Narrowing the field in elections: the next-two rule

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Abstract

We suggest a new approach to narrowing the field in elections, based on the *deservingness* of candidates to be contenders in a runoff, or to be declared one of several winners. Instead of specifying some minimum percentage (e.g., 50) that the leading candidate must surpass to avoid a runoff (usually between the top two candidates), we propose that the number of contenders depend on the distribution of votes among candidates. Divisor methods of apportionment proposed by Jefferson and Webster, among others, provide measures of deservingness, but they can prescribe a runoff even when one candidate receives more than 50 percent of the vote.

We propose a new measure of deservingness, called the Next-Two rule, which compares the performance of candidates to the two that immediately follow them. It never prescribes a runoff when one candidate receives more than 50 percent of the vote. More generally, it identifies as contenders candidates who are bunched together near the top and, unlike the Jefferson and Webster methods, never declares that all candidates are contenders. We apply the Next-Two rule to several empirical examples, including one (elections to major league baseball's Hall of Fame) in which more than one candidate can be elected.

Narrowing the Field in Elections: The Next-Two Rule[1]

1. Introduction

In competitive elections with more than two candidates, it is often desirable to narrow the field to a few top contenders. This is usually done by establishing a minimum percentage, which in different jurisdictions has been set at 50, 45, or 40, that the leading candidate must exceed in order to win outright.[2] If no candidate achieves this threshold, then a runoff is held in which the top two, or sometimes more, candidates continue onto a next round of competition (Grofman, 2008).

But why, as is usually the case, should only the top two candidates make the runoff? If four candidates obtain 31, 30, 29, and 10 percent of the vote, should only the first two advance? In this situation, it would seem sensible that the top three should do so.

To be sure, limiting a runoff to the top two candidates ensures that one candidate will win a majority, except in the unlikely event of a tie, in the runoff. But if the third-place finisher is a centrist, sandwiched between a left and a right candidate, he or she may well be the Condorcet winner—able to defeat each of his or her opponents in separate pairwise contests—so there is little justification for eliminating him or her.[3]

In this paper, we argue for a theoretically justified set of rules, which depend on the *distribution* of all votes, instead of rules that specify (i) a minimum percentage of the

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[1] We thank Christian Klamler, Samuel Merrill III, and Scott Moser for their valuable comments on an earlier version of this paper.

[2] In its 1996 presidential election in Sierra Leone, the threshold was 55 percent (Bouton, 2011). We know of no other instance of a threshold exceeding 50 percent.

[3] There is strong evidence that this was the case in the U.S. Senate race in New York in 1970, in which the centrist candidate, Charles E. Goodell, who received 24 percent of the vote, lost to James R. Buckley on his right, who received 39 percent, and Richard L. Ottinger on his left, who received 37 percent (Brams and Fishburn, 1983/2007, pp. 121-123).

vote that must be attained to win outright and (ii) a maximum number of candidates that are eligible to continue to the next round. Our ideas apply not only to single-winner but also to multiwinner elections, including those in which the number of winners is not specified in advance.

For example, to be elected to major league baseball's Hall of Fame, a player must have been retired at least five, and at most twenty, years and be the choice of at least 75 percent of the voters, who are professional sports writers who cover baseball. The number of inductees varies from year to year, which seems justified because the list of eligible candidates changes from year to year. Harder to justify is the arbitrariness of the 75-percent threshold, which is attested to by the fact that some candidates who do not make the grade in their early years of eligibility make it later, though their performance statistics are unchanged.

In section 2, we show how divisor methods of apportionment can be used to define the "deservingness" of candidates, just as they are used to determine the deservingness of U.S. states to receive the numbers of seats that they do in the House of Representatives—and, analogously, for political parties to receive proportional representation in parliaments. We focus on the two most commonly used divisor methods, originally proposed by Thomas Jefferson and Daniel Webster, and argue that both methods, especially in 3-candidate races, are not well suited to determining when there should be a runoff. Among other reasons, each may call for a runoff even when one candidate receives more than 50 percent, and each may prescribe that all candidates should be in a runoff.

In section 3, we propose a new rule, which we call the "Next-Two rule," which is based on a different principle. The candidate with the most votes is always a "contender." If the sum of the votes of the candidate with the next-most votes and the candidate following him or her is at least equal to the leading candidate's vote, then the candidate with the next-most votes is also a contender. This rule for designating contenders extends to lower-ranked candidates, stopping only when the sum of the votes of the next two candidates is less than the vote of the candidate ahead of them. We also consider an extension of this rule to more candidates, including the next three, and later a tightening of the Next-Two rule if more than one candidate can be elected (e.g., to baseball's Hall of Fame).

Next-Two tends to encourage lesser candidates not to drop out of a race, thereby fostering greater competition. However, we show that it may produce either fewer or more contenders than the Jefferson and Webster methods, depending on the distribution of votes.

In section 4, we apply the Next-Two rule to four empirical examples:

1. The 1824 presidential election, in which no candidate won a majority of electoral votes, so an election was held in the House of Representatives among the top three candidates.

2. The 1977 New York City Democratic mayoral primary, in which there were six candidates who received between 11 and 20 percent of the vote; it was decided by a runoff between the top two and then followed by a general election with multiple candidates.

3. The 2008 Democratic presidential races in the Iowa caucuses and New Hampshire primary, which led to a rapid narrowing of the field.

4. The 2010 and 2011 elections to baseball's Hall of Fame, in which one and two candidates, respectively, were elected.

In section 5, we draw several conclusions. Because of its iterative nature, we argue that the Next-Two rule is better able to select a subset of closely bunched candidates than the Webster or Jefferson methods, which may eliminate contenders within such a bunch. Because a runoff may include more than two contenders under the Next-Two rule, we recommend that when this happens, approval voting be used to find the most acceptable candidate.

2. Divisor Methods of Apportionment

Divisor methods of apportionment have been used for more than 200 years to assign seats in the U.S. House of Representatives to the states, based on their populations, and seats in parliaments to political parties, based on the votes they receive. These methods ask: Which entity (e.g., state or political party) deserves the next seat to be awarded? But they differ on how they define "deservingness."[4]

We ask a related question to determine which candidates are "contenders," who have done well enough in an election that they deserve to become candidates in a runoff. But unlike seats that must be apportioned, we do not assume there is a prespecified number of contenders; instead, this number is determined endogenously. We define

[4] Another commonly used apportionment method, proposed by Alexander Hamilton, does not award seats sequentially but instead all at once, according to each entity's "quota," or the exact number of seats to which it is entitled. Under the Hamilton method, one cannot determine, at each step, the deservingness of entities to receive a seat, as one can under the divisor methods.

contenders to be the candidates who deserve first "seats," according to an apportionment method, before any candidate deserves a second "seat."

