MAGKS

Joint Discussion Paper Series in Economics

by the Universities of

Aachen · Gießen · Göttingen
Kassel · Marburg · Siegen

ISSN 1867-3678

No. 36-2010

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This paper can be downloaded from
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A One-Sector Model with Learning-by-doing, Investment, Leisure, and Optimal Growth

*by Matthias Göcke **

Abstract: *A one-sector model of endogenous growth based on the accumulation of real capital by saving/investing and accumulation of human capital via learning-by-doing is presented. Experience is measured by means of production output aggregated over time. Explicitly separating learning and real capital accumulation allows for an independent control of the learning process via working time. Though based on a simple one-sector model, accumulation of both types of capital is endogenously determined and a simultaneous dynamic optimisation of leisure/working time and of consumption/saving is executed. Transitional dynamics are derived and numerical simulations performed.*

JEL-Code: *C61, D90, O40*

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The author would like to thank Eric C. Meyer for helpful comments.

A One-Sector Model with Learning-by-doing, Investment, Leisure, and Optimal Growth

1. Learning-by-doing and real capital accumulation

Via learning-by-doing a certain kind of human capital – *experience* – is accumulated as a *joint product* of production activity (Rosen, 1972). Thus accumulation of experience differs from real capital accumulation or from human capital based on explicit schooling, since in these cases the accumulation of capital is the *main* product, and since the accumulation process can be controlled independently – by real investment or by schooling decisions, respectively.

In the literature a bundle of methods of modelling accumulation of experience is applied. The prevailing practice is to relate learning to the accumulation of real physical capital ("*learning-by-investing*": Arrow, 1962, Levhari, 1966a, 1966b, and Sheshinski, 1967, for more recent examples, Romer, 1986, pp. 1018 ff., and Greiner, 1996, 2003). It is argued that increases in experience are especially based on using newly installed capital, so learning is positively influenced by investments. However, directly binding learning to real capital accumulation – and with this implicitly modelling a 'composite capital' stock comprising both, real capital and experience – has the disadvantage of a lack of separation of real capital and experience. As a result, a mismatch problem occurs if intertemporal optimisation of (A) consumption versus investing and simultaneously (B) leisure versus working is performed based on a model with merely one composite stock of capital: With only one type of capital, only a single shadow price (costate) exists, while there are two controls, consumption and leisure (or saving and working, respectively). Multiplicity of optimal growth paths and steady states may be the result of this mismatch problem (de Hek, 1998).

However, in this paper a one sector growth model is presented with human capital accumulation due to learning-by-doing as a by-product of production and real capital accumulation based on saving/investing. Accumulation of experience is based on the concept of the experience curve. The main implication of this concept is that with every doubling of the sum of the output added up over time, production costs are reduced by a constant rate (Lucas, 1993, pp. 259 ff.). I.e. the stock of experience of an economy is described by means of production output which is aggregated over time. Separating real capital accumulation from learning-by-working prevents the mismatch problem outlined above. We explicitly differentiate between real capital which is accumulated by saving and experience which results from working in the production of goods. Thus we have two different types of capital, (1) real capital and (2) production experience, and correspondingly two costates (shadow prices). This is sufficient to determine two control variables, saving and working, uniquely, and allows a simultaneous intertemporal optimisation of (1) investment vs. consumption and of (2) working vs. leisure.

From a technical point of view, we extend a model of Göcke (2002) by a real capital stock and by investment decisions. Göcke (2002) presents a one sector model with experience as the single source of growth, neglecting real capital. The Göcke (2002) model has only one stock and one costate, and serves as a base for a dynamic optimisation of leisure vs. working. However, our extended model has two types of capital and correspondingly two instead of one control, thus, the dimensions are doubled. The inclusion of two different types of capital has two advantages. First, we can explain sustained endogenous growth, though real capital and experience partially show diminishing returns each, as long as both types of capital together show non-diminishing returns to scale. This is an advantage compared to one-sector AK-type models with a single capital stock (even if it is seen as a composite), since in these models a linear influence of this single capital stock is required in order to avoid diminishing returns to capital, i.e. in order to yield sustained endogenous growth. Secondly, separating experience from real capital allows to analyse changes of the composition of the capital stocks: during the transition towards steady state the proportion of experience to real capital changes, and the transitional dynamics are determined by changes of this proportion. Thus, compared to an AK-type model our decomposition gives a better explanation of the transitional dynamics (especially of investment, working time, the return to both types of capital, and of the output growth).

Moreover, albeit we propose a simple one-sector model, the accumulation of both, real capital and experience (as a kind of human capital), is endogenously explained. Since a one-sector model is applied, this is a simplification compared to two-sector models of learning in the tradition of Uzawa (1965) and Lucas (1988).[1] In these models learning can independently be controlled as well – by a decision of allocation of time between working in the production sector versus learning in a schooling sector. But, since time utilisation between working and learning is rival, in these models actually "learning–*or*–doing" (Chamley, 1993) is modelled.

The outline of the paper is as follows: Dynamic optimisation based on a general formulation of the model is presented in section 2. In section 3 – as an introductory

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1 See Göcke (2004) for a two-sector model which combines learning-by-doing in a production sector with learning-by-schooling in an educational sector. However, real capital (accumulation) is not included in this two-sector model.

example – a Cobb-Douglas version of the model is presented without executing an optimisation of working and investing. In section 4 the optimisation results of the general model are applied to a simple example with a Cobb-Douglas type production function and a logarithmic utility function. Transitional dynamics in a situation with endogenous growth are illustrated by a numerical simulation. Section 5 concludes.

2. Intertemporal optimisation of real investment and working time

Population size is neglected for reasons of simplicity, and a generalised formulation of the one-sector model in per capita terms is presented. Per capita production output x_t is based on real p.c. capital k_t, on the "experience" (the p.c. human capital) ξ_t, and share q_t of the time potential which is spent on working ($0 < q_t \leq 1$).

