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PREFERENCE FOR FLEXIBILITY AND THE OPPORTUNITIES OF CHOICE [1]

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Abstract

A decision-maker exhibits preference for flexibility if he always prefers any set of alternatives to its subsets, even when two of them contain the same best element. Desire for flexibility can be explained as the consequence of the agent's uncertainty along a two-stage process, where he must first preselect a subset of alternatives from which to make a final choice later on. We investigate conditions on the rankings of subsets that are compatible with the following assumptions: (1) the agent is endowed with a VN-M utility function on alternatives, (2) the agent attaches a subjective probability to the survival of each subset of alternatives, and (3) the agent will make a best choice out of any set which becomes available, and ranks sets ex-ante in terms of the expected utility of the best choices within them.

We first prove that any total ordering respecting set inclusion is rationalizable in these terms. This result is essentially the same obtained by Kreps (1979) under an alternative interpretation. We also show that we cannot learn anything about the underlying utilities of agents unless we impose further restrictions on the admissible distributions of survival probabilities. Then we investigate the additional consequences of assuming that the survival probabilities of individual alternatives are independently distributed. We prove that this reduces significantly the class of set rankings which can be rationalized and that then one can infer some of the characteristics of the agent's preferences. We offer a full characterization for the case of three alternatives. We also provide necessary conditions for rationalizability in the general case.

1 Introduction

The idea that a decision-maker might prefer a set of opportunities to any of its subsets has a lot of appeal, and it admits different foundations. Some authors justify this preference as the result of an intrinsic desire for freedom; the same decision taken out of a larger set would have a higher value to the decision-maker, which could not be explained through the consequences of the choice but because of its circumstances (see Sen (1988), Pattanaik and Xu (1990)). Another justification for a decision-maker to prefer any set to its subsets appeals to the higher flexibility associated with a larger choice set, in the presence of some form of uncertainty. Preference for flexibility has a long tradition (Hart (1940), Koopmans (1964)), and it did receive a deep and elegant axiomatic treatment in Kreps (1979). There, the decision-maker faces uncertainty about his own future preferences. In the present paper, we adopt a very similar framework but we place the decision-maker's uncertainty on the availabilty of alternatives for choice.[1]

The framework in Kreps is as follows. An agent takes decisions in two stages. In the first stage, he chooses a subset of alternatives; in the second stage, he chooses one alternative out of the subset he decided upon on the first. When making his choice of a set in the first stage, the agent is uncertain about the preference relation which will govern his choice of a final alternative in the second stage. Under this interpretation, it is natural to represent the agent's uncertainty on preferences by assuming that he is endowed with a distribution over Von Neumann-Morgenstern utility functions, that this uncertainty is resolved in the second period, and that he chooses his utility maximizing alternative according to his second period utility among those that are available after the first stage choice. Under this interpretation, sets of alternatives can be evaluated in terms of the expected utility of the second-stage choices they allow, and such calculations would lead, for any specific lottery over VN-M utilities, to a ranking of sets. But not any ranking of sets can be explained as the result of an expected utility calculation of this kind. Kreps provides necessary and sufficient conditions for a total preordering over sets to be representable in terms of the model above.

We consider the same time structure as Kreps, and we restrict attention, for simplicity, to strict orders on sets. We propose an alternative source of uncertainty for the decision-maker: he knows that his future preferences will be the same as the present ones, but he is uncertain about the availability of the alternatives at the second stage. The decision maker must still select a set in the first period, and an alternative from this set in the second. But not all of the alternatives retained in the first period may be available in the second, and then the agent is bound to choose from the set of those alternatives that (1) were not discarded at the first stage, and (2) are still available at the time of a final decision. To motivate this scenario, think of actions with an environmental impact. Some alternatives may become definitely impossible after a first action is undertaken, while others may still be open for future choice. Yet, additional events like a natural disaster can narrow down

[1]For a suvey of these and other approaches on how to rank sets of objects see Barberà et al.(forthcoming).

the set of options for future choice even further. A similar though less dramatic example is that of a choice of restaurant. Those meals which are never available in a given restaurant are definitely discarded by the time one makes a reservation there. But the final selection of dishes will be made upon arrival at the restaurant, and then some items on the menu may also be unavailable at that time.

Under our interpretation, it is natural to assume that an agent will be endowed with some VN-M utility function, and also with a probability distribution over subsets of alternatives, indicating the subjective probability that the agent attaches, for each subset, to the event that the alternatives in this subset do survive, and the rest of alternatives don't. Such an agent can then attach an expected utility value to each subset of alternatives, wiewed as his screening choice. He can compute the probability with which he will have to choose from each of the subsets of his initial choice, and then the expected utility of his best choices on these subsets. This will induce a ranking on sets of alternatives, that we term an expected opportunity ranking. Again, while any VN-M utility function and any distribution of the survival probabilities for sets will give rise to a ranking of sets, not all rankings will admit a rationalization of this kind. Our purpose is to explore the restrictions on rankings of sets which arise from different specifications of our basic model of two-stage choice with certain preferences and uncertain survival probabilities for alternatives.

We first prove (Section 2) that a necessary and sufficient condition for a ranking to be representable in terms of expected opportunity is that any set should be strictly better than all of its subsets (the inclusion property). This is the same condition than Kreps' in our restricted setting. Indeed, our first model can be trated as a reinterpretation of Kreps', but this reintrpretation is what allows us to ask further relevant questions for research. Specifically, our constructive and simple proof of this first result shows that, in fact, we cannot learn anything about the utilities underlying a given ranking of sets satisfying the inclusion property. This is because there is so much freedom in choosing the companion distribution of survival probabilities that, in fact, one can rationalize any such ranking while assuming that all alternatives are indifferent. A continuity argument then allows for nearby rationalizations with any arbitrary utility orderings of alternatives.