Seats, as such, are not awarded to contenders. Instead, a divisor apportionment method is used to determine whether the votes that a candidate receives entitle it to a first seat—were it a state or political party—before the leading candidate is entitled to a second seat. In other words, does a candidate do "well enough," compared to the leading candidate, to be considered a contender?

Call the candidates $c_1, c_2, c_3, \ldots$, and suppose that the score of candidate c_i is s_i. (The score of a candidate may be the number of votes received, or the percentage of the vote, or any appropriate measure of success in an election.) Assume that the candidates are renumbered, if necessary, so that $s_1 \geq s_2 \geq s_3 \ldots$. Let $S = s_1 + s_2 + s_3 + \ldots$ equal the total score of all candidates in the election. Mostly we assume these scores are percentages of the total vote.

The Webster and Jefferson methods allocate seats sequentially. Suppose that, at some stage, the number of seats assigned to candidate c_i is a_i. Then the allocation of the next seat is based on formulas that depend on the value of s_i and the *current* value of a_i.

Webster Method

The next seat is assigned to a candidate, c_i, who maximizes $W_i = \dfrac{s_i}{a_i + 1/2}$.[5] For

[5] The 1/2 in the denominator of W_i reflects the rounding of fractions equal to or greater than 1/2 under the Webster method. Under the Jefferson method, as we will see, the 1/2 is replaced by 1. These constants, when added to a_i in the denominators of W_i (and later J_i), lead to *stable allocations*: No transfer of one seat from one candidate to another reduces the inequality in representation among candidates, based on different measures of inequality. The three divisor methods, in addition to those of Webster and Jefferson, that produce stable allocations are all based on different criteria of deservingness; all favor smaller parties in varying degrees (Balinski and Young, 2001; Marshall, Olkin and Pukelsheim, 2002). Their deservingness measures (we drop the subscript *i* used in W_i and J_i) are $s/[a(a + 1)]^{1/2}$ for Hill or "equal proportions,"

any candidate c_i who has not yet received a seat, $a_i = 0$, so $W_i = 2s_i$, whereas for any candidate c_i who has already received one seat, $W_i = 2s_i/3$. Because all candidates begin with $a_i = 0$ seats, candidate c_1 is the first candidate to be allocated a seat. Thus, a contender is any candidate c_k for whom $s_k \geq s_1/3$.

Example 1. $(s_1, s_2, s_3) = (45, 35, 20)$. Because $s_3 > s_1/3 = 15$, all three candidates are contenders according to the Webster method.

Example 2. $(s_1, s_2, s_3) = (57, 22, 21)$. Because $s_3 > s_1/3 = 19$, it again follows that all three candidates are contenders according to the Webster method. But note that candidate c_1 receives more than 50 percent of the vote in this example.

Proposition 1. *Under the Webster method, there are at most* $\left\lfloor \frac{3S}{s_1} \right\rfloor - 2$ *contenders.*

Proof. Suppose there are exactly k contenders. Then

$$S = s_1 + s_2 + \ldots \geq s_1 + s_2 + \ldots + s_k \geq s_1 + (k-1)(s_1/3) = \frac{(k+2)s_1}{3},$$

from which $k \leq \frac{3S}{s_1} - 2$. The result follows because k must be an integer. ■

Corollary 1. *If* $s_1 > S/2$*, then there are at most 3 contenders.*

$s/[(2a(a + 1)/(2a + 1)]$ for Dean or "harmonic mean," and s/a for Adams or "smallest divisors." Whereas the Webster method (also known as Sainte-Laguë) is used in four Scandinavian countries, and the Jefferson method (also known as d'Hondt) in 18 other countries, none of the other three divisor methods is used, with the exception of the Hill or equal-proportions method, that is currently used to apportion seats to states in the U.S. House of Representatives. The nondivisor Hamilton method, also called "largest remainders" (see note 2), is used in nine countries (Blais and Massicotte, 2002; Cox, 1997).

Examples 2 shows that the upper bound of Corollary 1 can be achieved. If $s_1 = S/2$, then the upper bound of Corollary 1 must be increased by 1. For example, if $(s_1, s_2, s_3, s_4) = (30, 10, 10, 10)$, then all four candidates are contenders, according to the Webster method.

Similar corollaries can be obtained using other fractions. For instance, it is easy to show from Proposition 1 that if $s_1 > S/3$, there are at most 6 contenders.

Jefferson Method

The next seat is assigned to a candidate, c_i, who maximizes $J_i = \frac{s_i}{a_i + 1}$. For a candidate c_i who has not yet received a seat, $J_i = s_i$, whereas for any candidate c_i who has already received a seat, $J_i = s_i/2$. As with the Webster method, all candidates begin with $a_i = 0$ seats, so candidate c_1 receives the first seat. Thus, a contender is any candidate c_k for whom $s_k \geq s_1/2$.

Example 1. $(s_1, s_2, s_3) = (45, 35, 20)$. Because $s_3 < s_1/2 = 22.5 \leq s_2$, it follows that only candidates c_1 and c_2 are contenders according to the Jefferson method (recall that all three candidates were contenders under the Webster method).

Example 2. $(s_1, s_2, s_3) = (57, 22, 21)$. Because $s_3 < s_2 < s_1/2 = 28.5$, it follows that only candidate c_1 is a contender according to the Jefferson method (recall that all three candidates were contenders under the Webster method).

Proposition 2. *Under the Jefferson method, there are at most* $\left\lfloor \frac{2S}{s_1} \right\rfloor - 1$

contenders.

Proof. Suppose there are exactly k contenders. Then

$$S = s_1 + s_2 + \ldots \geq s_1 + s_2 + \ldots + s_k \geq s_1 + (k-1)(s_1/2) = \frac{(k+1)s_1}{2},$$

from which $k \leq \frac{2S}{s_1} - 1$. The result follows because k must be an integer. ■

Corollary 2. *If* $s_1 > S/2$*, then there are at most 2 contenders.*

The example, $(s_1, s_2, s_3) = (65, 34, 1)$, shows that the upper bound in Corollary 2 can be achieved, even though c_1 receives almost twice as many votes as c_2. If $s_1 = S/2$, then the upper bound in Corollary 2 must be increased by 1. For example, if $(s_1, s_2, s_3) = (50, 25, 25)$, then all three candidates are contenders according to the Jefferson method.

