HITOTSUBASHI UNIVERSITY

Global COE Hi-Stat Discussion Paper Series 078

Research Unit for Statistical and Empirical Analysis in Social Sciences (Hi-Stat)

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July 2009

Hi-Stat Discussion Paper

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LIBERAL EGALITARIANISM AND THE HARM PRINCIPLE

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ABSTRACT. This paper analyses Rawls's celebrated difference principle, and its lexicographic extension, in societies with a finite and an infinite number of agents. A unified framework of analysis is set up, which allows one to characterise Rawlsian egalitarian principles by means of a weaker version of a new axiom - the Harm Principle - recently proposed by [13]. This is quite surprising, because the Harm principle is meant to capture a liberal requirement of noninterference and it incorporates no obvious egalitarian content. A set of new characterisations of the maximin and of its lexicographic refinement are derived, including in the intergenerational context with an infinite number of agents.

JEL classification. D63 (Equity, Justice, Inequality, and Other Normative Criteria and Measurement); D70 (Analysis of Collective Decision-Making); Q01 (Sustainable development).

Keywords. Difference principle, leximin, weak harm principle, infinite utility streams.

Date: JULY 16, 2009.

1. Introduction

Almost four decades after its publication, *A Theory of Justice* ([15]) maintains a prominent role in political philosophy, economics, and social choice. Among the most influential contributions of the book is the *difference principle* contained in Rawls's second principle of justice, according to which inequalities should be allowed only insofar as they benefit the worst-off members of society ([15], p.303). Both the difference principle, formally captured by the well-known *maximin* social welfare relation, and especially its lexicographic extension, the *leximin* social welfare relation, have generated a vast literature across disciplinary borders.

Rawls's difference principle and its extension are usually considered to have a strong egalitarian bias and are taken to represent the main alternative to libertarian and utilitarian approaches (see, e.g., the discussion in [17]). The classic characterisation of leximin, in fact, is due to Hammond ([10], [11]) and it requires an axiom (the so-called *Hammond Equity axiom*) with a marked egalitarian content: in a welfaristic framework, Hammond Equity asserts that if $x_i < y_i < y_j < x_j$ for two utility profiles x and y, for which $x_h = y_h$ for all agents $h \neq i, j$, then y should be (weakly) socially preferred to x. In a recent contribution, however, Mariotti and Veneziani ([13]) show that the leximin can be characterised using an axiom - the Harm Principle - that incorporates a liberal view of non-interference, without any explicit egalitarian content. This result is surprising and it raises a number of interesting issues for liberal approaches emphasising notions of individual autonomy or freedom in political philosophy and social choice, but it also sheds new light on the normative foundations of standard egalitarian principles.

This paper extends the analysis of the implications of liberal views of non-interference, as expressed in the Harm Principle, and it generalises ([13]) in a number of directions. Formally, it is shown that a weaker version of the Harm Principle, together with standard axioms in social choice, provides a unified axiomatic framework to analyse a set of social welfare relations originating from the difference principle in a welfaristic framework. Theoretically, the analysis provides a novel statement, based on liberal principles, of the ethical intuitions behind a family of normative principles stemming from Rawls's seminal work. On the one hand, the Harm Principle is different from standard informational invariance axioms in that it has a clear normative content. On the other hand, unlike the Hammond Equity axiom, the Harm principle does not incorporate an egalitarian intuition. Therefore, quite surprisingly, the ethical foundations of two social welfare relations traditionally considered as rather egalitarian - the difference principle and its lexicographic extension - rest only on the two standard axioms of Anonymity and Pareto efficiency, and on a liberal principle incorporating a noninterfering view. No axiom with a clear egalitarian content is necessary, and indeed our analysis provides a new meaning to the label 'liberal egalitarianism' usually associated to Rawls's approach. Actually, our analysis sheds new light on the importance of the notion of justice as impartiality incorporated in the Anonymity axiom in egalitarian approaches. This is particularly clear in societies with a finite number of agents: the Harm principle and the Pareto principle are consistent with some of the least egalitarian social welfare orderings (e.g. the lexicographic dictatorships), and the Anonymity axiom plays a pivotal role in determining the egalitarian outcome. Our analysis also raises some interesting issues concerning the actual implications of liberal approaches emphasising a notion of individual autonomy, or freedom: if one endorses some standard axioms - such as Anonymity and the Pareto principle - the adoption of an arguably weak liberal view of noninterference leads straight to an egalitarian social welfare relation. As noted by Mariotti and Veneziani ([13]), liberal noninterference implies equality, an insight that is proved to be quite robust in this paper.

To be specific, first of all, in economies with a finite number of agents, it is shown that a weaker version of the Harm Principle proposed by Mariotti and Veneziani ([13]) is sufficient to characterise the leximin social welfare ordering. This result is interesting because the weak Harm Principle captures liberal, noninterfering views even more clearly than the original Harm Principle. Further, based on the weak Harm Principle, a new characterisation of the *maximin* social welfare ordering is provided.

Second, this paper analyses the maximin and the leximin in the context of societies with an infinite number of agents. This is arguably a crucial task for egalitarians. In fact, despite Rawls's claims to the contrary, there is no compelling reason to restrict the application of the difference principle to intra-generational justice. In the intergenerational context, a basic concern for impartiality arguably implies that principles of justice be applied to all present and future generations. The extension to the case with an infinite number of generations, however, is problematic for *all* the main approaches, and indeed impossibility results easily obtain, for there exists no social welfare ordering that satisfies the standard axioms of Anonymity and Strong Pareto (see [12]). A number of recent contributions have provided characterisation results for social welfare relations by dropping either completeness (see, among the others, [4], [1], [5], [8], [3]) or transitivity (see, e.g.,[19]). In this tradition, this paper provides various new characterisations of the maximin and the leximin social welfare relations, based on the weak Harm principle in economies with an infinite number of agents. Although various formal frameworks and definitions have been proposed to analyse infinitely-lived societies, it is shown that the weak Harm Principle can be used to derive interesting results in all the main approaches.

The rest of the paper is structured as follows. Section 2 lays out the basic framework of analysis. Section 3 characterises the leximin and the maximin social welfare orderings in economies with a finite number of agents. Section 4 provides a number of characterisation results for leximin and maximin social welfare relations in societies with an infinite number of agents, in various different frameworks. Section 5 briefly concludes.