(1) $$x_t = x(k_t, \xi_t, q_t) \quad \text{with: } \frac{\partial x}{\partial k}, \frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial q} > 0 \quad \text{(production function)}$$

(If capital widening due to population growth is neglected) the accumulation of the real p.c. capital stock is given by the p.c. production output x_t minus p.c. consumption c_t, corrected for depreciation with a constant rate ($\mu \cdot k_t$, with $0 \leq \mu \leq 1$). (A dot "°" indicates the derivative with respect to time.)

(2) $$\mathring{k}_t \equiv \frac{dk}{dt} = x(k_t,\xi_t,q_t) - c_t - \mu \cdot k_t \quad \text{(real capital accumulation)}$$

Experience is accumulated based on 'new learning' $a[x_t,q_t]$ due to production activity x_t during working time q_t, minus a depreciation (i.e. via forgetting) with a constant rate μ. For simplicity, depreciation rate μ on both, real and human capital, is the same.

(3) $$\mathring{\xi}_t \equiv \frac{d\xi}{dt} = a[x(k_t,\xi_t,q_t),q_t] - \mu \cdot \xi_t \quad \text{with: } \frac{\partial a}{\partial x} > 0; \ \frac{\partial a}{\partial q} \gtreqless 0 \quad \text{(accumul. of experience)}$$

Dynamic optimisation is done via determining the time path of consumption c_t and of working time q_t based on a representative individual's time separable utility function. Utility comes from the p.c. consumption of goods and from leisure. Since leisure time is the residual of working time q_t, the share of leisure time is $(1-q_t)$. Overall utility U is the intertemporal aggregation of instantaneous utility at time t (u_t) applying the rate of time preference ρ as the discount rate:

(4) $$U = \int_0^\infty u_t[c_t,q_t] \cdot e^{-\rho \cdot t}\, dt \quad \text{with: } \frac{\partial u}{\partial c} > 0, \ \frac{\partial u}{\partial q} < 0$$

The (present-value) Hamiltonian (5), the first order optimality conditions (6) and (7), the motion of the costates, i.e. of the shadow prices of capital/consumption goods λ_1 (8) and of experience λ_2 (9), and the transversality conditions (10) are:

(5) $$H = u_t \cdot e^{-\rho \cdot t} + \lambda_1(t) \cdot \mathring{k}_t + \lambda_2(t) \cdot \mathring{\xi}_t \quad \Rightarrow$$

$$H = u_t[c_t, q_t] \cdot e^{-\rho \cdot t} + \lambda_1(t) \cdot (x(k_t, \xi_t, q_t) - c_t - \mu \cdot k_t) + \lambda_2(t) \cdot (a[x(\xi_t, q_t), q_t] - \mu \cdot \xi_t)$$

(6) $$\frac{\partial H}{\partial c} = 0 : \qquad \frac{\partial u}{\partial c} \cdot e^{-\rho \cdot t} - \lambda_1 = 0 \quad \Rightarrow$$

(6') $$\lambda_1 = \frac{\partial u}{\partial c} \cdot e^{-\rho \cdot t} = du_c \cdot e^{-\rho \cdot t} \qquad \text{with:} \quad du_c \equiv \frac{\partial u}{\partial c}$$

(7) $$\frac{\partial H}{\partial q} = 0 : \qquad \frac{\partial u}{\partial c} \cdot e^{-\rho \cdot t} + \lambda_1 \cdot \frac{\partial x}{\partial q} + \lambda_2 \cdot \left(\frac{\partial a}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial a}{\partial q} \right) = 0$$

(8) $$\frac{\partial H}{\partial k} = -\mathring{\lambda}_1 \qquad = \lambda_1 \cdot \left(\frac{\partial x}{\partial k} - \mu \right) + \lambda_2 \cdot \frac{\partial a}{\partial x} \cdot \frac{\partial x}{\partial k}$$

(9) $$\frac{\partial H}{\partial \xi} = -\mathring{\lambda}_2 \qquad = \lambda_1 \cdot \frac{\partial x}{\partial \xi} + \lambda_2 \cdot \left(\frac{\partial a}{\partial x} \cdot \frac{\partial x}{\partial \xi} - \mu \right)$$

(10) $$\lim_{t \to \infty} (\lambda_1(t) \cdot k_t) = 0 \qquad \text{and} \qquad \lim_{t \to \infty} (\lambda_2(t) \cdot \xi_t) = 0$$

The components of the FOC for the working time q in eq. (7) can be interpreted:

(11) $$\frac{\partial H}{\partial q} = 0 : \qquad \left(\frac{\partial u}{\partial c} \cdot \frac{\partial x}{\partial q} + \frac{\partial u}{\partial q} \right) \cdot e^{-\rho \cdot t} + \lambda_2 \cdot \left(\frac{\partial a}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial a}{\partial q} \right) = 0 \quad \Rightarrow$$

(11') $$\lambda_2 = e^{-\rho \cdot t} \cdot \frac{-du_q}{da_q} \qquad \text{with:} \quad du_q \equiv \frac{\partial u}{\partial c} \cdot \frac{\partial x}{\partial q} + \frac{\partial u}{\partial q} \quad \text{and} \quad da_q \equiv \frac{\partial a}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial a}{\partial q}$$

With an interpretation of the mathematical terms in eq. (11):

$\frac{\partial u}{\partial c} \cdot \frac{\partial x}{\partial q} > 0$:	marginal instantaneous utility from production due to increasing working time
$\frac{\partial u}{\partial q} < 0$:	dis-utility of less leisure time (of more working time)
$e^{-\rho \cdot t} > 0$:	discounting future utility (↔ 'present value')
$\lambda_2 > 0$:	shadow price of experience (with implicit discounting of future utility)
$\frac{\partial a}{\partial x} \cdot \frac{\partial x}{\partial q} > 0$:	marginal learning due to additional working time
$\frac{\partial a}{\partial q} \gtreqless 0$:	change of spillovers between different activities (possibly negative due to 'scattering'-effects)

The sum of the effects of a change in working time q on learning is typically positive: $da_q > 0$. Correspondingly, the dynamically optimal entire instantaneous marginal utility of working has to be negative ($du_q < 0$). Considering the positive intertemporal externality of working (and learning), a negative momentary marginal utility of working is dynamically optimal, while a purely static optimisation of leisure would imply marginal utility at every moment instantaneously to be balanced, i.e. $du_q = 0$.