Similar corollaries can be obtained using other scores. For instance, it is easy to show from Proposition 2 that when $s_1 > S/3$, there are at most 4 contenders.

To summarize, if the score of the leading candidate exceeds 50 percent, there can be up to three contenders under the Webster method and up to two under the Jefferson method (Corollaries 1 and 2).[6] In the case of the Webster method, such a candidate can receive more than twice the score of any other contender yet be faced with a runoff (Example 2); under the Jefferson method, there can be three contenders only if the score of the leading candidate is at most equal to double the scores of the other contenders, as

[6] The well-known Laaski-Taagepera (1979) index of the "effective number of parties," if used as a measure of the number of contenders, can also indicate there to be more than two contenders if one candidate receives more than 50 percent of the vote (see Feld and Grofman, 2007, for a different interpretation of this index). Thus in Example 2, this index, which is defined as $1/\sum(s_i/S)^2$, has a value of 2.4, suggesting there to be at least two contenders when the leading candidate receives 57 percent of the vote and the next-leading candidates less than half as much (22 percent). Because of such nonintutive results in the identification of significant parties (or, in our case, serious contenders), Taagepera (2005) proposed this index be supplemented, but not replaced, by an "index of balance," which can vary between 0 and 1; it does not seem to have a ready interpretation in the identification of contenders.

illustrated by the (50, 25, 25) example.

The Webster method, which allows for up to three contenders even when one candidate's score exceeds 50 percent, is more lenient than the Jefferson method. (The three other divisor methods described in note 3 are even more lenient, allowing still weaker candidates to be contenders.) Because we know of no voting system in public elections that fails to elect the leading candidate if he or she obtains a majority of the vote (we will give an example in section 4 where this is not the case in a private election), we henceforth do not consider the Webster and Jefferson methods as narrowing-down methods—except to compare them with the Next-Two rule and a related rule in section 3.

To include all candidates in a runoff, as can happen under all the divisor methods, is tantamount to repeating the election. As we will see, the Next-Two rule never allows this. The lowest-scoring candidate in an election is always eliminated, but there is a cost for this certainty: There is no guarantee that a Condorcet winner always qualifies as a contender (see note 3).[7]

3. The Next-Two Rule

Definition

Like the Webster, Jefferson, and other divisor methods, the Next-Two rule makes deservingness the criterion to be a contender. The Next-Two rule is applied to the candidates in sequence, until it identifies a candidate who is not a contender; then it stops.

As before, suppose that the candidates are $c_1, c_2, \ldots c_n$. Formally, the procedure is as follows:

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[7] *No* narrowing-down procedure based on vote totals, including those of Webster and Jefferson, can preclude this possibility, because a Condorcet winner is defined in terms of the preferences of voters, not the numbers of votes they receive. In particular, it is always possible that a Condorcet winner receives the fewest votes when preferences are not taken into account.

(1) Candidate c_1 is always a contender.

(2) If $2 \leq h < n$, and if candidates $c_2, c_3, \ldots c_{h-1}$ are contenders, then candidate c_h is a contender if and only if $s_{h-1} \leq s_h + s_{h+1}$—that is, the sum of the scores of the two candidates below candidate c_{h-1} is greater or equal to this candidate's score.

While candidate c_1 is always a contender, candidate c_n never is, so the Next-Two rule always eliminates at least one candidate.

We next illustrate the Next-Two rule with several examples, comparing it with the Jefferson and Webster methods and proving a number of propositions. We do this first for elections with exactly three candidates, showing graphically when each procedure prescribes two or three (in the case of the Jefferson and Webster methods) contenders. Then we extend the analysis to four or more candidates.

Elections with $n = 3$ Candidates

Consider the examples we presented earlier:

Example 1. $(s_1, s_2, s_3) = (45, 35, 20)$. Because $s_2 + s_3 \geq s_1$, it follows that candidates c_1 and c_2 are contenders according to the Next-Two rule, coinciding with those given by the Jefferson method. The Webster method makes all three candidates contenders.

Example 2. $(s_1, s_2, s_3) = (57, 22, 21)$. Because $s_1 > s_2 + s_3$, it follows that only candidate c_1 is a contender, which again agrees with the Jefferson method. And once again, the Webster method makes all three candidates contenders.

To gain more insight into similarities and differences in the three procedures, for convenience we rename candidates c_1, c_2, and c_3 to be candidates A, B, and C, respectively. Their fractional shares of the votes are $x = s_1/S$, $y = s_2/S$, and $z = s_3/S$.

The Next-Two rule prescribes that both candidates A and B are contenders if and only if $y + z \geq x$; otherwise, the only contender is candidate A. Note that $x + y + z = 1$. Then an equivalent way to state the Next-Two rule is that candidate B is a contender if and only if

$$y + (1 - x - y) \geq x,$$

which is equivalent to $x \leq 1/2$.[8]

Thus, if candidate A receives more than 50 percent of the vote, then he or she is the only contender—and the outright winner of the election—and there will be no runoff. The theoretical justification for this requirement is that candidate B must at least be able to tie candidate A if (i) candidate C drops out and (ii) all of candidate C's supporters shift their support to candidate B.

Graphical Analysis

To obtain a more precise idea of how the Next-Two rule compares with the Jefferson and Webster methods for determining deservingness, we compare the regions in the x-y plane in which each procedure prescribes a runoff between two or three (in the

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[8] A "double complement rule" proposed by Shugart and Taagepera (1994) also requires that $x \leq 50$. Their rule compares the performance of candidate A to that of candidate B by requiring that $x > y/2 + 25$ for the A to be the outright winner; otherwise, there will be a runoff between A and B. To illustrate, if $x = 40$, the candidate A will be the outright winner if candidate B receives $y < 30$, whereas if $y \geq 30$, both A and B will be contenders, leading to a runoff. Although this seems a reasonable rule to determine whether candidate B performs well enough to force a 2-candidate runoff, it does not address the question of how many additional candidates can qualify as contenders—and so be in a runoff—as do divisor methods and the Next-Two rule.

case of the Jefferson and Webster methods) candidates. For three candidates, this can be done graphically.

The fractions x, y, and z can be assumed to be non-negative and satisfy certain additional conditions. First, $x + y + z = 1$, or $z = 1 - x - y$, because there are only three candidates. Also, because of the assumed order of finish, $x \geq y \geq z$. In particular, note that $y \geq z$ is equivalent to $y \geq 1 - x - y$, or $1 - x - 2y \leq 0$.

