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Direct proofs of order independence

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Abstract

We establish a generic result concerning order independence of a dominance relation on finite games. It allows us to draw conclusions about order independence of various dominance relations in a direct and simple way.

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1 Introduction

In the literature on strategic games several dominance relations have been considered and for various of them order independence was established. Just to mention two well-known results. In Gilboa, Kalai and Zemel [1990] order independence of strict dominance by a pure strategy in finite games was established. In turn, in Osborne and Rubinstein [1994] order independence of strict dominance by a mixed strategy was proved.

A number of other order independence results have been proved. In Apt [2004] we provided a uniform exposition based on so-called abstract reduction systems, notably Newman's Lemma, and established some new order independence results. However, for each considered dominance relation some supplementary lemmas were needed.

The purpose of this paper is to provide a generic order independence result for finite games that allows us to prove order independence of each relevant dominance relation in a direct and simple way. The exposition still relies on Newman's Lemma, but is now directly linked with the dominance relations through a crucial notion of hereditarity. To check for order independence it suffices to show that the dominance relation is hereditary, a simple condition referring to a single reduction step. We show that in each case this is straightforward. In the conclusions we clarify what makes this approach simpler than the ones used in the literature.

2 Dominance relations

We assume the customary notions of a ***strategic game***, of ***strict dominance*** and ***weak dominance*** by a pure, respectively mixed strategy, see, e.g., Osborne and Rubinstein [1994]. We also use the standard notation. In particular, ΔS is the set of probabilities over the finite non-empty set S and for a joint strategy s, s_i is the strategy of player i and s_{-i} is the joint strategy of the opponents of player i. All considered games are assumed to be finite.

We assume an initial (finite) strategic game

$$G := (G_1, \ldots, G_n, p_1, \ldots, p_n).$$

where G_i is a non-empty set of strategies of player i and p_i his payoff function. Given non-empty sets of strategies $R_1, \ldots, R_n$ such that for all i, $R_i \subseteq G_i$ we say that $R := (R_1, \ldots, R_n, p_1, \ldots, p_n)$ is a ***restriction*** (of G). Here of course we view each p_i as a function on the subset $R_1 \times \ldots \times R_n$ of $G_1 \times \ldots \times G_n$.

In what follows, given a restriction R we denote by R_i the set of strategies of player i in R. Further, given two restrictions R and R' we write $R' \subseteq R$ when for all i, $R'_i \subseteq R_i$.

When reasoning about never best responses we want to carry out the argument for a number of alternatives in a uniform way. To this end by a ***set of beliefs*** of player i in a restriction R we mean one of the following sets

(i) $\mathcal{B}_i := R_{-i}$,

i.e., a belief is a joint pure strategy of the opponents,

(ii) $\mathcal{B}_i := \Pi_{j \neq i} \Delta R_j$,

i.e., a belief is a joint mixed strategy of the opponents,

(iii) $\mathcal{B}_i := \Delta R_{-i}$,

i.e., a belief is a probability distribution over the set of joint pure strategies of the opponents (so called ***correlated mixed strategy***).

In the second and third case the payoff function p_i is extended in a standard way to the ***expected payoff*** function $p_i : R_i \times \mathcal{B}_i \to \mathcal{R}$.

Consider now a restriction $R := (R_1, \ldots, R_n, p_1, \ldots, p_n)$ with for each player i a set of beliefs $\mathcal{B}_i(R)$. We say then that a strategy s_i in R is a ***never best response*** if

$$\forall \mu_i \in \mathcal{B}_i(R)\, \exists s'_i \in R_i\, p_i(s'_i, \mu_i) > p_i(s_i, \mu_i).$$

By a ***dominance relation*** D we mean a function that assigns to each restriction R a subset D_R of $\bigcup_{i=1}^n R_i$. Instead of writing $s_i \in D_R$ we say that s_i ***is*** D***-dominated in*** R. To avoid unnecessary complications we assume that for each restriction R and player i the set of his D-undominated strategies in R is non-empty, i.e., that for each i, $R_i \setminus D_R \neq \emptyset$. This natural assumption is satisfied by all considered dominance relations.

