the interaction kernel in the sense of Proposition 1 of that paper. This procedure is a departure from established practice as exemplified by Amson (1972, 1976) and Beckmann (1977).

First, Amson (1972, p. 437) and Beckmann (1977, p. 129) differentiate through to arrive at differential equations which, however, are generally unsolvable, while the direct inversion works. More precisely, Amson (1976, p. 104) and Beckmann (1977, p. 129) can solve for equilibrium only in singular cases, while the solution presented below holds for regular cases, in parameter space. Second, the present approach is valid irrespective of the functional form of the interaction kernel. Third, civic space need no longer be circular symmetric [see ten Raa (1984)].

The only assumption of Proposition 1 of ten Raa (1984) is that the total mass of the whole kernel is less than one: $\iint (k/K)\ d^+\ (.) < 1$. For example, if d^+ is the exponential decay kernel (see Section 1), the condition is $\iint (k/K)A \exp(-\beta r)rdrd\psi < 1$, using polar coordinates; $0 \le r < \infty$ and $0 \le \psi < 2\pi$. Simple evaluation yields $2\pi kA\beta^{-2} < K$. Since kA and β^{-2} are parameters which put weight on the global interaction effect, while K puts weight on the local repulsion effect, the condition is that the local pressure effect must outweigh the global interaction effect. Then the solution is as in Proposition 1 of ten Raa (1984) with its $a = (k/K)\ d^+$ and $x = -u_0/K$, and its properties are discussed there. Otherwise, the abstract city is torn apart by the global effects and zones of negative density emerge. Amson (1976, p. 99) provides an interpretation for such a phenomenon. To avoid heavy mathematics this paper restricts itself to the above implicit presentation of the equilibrium density τ.

4. UNIMODALITY

Tony Smith has told me that the modality of equilibrium densities in social interaction theory is an open issue. This can be dealt with now for the case under consideration, $\gamma = 2$. The equilibrium density is given by Proposition 1 of ten Raa (1984) which is, in fact, a sum of so-called convolution products of the interaction kernel with itself, at least under the condition of the previous section. The interaction kernel itself, i.e., the distance response function, is always assumed to be unimodal—weakly decreasing in distance. But Proposition 4 of ten Raa (1984) proves that this property is preserved under the convolution products, and its corollary is that the sum of those convolution products is also unimodal. Therefore, if the total mass of the whole interaction kernel is less than one, as in the previous section, then the equilibrium civic mass is unimodal. Note that this result has been derived under the condition $\gamma = 2$ in which the per capita dissatisfaction pressure is a linear function of civic density. The modality issue for the nonlinear case remains open.

5. CONCLUSIONS

(a) Equilibrium for Amson's plasma system can be formulated without recourse to the mathematical law of resultant forces.

(b) Equilibrium densities can be determined for spatial interaction models with non-Newtonian distance response functions, such as the exponential decay kernel whose theoretical superiority has recently been demonstrated by Smith.

(c) The solution is unimodal when per capita local repulsion is a linear function of density.

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