Combining these inequalities produces a triangle in the x-y plane that contains all possible 3-candidate elections. This triangle, shown in Figure 1, has vertices (1, 0), (1/2, 1/2), and (1/3, 1/3). The area of the triangle can be calculated by noting that it is a right triangle, with the right angle at (1/2, 1/2). The side from (1/2, 1/2) to (1, 0) has length $\sqrt{2}/2$, and the side from (1/2, 1/2) to (1/3, 1/3) has length $\sqrt{2}/6$, so the area of the triangle is $1/2 \times \sqrt{2}/2 \times \sqrt{2}/6 = 1/12$.

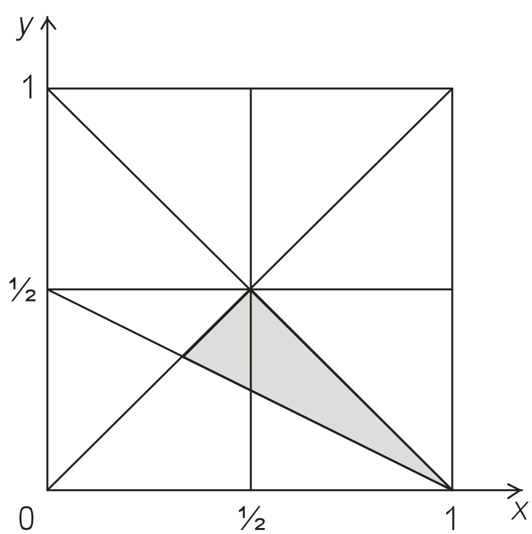

Figure 1. *Shaded Triangle Contains All 3-Candidate Elections in Which the Vote Fractions of Candidates A, B, and C Are x, y, and z = 1 – x – y*

The regions requiring runoffs, according to the Next-Two rule and both the Jefferson and Webster methods, are located inside this triangle, which is shown in Figure 2. The lighter-shaded regions indicate a runoff between two contenders, whereas the darker-shaded areas (for the Jefferson and Webster methods only) indicate a runoff among three contenders. For the Next-Two rule, as we showed earlier, there are a maximum of two contenders.

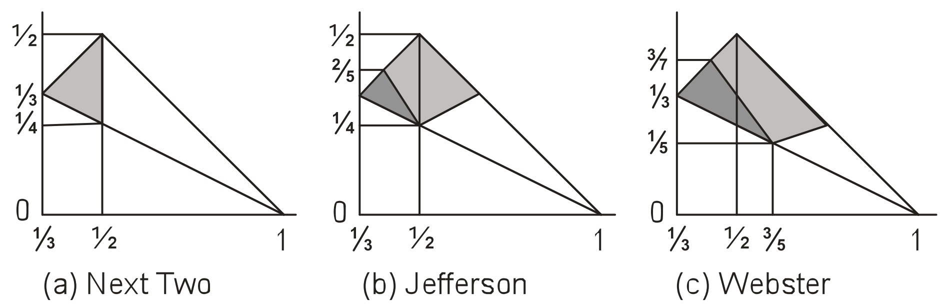

Figure 2. *Regions Requiring Runoffs: Lighter-Shaded Regions Indicate a Runoff between Two Contenders, and Darker-Shaded Regions a Runoff among Three Contenders*

Next-Two Rule

The Next-Two Rule requires a runoff if and only if $x \leq y + z$. As noted earlier, this condition is equivalent to $x \leq 1/2$, which is to say that candidate A does not receive a majority. The lighter-shaded triangle in Figure 2(a) shows this region and has vertices (1/2, 1/2), (1/3, 1/3), and (1/2, 1/4). To calculate its area, note that the distance from (1/3, 1/3) to (1/2, 1/4) is $\sqrt{5}/12$, whereas the distance from (1/3, 1/3) to (1, 0) is $\sqrt{5}/3$. Therefore, the area of the Next-Two triangle is 1/4 of the area of the triangle in Figure 1, or 1/48. If all possible values (x, y) are assumed to be equiprobable, this calculation implies a 2-candidate runoff in 1/4 of all elections under the Next-Two rule.

Jefferson Method

There are three contenders if $x/2 \le z/1$. Substituting $z = 1 - x - y$ produces the condition $2 - 3x - 2y \ge 0$, which is the darker-shaded triangle in Figure 2(b). This region has vertices (1/3, 1/3), (1/2, 1/4), and (2/3, 2/5). To calculate its area, note that the distance from (1/3, 1/3) to (2/5, 2/5) is $\sqrt{2}/15$, whereas, as already noted, the distance from (1/3, 1/3) to (1/2, 1/2) is $\sqrt{2}/6$. Therefore, the area of this region is 2/5 of the area of the Next-Two triangle in Figure 2(a) and so equals 1/120.

There are two contenders if $z < x/2 \le y$, which defines the lighter-shaded region in Figure 2(b) (adjacent to the darker-shaded triangle). The part of the quadrilateral that overlaps the Next-Two region is 3/5 of the area of the Next-Two triangle in Figure 2(a) and so equals 1/80. The remainder of the quadrilateral is in the region in which $x > 1/2$, wherein candidate A receives more than 50 percent of the vote.

Webster Method

As with the Jefferson method, there may be either two or three contenders in a runoff. We do not give the details here, but Figure 3(c) makes clear that whether there are three contenders (darker-shaded triangle) or two (lighter-shaded remainder of the quarilateral), candidate A can receive more than 50 percent of the vote.

Comparisons for *n* = 3 Candidates

The foregoing analysis proves the following proposition:

Proposition 3. *If there are three candidates, the conditions for a runoff are* *strictest for the Next-Two rule, followed by the Jefferson method and then the Webster method. Whereas the Next-Two rule always specifies a maximum of two contenders, both the Jefferson and Webster method allow for three contenders. However, only the Webster method allows for three contenders if candidate A receives more than 50 percent of the vote.*

Because a runoff among three contenders would be just a rerun of the original election, runoffs under the Jefferson and Webster methods could be restricted to the top two candidates, even when they prescribe three contenders. Given this restriction, however, each method still allows a runoff when candidate A receives more than 50 percent of the vote, which seems uncalled for to choose a winner, the case of Sierra Leone notwithstanding (see note 2).