In order to reveal the net internal marginal rate of return to real capital (r_k) and to experience (r_ξ), the movement of the costates (8) and (9) can be reformulated via the growth rates of the shadow prices ($\hat{\lambda}_1$, $\hat{\lambda}_2$) and using eqs. (6') and (11'). (A hat "^" indicates the growth rate of a variable.)

$$(12)\qquad r_k = -\hat{\lambda}_1 \equiv \frac{-\overset{\circ}{\lambda}_1}{\lambda_1} = \frac{\partial x}{\partial k} + \frac{\lambda_2}{\lambda_1}\cdot\frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial k} - \mu$$

$$= \frac{\partial x}{\partial k} - \frac{du_q}{du_c}\cdot\frac{da_k}{da_q} - \mu \qquad \text{with: } da_k \equiv \frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial k}$$

$$(13)\qquad r_\xi = -\hat{\lambda}_2 \equiv \frac{-\overset{\circ}{\lambda}_2}{\lambda_2} = \frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\lambda_1}{\lambda_2}\cdot\frac{\partial x}{\partial \xi} - \mu$$

$$= da_\xi - \frac{du_\xi \cdot da_q}{du_q} - \mu \qquad \text{with: } da_\xi \equiv \frac{\partial a}{\partial x}\cdot\frac{\partial x}{\partial \xi} \quad \text{and} \quad du_\xi \equiv \frac{\partial u}{\partial c}\cdot\frac{\partial x}{\partial \xi}$$

The derivative of the optimality conditions (6) and (7) with respect to time (and using $\overset{\circ}{\lambda}_1$, $\overset{\circ}{\lambda}_2$, $\overset{\circ}{k}$ and $\overset{\circ}{\xi}$) leads to differential equations for both control variables: $(dc/dt) \equiv \overset{\circ}{c} = \overset{\circ}{c}(k,\xi,c,q)$ and $(dq/dt) \equiv \overset{\circ}{q} = \overset{\circ}{q}(k,\xi,c,q)$. In combination with the motions of the state variables, the dynamics of the economy are given by a system of four differential equations. The first steps of this procedure are presented: the derivative of (6') and (11') with respect to time leads to eqs. (14) and (15). The growth rate of the shadow price of consumption/capital goods $\hat{\lambda}_1$ consists of the growth rate of the momentary marginal utility of consumption $d\hat{u}_c$ minus depreciation due to the rate of time preference ρ. The growth rate of the shadow price of experience $\hat{\lambda}_2$ is determined by the growth rate of marginal utility related to changes in working time q minus the growth of marginal learning due to changes of q (i.e. $d\hat{u}_q - d\hat{a}_q$).

$$(14)\qquad \hat{\lambda}_1 = \frac{d\overset{\circ}{u}_c}{du_c} - \rho = d\hat{u}_c - \rho$$

$$(15)\qquad \hat{\lambda}_2 = \frac{d\overset{\circ}{u}_q}{du_q} - \frac{d\overset{\circ}{a}_q}{da_q} - \rho = d\hat{u}_q - d\hat{a}_q - \rho$$

Combining (12) and (14) gives a Ramsey rule of optimal saving, extended by learning effects: Marginal return to real capital (LHS of (16)) consists of a direct production effect ($\partial x/\partial k$), but is extended by marginal learning based on capital input da_k which has to be converted by the price relation (λ_2/λ_1). This additional return has a positive effect on saving, i.e. a negative effect on present consumption.

(16) $$\frac{\partial x}{\partial k} + \frac{\lambda_2}{\lambda_1} \cdot da_k = \rho + \mu - d\hat{u}_c$$

Combining (13) and (15) leads to a Ramsey rule analogy of dynamically optimal working: an optimal decision on working time implies that the net return to experience (LHS of (17)) has to cover the sum of the discount rate and the 'shrinking rate' (i.e. negative growth rate) of marginal utility of working time (RHS of (17)). The decrease of marginal utility consists of the decrease of momentary marginal 'utility' of working ($-d\hat{u}_q$) extended by the growth rate of the marginal effects on learning ($d\hat{a}_q$).

(17) $$da_\xi + \frac{\lambda_1}{\lambda_2} \cdot \frac{\partial x}{\partial \xi} = \rho + \mu - (d\hat{u}_q - d\hat{a}_q)$$

3. A simple Cobb-Douglas version without optimisation

As a simple example, a Cobb-Douglas type version of p.c. production, real capital accumulation and learning-by-doing is presented – in a first step without dynamic optimisation of the potential control variables. In per capita terms, using the 'real capital intensity of experience' ψ, i.e. the ratio between real capital and accumulated experience, and disregarding time index "$_t$" the “supply side” of the model is as follows:

(18) $$\psi \equiv \frac{k}{\xi}$$ (capital-experience ratio, intensity)

(19) $$x = k^\alpha \cdot \xi^\gamma \cdot q^\beta = \psi^\alpha \cdot \xi^{(\alpha+\gamma)} \cdot q^\beta$$ (C.D.-type production per capita)

(20) $$\mathring{k} = x - c - \mu \cdot k = s \cdot x - \mu \cdot k$$ (accumulation of real capital p.c.)

$$= s \cdot \psi^\alpha \cdot \xi^{(\alpha+\gamma)} \cdot q^\beta - \mu \cdot \psi \cdot \xi$$ with: $c \equiv (1-s) \cdot x \Leftrightarrow s \equiv 1-(c/x)$

(21) $$\mathring{\xi} = \phi \cdot q^{(\eta-1)} \cdot x - \mu \cdot \xi$$ (learning by-doing; change of experience p.c.)

$$= \phi \cdot \psi^\alpha \cdot \xi^{\alpha+\gamma} \cdot q^{(\beta+\eta-1)} - \mu \cdot \xi$$