Because of the above, we turn to new questions which are specific to our interpretation and arise very naturally. Suppose that some further characteristics of the survival probabilities are known a priori and can be expressed as restrictions on the nature of their distribution. Say that a ranking over sets is rationalizable as an x-expected opportunity ranking if and only if there exist a survival probability distribution satisfying condition x, and a VN-M utility function generating the given ranking. One can then investigate, for any restriction x on the class of admissible distributions, what families of rankings of sets can still be rationalized. Moreover, one can then also go deeper on the features of the utility functions that are part of these rationalizations, and see whether something more can be learned about them in the unrestricted case.

In the rest of the paper we concentrate on the rationalizability of set rankings when the the survival probabilities of alternatives are independently distributed. By this assumption, we exclude the possibility that the decision-maker's screening of a given set of alternatives as his first period choice could have an influence on the probabilities that each of the retained alternatives remains available by the time of the final choice. Independence may be a natural requirement or not, depending on the context. Take the example of choice of restaurant, and consider the probability that fresh cod is available in restaurants that have it in the menu. Independence will be a natural way to express that the availability of fresh cod depends on the general conditions of the market, while it may be not be able to express the fact that restaurants with different menus can try more or less hard to make it available to their clients, even if they face the same market circumstances.

We find that the additional requirement of independence significantly restricts the set of rankings which can be rationalized. Moreover, expected opportunity rankings with independent survival probabilities must be rationalized by means of utility functions which are no longer arbitrary. In many cases, we can now infer some characteristics of the decision-maker's preferences on alternatives from his ranking of sets. We can offer a full characterization of the rankings which are rationalizable with indepndent probabilities for the three alternative case (Section 5). The analysis of these particular cases should give the reader a feeling for the general type of restrictions that independent rationalizability does impose on set rankings, and a measure of the difficulties that lie on the way of a general characterization that would apply to any number of alternatives.

2 The Model. A Representation Result

Let A be a finite set and $\succ$ a total order on 2^A. Recall that $\succ$ is a total order iff it is complete, transitive, and antisymmetric. Let $r(\succ, B) = \#\{C \in 2^A \mid C \succ B\}$ be the rank of B according to $\succ$. Clearly, $r(\succ, B) = 0$ iff B is $\succ$-maximal. [2]

Let $\mathfrak{L}$ be the set of lotteries on 2^A, i.e. $\ell \in \mathfrak{L}$ if $\ell : 2^A \to \mathbb{R}_+$ and $\sum_{E \in 2^A} \ell(E) = 1$. For each $\ell \in \mathfrak{L}$ and $C \in 2^A$ define the function $\ell_C : 2^A \to \mathbb{R}$ by $\ell_C(D) = \sum_{\{E \in 2^A \mid E \cap C = D\}} \ell(E)$. The interpretation of $\ell_C(D)$ is as follows: assume that the probabilities of the elements in 2^A is according to ℓ and that the agent in the first stage has chosen the screening set C. Then the probability for the event that the agent has to choose an element in the set D is given by $\ell_C(D)$. Using the definition of $\ell_C(\cdot)$ we obtain:

Remark 1. *Let $\ell \in \mathfrak{L}$ and $C \in 2^A$. Then $\ell_C : 2^A \to \mathbb{R}$ satisfies:*

1. $\ell_C(\cdot) \in \mathfrak{L}$;
2. $\ell_C(D) = 0$ *if* $D \nsubseteq C$;
3. $\ell_C(\emptyset) = \sum_{D \in 2^A \setminus C} \ell(D)$;

[2]We use the following notation: Let $E, F \in 2^A$. Then $E \subset F$ if E is strictly contained in F, and $E \subseteq F$, if E is a subset of F. Moreover we shall use the convention that $\sum_{\emptyset}(\cdot) = 0$

Remark 2. *Let $\ell \in \mathfrak{L}$. Consider $B, C \in 2^A$ such that $B \subset C$ and the corresponding lotteries $\ell_B(\cdot)$ and $\ell_C(\cdot)$. Then for all $D \in 2^A$*

$$\ell_B(D) = \sum_{\{D' | D' \cap B = D\}} \ell_C(D').$$

Proof. By the definition $\ell_B(D) = \sum_{\{E|E \cap B = D\}} \ell(E)$. However, for each $E \in 2^A$ such that $E \cap B = D$ there exists one and only one D' such that $E \cap C = D'$ and $D' \cap B = D$. Hence, $\sum_{\{E|E \cap B = D\}} \ell(E) = \sum_{\{D'|D' \cap B = D\}} \sum_{\{E|E \cap C = D'\}} \ell(E) = \sum_{\{D'|D' \cap B = D\}} \ell_C(D')$, and the equality obtains. □

Definition 1. *A utility function on $A \cup \{\emptyset\}$ is a function $u : A \cup \{\emptyset\} \to \mathbb{R}$ such that $u(\{\emptyset\}) = 0$.*
Given a utility function u and a lottery $\ell \in \mathfrak{L}$, we define the expected opportunity function $V : 2^A \to \mathbb{R}$ by $V(C) = \sum_{D \in 2^A} \max_{x \in D \cup \{\emptyset\}} u(x) \ell_C(D)$.

The function $V(C)$ computes the expected utility for the agent if he screens the set C. Thus we assume that if D is the set of surviving alternatives in C he will choose the best alternative in D. Yet, we keep the alternative open, that should all the alternatives in D be worse than $\{\emptyset\}$, then the no choice opportunity $x = \{\emptyset\}$ is available for him. Hence, $\max_{x \in D \cup \{\emptyset\}} u(x)$ is the utility he obtains if the alternatives in $E \in 2^A$ survive and $E \cap C = D$.