Elections with $n \geq 4$ Candidates

Both the Next-Two rule, and the Jefferson and Webster methods, are easy to apply when there are more than three candidates. Recall that to assess the deservingness of candidates B, C, and subsequent candidates under the Jefferson or Webster methods, the candidate's score is compared to the score of candidate A. To assess the deservingness of a candidate under the Next-Two rule, the candidate's score, plus the score of the subsequent candidate, is compared to the preceding candidate's score.

Thus, the Next-Two rule has a *moving* yardstick, whereas the Jefferson and Webster rules have a *fixed* yardstick. But the Next-Two rule has an additional requirement: A candidate cannot be judged deserving unless all the preceding candidates (whose scores are at least as high) are also judged deserving.[9]

Example 3. $(s_1, s_2, s_3, s_4) = (31, 30, 29, 10)$, which was our original example. Here the Next-Two rule, and the Jefferson and Webster methods, all prescribe the top three candidates (A, B, and C) to be contenders, leaving out only candidate D. The Next-Three rule (see note 7) also shows these candidates to be the contenders.

Example 4. $(s_1, s_2, s_3, s_4) = (42, 22, 19, 17)$. According to the Next-Two rule, the only contender is candidate A. According to the Jefferson method and the Next-Three rule, the contenders are candidates A and B. According to the Webster method, all four candidates are contenders.

Example 5. $(s_1, s_2, s_3, s_4) = (50, 48, 1, 1)$. According to the Next-Two rule, the only contender is candidate A. According to the Jefferson and Webster methods as well as the Next-Three rule, the contenders are candidates A and B.

In both Examples 4 and 5, the Next-Two rule gives a different result from all the other procedures. Whereas candidate A stands apart from the other candidates in Example 4, it might seem less justifiable to single him or her out in Example 5.

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[9] If there are four or more candidates, a less stringent rule would be the Next-Three rule, whereby candidate B is a contender if the sum of the votes of the candidates B, C, and D is at least equal to the number of votes of the candidate A. Note that the Next-Two rule requires that, to be a contender, candidate B must get at least 1/2 as many votes as candidate A, whereas the Next-Three rule requires that candidate B get at least 1/3 as many. Next-Three could be extended to all remaining candidates, which would be even more lenient in permitting contenders; more restrictive would be to require that all remaining candidates must beat the frontrunner instead of the candidate just ahead of them. In fact, both these rules give the same contenders as does Next-Two if there are three candidates, but not if there are more than three. We focus on Next-Two subsequently because of (i) its simplicity, (ii) its flexibility in allowing fewer or more contenders than the Webster and Jefferson methods (Propositions 4 and 5), and (iii) the inducement it gives to nonleading candidates to encourage the next-lower candidate to stay in a race (Proposition 6), thereby increasing competitiveness.

However, note in this example that candidates B, C, and D, even together, do not receive more votes than candidate A, so it seems quite defensible to make candidate A, who is just short of a majority, the sole contender.

If the Next-Two rule and the Jefferson and Webster methods indicate a different number of contenders, is the Next-Two rule always more selective (as it is in the 3-candidate case, when the Jefferson and Webster methods prescribe three contenders but the Next-Two rule at most two)? Propositions 5 and 6 show that the answer is "no."

Proposition 4. *The Next-Two rule can select more contenders than the Jefferson method if and only if there are four or more candidates.*

Proof. To show that this proposition does not hold if there are three candidates, see Figure 2. To show next that the Next-Two rule can prescribe a larger number of contenders than the Jefferson method, consider the following example:

Example 6. $(s_1, s_2, s_3, s_4) = (45, 24, 22, 9)$. Because $24 + 22 \geq 45$, candidate B is a contender under the Next-Two rule. Because B is, so is candidate C, since $22 + 9 \geq 24$. Consequently, candidates A, B, and C are all contenders under the Next-Two rule. But under the Jefferson method, only candidates A and B are contenders since candidate C receives less than 1/2 the support of A. This example can readily be extended to examples with more than four candidates. ■

Thus, neither the Next-Two rule nor the Jefferson method necessarily prescribes fewer (or more) contenders than the other procedure when there are at least four candidates.

Proposition 5. *The Next-Two rule can select more contenders than the Webster method if and only if there are five or more candidates.*

Proof. To show that this proposition does not hold if there are four candidates, return to the (x, y, z, w) notation for the score fractions of the players, and suppose that (x, y, z, w) produces more contenders under the Next-Two rule than under Webster method. If the only contender under Webster method is candidate A, then we have $y < x/3$. But $y \geq z$, so $y + z < 2x/3 < x$; hence, the Next-Two rule also prescribes that the only contender is candidate A.

Now suppose that Webster method produces candidates A and B as contenders. Then $y \geq x/3$ while $w \leq z < x/3$. In order for the Next-Two rule to produce candidates A, B, and C as contenders, we must have $y + z \geq x$ and $z + w \geq y$. But the first condition implies that $y \geq x - z > 2x/3$, which in turn means that $z + w \leq 2z < 2x/3 < y$, contradicting the second condition. Therefore, if the Webster method produces a runoff between candidates A and B, the Next-Two rule cannot produce a runoff among candidates A, B, and C.

To show next that the Next-Two rule can prescribe a larger runoff than the Webster method, consider the following example:

Example 7. $(s_1, s_2, s_3, s_4, s_5) = (40, 22, 19, 10, 9)$. It is easy to verify that candidates A, B, C, and D are all contenders under the Next-Two rule. But under the Webster method, only candidates A, B, and C are contenders since candidate D receives fewer than 1/3 the votes of candidate A. This example can readily be extended to examples with more than five candidates. ■

Taken together, Propositions 4 and 5 show that, given a sufficient number of candidates, the Next-Two rule is neither more nor less selective than the Jefferson or Webster methods.

Proposition 6. *Under the Next-Two rule, if any nonleading candidate drops out and if his or her votes are either lost or distributed proportionately to the other candidates, then no noncontender can become a contender.*

Proof. Suppose that, initially, candidates $c_1, c_2, c_3, \ldots c_{h-1}$ are contenders and candidate c_h is not, where $h < n$. It must be the case that $s_{h-1} > s_h + s_{h+1}$. If nonleading candidate j drops out, then the vote counts can be considered to be $s_1, s_2, \ldots s_n$, with s_j missing. If $j < h$, the contenders will be $c_1, c_2, \ldots c_{j-1}$ if $s_{j-1} > s_{j+1} + s_{j+2}$, or $c_1, c_2, \ldots c_{h-1}$ otherwise. If $j \geq h$, the contenders will remain $c_1, c_2, c_3, \ldots c_{h-1}$, because it will still be the case that s_{h-1} is strictly greater than the sum of the vote counts of the two candidates immediately following candidate c_{h-1}, one of whom—if $j = h$ or $j = h + 1$—will be replaced by an even lower -scoring candidate. ■

We conclude that the Next-Two rule gives nonleading candidates an incentive to encourage candidates below them *not* to drop out.