(22) $$\mathring{\psi} = \psi^\alpha \cdot \xi^{(\alpha+\gamma-1)} \cdot q^\beta \cdot [s - \phi \cdot \psi \cdot q^{(\eta-1)}]$$

with:
- s : savings-income ratio, propensity to invest (with $0 \le s \le 1$)
- α : production elasticity of real capital ($0 \le \alpha \le 1$)
- β : production elasticity of labour ($0 \le \beta \le 1$)
- γ : production elasticity of experience ($0 \le \gamma \le 1$)
- ϕ : productivity parameter in the learning function ($\phi > 0$)
- η : spillover parameter in the learning function ($0 \le \eta \le 1$)

We apply a 'spillover correction' $q^{(\eta-1)}$. This factor vanishes for $\eta=1$. In this case $\mathring{\xi}$ is determined by total output x, i.e. for $\eta=1$ we have a complete learning spillover between different production activities. In the opposite case of $\eta=0$ the change in the stock of experience is $\mathring{\Xi}=\phi\cdot(x/q)-\mu$. Thus, for $\eta=0$ we have no spillovers between different activities at all, since new learning is determined by per capita production per unit of time (i.e. by a single activity) only. However, in our model the property of endogenous growth does not depend on the size of the spillover parameter η. Hence, spillovers do not have the central importance they have in some endogenous growth models, where (asymptotically) non-diminishing returns to capital are based on spillovers (see e.g. Clemhout & Wan, 1970, Stokey, 1986, Kohn & Marion, 1993, and Dehejia, 1993).

Without intertemporal optimisation the share of working time q and the propensity to save s can be regarded as exogenously constant and the growth of the economy is determined by the two differential equations (20) and (21). For the special case of $\beta+\eta=1$ the influence of q on learning $\mathring{\xi}$ vanishes. For $\beta<1$ a higher q (i.e. an increasing labour input in production) leads to a higher output and with this to more learning, but only under-proportionally. For $\eta<1$ only partial spillovers between different activities (i.e. 'scattering'-effects) occur. For $\beta+\eta=1$ these effects and a diminishing marginal return to labour lead to an exact compensation of the effects of working time q on learning. We assume, that $1<\beta+\eta\leq 2$ is valid. Thus, an increase of working time q will bring about an improved learning-by-doing.

A steady state is characterised by constant intensities and constant growth rates, which may be zero or not, depending on the magnitude of the sum of production elasticities $(\alpha+\gamma)$. The steady state capital intensity of experience ψ_{st} is given by:

(23) $$\mathring{\psi} = 0 \;\Leftrightarrow\; \psi_{st} = \frac{s}{\phi}\cdot q^{1-\eta}$$

For $(\alpha+\gamma<1)$, due to diminishing returns, the system shows no sustained growth but constant steady state *levels* of p.c. experience and p.c. production (ξ_{st}, x_{st}):

(24) $$\alpha+\gamma<1: \;\; \mathring{\xi} = 0 \;\text{ and }\; \mathring{k} = 0 \;\text{ for}$$

$$\xi_{st} = \left(\frac{\phi^{1-\alpha}\cdot s^{\alpha}}{\mu}\cdot q^{(\alpha+\beta+\eta-\alpha\eta-1)}\right)^{\left(\frac{1}{1-\alpha-\gamma}\right)}$$

$$x_{st} = \left(\left(\frac{\phi^{\gamma}\cdot s^{\alpha}}{\mu^{\alpha+\gamma}}\right)^{\gamma}\cdot q^{(\beta-\gamma+\gamma\eta)}\right)^{\left(\frac{1}{1-\alpha-\gamma}\right)}$$

In the case of $(\alpha+\gamma=\mathbf{1})$ non-diminishing returns to accumulated factors occur. This brings about sustained endogenous growth with a constant non-zero steady state *growth rate*. In this case, accumulation of both factors is:

(25) $\alpha+\gamma=1: \qquad \mathring{k} = s\cdot\psi^{\alpha}\cdot\xi\cdot q^{\beta} - \mu\cdot\psi\cdot\xi$

$$\hat{\xi} \equiv \frac{\mathring{\xi}}{\xi} = \phi\cdot\psi^{\alpha}\cdot q^{(\beta+\eta-1)} - \mu$$

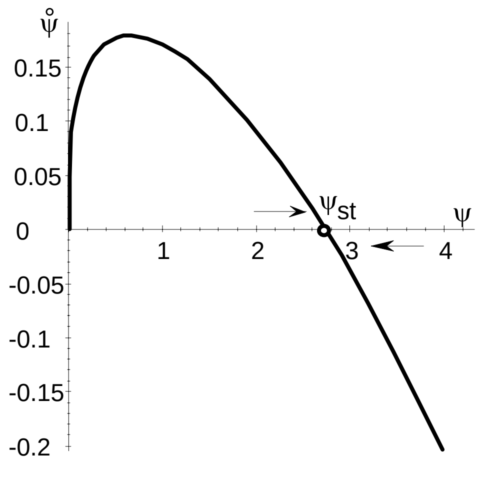
$$\mathring{\psi} = \psi^{\alpha}\cdot q^{\beta}\cdot[s - \phi\cdot\psi\cdot q^{(\eta-1)}]$$

After reaching steady state intensity ψ_{st} we see a common constant growth rate ($\hat{x}_{st}$):

(26) $\alpha+\gamma=1: \quad \hat{x}_{st} = \hat{k}_{st} = \hat{\xi}_{st} = \phi^{1-\alpha}\cdot s^{\alpha}\cdot q^{(\alpha+\beta+\eta-\alpha\eta-1)} - \mu$

In the case of endogenous growth $(\alpha+\gamma=1)$, as eq. (25) shows, the dynamics of the model are determined solely by the dynamics of the intensity ψ. A simulation of the corresponding one-dimensional phase diagram for ψ is given in Fig. 1. For constant behavioural parameters s and q, the system converges towards a *stable* steady state (ψ_{st}).