Definition 2. *The order $\succ$ on 2^A is an expected opportunity ranking iff there exist a lottery $\ell \in \mathfrak{L}$ and a utility function $u : A \cup \{\emptyset\} \to \mathbb{R}$ such that the corresponding expected opportunity function $V : 2^A \to \mathbb{R}$ satisfies*

$$B \succ C \iff V(B) > V(C).$$

Remark 3. *Assume the total ordering $\succ$ on 2^A is an expected opportunity ranking with lottery $\ell \in \mathfrak{L}$ and utility function u. Then $V(C) \geq 0$ for all $C \in 2^A$. Moreover, for all $x \in A$, $u(x) > 0$ and there exists $E \in 2^A$ such that $x \in E$ and $\ell(E) > 0$.*

Proof. Clearly $V(C) \geq 0$ since $u(\{\emptyset\}) = 0$. Moreover $V\{(\emptyset\}) = 0$. Hence, since $\succ$ is a total ordering, $V(E) > 0$ if $E \neq \emptyset$. Consequently, $V(\{x\}) = \sum_{D \in 2^A} max_{y \in D \cup \{\emptyset\}} u(y) \ell_{\{x\}}(D) = \max(0, u(x)) \sum_{x \in E} \ell(E) > 0$ for all x, and the conclusion obtains. □

We now state the condition that characterizes total orderings on 2^A which are expected opportunity rankings.

Definition 3. *An order $\succ$ on 2^A satisfies the inclusion property iff for all $B, C \in 2^A$:*

$$C \supset B \;\Rightarrow\; C \succ B.$$

Theorem 1. *Let A be a finite set and let $\succ$ be a total ordering on 2^A. Then the ordering $\succ$ is an expected opportunity ranking if and only if $\succ$ satisfies the inclusion property.*

Proof. Assume that $\succ$ is an expected opportunity ranking and assume that $B \subset C$. Then

$$V(C) = \sum_{D' \in 2^A} max_{x \in D' \cup \{\emptyset\}} u(x) \ell_C(D') = \sum_{D \subseteq B} \sum_{\{D' | D' \cap B = D\}} max_{x \in D' \cup \{\emptyset\}} u(x) \ell_C(D') \geq$$

$$\sum_{D \subseteq B} \sum_{\{D' | D' \cap B = D\}} max_{x \in D \cup \{\emptyset\}} u(x) \ell_C(D') = \sum_{D \subseteq B} max_{x \in D \cup \{\emptyset\}} u(x) \sum_{\{D' | D' \cap B = D\}} \ell_C(D') =$$

$$\sum_{D \subseteq B} max_{x \in D \cup \{\emptyset\}} u(x) \ell_B(D) = V(B)$$

where the next to last equality follows from Remark 2. Since we know that $V(C) \neq V(B)$ we obtain that $V(C) > V(B)$, and hence $C \succ B$.

We shall now show that a total ordering which satisfies the inclusion property is an expected opportunity ranking corresponding to some $\ell \in \mathfrak{L}$ and the utility function u given by $u(x) = 1$ for all $x \in A$ and $u(\{\emptyset\}) = 0$. Notice, that the expected opportunity function V corresponding to u and ℓ is given by is given by $V(E) = 1 \cdot (1 - \ell_E(\emptyset)) + 0 \cdot \ell_E(\emptyset)$. Hence V represents $\succ$ iff ℓ satisfies $E \succ F \Leftrightarrow \ell_E(\emptyset) < \ell_F(\emptyset)$. Thus, we have to construct ℓ such that $E \succ F \Leftrightarrow \sum_{\{D \subseteq A \setminus E\}} \ell(D) < \sum_{\{D \subseteq A \setminus F\}} \ell(D)$.

Corresponding to $\succ$ define the complement $\succ^*$ by

$$E \succ^* F \text{ iff } A \setminus E \succ A \setminus F.$$

We now define two functions $s, t : 2^A \to \mathbb{R}$ recursively after the rank of the sets according to $\succ^*$. Since $\succ$ satisfies the inclusion property $r(\succ^*, \emptyset)$=0. We define $s(\emptyset) = 0$ and $t(\emptyset) = 0$. Now let $E \in 2^A$ be the set with $r(\succ^*, E) = 1$. Since $\succ$ satisfies the inclusion property, there exists $x \in A$ such that $E = \{x\}$. Let $s(E) = r(\succ^*, E) = 1$ and let $t(E) = s(E) - \sum_{\{F | E \succ^* F\}} s(F) = 1 - 0 = 1$. Now assume that s and t have been defined for all $B \in 2^A$ with $r(\succ^*, B) \leq k$, where $k \geq 1$. Let D, E be such that $r(\succ^*, D) = k$ and $r(\succ^*, E) = k + 1$. Notice that s, t are already defined for all sets B such that $B \subset E$, since $\succ$ satisfies the inclusion property . We now define

$$s(E) = max(s(D) + 1, \sum_{B \subset E} t(B)) \text{ and } t(E) = s(E) - \sum_{B \subset E} t(B).$$

Since 2^A is finite the functions s, t are now defined.

Clearly, the construction of s gives

$$s(B) > s(C) \Leftrightarrow C \succ^* B \Leftrightarrow A \setminus C \succ A \setminus B \text{ and hence } C \succ B \Leftrightarrow s(A \setminus B) > s(A \setminus C)$$

Moreover the construction of s, t yields that for every $B \in 2^A$:

Table 1. An example of the construction in the proof of Theorem 1

$\succ$	$\succ^*$	$s(\cdot)$	$t(\cdot)$	expected utility (with utility kt=1)
$xyzw$	$\emptyset$	$s(\emptyset) = 0$	$t(\emptyset) = 0$	$1 = V(xyzw)$
xyz	w	$s(w) = 1$	$t(w) = 1$	$28/29 = V(xyz)$
xyw	z	$s(z) = 2$	$t(z) = 2$	$27/29 = V(xyw)$
xzw	y	$s(y) = 3$	$t(y) = 3$	$26/29 = V(xzw)$
xy	zw	$s(zw) = 4$	$t(zw) = 1$	$25/29 = V(xy)$
xw	yz	$s(yz) = 5$	$t(yz) = 0$	$24/29 = V(xw)$
yzw	x	$s(x) = 6$	$t(x) = 6$	$23/29 = V(yzw)$
xz	yw	$s(yw) = 7$	$t(yw) = 3$	$22/29 = V(xz)$
x	yzw	$s(yzw) = 10$	$t(yzw) = 0$	$19/29 = V(x)$
yw	xz	$s(xz) = 11$	$t(xz) = 3$	$18/29 = V(yw)$
yz	xw	$s(xw) = 12$	$t(xw) = 5$	$17/29 = V(yz)$
zw	xy	$s(xy) = 13$	$t(xy) = 4$	$16/29 = V(zw)$
w	xyz	$s(xyz) = 18$	$t(xyz) = 0$	$11/29 = V(w)$
y	xzw	$s(xzw) = 19$	$t(xzw) = 1$	$10/29 = V(y)$
z	xyw	$s(xyw) = 22$	$t(xyw) = 0$	$7/29 = V(z)$
$\emptyset$	$xyzw$	$s(xyzw) = 29$	$t(xyzw) = 1$	$0 = V(\emptyset)$
			$29 = T$	

$$s(B) = \sum_{C \subseteq B} t(C),$$

Now define the lottery $\ell : 2^A \to \mathbb{R}$ by $\ell(B) = \frac{t(B)}{T}$ for all $B \in 2^A$ where $T = \sum_{B \in 2^A} t(B)$. We shall show that ℓ has the wanted property i.e. $B \succ C \Leftrightarrow \ell_C(\emptyset) > \ell_B(\emptyset)$. By Remark 1 and the definition of ℓ we get

$$\ell_C(\emptyset) = \sum_{D \in 2^{A \setminus C}} \ell(D) = \sum_{D \in 2^{A \setminus C}} \frac{t(D)}{T}$$

Thus, since $s(B) = \sum_{C \subseteq B} t(B)$ we obtain

$$\ell_C(\emptyset) = \frac{s(A \setminus C)}{T}.$$

Since the construction of s yields $B \succ C \Leftrightarrow s(A \setminus C) > s(A \setminus B)$ the conclusion $B \succ C \Leftrightarrow \ell_C(\emptyset) > \ell_B(\emptyset)$ obtains.

$\square$

Table 1 provides an example of how is construct a function $V(\cdot)$ representing a given order. Clearly, the utility function u and the lottery ℓ in Theorem 1 are not uniquely defined. In fact, our proof shows that we cannot infer anything about the agent's preferences on alternatives from knowledge of its ranking of sets. To see this, consider any complete preordering R on the elements in A. We can now find a utility function $\tilde{u} : A \cup \{\emptyset\} \to \mathbb{R}$ and a lottery ℓ such that the corresponding opportunity function represents $\succ$ and moreover $\tilde{u}'s$ restriction to A represents R. Indeed, let u and ℓ be as in the proof of Theorem 1 and let $v : A \to \mathbb{R}$ be any representation of R. Clearly, for $\epsilon > 0$ the function $\tilde{u} : A \cup \{\emptyset\} \to \mathbb{R}$ defined by $\tilde{u}(x) = u(x) + \epsilon v(x)$ for $x \in A$ and $\tilde{u}(\{\emptyset\}$=0 shall also represents R, as u is constant. However, as $\succ$ is antisymmetric then for sufficiently small ϵ, $\epsilon > 0$, $\succ$ can also be represented by the opportunity function corresponding to $\tilde{u}$ and ℓ.

3 Uncertainty about Preferences

In Kreps (1979) the desire for flexibility is explained by uncertainty about the preference relation in the second stage. We shall now show that a representation of the total order $\succ$ with the expected opportunity function V can be interpreted within the framework of Kreps (1979) Assume that $V(B) = \sum_{D \subset B} max_{x \in D \cup \{\emptyset\}} u(x)\ell_B(D)$ represents $\succ$. We know from the definition of a utility function that $u(\{\emptyset\}) = 0$ and from Remark 2 that $u(x) > 0$ for all $x \in A$. Hence,

$$V(B) = \sum_{D \in 2^A} max_{x \in D \cup \{\emptyset\}} u(x)\ell_B(D) = \sum_{\{D | D \neq \emptyset,\, D \subseteq B\}} max_{x \in D} u(x)\ell_B(E) =$$

$$\sum_{\{E | B \cap E \neq \emptyset\}} max_{x \in B \cap E} u(x)\ell(E)$$

We shall now reinterpret the expected opportunity function V in terms of uncertainty about the preferences in the second stage. Let the set of states in the second stage be $S = 2^A$ and let ℓ be the probability measure on S. Define the state contingent utility function $\tilde{u} : A \times 2^A \to \mathbb{R}$ by

$$\tilde{u}(x, E) = \begin{cases} u(x) & \text{for } x \in E \\ 0 & \text{for } x \notin E \end{cases} \tag{1}$$

Now, assume that the agent in each state E maximizes $\tilde{u}(\cdot, E)$. The expected utility ranking of the subsets will then be the function $K(\cdot) : 2^A \to \mathbb{R}$, where

$$K(B) = \sum_{E} max_{x \in B} \tilde{u}(x, E)\ell(E) = \sum_{\{E | B \cap E \neq \emptyset\}} max_{x \in B \cap E} \tilde{u}(x, E)\ell(E) =$$

$$\sum_{\{E | B \cap E \neq \emptyset\}} max_{x \in B \cap E} u(x)\ell(E) =$$

$$\sum_{D \neq \emptyset \text{ and } D \subseteq B} \sum_{\{E | B \cap E = D\}} max_{x \in D} u(x) \ell(E) = V(B)$$

$$\sum_{D \subseteq B} max_{x \in D \cup \{\emptyset\}} u(x) \ell_B(D) = V(B).$$

Hence the expected opportunity function V can be reinterpreted as a representation in terms of uncertainty of second stage preferences with the state space $S = 2^A$ and the state contingent utility function $\tilde{u}$.