Example 6 illustrates Proposition 6. If the 22-candidate drops out, the 24-candidate will no longer be a contender. Likewise, if the 9-candidate drops out, the 22-candidate will no longer be a contender. Thus, nonleading candidates have an incentive to encourage the next-lower candidate to stay in the race, because they can lose their contender status if he or she drops out. (This is not true of the Webster or Jefferson methods, because a contender's status does not depend on whether another candidate stays in or drops out but only on his or her votes compared to those of the leading candidate.) Thus, Next-Two fosters competition, because every candidate—except for the leading candidate, whose contender status does not depend on lesser candidates—will be encouraged by the next-higher candidate to stay in the race.

In section 5 we apply the Next-Two rule to a several empirical examples, in all of which there were at least two contenders. In the last example, more than one candidate can be elected. For this example, we suggest a possible modification to the Next-Two rule to make the choice of contenders more selective.

4. Empirical Examples

1824 Presidential Election

This is the only U.S. presidential election in which, based on electoral votes, there were two contenders, and almost a third, according to the Next-Two rule:[10]

Candidates	Andrew Jackson	John Quincy Adams	William H. Crawford	Henry Clay
Elect. Votes	99 (38%)	84 (32%)	41 (16%)	37 (14%)

The Next-Two rule fails to make Crawford a contender, because the sum of his votes and Clay's votes (78) are fewer than Adams's votes (84)—but not by much.

Because no candidate won a majority (131) of the 261 electoral votes, the election was thrown into the House of Representatives, which the Twelfth Amendment (1804) prescribes must not exceed three candidates.[11] The Twelfth Amendment also prescribes

[10] If popular votes were the measure of performance, then there was one presidential election in which there were three contenders. In 1912, the top four candidates were Woodrow Wilson (42 percent), Theodore Roosevelt (27 percent), William Howard Taft (23 percent), and Eugene Debs (6 percent), with less than 2 percent going to lesser candidates. According to the Next-Two rule, the top three candidates were contenders and, hence, should compete in a runoff. But because Wilson won a majority of electoral votes (82 percent), this election, unlike that of 1824, never went to the House of Representatives. This was also true of the 2000 election, but only after George W. Bush was awarded all of Florida's electoral votes on December 12. In the two other elections in which no candidate received a majority of electoral votes initially, one was decided in the House (1800) and the other by a specially appointed commission (1876), but each of these involved only two contenders.

[11] It is noteworthy that the Founding Fathers recognized that a runoff need not be between just the top two candidates, even though this is the norm, almost without exception, today. It is also noteworthy that in one earlier presidential election—that of 1800—it took 36 ballots in the House of Representatives before one of the two contenders, Thomas Jefferson, defeated the other, John Adams (who had previously been

that each of the states (there were 24 at the time) cast one vote in the House, so state delegations had to decide, usually by a majority vote within the state delegation, how they would cast their single vote in the House election.

The question in this election became which contender the eliminated candidate, Clay, would favor (he won 3 states; Crawford won 4, Jackson 7, and Adams 10).[12] Clay, who detested Jackson, threw his support to Adams in the so-called corrupt bargain (Clay became the secretary of state under Adams), giving Adams exactly the 13 votes he needed to win with a bare majority of the 24 votes.[13]

1977 New York City Democratic Mayoral Primary and General Election

In the 1977 New York City Democratic mayoral primary, there were six major candidates, five of whom are contenders according to the Next-Two rule:

Candidates	Edward I. Koch	Mario Cuomo	Abraham D. Beame	Bella Abzug	Percy Sutton	Herman Badillo
% Votes	19.8	18.7	18.0	16.6	14.4	11.0

A scattering of the remaining votes (2.6 percent) went to other candidates, but these—combined with Badillo's 11.0 percent—would fail to make Badillo a contender.

Rarely, of course, are there so many contenders, whichever narrowing procedure is used, which raises the question of what voting procedure should be used to select a winner in a crowded runoff (we will return to this question in the concluding section). In

president). A resolution of this election was complicated by the votes cast for each party's vice-presidential candidates.

[12] Note that if Clay were able to be a candidate in the House, Crawford would be a contender according to the Next-Two rule, because the sum of his votes and Clay's (7) equals Jackson's total.

[13] For two different rational-choice explanations of why the states that Clay controlled went for Adams over Jackson—apart from Clay's endorsement of Adams and the possibility of a corrupt bargain between them—see Riker (1962, pp. 149-158) and Brams (1978/2007, pp. 158-166/108-114).

the actual runoff between the top two candidates two weeks after the primary, Koch beat Cuomo 55-45 percent.

Six weeks later Koch won the general election with 49.99 percent of the vote, beating Cuomo, who chose to run again as the Liberal party nominee and who received 40.97 percent of the vote; a Republican and a Conservative party candidate trailed far behind, each obtaining about 4 percent. This time, however, even though Koch failed to win a majority, he became the winner since there was no provision for a runoff in the general election.

2008 Presidential Election: Democratic Races in Iowa Caucuses and New Hampshire Primary

The main contestants in these two races were Barack Obama (O), Hillary Clinton (C), John Edwards (E), and John Richardson (R). On January 3 in Iowa, (O, C, E, R) received (38, 30, 29, 2) percent of the votes in the caucuses, leaving only 1 percent for other candidates. Under the Next-Two rule, Obama, Clinton, and Edwards are the contenders.

In fact, all four candidates continued onto New Hampshire, which had a primary on January 8. The returns there were (C, O, E, R) = (39, 36, 17, 5) percent of the votes, leaving 3 percent for minor candidates. In this election, the Next-Two rule shows only Clinton and Obama to be the contenders.

Richardson dropped out after New Hampshire, whereas Edwards continued for three more weeks, after which he "suspended" his campaign after doing poorly in caucuses and primaries in four more states. While both candidates held out somewhat longer than predicted by the Next-Two rule, the reality of their nearly hopeless situations eventually pushed both out of the race.