Fig. 1: Phase diagram for the capital intensity of experience ψ
in the case of endogenous growth $(\alpha+\gamma=1)$ and constant q and s



ψ_{st} = 2.721655 ; $\hat{\xi}_{st} = \hat{k}_{st} = \hat{x}_{st}$ = 0.0396189514

(s = 1/3 ; q = 2/3 ; α = 1/3 ; γ = 2/3 ; β = 1/2 ; ϕ = 1/10 ; μ = 1/10 ; η = 1/2)

4. Cobb-Douglas production, logarithmic utility, and dynamic optimisation

4.1 The model and some simplifications

Now the dynamics based on optimisation are demonstrated for the Cobb-Douglas-type production technology as a simple example:

(19') $x = k^{\alpha} \cdot \xi^{\gamma} \cdot q^{\beta}$ (p.c. production)

(20') $\overset{\circ}{k} = s \cdot x - \mu \cdot k \;=\; s \cdot k^{\alpha} \cdot \xi^{\gamma} \cdot q^{\beta} - \mu \cdot k$ (real capital accumulation)

(21') $\overset{\circ}{\xi} = \phi \cdot q^{(\eta-1)} \cdot x - \mu \cdot \xi \;=\; \phi \cdot q^{(\beta+\eta-1)} \cdot \xi^{\gamma} - \mu \cdot \xi$ (learning-by-doing)

In our example, utility shows a constant elasticity of *intra*temporal substitution (concerning the *momentary* choice between leisure and consumption of goods and at *each* point t in time) and a constant *inter*temporal elasticity of substitution (concerning the weight of utility between *different points in time*). For reasons of simplicity, both elasticities are assumed as one, so a logarithmic instantaneous utility u_t results:

(27) $U = \int_0^{\infty} u_t \cdot e^{-\rho \cdot t}\, dt$ with $u_t = b \cdot \ln(c) + (1-b) \cdot \ln(1-q)$

Assuming a log-utility function has non-trivial implications. Even if the production technology is able to support sustained growth due to non-diminishing returns, the introduction of a choice between leisure and working may result in a situation without endogenous growth, if marginal utility of the consumption of goods is decreasing faster than the production productivity of accumulated factors is growing. The result would be, that productivity increases are mainly utilised for an expansion of leisure time – i.e. an ever decreasing share of working time due to consumption saturation.[2] Göcke (2002) shows that endogenous growth only results if the *intra*temporal elasticity of substitution between *momentary* consumption of goods and leisure is exactly one. This is guaranteed by log-utility, so endogenous growth is feasible.

Using savings-income ratio $s \equiv 1-(c/x)$ simplifies the following expressions, since its range is limited $(0 \leq s < 1)$ and because s, as a ratio, converges towards a constant steady state level. The Hamiltonian and the FOCs of our problem now are:

(28) $H = [\, b \cdot \ln(c) + (1-b) \cdot \ln(1-q) \,] \cdot e^{-\rho \cdot t} + \lambda_1(t) \cdot [\, s \cdot k^{\alpha} \cdot \xi^{\gamma} \cdot q^{\beta} - \mu \cdot k \,]$
$\qquad + \lambda_2(t) \cdot [\, \phi \cdot q^{(\eta+\beta-1)} \cdot \xi^{\gamma} - \mu \cdot \xi \,]$

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2 For a discussion of this general problem of endogenous growth models allowing the choice between leisure and consumption see Baldassarri, De Santis, Moscarini (1994).

(29) $$\frac{\partial H}{\partial c} = 0 \quad \Rightarrow \quad \lambda_1 = \frac{b}{c}\cdot e^{-\rho\cdot t} = \frac{b\cdot e^{-\rho\cdot t}}{(1-s)\cdot k^{\alpha}\cdot \xi^{\gamma}\cdot q^{\beta}}$$

(30) $$\frac{\partial H}{\partial q} = 0 = \left(\frac{b\cdot\beta}{(1-s)\cdot q}+\frac{b-1}{1-q}\right)\cdot e^{-\rho\cdot t}+\lambda_2\cdot[\beta\cdot\phi\cdot q^{\eta+\beta-2}\cdot k^{\alpha}\cdot\xi^{\gamma}+(\eta-1)\cdot\phi\cdot q^{\eta+\beta-2}\cdot k^{\alpha}\cdot\xi^{\gamma}]$$

The motions of the costates in the C.-D.-production / log-utility example are:

(31) $$\frac{\partial H}{\partial k} = -\overset{\circ}{\lambda}_1 = \lambda_1\cdot(\alpha\cdot k^{\alpha-1}\cdot\xi^{\gamma}\cdot q^{\beta} - \mu) + \lambda_2\cdot(\phi\cdot\alpha\cdot k^{\alpha-1}\cdot\xi^{\gamma}\cdot q^{\beta+\eta-1})$$

(32) $$\frac{\partial H}{\partial \xi} = -\overset{\circ}{\lambda}_2 = -\frac{b\cdot\gamma}{(1-s)\cdot\xi}\cdot e^{-\rho\cdot t} - \lambda_2\cdot[\gamma\cdot\phi\cdot k^{\alpha}\cdot\xi^{\gamma-1}\cdot q^{\beta+\eta-1} - \mu]$$

Since we have four differential equations to handle, to avoid non-linearities some *simplifications* are introduced only for computational reasons. In order to get linear functions the labour is assumed to have a non-diminishing marginal product ($\boldsymbol{\beta = 1}$) and spillovers of learning are complete ($\boldsymbol{\eta = 1}$). With this, the following expressions for the shadow prices of real goods λ_1 and experience λ_2 result:

(33) $$\lambda_1 = \frac{b\cdot e^{-\rho\cdot t}}{c} = \frac{b\cdot e^{-\rho\cdot t}}{(1-s)\cdot k^{\alpha}\cdot\xi^{\gamma}\cdot q}$$

(34) $$\lambda_2 = \frac{[(1-b)\cdot(1-s)\cdot q - b\cdot(1-q)]\cdot e^{-\rho\cdot t}}{\phi\cdot(1-q)\cdot c}$$

Via the costate motions (8) and (9) the marginal net rate of return to real capital (r_k) and to experience (r_ξ) can be calculated:

(35) $$r_k = -\hat{\lambda}_1 = \frac{\alpha\cdot(1-b)\cdot(1-s)\cdot q^2}{b\cdot(1-q)}\cdot k^{\alpha-1}\cdot\xi^{\gamma} - \mu$$