An advantage of explaining desire for flexibility by opportunities of choice is that it is not based on an arbitrary construction of the state space. Moreover as we shall see in the next section, an interpretation of flexibility based on expected opportunity gives rise to additional interesting questions.

4 Independent survival Probabilities

In Theorem 1 the lottery $\ell(\cdot)$ was an arbitrary lottery on 2^A. It is natural to ask whether it is also possible to get a representation theorem where $\ell(B)$ can be calculated by using independent survival probabilities for each alternative in A.

Definition 4. *Let $\hat{\ell} : A \to [0, 1]$ be such that $\ell(B) = \prod_{b \in B} \hat{\ell}(b) \prod_{c \notin B}(1 - \hat{\ell}(c))$ for all $B \in 2^A$. We say that $\succ$ is an expected opportunity ranking with independent survival probabilities if there exists a utility function u and independent survival probabilities $\hat{\ell}$ such that the corresponding opportunity function represents $\succ$.*

The following definition of reversals within a ranking $\succ$ is crucial in what follows: Given two sets E, F and a third set B with no elements in common with any of in the first two, we say that B reverses E and F in $\succ$ if the ranking of E and F is opposite to that of $E \cup B$ and $F \cup B$.

Reversals, whose formal definition follows, will be warning signals that some rankings of sets are not representable, and will help to impose conditions on the utilities and the probabilities to be used in representations, when one exists.

Definition 5. *An order $\succ$ is said to have no reversals if for all $E, F, B \in 2^A$ with $B \cap (E \cup F) = \emptyset$: $E \succ F$ implies $E \cup B \succ F \cup B$.*
Let $E, F \in 2^A$. A set $B \in 2^A$ with $B \cap (E \cup F) = \emptyset$ reverses $E \succ F$ if $F \cup B \succ E \cup B$.

Lemma 1 expresses the first of a series of restrictions on rankings that are imposed by rationalizability with independent probabilities. Specifically, it states a set B which reverses the sets E and F can not consist of elements which all have larger utility than the elements in E and F.

Lemma 1. *Assume that the total ordering $\succ$ on 2^A is an expected opportunity ranking which can be represented by $\hat{\ell}$ and u. Consider any sets $E, F, B \in 2^A$. Assume that $E \succ F$ and that $u(b) > u(x)$ for all $b \in B$ and all $x \in E \cup F$. Then $E \cup B \succ F \cup B$.*

Proof. Since $E \succ F$ and $u(b) > u(x)$ for all $b \in B$ and all $x \in E \cup F$, $V(E \cup B) = V(B) + \prod_{b \in B}(1-\hat{\ell}(b))V(E) > V(B)+\prod_{b \in B}(1-\hat{\ell}(b))V(F) = V(B \cup F)$. Hence $E \cup B \succ F \cup B$. $\square$

Corollary 1. *If $\succ$ is representable as an expected opportunity ranking with independent probabilities and $x, y, z \in X$ are such that $u(x) > u(y)$ and $u(x) > u(z)$, then $\{x\}$ cannot reverse the order of $\{y\}$ and $\{z\}$ in $\succ$.*

Before we proceed investigating further requirements imposed by this type of representability, let us elaborate on the reasons why reversals can actually occur, by examining the expression that determines the expected opportunity value.

To begin with we characterize the orderings that do not present reversals.

Consider an alternative $x \in A$ and the corresponding set $\{x\}$. Adding an alternative z to $\{x\}$ has two effects, a utility gain and an insurance gain. Indeed, we have that

$$V(\{x, z\}) - V(\{x\}) = max\{(u(z) - u(x))\hat{\ell}(x)\hat{\ell}(z), 0\} + (1 - \hat{\ell}(x))V(\{z\}).$$

The first term is the utility gain, which is obtained when both alternatives x, z survive. Clearly the utility gain is 0 iff $u(x) > u(z)$ as $\ell(x) > 0$ and $\ell(z) > 0$ by Remark 3. The other term is the pure insurance gain which is obtained in the event where z survives but not x. This is positive iff $\hat{\ell}(x) < 1$ since $V(\{z\}) > 0$. Next consider two alternatives x and y and add an alternative z such that $u(x) > u(z)$ and $u(y) > u(z)$ to both $\{x\}$ and $\{y\}$. Then in both cases there is only an insurance gain. If z reverses, the insurance gain from adding z to $\{y\}$ must be higher than the insurance gain from adding z to $\{x\}$. Therefore y is a more risky alternative than x i.e. $\hat{\ell}(x) > \hat{\ell}(y)$.

Remark 4. *Let A be a finite set and let $\succ$ be a total ordering on 2^A. Then $\succ$ is an expected opportunity ranking with constant utility function u and independent survival probabilities iff $\succ$ satisfies the inclusion property and there are no reversals.*

To see the force of Corollary 1 alone, consider the following proof that some rankings of sets satisfying the inclusion property cannot be represented as expected opportunity rankings (a more complete statement of what rankings are representable will come later, but using additional lemmas that are not needed here).

Remark 5. *The ranking*

$$\{x, y, z\} \succ \{y, z\} \succ \{x, z\} \succ \{x, y\} \succ \{x\} \succ \{y\} \succ \{z\} \succ \phi.$$

cannot be represented as an expected opportunity ranking with independent probabilities. Clearly, the ranking satisfies the inclusion property. Notice that $\{x\}$ reverses $\{z\}$ and $\{y\}$, $\{y\}$ reverses $\{x\}$ and $\{z\}$, and $\{z\}$ reverses $\{x\}$ and $\{y\}$. Therefore, neither $\{x\}$, nor $\{y\}$, nor $\{z\}$ can have the higher utility among these three alternatives. Hence, no utitity function assigning different values to x,y and z can be chosen as part of a representation of $\succ$. Since our ranking presents reversals then Remark 4 completes the proof that no representation at all can be found for this ranking with independent survival probabilities.

We now retun to our study of restrictions imposed by representability with independent probabilities.