2. The framework

Let $X \equiv \mathbb{R}^{\mathbb{N}}$ be the set of countably infinite utility streams, where $\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ is the set of natural numbers. An element of X is ${}_1u = (u_1, u_2, ...)$ and, for $t \in \mathbb{N}$, u_t is the utility level of a representative member of generation t. For $T \in \mathbb{N}$, ${}_1u_T = (u_1, ..., u_T)$ denotes the T-head of ${}_1u$ and ${}_{T+1}u = (u_{T+1}, u_{T+2}, ...)$ denotes the T-tail of ${}_1u$, so that ${}_1u = ({}_1u_T, {}_{T+1}u)$; ${}_1\underline{u}_T$ denotes the welfare level of the worst-off generation of the T-head of ${}_1u$. For $\epsilon \in \mathbb{R}$, ${}_{con}\epsilon$ denotes the stream of constant level of well-being equal to ϵ. A permutation π is a bijective mapping of $\mathbb{N}$ onto itself. A permutation π of $\mathbb{N}$ is finite if there is $T \in \mathbb{N}$ such that $\pi(t) = t$ for all $t > T$. For any ${}_1u \in X$ and any permutation π, let $\pi({}_1u) = \left(u_{\pi(t)}\right)_{t \in \mathbb{N}}$ be a permutation of ${}_1u$. For any $T \in \mathbb{N}$ and ${}_1u \in X$, ${}_1\bar{u}_T$ is a permutation of ${}_1u_T$ such that the components are ranked in ascending order.

For any ${}_1u, {}_1v \in X$, we write ${}_1u \geq {}_1v$ to mean $u_t \geq v_t$ for all $t \in \mathbb{N}$; ${}_1u > {}_1v$ to mean ${}_1u \geq {}_1v$ and ${}_1u \neq {}_1v$; and ${}_1u \gg {}_1v$ to mean $u_t > v_t$ for any $t \in \mathbb{N}$.

Let $\succcurlyeq$ be a (binary) relation over X. For any ${}_1u, {}_1v \in X$, we write ${}_1u \succcurlyeq {}_1v$ for $({}_1u, {}_1v) \in \succcurlyeq$ and ${}_1u \not\succcurlyeq {}_1v$ for $({}_1u, {}_1v) \notin \succcurlyeq$; $\succcurlyeq$ stands for "at least as good as". The asymmetric factor $\succ$ of $\succcurlyeq$ is defined by ${}_1u \succ {}_1v$ if and only if

${}_1u \succcurlyeq {}_1v$ and ${}_1v \not\succcurlyeq {}_1u$, and the symmetric part $\sim$ of $\succcurlyeq$ is defined by ${}_1u \sim {}_1v$ if and only if ${}_1u \succcurlyeq {}_1v$ and ${}_1v \succcurlyeq {}_1u$. They stand, respectively, for "strictly better than" and "indifferent to". A relation $\succcurlyeq$ on X is said to be: *reflexive* if, for any ${}_1u \in X$, ${}_1u \succcurlyeq {}_1u$; *complete* if, for any ${}_1u,{}_1v \in X$, ${}_1u \neq {}_1v$ implies ${}_1u \succcurlyeq {}_1v$ or ${}_1v \succcurlyeq {}_1u$; *transitive* if, for any ${}_1u,{}_1v,{}_1w \in X$, ${}_1u \succcurlyeq {}_1v \succcurlyeq {}_1w$ implies ${}_1u \succcurlyeq {}_1w$. $\succcurlyeq$ is a quasi-ordering if it is reflexive and transitive, while $\succcurlyeq$ is an ordering if it is a complete quasi-ordering. Let $\succcurlyeq$ and $\succcurlyeq'$ be relations on X. $\succcurlyeq'$ is an extension of $\succcurlyeq$ if $\succcurlyeq \subseteq \succcurlyeq'$ and $\succ \subseteq \succ'$.

If there are only a finite set $\{1, ..., T\} = N \subset \mathbb{N}$ of agents, or generations, X_T denotes the set of utility streams of X truncated at $T = |N|$, where $|N|$ is the cardinality of N. In order to simplify the notation, in economies with a finite number of agents we write u for ${}_1u_T$. With the obvious adaptations the notation spelled out above is carried over utility streams in X_T.

3. The Economy with a Finite Number of Agents

In this section, we analyse liberal egalitarianism in societies with a finite number of agents. First, we generalise the characterisation of the leximin social welfare ordering (SWO) provided by Mariotti and Veneziani ([13], Theorem 1, p.126) by weakening the main axiom incorporating a liberal view of noninterference, the Harm Principle. Then, we provide a novel characterisation of Rawls's original difference principle, as formalised in the maximin SWO based on the weak Harm Principle.

3.1. **The Leximin.**

As is well-known, the leximin states that society should lexicographically maximise the welfare of its worst-off members. Formally, the *leximin relation* $\succcurlyeq^{LM} = \succ^{LM} \cup \sim^{LM}$ on X_T is defined as follows. The asymmetric factor $\succ^{LM}$ of $\succcurlyeq^{LM}$ is defined by:

$$u \succ^{LM} v \Leftrightarrow \bar{u}_1 > \bar{v}_1 \text{ or } [\exists i \in N \backslash \{1\} : \bar{u}_j = \bar{v}_j \, (\forall j \in N : j < i) \text{ and } \bar{u}_i > \bar{v}_i].$$

The symmetric factor $\sim^{LM}$ of $\succcurlyeq^{LM}$ is defined by:

$$u \sim^{LM} v \Leftrightarrow \bar{u}_i = \bar{v}_i, \forall i \in N.$$

$\succcurlyeq^{LM}$ is also easily shown to be an ordering. Classic analyses of the leximin social welfare ordering (SWO) typically involve the following three axioms (see [10]).

STRONG PARETO OPTIMALITY, **SPO**: $\forall u, v \in X_T : u > v \Rightarrow u \succ v$.

ANONYMITY, **A**: $\forall u, v \in X_T : u = \pi(v) \; \exists \pi$ of $N \Rightarrow u \sim v$.

Hammond Equity, **HE**. $\forall u, v \in X_T : u_i < v_i < v_j < u_j$ $\exists i, j \in N$, $u_k = v_k$ $\forall k \in N \backslash \{i, j\} \Rightarrow v \succcurlyeq u$.

The first two axioms are standard in social choice theory and need no further comment. It is important to note, instead, that **HE** expresses a clear concern for equality in welfare distributions, for it asserts that among any two welfare allocations which differ only in two components, society should prefer the more egalitarian one. The classic characterisation by Hammond ([10]) states that a SWO is the leximin ordering if and only if it satisfies **SPO**, **A**, and **HE**.[1]

In a recent contribution, Mariotti and Veneziani ([13]) drop **HE** and introduce a new axiom, called the Harm Principle (**HP**), which is meant to capture a liberal view of noninterference whenever individual choices have no effect on others. To be precise, starting from two welfare allocations u and v for which u is socially preferred to v, consider two different welfare allocations u' and v' such that agent i is worse off at these than at the corresponding starting allocations, the other agents are equally well off, and agent i prefers u' to v'. The decrease in agent i's welfare level may be due to her negligence or her bad luck, but in any case **HP** states that society's preference over u' and v' should coincide with person i's preferences. In this sense, **HP** requires that having already suffered a welfare loss in both allocations, agent i should not be punished in the social welfare ordering by changing social preferences against i. This seems a rather intuitive way of capturing a liberal view of noninterference, and the name of the axiom is meant to echo John Stuart Mill's famous formulation in his essay *On Liberty* (see [22], and the discussion in [13]). Yet, although it has no explicit egalitarian content, quite surprisingly, Mariotti and Veneziani ([13], Theorem 1, p.126) prove that, jointly with **SPO** and **A**, **HP** characterises $\succcurlyeq^{LM}$.