Given two restrictions R and R' we write $R \to_D R'$ when $R \neq R', R' \subseteq R$ and

$$\text{each strategy } s_i \in (\textstyle\bigcup_{j=1}^n R_j) \setminus (\bigcup_{j=1}^n R'_j) \text{ is } D\text{-dominated in } R.$$

An ***outcome*** of an iteration of $\to_D$ starting in a game G is a restriction R that can be reached from G using $\to_D$ in finitely many steps and such that for no R', $R \to_D R'$ holds.

We call a dominance relation D

- ***order independent*** if for all initial games G all iterations of $\to_D$ starting in G yield the same final outcome,

- ***hereditary*** if for all initial games G, all restrictions R and R' such that $R \to_D R'$ and a strategy s_i in R'

$$s_i \text{ is } D\text{-dominated in } R \text{ implies that } s_i \text{ is } D\text{-dominated in } R',$$

- ***monotonic*** if for all initial games G, all restrictions R and R' such that $R' \subseteq R$ and a strategy s_i in R'

$$s_i \text{ is } D\text{-dominated in } R \text{ implies that } s_i \text{ is } D\text{-dominated in } R'.$$

Clearly every monotonic dominance relation is hereditary, but the converse does not need to hold.

3 Proofs of order independence

We shall establish the following result.

Theorem 1 *Every hereditary dominance relation D is order independent.*

We relegate the proof in the appendix. This theorem can be used to prove in a straightforward way order independence of various dominance relations. We illustrate it now by means of various examples.

3.1 Strict dominance

Recall that a strategy s_i is ***strictly dominated*** in a restriction $(R_1, \ldots, R_n, p_1, \ldots, p_n)$ if for some strategy $s'_i \in R_i$

$$\forall s_{-i} \in R_{-i}\, p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}).$$

Denote the reduction relation corresponding to strict dominance by a pure strategy by $\rightarrow_S$. To see that strict dominance by a pure strategy is hereditary suppose that $R \rightarrow_S R'$ and that a strategy s_i in R' is strictly dominated in R. The initial game is finite, so there exists in R a strategy s'_i that strictly dominates s_i in R and is not strictly dominated in R. Then s'_i is not eliminated in the step $R \rightarrow_S R'$ and hence is a strategy in R'. But $R' \subseteq R$, so s'_i also strictly dominates s_i in R'.

In contrast, strict dominance is not monotonic for the simple reason that no strategy is strictly dominated in a game in which each player has exactly one strategy. Order independence of strict dominance for finite games was originally proved in Gilboa, Kalai and Zemel [1990], by relying on the notion of monotonicity (called there hereditarity) used for binary dominance relations. This approach works since a slightly different reduction relation is used there, according to which every eliminated strategy is strictly dominated in the original restriction by a strategy that is *not* eliminated. It is straightforward to check that this reduction relation coincides with $\rightarrow_S$.

3.2 Global strict dominance

We say that a strategy s_i is ***globally strictly dominated*** in a restriction $(R_1, \ldots, R_n, p_1, \ldots, p_n)$ if for some strategy $s'_i \in G_i$ (so *not* $s'_i \in R_i$)

$$\forall s_{-i} \in R_{-i}\, p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}).$$

This notion of dominance was originally considered in Milgrom and Roberts [1990, pages 1264-1265]. Its order independence for finite games is a consequence of a more general result proved in Ritzberger [2002, pages 200-201] and also Chen, Long and Luo [2007], where this dominance relation was analyzed for arbitrary games.

Global strict dominance is clearly monotonic, so it is hereditary.

3.3 Never best response

Suppose that each player i in the initial game G has a set of beliefs $\mathcal{B}_i$. Then in each restriction R we choose the corresponding set of beliefs $\mathcal{B}_i(R)$ of player i. For instance, if $\mathcal{B}_i = \Pi_{j \neq i} \Delta G_j$ then we set $\mathcal{B}_i(R) := \Pi_{j \neq i} \Delta R_j$.

Note that if $R' \subseteq R$, then we can identify $\Pi_{j \neq i} \Delta R'_j$ with a subset of $\Pi_{j \neq i} \Delta R_j$ and $\Delta R'_{-i}$ with a subset of ΔR_{-i}. So we can assume that $R' \subseteq R$ implies that $\mathcal{B}_i(R') \subseteq \mathcal{B}_i(R)$.