(36) $$r_\xi = -\hat{\lambda}_2 = \frac{\phi\cdot\gamma\cdot(1-b)\cdot(1-s)\cdot q^2}{(1-b)\cdot(1-s)\cdot q - b\cdot(1-q)}\cdot k^{\alpha}\cdot\xi^{\gamma-1} - \mu$$

4.2 Dynamics without endogenous growth ($\alpha+\gamma<1$)

Performing the above mentioned calculation steps leads to differential equations for both controls, s and q:

(37) $$\overset{\circ}{s} = \frac{ds}{dt} = (1-s)\left\{\frac{\alpha\xi^{\alpha+\gamma-1}}{-b\psi^{1-\alpha}}\left[\frac{q(1-s)(q-b)}{(1-q)}+b(1-s-sq)\right]+q\xi^{\alpha+\gamma-1}\psi^{\alpha}\gamma\phi-\frac{(\alpha+\gamma)\mu-\rho-\mu}{q}\right\}$$

(38) $$\overset{\circ}{q} = \frac{dq}{dt} = [\,\rho + \mu\cdot(1-\alpha-\gamma) - (1-s)\cdot\alpha\cdot\psi^{\alpha-1}\cdot\xi^{\alpha+\gamma-1}\cdot q\,]\cdot(1-q)$$

With the sum of production elasticities of ($\alpha+\gamma<\mathbf{1}$) no sustained growth, but a steady state results with constant levels and constant controls. This steady state can be derived applying $\overset{\circ}{q}=0$, $\overset{\circ}{s}=0$, $\overset{\circ}{\xi}=0$ and $\overset{\circ}{k}=0$. The long run level of the propensity to save s_{st} and of the share of working time q_{st} result as:

(39) $$s_{st} = \frac{\alpha\cdot\mu}{\rho+\mu-\gamma\cdot\mu} \quad \text{and} \quad q_{st} = \frac{(\mu+\rho)\cdot b}{\rho+\mu\cdot[1-(\alpha+\gamma)\cdot(1-b)]}$$

Since both controls are constant in steady state, the results from (23) and (24) apply:

(40) $$\psi_{st} = \frac{s_{st}}{\phi} \quad \Rightarrow \quad \xi_{st} = \left(\frac{\phi^{1-\alpha}\cdot s_{st}^{\alpha}}{\mu}\cdot q_{st}\right)^{\left(\frac{1}{1-\alpha-\gamma}\right)}$$

If these steady state results are applied to (35) and (36), we see that a process of arbitrage between real capital and experience leads to a common steady state rate of return to both types of capital. Since the steady state rate of return does not exceed the rate of time preference ρ, the incentive to accumulate any kind of capital and with this economic growth vanishes in the long run.

(41) $$\alpha+\gamma<1: \quad r_{k,st} = r_{\xi,st} = \rho$$

4.3 Investment, learning and endogenous growth ($\alpha+\gamma=\mathbf{1}$)

Linearity in accumulated factors as a whole (i.e. $\alpha+\gamma=\mathbf{1}$) results in non-diminishing returns and endogenous growth. In this case the dynamics can be written using only the capital intensity of experience ($\psi\equiv k/\xi$) and substituting for both single stocks k and ξ:

(42) $$\overset{\circ}{s} = \frac{ds}{dt} = (1-s)\left\{\frac{-\alpha}{b\psi^{1-\alpha}}\left[\frac{q(1-s)(q-b)}{(1-q)} + b(1-s-sq)\right] + (1-\alpha)q\psi^{\alpha}\phi + \frac{\rho}{q}\right\}$$

(43) $$\overset{\circ}{q} = \frac{dq}{dt} = [\,\rho-(1-s)\cdot\alpha\cdot\psi^{\alpha-1}\cdot q\,]\cdot(1-q)$$

(44) $$\overset{\circ}{\psi} = \psi^{\alpha}\cdot q\cdot(s-\phi\cdot\psi)$$

(45) $$\overset{\circ}{k} = s_t\cdot\psi^{\alpha}\cdot\xi\cdot q_t - \mu\cdot\psi\cdot\xi$$

(46) $$\hat{\xi} \equiv \frac{\overset{\circ}{\xi}}{\xi} = \phi\cdot\psi^{\alpha}\cdot q_t - \mu$$

The dynamics of the system are completely determined by $\overset{\circ}{s}$, $\overset{\circ}{q}$ and $\overset{\circ}{\psi}$. The time path of the growth rate of production $\hat{x}$ can be calculated based on the dynamics of ψ, q and s:

(47) $$x = k^{\alpha}\cdot\xi^{\gamma}\cdot q^{\beta} \quad\Rightarrow\quad \hat{x} = \alpha\cdot\hat{\psi} + \hat{\xi} + \hat{q} \quad\Rightarrow$$
$$\hat{x} = \alpha\cdot\psi^{\alpha-1}\cdot q\cdot(s-\phi\cdot\psi) + \phi\cdot\psi^{\alpha}\cdot q + [\frac{\rho}{q} - (1-s)\cdot\alpha\cdot\psi^{\alpha-1}]\cdot(1-q) - \mu$$

In the case of $(\alpha+\gamma=1)$ both rates of return depend on s, q and on the intensity ψ:

(35') $$r_k = \frac{\alpha\cdot(1-b)\cdot(1-s)\cdot q^2}{b\cdot(1-q)}\cdot\psi^{-\gamma} - \mu$$

(36') $$r_{\xi} = \frac{\phi\cdot\gamma\cdot(1-b)\cdot(1-s)\cdot q^2}{(1-b)\cdot(1-s)\cdot q - b\cdot(1-q)}\cdot\psi^{\alpha} - \mu$$

The steady state under endogenous growth is characterised by a constant non-zero growth rate of both stocks ξ and k, but by a constant intensity ψ_{st} and by constant control ratios s_{st} and q_{st}. A constant saving ratio leads to:

(48) $\overset{\circ}{s} = 0$:

$$s_{st,1} = \frac{q^3\alpha - q^2\phi b\psi - 2\,b\alpha q^2 + \alpha b q - q^3\phi\alpha b\psi + q^2\phi\alpha b\psi - \psi^{1-\alpha}b\rho + \psi^{1-\alpha}\rho b q + q^3\phi b\psi}{-\alpha\, q(-q^2 + q\,b - b + q^2 b)}$$