In this paper, we shall explore further the implications of liberal, noninterfering views in social choice. As a first step, though, we shall formulate a weaker version of **HP**, which can be formally stated as follows.

Weak Harm Principle, **WHP**: $\forall u, v, u', v' \in X_T : u \succ v$ and u' and v'

[1] See also the related Hammond [11] and the generalisation in Tungodden [21].

are such that, $\exists i \in N$,

$$\begin{aligned} u'_i &< u_i \\ v'_i &< v_i \\ u'_j &= u_j \ \forall j \in N \backslash \{i\} \\ v'_j &= v_j \ \forall j \in N \backslash \{i\} \end{aligned}$$

implies $u' \succcurlyeq v'$ whenever $u'_i > v'_i$.

WHP weakens the axiom proposed by Mariotti and Veneziani ([13]) in that it does not require that society's preferences over u' and v' be identical with agent i's, but only that agent i's preferences should not be completely overridden. In this sense, the liberal content of **WHP**, and the requirement that agent i should not be punished in the social welfare ordering by changing social preferences against her, is even clearer, especially if one notes that the last part of the axiom could equivalently be written as $v' \not\succ u'$ whenever $u'_i > v'_i$. The surprising characterisation result proved by Mariotti and Veneziani ([13]) can then be strengthened.

Theorem 3.1. *A* SWO $\succcurlyeq$ *on* X_T *is the leximin ordering if and only if it satisfies Anonymity (**A**), Strong Pareto Optimality (**SPO**), and the Weak Harm Principle (**WHP**).*

Proof. ($\Rightarrow$) Let $\succcurlyeq$ on X_T be the leximin ordering, i.e., $\succcurlyeq = \succcurlyeq^{LM}$. Since **WHP** is weaker than **HP**, the proof that $\succcurlyeq^{LM}$ on X_T meets **SPO**, **A**, and **WHP** follows from the proof of necessity in ([13], Theorem 1, p.126).

($\Leftarrow$) Let $\succcurlyeq$ on X_T be a SWO satisfying **SPO**, **A**, and **WHP**. We show that $\succcurlyeq$ on X_T is the leximin SWO. Thus, we should prove that, $\forall u, v \in X_T$,

$$u \sim^{LM} v \Leftrightarrow u \sim v \tag{3.1}$$

and

$$u \succ^{LM} v \Leftrightarrow u \succ v \tag{3.2}$$

First, we prove the implication $\Rightarrow$ of 3.1. If $u \sim^{LM} v$, then $\bar{u} = \bar{v}$, and so $u \sim v$, by **A**.

Next, we prove the implication $\Rightarrow$ of 3.2. Suppose that $u \succ^{LM} v$, and so, by definition $\exists t \in \{1, ..., T\}$ such that $\bar{u}_s = \bar{v}_s \ \forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$. Suppose, by contradiction, that $v \succ u$. Note that since $\succcurlyeq$ satisfies **A**, in what follows we can focus, without loss of generality, either on u and v, or on the ranked vectors $\bar{u}$ and $\bar{v}$. Therefore, suppose $\bar{v} \succ \bar{u}$. As **SPO** holds it must be the case that $\bar{v}_l > \bar{u}_l$ for some $l > t$. Let

$$k = \min\{t < l \leq T | \bar{v}_l > \bar{u}_l\}.$$

By **A**, let $v_i = \bar{v}_k$ and let $u_i = \bar{u}_{k-g}$, for some $1 \leq g < k$, where $\bar{u}_{k-g} > \bar{v}_{k-g}$. Then, let two real numbers $d_1, d_2 > 0$, and consider vectors u', v' and the corresponding ranked vectors $\bar{u}'$, $\bar{v}'$ in X formed from $\bar{u}$, $\bar{v}$ as follows: first, $\bar{u}_{k-g}$ is lowered to $\bar{u}_{k-g} - d_1$ such that $\bar{u}_{k-g} - d_1 > \bar{v}_{k-g}$; next, $\bar{v}_k$ is lowered to $\bar{v}_k - d_2$ such that $\bar{u}_k > \bar{v}_k - d_2 > \bar{u}_{k-g} - d_1$; finally, all other entries of $\bar{u}$ and $\bar{v}$ are unchanged. By construction $\bar{u}'_j \geq \bar{v}'_j$ for all $j \leq k$, with at least two inequalities, $\bar{u}'_{k-g} > \bar{v}'_{k-g}$ and $\bar{u}'_k > \bar{v}'_k$, whereas **WHP**, combined with **A**, implies $\bar{v}' \succcurlyeq \bar{u}'$. By **SPO**, $d_1, d_2 > 0$ can be chosen so that $\bar{v}' \succ \bar{u}'$, without loss of generality. Consider two cases:

a) Suppose that $\bar{v}_k > \bar{u}_k$, but $\bar{u}_l \geq \bar{v}_l$ for all $l > k$. It follows that $\bar{u}' > \bar{v}'$, and so **SPO** implies that $\bar{u}' \succ \bar{v}'$, a contradiction.

b) Suppose that $\bar{v}_l > \bar{u}_l$ for some $l > k$. Note that by construction $\bar{v}'_l = \bar{v}_l$ and $\bar{u}'_l = \bar{u}_l$ for all $l > k$. Then, let

$$k' = \min\{k < l \leq T | \bar{v}'_l > \bar{u}'_l\}.$$

where $k' > k$. The above argument can be applied to $\bar{u}'$, $\bar{v}'$ to derive vectors $\bar{u}''$, $\bar{v}''$ such that $\bar{u}''_j \geq \bar{v}''_j$ for all $j \leq k'$, whereas **WHP**, combined with **A** and **SPO**, implies $\bar{v}'' \succ \bar{u}''$. And so on. After a finite number of iterations s, two vectors $\bar{u}^s$, $\bar{v}^s$ can be derived such that, by **WHP**, combined with **A** and **SPO**, we have that $\bar{v}^s \succ \bar{u}^s$, but **SPO** implies $\bar{u}^s \succ \bar{v}^s$, yielding a contradiction.