Lemma 2. *Let the total ordering $\succ$ on 2^A be an expected opportunity ranking represented by $\hat{\ell}$ and u. Assume that $\{x\} \succ \{y\}$ and that $\{z\}$ reverses $\{x\} \succ \{y\}$. Then $u(y) > u(x)$ and $u(y) > u(z)$. Moreover, $\hat{\ell}(x) > \hat{\ell}(y)$.*

Proof. We have to prove that neither x nor z can be the highest in utility. By Lemma 1, z can not be highest in utility. Thus, assume that x is the highest in utility and hence that either $u(x) > u(z) > u(y)$ or $u(x) > u(y) > u(z)$. We show that both cases lead to a contradiction.
First assume that $u(x) > u(z) > u(y)$. Then $\{z\}$ can not create a reversal to $\{x\} \succ \{y\}$ as

$$V(\{xz\}) = V(\{x\}) + (1 - \hat{\ell}(x))V(\{z\})) =$$

$$V(\{x\}) + V(\{z\}) - \hat{\ell}(x)\hat{\ell}(z)u(z) \geq V(\{x\}) + V(\{z\}) - \hat{\ell}(x)\hat{\ell}(z)u(x) =$$

$$V(\{z\}) + (1 - \hat{\ell}(z))V(\{x\}) \geq V(\{z\}) + (1 - \hat{\ell}(z))V(\{y\}) = V(\{yz\}).$$

Next assume that $u(x) > u(y) > u(z)$. First notice that if z creates a reversal to $\{x\} \succ \{y\}$ then as z is lowest in utility we have $\hat{\ell}(y) < \hat{\ell}(x)$. Indeed, $V(\{y\}) + (1 - \hat{\ell}(y))V(\{z\}) = V(\{zy\}) > V(\{xz\}) = V(\{x\}) + (1 - \hat{\ell}(x))V(\{z\})$ and $V(\{x\}) > V(\{y\})$ imply $\hat{\ell}(y) < \hat{\ell}(x)$. We now show that if $\hat{\ell}(y) < \hat{\ell}(x)$ and also $u(x) > u(y) > u(z)$ then z does not create a reversal and hence we again have a contradiction. Indeed, we have $V(\{xz\}) = V(\{x\}) + (1 - \hat{\ell}(x))V(\{z\}) = \hat{\ell}(x)u(x) + (1 - \hat{\ell}(x))V(\{z\}) > \hat{\ell}(x)u(y) + (1 - \hat{\ell}(x))V(\{z\}) > \hat{\ell}(y)u(y) + (1 - \hat{\ell}(y))V(\{z\}) = V(\{yz\})$; where the last inequality follows from the facts that $u(y) > u(z) \geq V(\{z\})$ and $\hat{\ell}(y) < \hat{\ell}(x)$. Thus z does not create a reversal to $\{x\} \succ \{y\}$. We conclude that y is highest in utility and the first part of the lemma obtains.

The second part follows trivially, as $\{x\} \succ \{y\}$ and $u(y) > u(x)$. □

Lemma 1 and Lemma 2 imply

Corollary 2. *Assume the total ordering $\succ$ on 2^A is an expected opportunity ranking with independent survival probabilities. Then for all $x, y, z \in A$, if $\{y\}$ reverses $\{x\} \succ \{z\}$ then $\{z\}$ does not reverse $\{x\} \succ \{y\}$ and does not reverse $\{y\} \succ \{x\}$.*

Proof. Assume that $\succ$ can be represented by u and $\hat{\ell}$ and that $\{y\}$ reverses $\{x\} \succ \{z\}$. Then by Lemma 2, $u(z) > u(x)$ and $u(z) > u(y)$ and thus Lemma 1 gives the conclusion. □

Lemma 3. *Assume that $\succ$ is an expected opportunity ranking with utility function u and independent survival probabilities $\hat{\ell}$. Let $x, y \in A$ satisfy $u(z) > u(y) > u(x)$ for all $z \in A \setminus \{x, y\}$. Then*

(i) the expected opportunity function V corresponding to u and $\hat{\ell}$ only depend on $u(x)$ and $\hat{\ell}(x)$ through $u(x)\hat{\ell}(x)$,

(ii) *there exists a representatation of $\succ$ as an expected opportunity ranking with utility function $\tilde{u}$ and independent lotteries $\tilde{\ell}$ such that $\tilde{u}(z) > \tilde{u}(x) = \tilde{u}(y)$ for all $z \in A \setminus \{x, y\}$,*

(iii) *there exists a representatation of $\succ$ as an expected opportunity ranking with utility function $\tilde{u}$ and independent lotteries $\tilde{\ell}$ such that $\tilde{u}(z) > \tilde{u}(x) > \tilde{u}(y)$ for all $z \in A \setminus \{x, y\}$ and $\tilde{\ell}(y) = 1$.*

Proof. Let V be the representation using u and $\hat{\ell}$. For all $E \in 2^A$ we have $V(E) = V(E \setminus \{x\}) + (\prod_{\{z \in E \setminus \{x\}\}}(1 - \hat{\ell}(z)))V(\{x\})$. This proves (i).

(ii) follows from (i) as we simply adjust the probality and the utility of x such that $\tilde{\ell}(x)\tilde{u}(x) = V(x)$ and $\tilde{u}(x) = u(y)$ and keep the utility and the probability of all other elements as specified with u and $\hat{\ell}$.

To prove (iii) we first use the representation from (ii) and make a sufficiently small increase of the utility of x keeping the probability of x, and the utility and probability of all alternatives $z \neq x$. Secondly, we use (i) and put the survival probability of $\tilde{\ell}(y) = 1$ and $\tilde{u}(y) = V(y)$. This only can lower the utility of y and (iii) obtains. □

Lemma 3 shows that if we have a representation of $\succ$ as an expected opportunity ranking with independent utilities and we know which two elements have the lowest utility, then we can find representations where any of these two elements has the lowest utility.