Major League Baseball of Fame

As we indicated in section 1, entry into major league baseball's Hall of Fame requires the support of at least 75 percent of the voters, who are at least 10-year veterans of the Baseball Writers Association of America (BBWAA). The ballot includes players who (i) retired not more than 20 years earlier, were candidates from the previous year's ballot, and received at least 5 percent of the vote on that ballot but were not elected and (ii) selected players, chosen by a screening committee, who retired exactly five years earlier. Each writer can vote for a maximum of 10 candidates.

In the balloting for 2011, there were 581 voters, who cast an average of 5.98 votes. Two candidates, Roberto Alomar and Bert Blyleven, were elected with the support of 90.0 and 79.7 percent of voters, respectively (Baseball Hall of Fame Balloting, 2010, 2011).

Curiously, Alomar and Blyleven just missed the 75-percent mark in the 2010 election, obtaining, respectively, 73.7 and 74.2 percent. They were bested that year by one player, Andre Dawson, who was supported by 77.9 percent and was the only player elected in 2010 (Dawson had previously failed to attain 75 percent in 2008 and 2009). In rare elections, the most recent being 1996, no players have been elected.

Instead of assuming that players must reach the 75-percent mark, consider what the Next-Two rule would prescribe. In 2011, 14 players would have been elected; the 14^{th} player, who received 11.7 percent support, was the first player to receive a higher percentage than the combined percentages of the two players that followed him (with 6.1

and 4.1 percent, respectively), thereby precluding any lower-ranked players from being contenders. In 2010, the top 16 players would have made the grade, with the 16th player obtaining 11.0 percent support.

In making these projections for 2010 and 2011, we assumed that the voting behavior of BBWAA members would not have changed if the Next-Two rule had been in place. But surely it would have, assuming that most members do not consider 14-16 new players worthy of membership in the Hall of Fame.

Might the Next-Two rule be modified to allow for admission of only a few of the most deserving players? At the same time, might the modified procedure solve the "problem" of 2010, when only one player was elected but two others, based on their support, were almost as deserving?

In particular, it seems unfair that only Dawson (77.9 percent)—and not also Blyleven (74.2 percent) or Alomar (73.7 percent)—was elected when the 4th candidate (Jack Morris, with 52.3 percent) was far below the top three vote-getters, who were separated by less than 5 percentage points. To be sure, the apparent injustice to Blyleven and Alomar in 2010 was rectified by their election in 2011, but it seems preferable to have a procedure that, when it elects one player, also elects others who get nearly as much support. On the other hand, when one player stands out by getting far more support than any others, shouldn't he alone be elected?

Clearly, the Next-Two rule, by electing 16 and 14 players in 2010 and 2011 if the voting behavior of the voters had not changed, would not have done this. Can it be made more selective? We suggest that instead of assuming that the sum of the votes of candidates B and C must at least equal the votes of candidate A in order that B be a contender, assume that a fraction $\lambda < 1$ of this sum is at least equal to the vote of candidate A.

To illustrate in 2010, for the top triple,

$$\lambda[74.2 \text{ (Blyleven)} + 73.7 \text{ (Alomar)}] \geq 77.9 \text{ (Dawson)}, \tag{1}$$

or $\lambda \geq 0.53$, in order for Blyleven to be a contender after Dawson, who received more than 75 percent support, is elected. For the next triple,

$$\lambda[73.7 \text{ (Alomar)} + 52.3 \text{ (Morris)}](k) \geq 74.2 \text{ (Blyleven)}, \tag{2}$$

or $\lambda \geq 0.59$, in order for Alomar to be a contender when Blyleven is. And for the next triple,

$$\lambda[52.3 \text{ (Morris)} + 47.3 \text{ (Lee Smith)}] \geq 73.7 \text{ (Alomar)}, \tag{3}$$

or $\lambda \geq 0.74$, in order for Morris to be a contender when Alomar is. Inequalities (1) and (2) show that the 2nd and 3rd candidates in 2010—all of whom received more than 70 percent support, but only one of whom passed the threshold of 75 percent—would have made the grade if $\lambda = 0.59$, but the 4th candidate, Morris, who was considerably lower at 52.3 percent, would not have, because $\lambda \geq 0.74$ is necessary in order to satisfy inequality (3).

Raising 0.59 to 0.60 for simplicity, we define a "60 percent" Next-Two rule, using the 2010 election as an example. When $\lambda = 0.60$, then inequalities (1) and (2) are satisfied, but (3) is not, so the top three candidates would be elected.

Contrast this result with one in which the leading candidate receives 100 percent support, which in fact has never happened (99 percent support has been achieved only a few times). If Dawson came in second with the 77.9 percent—the amount he actually received—and Blyleven had followed suit, then Dawson would not have made the grade, because

$$(0.60)[77.9\text{ (Dawson)} + 74.2\text{ (Blyleven)}] = 91.3 < 100.$$

In this case, only the 100-percent candidate would have been elected, because the gap that separates him from the next two candidates is too great.

More generally, assume that major league baseball retains the 75 percent threshold, but only for the election of the top vote-getter. Given his election, assume it uses the "60 percent" Next-Two rule—or some other value of $\lambda < 1$—to determine who else, if anybody, is also elected. Other players would then be elected if they did almost as well as the top player, as happened in 2010, but no others would be elected if they didn't come close, as in our preceding example in which only the top player, with 100 percent support, would have been elected.

The modified Next-Two rule, because of its iterative nature, recognizes the closeness of players like Blyleven and Alomar in 2010 and selects both. By the same token, it recognizes the disparity between Dawson and a hypothetical 100 percent player and would select only the 100 percent player. On the other hand, the Webster and Jefferson methods, by asking whether candidates are within 1/3 or 1/2 of the percentage support of the top player (at or above 75 percent), would elect candidates just above this mark but not candidates just below it.

When more than one candidate, such as to baseball's Hall of Fame, can be elected, the Next-Two rule takes better account of the distribution of top players than any of the divisor methods of apportionment. Because Next-Two rule is iterative, it stops selection when there is a substantial gap in the distribution, whereas the divisor methods set the deservingness threshold without regard to gaps between adjacent players, because only the top candidate's performance is the standard.

5. Summary and Conclusions

We suggested a new approach to narrowing the field in elections, based on the deservingness of candidates to be contenders. Instead of specifying in advance the number of candidates to be in a runoff (usually two) if the leading candidate does not surpass a particular percentage (usually 50), we proposed that the number in the runoff depend on the distribution of votes among candidates. If one candidate stands well above the others, he or she should be the sole contender, but if there are two or more candidates who are bunched together at or near the top, most or all should be contenders.