We have proved that if $u \succ^{LM} v$ then $u \succcurlyeq v$. Suppose now, by contradiction, that $v \sim u$, or equivalently $\bar{v} \sim \bar{u}$. Since, by our supposition, $\bar{v}_t < \bar{u}_t$, there exists $\epsilon > 0$ such that $\bar{v}_t < \bar{u}_t - \epsilon < \bar{u}_t$. Let ${}_1\bar{u}^\epsilon \in X$ be a vector such that $\bar{u}^\epsilon_t = \bar{u}_t - \epsilon$ and $\bar{u}^\epsilon_j = \bar{u}_j$ for all $j \neq t$. It follows that $\bar{u}^\epsilon \succ^{LM} \bar{v}$ but $\bar{v} \succ \bar{u}^\epsilon$ by **SPO** and the transitivity of $\succcurlyeq$. Hence, the above argument can be applied to $\bar{v}$ and $\bar{u}^\epsilon$, yielding the desired contradiction. □

The properties in Theorem 3.1 are clearly independent.

Theorem 3.1 has a number of interesting theoretical implications. First of all, Theorem 3.1 implies that **HE** and **WHP** equivalent in the presence of **A** and **SPO**. Yet, the weakening of **HP** makes it even clearer that the two axioms are completely independent. Actually, it can be proved that if $N = \{1, 2\}$, then in the presence of **SPO**, **HE** implies **WHP**, but the converse is not true. This implies that the above characterisation is far from trivial, given that, at least in some cases, and if **SPO** is assumed, **HE** is actually stronger than **WHP**. Third, and perhaps more interesting, Theorem 3.1 puts the normative foundations of leximin under a rather different light. For, unlike in standard results, the egalitarian SWO is characterised without appealing to any axioms with a clear egalitarian content. Actually, it is easily shown that **SPO** and **WHP** alone are compatible with some of the least egalitarian SWOs, namely the lexicographic dictatorships, which forcefully shows that **WHP** imposes no significant egalitarian restriction. As a result, Theorem 3.1 interestingly shows the normative strength of the Anonymity axiom in determining the egalitarian outcome, an important insight which is not obvious in standard characterisations based on **HE**.

The main implication of Theorem 3.1, however, is that it suggests that the core intuition of Mariotti and Veneziani ([13], Theorem 1, p.126) concerning the implications of liberal noninterfering views is robust: a strongly egalitarian SWO can be characterised with an even weaker axiom that only incorporates a liberal view of non-interference. In the next sections, this intuition is extended further and it is shown that the counterintuitive implications of liberal noninterfering principles in terms of egalitarian orderings are quite general and robust. Analogous characterisations of a whole *family* of principles inspired by Rawls's theory are obtained in societies with both finite and infinite populations, based on the weak Harm Principle.

3.2. **The Difference Principle.** The *maximin relation* $\succcurlyeq^M = \succ^M \cup \sim^M$ on X_T is defined as follows. The asymmetric factor $\succ^M$ of $\succcurlyeq^M$ is defined by:

$$u \succ^M v \Leftrightarrow \bar{u}_1 > \bar{v}_1.$$

The symmetric factor $\sim^M$ of $\succcurlyeq^M$ is defined by:

$$u \sim^M v \Leftrightarrow \bar{u}_1 = \bar{v}_1.$$

$\succcurlyeq^M$ is easily shown to be an ordering. The maximin SWO formalises Rawls's difference principle. As is well-known, the maximin does not satisfy **SPO**, and therefore the following, weaker axiom is imposed on the SWO.

WEAK PARETO OPTIMALITY, **WPO**: $\forall u, v \in X_T : u \gg v \Rightarrow u \succ v$.

Second, a continuity axiom is imposed, which represents a standard interprofile condition requiring the SWO to vary continuosly with variations in utility streams. This axiom is common in characterisations of the maximin SWO (see, e.g., [7]).

CONTINUITY, **C**: $\forall u \in X_T$, $\{v \in X_T | v \succcurlyeq u\}$ is closed and $\{v \in X_T | u \succcurlyeq v\}$ is closed.

The next Theorem shows that the combination of Anonymity (**A**), Weak Pareto Optimality (**WPO**), Continuity (**C**), and the Weak Harm Principle (**WHP**) characterises the maximin SWO.

Theorem 3.2. *A* SWO $\succcurlyeq$ *on* X_T *is the maximin ordering if and only if it satisfies Anonymity (**A**), Weak Pareto Optimality (**WPO**), Continuity (**C**), and the Weak Harm Principle (**WHP**).*

Proof. ($\Rightarrow$) Let $\succcurlyeq$ on X_T be the maximin ordering, i.e., $\succcurlyeq = \succcurlyeq^M$. It can be easily verified that $\succcurlyeq^M$ on X_T satisfies **WPO**, **A**, **C**, and **WHP**.

($\Leftarrow$) Let $\succcurlyeq$ on X_T be a SWO satisfying **A**, **WPO**, **WHP**, and **C**. We show that $\succcurlyeq$ is the maximin SWO. We shall prove that, $\forall u, v \in X_T$,

$$u \succ^M v \Leftrightarrow u \succ v \tag{3.3}$$

and

$$u \sim^M v \Leftrightarrow u \sim v. \tag{3.4}$$

Note that as $\succcurlyeq$ on X_T satisfies **A**, in what follows we can focus either on u and v, or on the ranked vectors $\bar{u}$ and $\bar{v}$, without loss of generality.

First, we show that the implication ($\Rightarrow$) of (3.3) is satisfied. Take any $u, v \in X_T$. Suppose that $u \succ^M v \Leftrightarrow \bar{u}_1 > \bar{v}_1$ and assume, by contradiction, that $v \succ u$, or equivalently, $\bar{v} \succ \bar{u}$. As **WPO** holds, $\bar{v}_j \geq \bar{u}_j$ for some $j \in N$, otherwise a contradiction immediately obtains. We prooceed according the following steps.

Step 1. Let

$$k = \min \{l \in N | \bar{v}_l \geq \bar{u}_l\}.$$

By **A**, let $v_i = \bar{v}_k$ and let $u_i = \bar{u}_1$. Then, consider two real numbers $d_1, d_2 > 0$, and two vectors u', v^* - together with the corresponding ranked vectors $\bar{u}', \bar{v}^* \in X_T$ - formed from $\bar{u}, \bar{v}$ as follows: $\bar{u}_1$ is lowered to $\bar{u}_1 - d_1 > \bar{v}_1$; $\bar{v}_k$ is lowered to $\bar{u}_k > \bar{v}_k - d_2 > \bar{u}_1 - d_1$; and all other entries of $\bar{u}$ and $\bar{v}$ are unchanged. By construction $\bar{u}'_j > \bar{v}^*_j$ for all $j \leq k$, whereas by **WHP** and **A**, we have $\bar{v}^* \succcurlyeq \bar{u}'$.