5 The case n=3

In this section we provide a full characterization of total orders which are representable as expected opportunity rankings with independent survival probabilities when there are only three alternatives. To do that, we show that the necessary condition in Corollary 1 becomes also sufficient in that particular case.

Theorem 2. *Assume that $\#A = 3$. Then the total ordering $\succ$ on 2^A can be represented as an expected opportunity ranking with independent survival probabilities iff $\succ$ satisfies*

1. *the inclusion property, and*

2. *for any labeling x, y, z of the three alternatives in A if $\{y\}$ reverses $\{x\} \succ \{z\}$ then $\{z\}$ does not reverse $\{x\} \succ \{y\}$ and does not reverse $\{y\} \succ \{x\}$.*

The theorem states that in the case with $\#A = 3$ the orderings which can not be represented as expected opportunity ranking with independent survival probabilities are exactly the ones where the two worst singletons in the ordering $\succ$ create reversals. Indeed, let $A = \{x, y, z\}$. From now on, and without loss of generality, we assume that $\{x\} \succ \{y\} \succ \{z\}$. Also we shall for short use the terminology that e.g. $\{y\}$ reverses if $\{z, y\} \succ \{x, y\}$. With this terminology the condition in Theorem 2 states that the orderings which can not be represented with independent survival probabilities are the ones where $\{z\}$ and $\{y\}$ reverse or where $\{x\}$ and $\{z\}$ reverse. However, notice that if x and z reverse, that is $\{xz\} \succ \{xy\}$ and $\{yz\} \succ \{xz\}$, then by transitivity of $\succ$ also $\{yz\} \succ \{xy\}$ and thus also y reverses. Hence the orderings which can not be represented with independent probabilities are the two orderings where $\{y\}$ and $\{z\}$ both reverse.

Proof. We already saw in Theorem 1 and Corollary 2 that the conditions are necessary. We now prove that they are sufficient. Clearly, we can always multiply all utilities with the same positive constant and hence we can use the normalization that $V(\{x\}) = 1$ in our representations. First, if $\succ$ has no reversals, Remark 4 implies that $\succ$ can be represented using a constant utility function.

Now assume there is at least one reversal, for example $\{x\}$ reverses $\{y\} \succ \{z\}$. The principles we shall use in the construction of the expected opportunity function are as follows. From Lemma 2 we know that in any representation z has the highest utility of the three alternatives x, y, z. Moreover, from Lemma 3 we know that if there is a representation then there is one where x has the lowest utility. Hence, we can restrict attention to utility functions u, such that $u(z) > u(y) > u(x)$. Lemma 3 also allows us to choose $\hat{\ell}(x) = 1$. Hence, we look at z as the risky alternative (with high utility and small survival probability) and at x, the alternative which creates the reversal, as the safe alternative. We shall in the construction let the survival probability of z be very small. Indeed, we first construct a representation $\tilde{V} : 2^A \to \mathbb{R}$ of $\succ$ such that $\tilde{V}(E) = \tilde{V}(E \setminus \{z\}) + \tilde{V}(\{z\})$ for all $E \in 2^A$ and where the restriction of $\tilde{V}$ to $\{E \in 2^A | z \notin E\}$ is an expected opportunity ranking with independent survival probabilities. Clearly, if such $\tilde{V}$ has been obtained we can find $u(z)$ sufficiently large and $\hat{\ell}(z)$ sufficiently small with $u(z)\hat{\ell}(z) = \tilde{V}(z)$ such that the function $V : 2^A \to \mathbb{R}$ where $V(E) = \tilde{V}(E)$ if $z \notin E$ and $V(E) = \tilde{V}(\{z\}) + (1 - \ell(z))\tilde{V}(E \setminus \{z\})$ if $z \in E$, also represents $\succ$. Obviously, V is an expected opportunity function with independent survival probabilities.

We now show that all rankings satisfying (1) and (2) and where there are reversals can be represented by a function $\tilde{V}$ as described above. First assume that only $\{x\}$ reverses. Together with the inclusion property, this identifies the two rankings:

$$\{x, y, z\} \succ \{x, z\} \succ \{x, y\} \succ \{y, z\} \succ \{x\} \succ \{y\} \succ \{z\} \succ \emptyset$$

and

$$\{x, y, z\} \succ \{x, z\} \succ \{x, y\} \succ \{x\} \succ \{y, z\} \succ \{y\} \succ \{z\} \succ \emptyset.$$

We know that z is highest in utility and we assume that x is lowest. Let $\tilde{V}(\{x\}) = 1$ and choose $\tilde{V}(\{y\})$ such that $\tilde{V}(\{x\}) > \tilde{V}(\{y\}) > \tilde{V}(\{x\})/2$. Now determine $\bar{V}(\{z\})$ by $\bar{V}(\{z\}) + \tilde{V}(\{y\}) = \tilde{V}(\{x\})$. Notice that $\tilde{V}(\{y\}) > \bar{V}(\{z\})$. Moreover, choose $\hat{\ell}(y)$ as the uniquely defined $\hat{\ell}(y) \in]0, 1[$ such that $\tilde{V}(\{x\}) + \bar{V}(\{z\}) = \tilde{V}(\{y\}) + (1 - \hat{\ell}(y))\tilde{V}(\{x\})$.

Now consider the first of the two rankings and let $\tilde{V}(\{z\}) = \bar{V}(\{z\}) + \epsilon$, where $\epsilon > 0$. When ϵ is sufficiently small we obtain

$$\begin{aligned} \tilde{V}(\{x\}) + \tilde{V}(\{z\}) &> \tilde{V}(\{y\}) + (1 - \hat{\ell}(\{y\}))\tilde{V}(\{x\}) > \\ \tilde{V}(\{y\}) + \tilde{V}(\{z\}) &> \tilde{V}(\{x\}) > \tilde{V}(\{y\}) > \tilde{V}(\{z\}). \end{aligned}$$

Now let $u(y) = \frac{\tilde{V}(\{y\})}{\hat{\ell}(y)}$, $\hat{\ell}(x) = 1$, and $u(x) = \tilde{V}(\{x\})$. As $0 < \tilde{V}(\{z\}) = \tilde{V}(\{y\}) - \hat{\ell}(y)\tilde{V}(\{x\})$ by construction of $\hat{\ell}(y)$, we clearly have that $u(y) > u(x)$. Hence we have constructed a representation $\tilde{V}$ with the wanted properties.