To prove that being a never best response is a hereditary dominance relation consider the corresponding reduction relation between restrictions that we denote by $\rightarrow_N$. Suppose that $R \rightarrow_N R'$ and that a strategy s_i in R' is a never best response in R. Assume by contradiction that for some $\mu_i \in \mathcal{B}_i(R')$, s_i is a best response to μ_i in R', i.e.,

$$\forall s'_i \in R'_i \; p_i(s_i, \mu_i) \geq p_i(s'_i, \mu_i).$$

We have $\mu_i \in \mathcal{B}_i(R)$ since $\mathcal{B}_i(R') \subseteq \mathcal{B}_i(R)$. Take a best response s'_i to μ_i in R. Then s'_i is not eliminated in the step $R \rightarrow_N R'$ and hence is a strategy in R'. But by the choice of s_i and s'_i

$$p_i(s'_i, \mu_i) > p_i(s_i, \mu_i),$$

so we reached a contradiction.

Order independence of iterated elimination of never best responses was originally proved in Apt [2005] by comparing it with the iterated elimination of global never best responses, a notion we discuss next.

3.4 Global never best response

Suppose again that each player i in the initial game G has a set of beliefs $\mathcal{B}_i$. Choose in each restriction R the corresponding set of beliefs $\mathcal{B}_i(R)$ of player i.

We say that a strategy s_i in a restriction R is a ***global never best response*** if

$$\forall \mu_i \in \mathcal{B}_i(R) \; \exists s'_i \in G_i \; p_i(s'_i, \mu_i) > p_i(s_i, \mu_i).$$

So in defining a global never best response we compare the given strategy with all strategies in the *initial* game and not the *current* restriction. Note that iterated elimination of global never best responses, when performed 'at full speed' yields the set of ***rationalizable strategies*** as defined in Bernheim [1984].

The property of being a global never best response is clearly monotonic, so it is hereditary. Order independence of this dominance relation was originally proved in Apt [2005] for arbitrary, so possibly infinite, games.

3.5 Strict dominance by a mixed strategy

Denote the corresponding reduction relation between restrictions by $\rightarrow_{SM}$. Given two mixed strategies m_i, m'_i and a strategy s_i we denote by $m_i[s_i/m'_i]$ the mixed strategy obtained from m_i by substituting the strategy s_i by m'_i and by 'normalizing' the resulting sum. First, we establish the following auxiliary lemma.

Lemma 1 (Persistence) *Assume that $R \rightarrow_{SM} R'$ and that a strategy s_i in R is strictly dominated in R by a mixed strategy from R. Then s_i is strictly dominated in R by a mixed strategy from R'.*

Proof. We shall use the following obvious properties of strict dominance by a mixed strategy in a given restriction:

(a) for all $\alpha \in (0,1]$, if s_i is strictly dominated by $(1-\alpha)s_i + \alpha\, m_i$, then s_i is strictly dominated by m_i,

(b) if s_i is strictly dominated by m_i and s'_i is strictly dominated by m'_i, then s_i is strictly dominated by $m_i[s'_i/m'_i]$.

Suppose that $R_i \setminus R'_i = \{t_i^1, \ldots, t_i^k\}$. By definition for all $j \in \{1, \ldots, k\}$ there exists in R a mixed strategy m_i^j such that t_i^j is strictly dominated in R by m_i^j. We first prove by complete induction that for all $j \in \{1, \ldots, k\}$ there exists in R a mixed strategy n_i^j such that

$$t_i^j \text{ is strictly dominated in } R \text{ by } n_i^j \text{ and } \mathit{support}(n_i^j) \cap \{t_i^1, \ldots, t_i^j\} = \emptyset. \tag{1}$$

For some $\alpha \in (0,1]$ and a mixed strategy n_i^1 with $t_i^1 \notin \mathit{support}(n_i^1)$ we have

$$m_i^1 = (1-\alpha)t_i^1 + \alpha\, n_i^1.$$

By assumption t_i^1 is strictly dominated in R by m_i^1, so by property (a) t_i^1 is strictly dominated in R by n_i^1, which proves (1) for $j = 1$.

Assume now that $\ell < k$ and that (1) holds for all $j \in \{1, \ldots, \ell\}$. By assumption $t_i^{\ell+1}$ is strictly dominated in R by $m_i^{\ell+1}$.