Step 2. Let

$$0 < \epsilon < \min\{\bar{u}'_j - \bar{v}^*_j | \forall j \leq k\}$$

and define $\bar{v}' = \bar{v}^* + {}_{con}\epsilon$. By construction, $\bar{v}^* \ll \bar{v}'$, and $\bar{v}'_j < \bar{u}'_j$ for all $j \leq k$. **WPO** implies $\bar{v}' \succ \bar{v}^*$. As $\bar{v}^* \succcurlyeq \bar{u}'$, by *step* 1, the transitivity of $\succcurlyeq$

implies $\bar{v}' \succ \bar{u}'$.

If $\bar{u}'_j > \bar{v}'_j$ for all $j \in N$, **WPO** implies $\bar{u}' \succ \bar{v}'$, a contradiction. Otherwise, let $\bar{v}'_l \geq \bar{u}'_l$ for some $l > k$. Then, let

$$k' = \min\left\{l \in N | \ \bar{v}'_l \geq \bar{u}'_l\right\}$$

where $k' > k$.
The above steps 1-2 can be applied to $\bar{u}', \bar{v}'$ to derive vectors $\bar{u}'', \bar{v}''$ such that $\bar{u}''_j > \bar{v}''_j$ for all $j \leq k'$, whereas $\bar{v}'' \succ \bar{u}''$. By **WPO**, a contradiction is obtained whenever $\bar{u}''_j > \bar{v}''_j$ for all $j \in N$. Otherwise, let $\bar{v}''_l \geq \bar{u}''_l$ for some $l > k$. And so on. After a finite number s of iterations, two vectors $\bar{u}^s, \bar{v}^s$ can be derived such that $\bar{v}^s \succ \bar{u}^s$, by steps 1-2, but $\bar{u}^s \succ \bar{v}^s$, by **WPO**, a contradiction.
Therefore, it must be $\bar{u} \succcurlyeq \bar{v}$ whenever $\bar{u} \succ^M \bar{v}$. We have to rule out the possibility that $\bar{u} \sim \bar{v}$. We proceed by contradiction. Suppose that $\bar{u} \sim \bar{v}$. Since $\bar{v}_1 < \bar{u}_1$, there exists $\epsilon > 0$ such that $\bar{v}^\epsilon = \bar{v} + {}_{con}\epsilon$ and $\bar{v}^\epsilon_1 < \bar{u}_1$ so that $\bar{u} \succ^M \bar{v}^\epsilon$. However, by **WPO** and transitivity of $\succcurlyeq$ it follows that $\bar{v}^\epsilon \succ \bar{u}$. Then the above reasoning can be applied to vectors $\bar{v}^\epsilon$ and $\bar{u}$ to obtain the desired contradiction.
Now, we show that the implication ($\Rightarrow$) of (3.4) is met as well. Suppose that $\bar{u} \sim^M \bar{v} \Leftrightarrow \bar{u}_1 = \bar{v}_1$. Assume, to the contrary, that $u \not\sim v$. Without loss of generality, let $\bar{u} \succ \bar{v}$. By **A**, it must be $\bar{u} \neq \bar{v}$. As $\bar{u} \succ \bar{v}$, it follows from **C** that there exists neighborhoods $S(\bar{u})$ and $S(\bar{v})$ of $\bar{u}$ and $\bar{v}$ such that $u' \succ v'$ for all $u' \in S(\bar{u})$ and for all $v' \in S(\bar{v})$. Then, there exists $v' \in S(\bar{v})$ such that $v' \gg \bar{v}$ and $\bar{u} \succ v' \sim \bar{v}'$, so that $\bar{u} \succ \bar{v}'$ but $\bar{v}' \succ^M \bar{u}$. By the implication (3.3) proved above, it follows that $\bar{v}' \succ \bar{u}$, a contradiction. □

The properties in Theorem 3.2 are clearly independent.

The theoretical relevance of the latter result can be appreciated if Theorem 3.2 is compared with alternative characterisations. On the one hand, unlike axioms on informational comparability often used in the literature (see, e.g., [16]), the weak Harm Principle has a clear ethical foundation, but, as noted above it has no obvious egalitarian implication. In a recent contribution, Bosmans and Ooghe ([7]) characterise the maximin SWO using Anonymity (**A**), Weak Pareto Optimality (**WPO**), Continuity (**C**), and Hammond Equity (**HE**). Instead, as in the case of the leximin ordering analysed above, Theorem 3.2 characterises an egalitarian SWO such as the maximin without appealing to an axiom like **HE**, which arguably has a marked egalitarian content, and using instead **WHP**, which only incorporates a liberal, noninterfering view of society.

3.3. **Egalitarian Principles in the Infinitely-Lived Society.** In this section, the axiomatic analysis of the difference principle and its main refinement, the leximin, is extended to economies with an infinite number of agents, focusing on the role of liberal views of noninterference as formulated in the weak Harm Principle. As is well-known, the case with an infinite number of agents raises a number of issues concerning the existence and the characterisation of SWOs and different definitions can be provided of social welfare relations (SWRs) in order to compare (countably) infinite utility streams. In this section, we first adopt the framework proposed by Asheim and Tungodden ([1]) for the leximin, and provide an alternative characterisation of the leximin SWR. Then, we provide a new characterisation of an infinite-horizon ordering extension of a leximin SWR in the framework recently proposed by Bossert, Sprumont and Suzumura ([8]). Finally, we extend the framework of Asheim and Tungodden to analyse the maximin SWR, and propose a new characterisation of the difference principle in the context of infinitely-lived economies.

3.3.1. *The Leximin SWR.* Following Asheim and Tungodden, there are two different ways to formally define the leximin SWR. The first one is the so-called weak leximin, or W-Leximin and can be formalised as follows.

Definition 3.3. (Definition 2, [1], p. 224) For all ${}_1u, {}_1v \in X$, ${}_1u \sim^{LM^*} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T}$: ${}_1\bar{u}_T = {}_1\bar{v}_T$, and ${}_1u \succ^{LM^*} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T}$ $\exists t \in \{1, ..., T\}$ $\bar{u}_s = \bar{v}_s$ $\forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$.

The characterisation results are based on a number of standard axioms. The first three axioms are similar to those used in the finite case, and need no further comment, except possibly noting that in this context, **WHP** is weaker than the version in Section 2 above, since it only holds for vectors with the same tail.

Finite Anonymity, **FA**: $\forall {}_1u \in X$ and $\forall$ finite permutation π of $\mathbb{N}$, $\pi({}_1u) \sim {}_1u$.