Steady state constancy of the share of working time results in:

(49) $$\overset{\circ}{q} = 0: \quad s_{st,2} = \frac{\alpha\cdot q - \rho\cdot\psi^{1-\alpha}}{\alpha\cdot q}$$

From (23) it follows that:

(50) $$\overset{\circ}{\psi} = 0: \quad s_{st,3} = \phi\cdot\psi$$

Combining these three different results for s_{st} gives two different expressions for the working time share in the steady state:

(51) $$s_{st,2} = s_{st,3}: \quad q_{st,1} = \frac{\rho\cdot\psi^{1-\alpha}}{\alpha\cdot(1-\phi\cdot\psi)}$$

(52) $$s_{st,1} = s_{st,2}: \quad q_{st,2} = \frac{\phi\,b\,\psi + \alpha\,b - \phi\,b\,\psi\,\alpha - \psi^{1-\alpha}\rho + \psi^{1-\alpha}b\,\rho}{-b\,(\phi\,\psi\,\alpha - \phi\,\psi - \alpha)}$$

The steady state intensity now can be calculated by solving the following problem:

(53) $q_{st,1} = q_{st,2}$: $\psi_{st} = e^Z$ with Z as solution of:

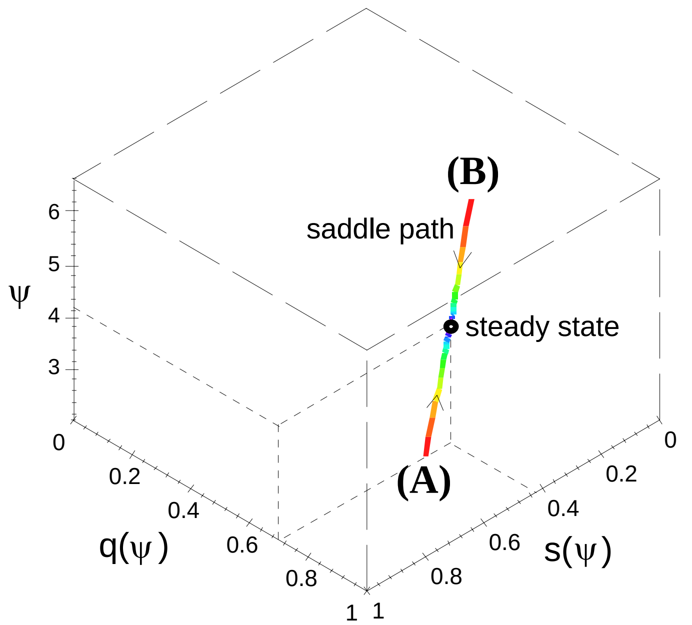
$$Z\alpha - Z - \ln\left(\frac{-\rho\,(-e^Z b\,\phi - \alpha + \phi\, e^Z \alpha)}{\alpha\, b\,(-1 + \phi\, e^Z)\,(\phi\, e^Z \alpha - \phi\, e^Z - \alpha)}\right) = 0$$

Local stability surrounding the steady state of the system, set up by $\mathring{s}$, $\mathring{q}$ and $\mathring{\psi}$, can be analysed by means of the characteristic roots of the coefficient matrix based on a first order Taylor expansion around steady state. If 'sensible' parameter values are chosen (e.g. excluding unbounded utility), the system shows one eigenvalue with a negative (corresponding to the predetermined variable ψ) and two eigenvalues with a positive real part (corresponding to both jump variables/controls). Thus we observe saddle path stability, since a typical optimal control problem with infinite horizon is analysed.

4.4 A numerical example

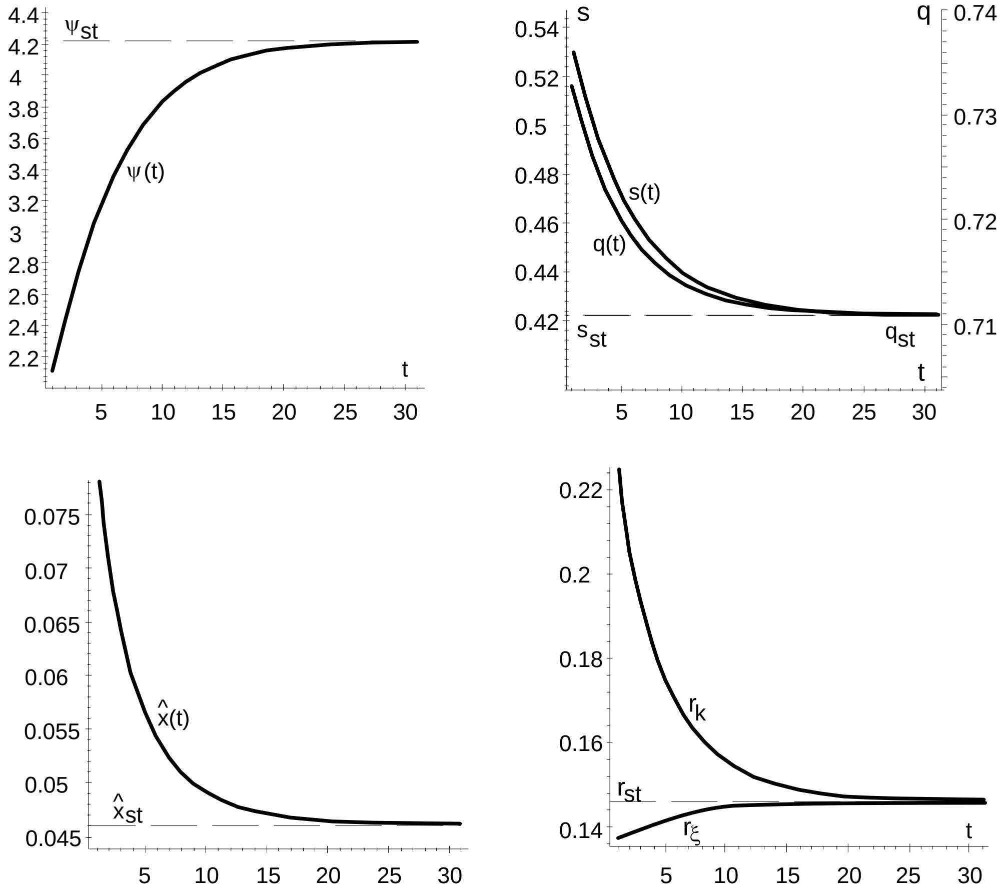
As a numerical example, the parameters are chosen as $\phi = 1/10$; $b = 1/2$; $\rho = 1/10$; $\mu = 1/10$, $\alpha = 1/2$ and $\gamma = 1/2$. Two alternative initial points on the saddle path, 50 percent below and above steady state intensity $\psi_{st} = 4.2212$ were calculated using the time-elimination-method (see Barro & Sala-i-Martin, 1995, pp. 490 f.; the software MAPLE was used for the simulation). The stable saddle path is depicted in Fig. 2 as a trajectory in the (ψ,q,s)-space leading to the steady state point determined by ψ_{st}, q_{st} and s_{st}.