Let

$$m''_i := m_i^{\ell+1}[t_i^1/n_i^1]\ldots[t_i^\ell/n_i^\ell].$$

By the induction hypothesis and property (b) $t_i^{\ell+1}$ is strictly dominated in R by m''_i and $\mathit{support}(m''_i) \cap \{t_i^1, \ldots, t_i^\ell\} = \emptyset$.

For some $\alpha \in (0,1]$ and a mixed strategy $n_i^{\ell+1}$ with $t_i^{\ell+1} \notin \mathit{support}(n_i^{\ell+1})$ we have

$$m''_i = (1-\alpha)t_i^{\ell+1} + \alpha\, n_i^{\ell+1}.$$

By (a) $t_i^{\ell+1}$ is strictly dominated in R by $n_i^{\ell+1}$. Also $\mathit{support}(n_i^{\ell+1}) \cap \{t_i^1, \ldots, t_i^{\ell+1}\} = \emptyset$, which proves (1) for $j = \ell + 1$.

Suppose now that the strategy s_i is strictly dominated in R by a mixed strategy m_i from R. Define

$$m'_i := m_i[t_i^1/n_i^1]\ldots[t_i^k/n_i^k].$$

Then by property (b) and (1) s_i is strictly dominated in R by m'_i and $\mathit{support}(m'_i) \subseteq R'_i$, i.e., m'_i is a mixed strategy in R'. □

Hereditarity of $\rightarrow_{SM}$ is now an immediate consequence of the Persistence Lemma 1. Indeed, suppose that $R \rightarrow_{SM} R'$ and that $s_i \in R'_i$ is strictly dominated in R by a mixed strategy in R. By the Persistence Lemma 1 s_i is strictly dominated in R by a mixed strategy in R'. So s_i is also strictly dominated in R' by a mixed strategy in R'.

The proof of order independence of strict dominance by a mixed strategy due to Osborne and Rubinstein [1994, pages 61-62] relied on the existence of Nash equilibrium in strictly competitive games.

3.6 Global strict dominance by a mixed strategy

We say that a strategy s_i is ***globally strictly dominated by a mixed strategy*** in a restriction $(R_1, \ldots, R_n, p_1, \ldots, p_n)$ if for some mixed strategy m'_i in G_i (so *not* m'_i in R_i)

$$\forall s_{-i} \in R_{-i}\, p_i(m'_i, s_{-i}) > p_i(s_i, s_{-i}).$$

This notion of dominance was studied in Brandenburger, Friedenberg and Keisler [2006] (it is their operator Φ) and in Apt [2007]. It is obviously monotonic and hence hereditary.

The proof of order independence of this relation is implicit in Apt [2007]. Dominance relations are viewed there as operators on the set of all restrictions. Its Theorem 1 states that monotonic operators are order independent. Monotonicity property of the operator corresponding to global strict dominance by a mixed strategy is noted there in Section 10.

3.7 Inherent dominance

Consider a restriction $(R_1, \ldots, R_n, p_1, \ldots, p_n)$. We say that a strategy s_i is ***dominated given*** $\tilde{R}_{-i} \subseteq R_{-i}$, where $\tilde{R}_{-i}$ is non-empty, if s_i is weakly dominated in the restriction $(R_i, \tilde{R}_{-i}, p_1, \ldots, p_n)$. Then we say that a strategy s_i is ***inherently dominated*** if for every non-empty subset $\tilde{R}_{-i}$ of R_{-i} it is weakly dominated given $\tilde{R}_{-i}$.

This notion of dominance was introduced in Börgers [1990], where its order independence was proved by establishing a connection between inherent dominance and rationalizability. In Börgers [1993] parts of Börgers [1990] were published, but not the proof of order independence. Denote by $\rightarrow_I$ the corresponding reduction relation.

To prove that inherent dominance is hereditary suppose that $R \rightarrow_I R'$ and that a strategy s_i in R' is inherently dominated in R. Fix a non-empty subset $\tilde{R}_{-i}$ of R'_{-i}.

The initial game is finite, so there exists in R_i a strategy s'_i that weakly dominates s_i in $(R_i, \tilde{R}_{-i}, p_1, \ldots, p_n)$ and is not weakly dominated in the restriction $(R_i, \tilde{R}_{-i}, p_1, \ldots, p_n)$. Then s'_i is not eliminated in the step $R \rightarrow_I R'$ and hence is a strategy in R'_i. So s_i is weakly dominated in $(R'_i, \tilde{R}_{-i}, p_1, \ldots, p_n)$ by s'_i. This proves hereditarity.