Strong Pareto Optimality, **SPO**: $\forall {}_1u, {}_1v \in X : {}_1u > {}_1v \Rightarrow {}_1u \succ {}_1v$.

Weak Harm Principle, **WHP***: $\forall {}_1u, {}_1v, {}_1u', {}_1v' \in X : \exists T \geq 1$ ${}_1u =$

$({}_1u_T,{}_{T+1}v) \succ {}_1v$, and ${}_1u'$ and ${}_1v'$ are such that, $\exists i \leq T$,

$$\begin{aligned} u'_i &< u_i \\ v'_i &< v_i \\ u'_j &= u_j \ \forall j \neq i \\ v'_j &= v_j \ \forall j \neq i \end{aligned}$$

implies ${}_1u' \succcurlyeq {}_1v'$ whenever $u'_i > v'_i$.

Next, following Asheim and Tungodden ([1], p. 223), an axiom is imposed, which represents a mainly technical requirement to deal with infinite dimensional vectors.

Weak Preference Continuity, **WPC**: $\forall {}_1u, {}_1v \in X: \ \exists \tilde{T} \geq 1$ such that $({}_1u_T,{}_{T+1}v) \succ {}_1v \ \forall T \geq \tilde{T} \Rightarrow {}_1u \succ {}_1v$.

Finally, the next axiom states that the SWR should at least be able to compare (infinite-dimensional) vectors with the same tail, which seems an obviously desirable property.

Weak Completeness, **WC**: $\forall {}_1u, \ {}_1v \in X, \ \exists T \geq 1 \ \ \pi(({}_1u_T,{}_{T+1}v)) \neq {}_1v \ \ \forall$ finite permutation of $\mathbb{N} \Rightarrow ({}_1u_T,{}_{T+1}v) \succcurlyeq {}_1v$ or ${}_1v \succcurlyeq ({}_1u_T,{}_{T+1}v)$.

The next Theorem proves that the combination of the Finite Anonymity (**FA**), Strong Pareto Optimality (**SPO**), Weak Harm Principle (**WHP***), Weak Preference Continuity (**WPC**), and Weak Completeness (**WC**), characterises the leximin SWR.

Theorem 3.4. *$\succcurlyeq$ is an extension of $\succcurlyeq^{LM^*}$ if and only if $\succcurlyeq$ satisfies Finite Anonymity (**FA**), Strong Pareto Optimality (**SPO**), Weak Harm Principle (**WHP***), Weak Preference Continuity (**WPC**), and Weak Completeness (**WC**).*

Proof. ($\Rightarrow$) Let $\succcurlyeq^{LM^*} \subseteq \succcurlyeq$. It is easy to see that $\succcurlyeq$ meets **FA** and **SPO**. By observing that $\succcurlyeq^{LM^*}$ is complete for comparisons between utility streams having the same tail it is also easy to see that $\succcurlyeq$ satisfies **WC** and **WPC**. We show that $\succcurlyeq$ meets **WHP***. For, take any ${}_1u, {}_1v, {}_1u', {}_1v' \in X$ such that $\exists T \geq 1 \ {}_1u = ({}_1u_T,{}_{T+1}v) \succ {}_1v$, and ${}_1u'$ and ${}_1v'$ are such that, $\exists i \leq T$, $u'_i < u_i$, $v'_i < v_i$, $u'_j = u_j \ \forall j \neq i$, $v'_j = v_j \ \forall j \neq i$. We show that ${}_1u' \succ {}_1v'$ whenever $u'_i > v'_i$. As $\succcurlyeq^{LM^*}$ is complete for comparisons between utility streams having the same tail, it must be true that ${}_1u \succ^{LM^*} {}_1v$. Therefore, by definition, $\exists \tilde{T} \geq 1$ such that $\forall T' \geq \tilde{T} \ \ \exists t \in \{1, ..., T'\} \ \bar{u}_s = \bar{v}_s \ \forall 1 \leq s < t$

and $\bar{u}_t > \bar{v}_t$. Take any $T' \geq \tilde{T}$. As $T' < \infty$ it follows from Mariotti and Veneziani's Theorem [[13], Theorem 1, p. 126] that there exists $t^* \leq t \leq T'$ such that $\bar{u}'_s = \bar{v}'_s \ \forall 1 \leq s < t^*$ and $\bar{v}'_{t^*} < \bar{u}'_{t^*}$. As it holds true for any $T' \geq \tilde{T}$ it follows that ${}_1u' \succ {}_1v'$ as $\succcurlyeq^{LM^*} \subseteq \succcurlyeq$.

($\Leftarrow$) Suppose that $\succcurlyeq$ satisfies **FA**, **SPO**, **WHP***, **WPC**, and **WC**. We show that $\sim^{LM^*} \subseteq \sim$ and $\succ^{LM^*} \subseteq \succ$. Take any ${}_1u, {}_1v \in X$.

Assume that ${}_1u \sim^{LM^*} {}_1v$. By definition, $\exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T}$ ${}_1\bar{u}_T = {}_1\bar{v}_T$, and so ${}_{T+1}u = {}_{T+1}v$, for any $T \geq \tilde{T}$. It follows that ${}_1u \sim {}_1v$, by **FA**.

Next, suppose that ${}_1u \succ^{LM^*} {}_1v$, and so, by definition, $\exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T}$ $\exists t \in \{1, ..., T\}$ such that $\bar{u}_s = \bar{v}_s \ \forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$. Take any such T and consider the vector ${}_1w \equiv ({}_1u_T, {}_{T+1}v)$. We want to show that ${}_1w \succ {}_1v$. By **FA** and transitivity, we can consider ${}_1\bar{w} \equiv ({}_1\bar{u}_T, {}_{T+1}v)$ and ${}_1\bar{v} \equiv ({}_1\bar{v}_T, {}_{T+1}v)$. Suppose that ${}_1\bar{v} \succcurlyeq {}_1\bar{w}$. We distinguish two cases.