The apportionment methods of Webster and Jefferson are based on different measures of deservingness. We rejected both methods in the end, because they allow candidate B to be a contender even if (i) candidate A receives more than 50 percent of the vote or (ii) this percentage is twice or thrice as much as that received by each of the other candidates.

The procedure we introduced that was not an apportionment method, the Next-Two rule, uses an iterative criterion to determine contenders. Starting with the top triple of candidates, the sum of the votes of candidates B and C must be at least equal to the vote of candidate A for B to be a contender. When, for some triple, this chain is broken as one goes down the list of candidates, then no additional candidates are contenders.

This rule allows for at most $n - 1$ contenders if there are n candidates. Thus, if there are three candidates, at most two can be contenders, whereas the Webster and

Jefferson methods can permit all three candidates to be contenders, which would mean just a rerun of the original election.

Although the Jefferson and Webster methods allow for more contenders than the Next-Two rule if there are three candidates, this result is not general. The Next-Two rule can select more contenders than the Jefferson and Webster methods if and only if there are at least four (Jefferson) or five (Webster) candidates. Thus, given a sufficient number of candidates, the Next-Two rule is neither more nor less selective than the Jefferson or Webster methods. It also induces candidates to discourage lesser candidates from dropping out, making races more competitive.

In effect, apportionment methods use only the performance of the top candidate as a yardstick—do other candidates receive at least some fraction as many votes as he or she does? The Next-Two rule compares the performance of a candidate to the two below him or her—can the latter, together, defeat him or her? We indicated that the Next-Two rule could be loosened to allow for more lower-ranked candidates (e.g., the next three), whose supporters, combined, would be able to defeat the next-higher candidate.

If there are more than two contenders in a runoff, it is possible that no candidate will win a majority and, hence, become the outright winner according to the Next-Two rule. In this case, we recommend that approval voting (Brams and Fishburn, 1983/2007; Brams, 1978/2007, 2008) be used in the runoff (it could be used in the initial election as well) to increase the probability that the leading candidate wins a majority in the runoff.[14]

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[14] Assuming that votes of eliminated candidates are lost or distributed proportionately, repeatedly applying the Next-Two rule to a diminishing set of contenders eventually makes the leading candidate the winner. Thus, its repeated use is not effective in choosing other than the plurality winner. Instead, it is better used to identify a set of contenders at the outset. If there are more than two, then some other procedure (e.g., approval voting) can be used to choose a winner—or perhaps several winners—from amongst the contenders.

Winning majority approval, or the greatest majority approval if two or more candidates are majority-approved, would enhance the legitimacy of the leading candidate's victory.

Our empirical examples illustrated how the Next-Two rule would have worked in some presidential elections, including a presidential caucus and a presidential primary. A mayoral primary showed how it would have somewhat narrowed the field, in which case approval voting would seem an appropriate way to single out a winner in a runoff with several contenders.

We also showed the applicability of the Next-Two rule to elections, like that to baseball's Hall of Fame, in which there are a variable number of winners. To ensure that this number is not too large, we illustrated how it could be modified to make it more selective. A future task will be to analyze equilibrium strategies when, as with Next-Two, the number of contenders is determined by a moving yardstick rather than one that is fixed, as Bouton (2011) assumes in his strategic analysis.

In conclusion, the Next-Two rule seems a better way to narrow the field than setting some arbitrary vote threshold or stipulating a certain number of finalists. It takes seriously the distribution of votes among candidates, which present practices do not do, and draws the line where there is a sufficiently large gap among the top candidates.

References

Balinski, Michel L, and H. Peyton Young (1982/2001). *Fair Representation: Meeting the Ideal of One Man, One Vote*, 1st ed./2nd ed. New Haven, CT: Yale University Press/Washington, DC: Brookings Institution.

Baseball Hall of Fame Balloting (2010, 2011). http://en.wikipedia.org/wiki/Baseball_Hall_of_Fame_balloting,_2011

Blais, André, and Louis Massicotte (2002). "Electoral Systems." In Lawrence LeDuc, Richard S. Niemi, and Pippa Norris (eds.), *Comparing Democracies 2: New Challenges in the Study of Elections and Voting*. London: Sage, pp. 40-69.

Bouton, Laurent (2011). "A Theory of Strategic Voting in Runoff Elections." Preprint, Department of Economics, Boston University.

Brams, Steven J. (1978/2007). *The Presidential Election Game,* 1st ed./2nd ed. New Haven, CT: Yale University Press/Natick, MA: A K Peters.

Brams, Steven J. (2008). *Mathematics and Democracy: Designing Better Voting and Fair-Division Procedures*. Princeton, NJ: Princeton University Press.

Brams, Steven J. and Peter C. Fishburn (1978). "Approval Voting," *American Political Science Review* 72, no. 3 (September): 831-847.

Brams, Steven J., and Peter C. Fishburn (1983/2007). *Approval Voting*, 1st ed./2nd ed. Cambridge, MA: Birkhäuser Boston/New York: Springer.

Cox, Gary W. (1997). *Making Votes Count: Strategic Coordination in the World's Electoral Systems*. Cambridge, UK: Cambridge University Press.

Feld, Scott L., and Bernard Grofman (2007). "The Laakso-Taagepera Index in a Mean and Variance Framework, *Journal of Theoretical Politics* 19, no. 1: 101-106.

Grofman, Bernard (2008). "A Taxonomy of Runoff Methods," *Electoral Studies* 27, no. 3: 395-399.

Laasko, Markku,, and Rein Taagepera (1979). "'Effective' Number of Parties: A Measure with Application to West Europe," *Comparative Political Studies* 12, 1 (April): 3-27.

Marshall, Albert W., Ingram Olkin, and Friedrich Pukelsheim (2002). "A Majorization Comparison of Apportionment Methods in Proportional Representation," *Social Choice and Welfare* 19, no 4 (October): 885-900.

Riker, William H. (1962). *The Theory of Political Coalitions*. New Haven, CT: Yale University Press.

Shugart, Matthew Sobereg, and Rein Taagepera (1994). "Plurality Versus Majority Election of Presidents: A Proposal for a 'Double Complement Rule,'" *Comparative Political Studies* 27, no. 3: 323-348.

Taagepera, Rein (2005). "Conservation of Balance in the Size of Parties," *Party Politics* 11, no. 3: 283-298.