4 Conclusions

We established here several order independence results. They were all proved by just checking a single property of the dominance relation, namely hereditarity. This approach works because of a combination of factors. First, as in Apt [2004], we used abstract reduction systems. This allowed us to decouple one part of the argument from the study of the actual dominance relations.

Second, we viewed the dominance relations as *unary* relations, whereas the common approach in the literature is to view them as *binary* relations. Finally, we relied on the notion of hereditarity that is weaker than monotonicity. These changes allowed us to treat various forms of strict dominance and of being a never response, and inherent dominance in a uniform way.

We conclude by offering the following observation. It is clear how to define the intersection of dominance relations. The intersection of order independent dominance relations does not need to be order independent. On the other hand, the intersection of hereditary dominance relations is clearly hereditary. So our approach also allows us to draw conclusions about order independence of intersections of the discussed dominance relations.

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Appendix

We present here the proof of Theorem 1. As in Apt [2004] we shall use the notion of an ***abstract reduction system***, extensively studied in Terese [2003]. It is simply a pair $(A, \rightarrow)$ where A is a set and $\rightarrow$ is a binary relation on A. Let $\rightarrow^*$ denote the transitive reflexive closure of $\rightarrow$. So in particular, if $a = b$, then $a \rightarrow^* b$.

We say that b is a $\rightarrow$***-normal form of*** a if $a \rightarrow^* b$ and no c exists such that $b \rightarrow c$, and omit the reference to $\rightarrow$ if it is clear from the context. If every element of A has a unique normal form, we say that $(A, \rightarrow)$ (or just $\rightarrow$ if A is clear from the context) satisfies the ***unique normal form property***.

We say that $\rightarrow$ is ***weakly confluent*** if for all $a, b, c \in A$

$$\begin{array}{ccc} & a & \\ \swarrow & & \searrow \\ b & & c \end{array}$$

implies that for some $d \in A$

$$\begin{array}{ccc} b & & c \\ \searrow^{*} & & {}^{*}\swarrow \\ & d & \end{array}$$

The following crucial lemma is due to Newman [1942].

Lemma 2 (Newman) *Consider an abstract reduction system $(A, \rightarrow)$ such that*

- *no infinite $\rightarrow$ sequences exist,*
- *$\rightarrow$ is weakly confluent.*

Then $\rightarrow$ satisfies the unique normal form property.

Proof. By the first assumption every element of A has a normal form. To prove uniqueness call an element a *ambiguous* if it has at least two different normal forms. We show that for every ambiguous a some ambiguous b exists such that $a \rightarrow b$. This proves absence of ambiguous elements by the first assumption.

So suppose that some element a has two distinct normal forms n_1 and n_2. Then for some b, c we have $a \rightarrow b \rightarrow^* n_1$ and $a \rightarrow c \rightarrow^* n_2$. By weak confluence some d exists such that $b \rightarrow^* d$ and $c \rightarrow^* d$. Let n_3 be a normal form of d. It is also a normal form of b and of c. Moreover $n_3 \neq n_1$ or $n_3 \neq n_2$. If $n_3 \neq n_1$, then b is ambiguous and $a \rightarrow b$. And if $n_3 \neq n_2$, then c is ambiguous and $a \rightarrow c$. $\square$

Clearly, order independence of a dominance relation D is equivalent to the statement that for all initial games G the reduction relation $\rightarrow_D$ satisfies the unique normal form property on the set of all restrictions.

Proof of Theorem 1.

Consider a restriction R. Suppose that $R \rightarrow_D R'$ for some restriction R'. Let R'' be the restriction of R obtained by removing all strategies that are D-dominated in R.

We have $R'' \subseteq R'$. Assume that $R' \neq R''$. Choose an arbitrary strategy s_i such that $s_i \in R'_i \setminus R''_i$. So s_i is D-dominated in R. By the hereditarity of D, s_i is also D-dominated in R'. This shows that $R' \rightarrow_D R''$.

So we proved that either $R' = R''$ or $R' \rightarrow_D R''$, i.e., that $R' \rightarrow_D^* R''$. This implies that $\rightarrow_D$ is weakly confluent. It suffices now to apply Newman's Lemma 2. $\square$