Case 1. ${}_1\bar{v} \succ {}_1\bar{w}$

As **SPO** holds it must be the case that $\bar{v}_l > \bar{w}_l$ for some $l > t$. Let

$$k = \min\{t < l \leq T | \bar{v}_l > \bar{w}_l\}.$$

Let $v_i = \bar{v}_k$ and let $w_i = \bar{w}_{k-g}$, for some $1 \leq g < k$, where $\bar{w}_{k-g} > \bar{v}_{k-g}$. Then, let two real numbers $d_1, d_2 > 0$, and consider vectors ${}_1w'$, ${}_1v'$ formed from ${}_1\bar{w}$, ${}_1\bar{v}$ as follows: $\bar{w}_{k-g}$ is lowered to $\bar{w}_{k-g} - d_1$ such that $\bar{w}_{k-g} - d_1 > \bar{v}_{k-g}$; $\bar{v}_k$ is lowered to $\bar{v}_k - d_2$ such that $\bar{w}_k > \bar{v}_k - d_2 > \bar{w}_{k-g} - d_1$; and all other entries of ${}_1\bar{w}$ and ${}_1\bar{v}$ are unchanged. By **FA**, consider ${}_1\bar{w}' = ({}_1\bar{w}'_T, {}_{T+1}v)$ and ${}_1\bar{v}' = ({}_1\bar{v}'_T, {}_{T+1}v)$. By construction $\bar{w}'_j \geq \bar{v}'_j$ for all $j \leq k$, with at least two inequalities, $\bar{w}'_{k-g} > \bar{v}'_{k-g}$ and $\bar{w}'_k > \bar{v}'_k$, whereas **WHP***, combined with **FA**, implies $\bar{v}' \succcurlyeq \bar{w}'$. Furthermore, by **SPO**, it is possible to choose $d_1, d_2 > 0$, such that $\bar{v}' \succ \bar{w}'$, without loss of generality. Consider two cases:

a) Suppose that $\bar{v}_k > \bar{w}_k$, but $\bar{w}_l \geq \bar{v}_l$ for all $l > k$. It follows that ${}_1\bar{w}' > {}_1\bar{v}'$, and so **SPO** implies that ${}_1\bar{w}' \succ {}_1\bar{v}'$, a contradiction.

b) Suppose that $\bar{v}_l > \bar{w}_l$ for some $l > k$. Note that by construction $\bar{v}'_l = \bar{v}_l$ and $\bar{w}'_l = \bar{w}_l$ for all $l > k$. Then, let

$$k' = \min\{k < l \leq T | \bar{v}'_l > \bar{w}'_l\}.$$

where $k' > k$. The above argument can be applied to ${}_1\bar{w}'$, ${}_1\bar{v}'$ to derive vectors ${}_1\bar{w}''$, ${}_1\bar{v}''$ such that $\bar{w}''_j \geq \bar{v}''_j$ for all $j \leq k'$, whereas **WHP***, combined with **FA** and **SPO**, implies ${}_1\bar{v}'' \succ {}_1\bar{w}''$. And so on. After a finite number of iterations s, two vectors ${}_1\bar{w}^s$, ${}_1\bar{v}^s$ can be derived such that, by **WHP***, combined with **FA** and **SPO**, we have that ${}_1\bar{v}^s \succ {}_1\bar{w}^s$, but **SPO** implies ${}_1\bar{w}^s \succ {}_1\bar{v}^s$, yielding a contradiction.

Case 2. $\bar{v} \sim \bar{w}$
Since, by our supposition, $\bar{v}_t < \bar{u}_t \equiv \bar{w}_t$, there exists $\epsilon > 0$ such that $\bar{v}_t < \bar{w}_t - \epsilon < \bar{w}_t$. Let ${}_1\bar{w}^\epsilon \in X$ be a vector such that $\bar{w}_t^\epsilon = \bar{w}_t - \epsilon$ and $\bar{w}_j^\epsilon = \bar{w}_j$ for all $j \neq t$. It follows that ${}_1\bar{w}^\epsilon \succ^{LM^*} {}_1\bar{v}$ but ${}_1\bar{v} \succ {}_1\bar{w}^\epsilon$ by **SPO** and the transitivity of $\succcurlyeq$. Hence, the argument of *Case* 1 above can be applied to ${}_1\bar{v}$ and ${}_1\bar{w}^\epsilon$, yielding the desired contradiction.
It follows from **WC** that ${}_1\bar{w} \succ {}_1\bar{v}$. **FA**, combined with the transitivity of $\succcurlyeq$, implies that $({}_1u_T,\, {}_{T+1}v) \succ {}_1v$. Since it holds true for any $T \geq \tilde{T}$, **WPC** implies ${}_1u \succ {}_1v$, as desired. □

The properties in Theorem 3.4 are easily shown to be independent (see Appendix).

It is worth stressing again that in societies with an infinite number of agents, or generations, there is no obvious, and unanimously accepted, definition of the leximin SWR. Asheim and Tungodden ([1], p. 224), for example, provide an alternative, stronger definition of the leximin - the S-Leximin - that can be formalised as follows.

Definition 3.5. (Definition 1, [1], p. 224) For all ${}_1u, {}_1v \in X$, ${}_1u \succcurlyeq_S^{LM^*} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T}$: either ${}_1\bar{u}_T = {}_1\bar{v}_T$ or $\exists t \in \{1, ..., T\}$: $\bar{u}_s = \bar{v}_s$ $\forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$.

In the above analysis, we have focused on the W-Leximin because we think that the continuity axiom **WPC** is more appealing than the Strong Preference Continuity property adopted by Asheim and Tungodden ([1], p. 223) to characterise the S-leximin, which seems a rather strong requirement (as forcefully argued, for example, by Basu and Mitra [5], p. 358). Strong Preference Continuity can be formalised as follows.

Strong Preference Continuity, **SPC**: $\forall {}_1u, {}_1v \in X$: $\exists \tilde{T} \geq 1$ such that $({}_1u_T, {}_{T+1}v) \succcurlyeq {}_1v$ $\forall T \geq \tilde{T}$, and $\forall \tilde{T} \geq 1$ $\exists T \geq \tilde{T}$ such that $({}_1u_T, {}_{T+1}v) \succ {}_1v \Rightarrow {}_1u \succ {}_1v$.

A result analogous to Theorem 3.4 can be established for the stronger definition 3.5 by replacing **WPC** with the Strong Preference Continuity (**SPC**). It can be easily obtained through a trivial modification of the parts of the proof of Theorem 3.4 that involve **WPC**, and by observing that the necessity of **WHP*** can be easily established along the same lines as in Theorem 3.4.

Theorem 3.6. *$\succcurlyeq$ is an extension of $\succcurlyeq_S^{LM^*}$ if and only if $\succcurlyeq$ satisfies Finite Anonymity (**FA**), Strong Pareto Optimality (**SPO**), Weak Harm Principle (**WHP***), Strong Preference Continuity (**SPC**), and Weak Completeness (**WC**).*

The properties in Theorem 3.6 are are easily shown to be independent (see Appendix).