Fig. 2: Stable saddle path



Steady state: $\psi_{st} = 4.2212$; $s_{st} = 0.42212$; $q_{st} = 0.71106$;
trajectory (A): $\psi(t{=}0) = 2.1106 \;\Rightarrow\; s(t{=}0) = 0.52986$ and $q(t{=}0) = 0.73281$;
trajectory (B): $\psi(t{=}0) = 6.3317 \;\Rightarrow\; s(t{=}0) = 0.34068$ and $q(t{=}0) = 0.69818$.

Fig. 3: Time paths of real capital intensity ψ, savings-ratio s, working time share q, production growth, and of both net rates of return



Steady state: $\hat{x}_{st} = \hat{c}_{st} = \hat{k}_{st} = \hat{\xi}_{st} = 0.04609$; $r_{st} = 0.146$

In Fig. 3 the dynamics are explicitly illustrated for the lower arm of the saddle path, i.e. for trajectory (A), with a low initial intensity of $\psi(t{=}0) = 2.11$. While s is declining from about 0.53 to 0.42 and q from 0.733 to 0.711, the real capital intensity ψ even doubles its initial value during transition. Consequently, the dynamics of the ratio $\psi \equiv (k/\xi)$ dominates the dynamics of the entire system. As demonstrated by eqs. (35') and (36'), the marginal rate of return to both types of capital is determined by ψ. For a lower intensity the rate of return to (relatively scarce) real capital r_k is high, while the rate of return to experience r_ξ is lower. In the long run, a process of arbitrage is induced by dynamically optimal consumption and working time decisions, leading to an 'interest parity' between both accumulated factors (with a common steady state rate of return, $r_{st} = 0.146$). For the case of endogenous growth, this long run rate is higher than the discounting of utility via time preference ($\rho = 1/10$), so the incentives to accumulate factors by investing or by learning sustain. The dynamics of the control variables can be explained based on both rates of return. Since r_k is initially relatively high (due to a low capital intensity ψ) it is profitable to save a higher portion of the income compared to the steady state. With a high s the capital intensity rises, and so r_k and s converge towards their steady state levels. Because of the high rate of return to real capital, the price of goods is high in the beginning (for a low ψ). As a response, in order to produce more goods, the share of working time q exceeds the steady state level for a low ψ. However, since the return to experience r_k is relatively low, this expansion is dampened. So experience is accumulated slower than real capital, leading to an increasing intensity ψ. Due to the high level of the saving ratio s and the working time q as a consequence of the low real capital intensity ψ, we observe a high rate of output growth $\hat{x}$, descending towards its steady state rate $\hat{x}_{st}$ when the (increasing) intensity converges to its steady state ψ_{st}.

5. Conclusion

A macroeconomic one-sector growth model with real capital accumulation and learning-by-doing based on overall production activity is presented. Accumulation of human capital is measured by means of output aggregated over time, i.e. an experience curve analogy is applied. Compared to the prevailing modelling technique of 'learning-by-investment', separating the accumulation of experience from real capital accumulation allows (a) an independent control of learning by working versus leisure time, and (b) transitional dynamics can be explained by the relative size of both capital stocks. Dynamic optimisation of the choice between consumption vs. investment and working vs. leisure is performed simultaneously. Dependent on the sum of production elasticities of both accumulated factors (α for real capital and γ for experience), a situation with or without endogenous growth may result. Due to the existence of a positive feedback between real capital accumulation and learning, sustained endogenous growth occurs even in situations with partially diminishing returns to real capital (for $\alpha < 1$), since the explicit inclusion of experience means that both types of capital together show non-diminishing returns (for $\alpha + \gamma = 1$). Remarkably, both explicitly separated accumulation processes are based on an activity in the same sector: the production of goods.

The dynamics of leisure/working-time and saving, i.e. the accumulation of experience and real capital, and consequently, of output are derived for situations without and with endogenous growth. The long-run steady state behaviour as well as the transitional dynamics are driven by the dynamics of the marginal rates of return to both kinds of capital. However, both rates of return are determined by the relative size of both capital stocks, i.e. by the ratio ψ of real capital to experience. The intensity ψ influences the whole dynamics of the economy. In the case of a relatively scarce real capital (compared to experience), the partial marginal return to real capital is above the long run steady state level, and thus, real capital accumulation is fostered until the steady state ratio is reached. During the transition process, working (and producing goods for investment) exceeds its long run level, since a high return to capital results in a high shadow price of goods. Consequently, an economy which is – compared to production experience – scarcely endowed with real capital will during transition show an output growth rate which is above the long run steady state growth, due to a high saving ratio and due to a long working time,. This represents the "Wirtschaftswunder" situation in Germany or in Japan subsequent to the second world war. Furthermore, it may be an adequate description of the situation of strongly growing countries where the establishment of liberal goods and capital markets allows an optimal reaction of real capital accumulation and working time (i.e. larger saving/investment ratios and a higher working time) as a response to high rates of return to real capital. This may describe part of the exceptional growth of the Asian countries.

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