The second ranking can be representated exactly in the same way by defining $\tilde{V}(\{z\}) = \bar{V}(\{z\}) - \epsilon$.

Now we notice that since $\succ$ is transitive there is no ranking where only $\{y\}$ reverses. There is however one ranking where only $\{z\}$ reverses, namely the ranking:

$$\{x, y, z\} \succ \{x, y\} \succ \{y, z\} \succ \{x, z\} \succ \{x\} \succ \{y\} \succ \{z\} \succ \emptyset$$

We know that y is highest in utility and assume z is lowest. Let $\tilde{V}(\{x\}) = 1$ and let $\bar{V}(\{y\})$ be such that $\bar{V}(\{x\}) > \tilde{V}(\{y\}) > \tilde{V}(\{x\})/2$. Now choose $\tilde{V}(\{z\})$ such that $\bar{V}(\{y\}) + \tilde{V}(\{z\}) > \tilde{V}(\{x\})$ and $\bar{V}(\{y\}) > \tilde{V}(\{z\})$. Define the survival probility of the middle alternative x by the uniquely defined $\hat{\ell}(x) \in]0, 1[$ such that $\bar{V}(\{y\}) + \tilde{V}(\{z\}) = \tilde{V}(\{x\}) + (1 - \hat{\ell}(x))\tilde{V}(\{z\})$. The function $\tilde{V}$ now obtains by defining $\tilde{V}(\{y\}) = \bar{V}(\{y\}) + \epsilon$, where $\epsilon > 0$ is sufficiently small and by letting $u(x) = \frac{\tilde{V}(x)}{\hat{\ell}(x)}$, $\ell(z) = 1$, and $u(z) = \tilde{V}(z)$ as before.

Now we consider the rankings with two or three reversals. Because of part two in the theorem we know that if $\{z\}$ reverses then $\{y\}$ does not reverse, if $\{y\}$ reverses then $\{z\}$ does not reverse, and if $\{x\}$ reverses then $\{z\}$ does not reverse. Therefore the only ranking which is left is the ranking where $\{x\}$ reverses $\{y\} \succ \{z\}$ and $\{x\}$ reverses $\{y\} \succ \{z\}$ i.e. the ranking:

$$\{x, y, z\} \succ \{x, z\} \succ \{y, z\} \succ \{x, y\} \succ \{x\} \succ \{y\} \succ \{z\} \succ \emptyset$$

We know z that is highest in utility and by arbitrarily focusing on the reversal by $\{x\}$ we assume that x is lowest. Let $\tilde{V}(\{x\}) = 1$. Choose $\tilde{V}(\{y\})$ such that $\tilde{V}(\{x\}) > \tilde{V}(\{y\}) > \tilde{V}(\{x\})/2$, and $\bar{V}(\{z\})$ such that $\bar{V}(\{z\}) + \tilde{V}(\{y\}) > \tilde{V}(\{x\})$ and $\tilde{V}(\{y\}) > \tilde{V}(\{z\})$. Moreover, let as before the probability of the middle alternative y be the uniquely defined $\hat{\ell}(y) \in]0, 1[$ such that $\tilde{V}(\{y\}) + \bar{V}(\{z\}) = \tilde{V}(\{y\}) + (1 - \hat{\ell}(y))\tilde{V}(\{x\})$. Now again define $\tilde{V}(\{z\}) = \bar{V}(\{z\}) + \epsilon$ for sufficiently small $\epsilon > 0$ and we have the wanted function $\tilde{V}$. The representation of this ranking completes the proof. $\square$

6 Conclusions

Preferences for flexibility are plausible in many decision contexts, while not necessarily in others. We have provided what we think is an attractive justification of it in terms of uncertainty over the future availability of certain alternatives which are open today.

We have investigated whether this interpretation allows us to establish a link between set rankings (which can be interpreted as been revealed through choices) and the underlying (and presumably unobservable) preferences and probabilities attached to single alternatives. This formulation does not impose much of a restriction on the rankings of sets (only inclusion condition), and does not imply any particular connection between the utility of sets and direct utility of single alternatives.

We have pursued the analysis of further questions suggested by our model for the case where one has information on the shapes of utilities or on the probability distributions determining what alternatives might be available in the future. We have illustrated the potential of such additional analysis by proving that if we know that the survival probabilities of different alternatives are independent, then knowledge of the ranking of sets becomes informative about the underlying utilities and survival probabilities of single alternatives.

References

Barberà S., Bossert W., Pattanaik P.K.: Ranking Sets of Objects, in Barberà, Hammond and Seidl, Handbook of Utility Theory, vol. 2, Kluwer (forthcoming).

Hart, A. G: Antecipation, Uncertainty and Dynamic Planning, University of Chicago Press, Chicago, (1940).

Kreps D.: A Representation Theorem for "Preferences for Flexibility", Econometrica 47, 565-577, (1979)

Koopmans, T.C., On the Flexiblity of Future Preferences in:Shelly, M.W. and Bryan, G. L. 8eds), Human Judgments and Optimality, Wiley, New York, 243-254, (1964)

Pattanaik, P.K. and Y. Xu: On Ranking Opportunity Sets in Terms of Freedom of Choice, Recherches Economiques de Louvain 56, 383-390, (1990)

Sen A.: Freedom of Choice: Concepts and Content, European Economic review 32, 269-294, (1988)

Sen A.: Welfare, Preferences and Freedom, Journal of Econometrics 50, 15-29, (1991)