In Theorems 3.4-3.6, we have identified the class of leximin SWRs by postulating a continuity property on the quasi-ordering (respectively, **WPC** and **SPC**), which represents a mainly technical requirement in ranking infinite utility streams. As axioms such as **SPO** and **FA** may be considered ethically more defensible than continuity axioms, Bossert, Sprumont and Suzumura ([8]) have not postulated any continuity property on the quasi-ordering and have provided a characterisation of a subclass of the class of orderings satisfying **SPO**, **FA**, and an infinite version of **HE**. Formally, the relationship between the characterisation of the leximin by Bossert et al. ([8]) and that by Asheim and Tungodden ([1]) is analogous to the relationship between the characterisation of the utilitarian SWR by Basu and Mitra ([5]) and the characterisations of the more restrictive utilitarian SWR induced by the overtaking criterion. We explore this relationship by extending our analysis of **WHP*** to the framework developed by Bossert et al. ([8]).

For each $T \in \mathbb{N}$, let the leximin ordering on X_T be denoted as $\succcurlyeq_T^{LM}$. The definition of the leximin SWR proposed by Bossert et al. ([8]) can be formulated as follows. Define a relation $\succcurlyeq_T^{L} \subseteq X \times X$ by letting, for all ${}_1u, {}_1v \in X$,

$$ {}_1u \succcurlyeq_T^{L} {}_1v \Leftrightarrow {}_1u_T \succcurlyeq_T^{LM} {}_1v_T \text{ and } {}_{T+1}u \geq {}_{T+1}v. \tag{3.5}$$

The relation $\succcurlyeq_T^{L}$ can be shown to be reflexive and transitive for all $T \in \mathbb{N}$. Then the leximin SWR is $\succcurlyeq^{L} = \bigcup_{T \in \mathbb{N}} \succcurlyeq_T^{L}$ ([8], p. 586). The leximin SWR $\succcurlyeq^{L}$ is reflexive and transitive, but not necessarily complete. Moreover, Bossert et al. ([8]) show that $\succcurlyeq^{L}$ satisfies the following property ([8], p. 586, equation (14)):

$$ \forall {}_1u, {}_1v \in X : \exists T \in \mathbb{N} \text{ such that } {}_1u \succ_T^{L} {}_1v \Leftrightarrow {}_1u \succ^{L} {}_1v. \tag{3.6}$$

We show that the set of orderings extensions of $\succcurlyeq^{L}$ characterised by the next theorem, based on Finite Anonymity (**FA**), Strong Pareto Optimality (**SPO**), and the weak Harm Principle (**WHP***), is non-empty.

Theorem 3.7. *$\succcurlyeq$ is an extension of $\succcurlyeq^L$ if and only if $\succcurlyeq$ satisfies Finite Anonymity (**FA**), Strong Pareto Optimality (**SPO**), and Weak Harm Principle (**WHP***).*

Proof. ($\Rightarrow$) The proof that any ordering extension of $\succcurlyeq^L$ satisfies **FA** and **SPO** is as in ([8], Theorem 2, p. 586). We only need to prove that any ordering extension $\succcurlyeq$ of $\succcurlyeq^L$ satisfies **WHP***. Consider any ${}_1u$, ${}_1v$, ${}_1u'$, ${}_1v' \in X$ such that $\exists T \geq 1$ ${}_1u = ({}_1u_T, {}_{T+1}v) \succ {}_1v$, and ${}_1u'$ and ${}_1v'$ are such that, $\exists i \leq T$, $u'_i < u_i$, $v'_i < v_i$, $u'_j = u_j$ $\forall j \neq i$, $v'_j = v_j$ $\forall j \neq i$. We show that ${}_1u' \succ {}_1v'$ whenever $u'_i > v'_i$. Since $\succcurlyeq^{LM}_T$ is complete and ${}_{T+1}v = {}_{T+1}u$ it cannot be ${}_1v_T \succcurlyeq^{LM}_T {}_1u_T$, otherwise $({}_1v, {}_1u) \in \succcurlyeq^L \subseteq \succcurlyeq$ which contradicts ${}_1u \succ {}_1v$. Thus, we have that ${}_1u_T \succcurlyeq^{LM}_T {}_1v_T$, ${}_1v_T \not\succcurlyeq^{LM}_T {}_1u_T$, and ${}_{T+1}v = {}_{T+1}u$, so that $({}_1u, {}_1v) \in \succ^L_T$ by (3.5). It follows from (3.6) that $({}_1u, {}_1v) \in \succ^L$. As ${}_1u'$ and ${}_1v'$ are such that, $\exists i \leq T$, $u'_i < u_i$, $v'_i < v_i$, $u'_j = u_j$ $\forall j \neq i$, $v'_j = v_j$ $\forall j \neq i$, it can easily be shown, as in ([13]), that ${}_1u'_T \succ^{LM}_T {}_1v'_T$ whenever $u'_i > v'_i$. As ${}_{T+1}v' = {}_{T+1}u'$ and ${}_1u'_T \succ^{LM}_T {}_1v'_T$ it follows from (3.5) that ${}_1u' \succ^L_T {}_1v'$, and therefore ${}_1u' \succ^L {}_1v'$ by (3.6). But since $\succcurlyeq$ is an ordering extension of $\succcurlyeq^L$ it follows that ${}_1u' \succ {}_1v'$.

($\Leftarrow$) The proof is identical to ([8], Theorem 2, p. 587) using the characterisation of the T-person leximin in Theorem (3.1). □

Finally, it is worth noting that the Weak Harm Principle (**WHP***) can also be used to characterise the intergenerational version of the leximin ordering recently proposed by Sakai ([19]), which drops transitivity but retains completenss. In particular, if one replaces Hammond Equity with **WHP***, a modified version of his characterisation results ([19], Lemma 6, p.17; and Theorem 5, p.18) can easily be proved.

3.3.2. *The Maximin SWR.* In this subsection, we analyse Rawls's difference principle in the context of economies with an infinite number of agents. First of all, we focus on the subset of utility streams that reach a minimum in a finite period. Formally, define the following subset Y of X :

$$Y = \left\{ {}_1u \in X | \exists \tilde{T} \geq 1 \colon \; {}_1\underline{u}_T = \; {}_1\underline{u}_{\tilde{T}} \;\; \forall T \geq \tilde{T} \right\}.$$

Then, in the framework proposed by Asheim and Tungodden ([1]), the maximin SWR can be formally defined as follows.

Definition 3.8. For all ${}_1u, {}_1v \in Y$, ${}_1u \sim^{M^*} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that ${}_1\underline{u}_T = {}_1\underline{v}_T$ $\forall T \geq \tilde{T}$, and ${}_1u \succ^{M^*} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that ${}_1\underline{u}_T > {}_1\underline{v}_T$ $\forall T \geq \tilde{T}$.

Let $\succcurlyeq^{M^*} = \succ^{M^*} \bigcup \sim^{M^*}$. It is easy to show that $\succcurlyeq^{M^*}$ is a quasi-ordering on X and that $\succcurlyeq^{M^*}$ is complete for any ${}_1u, {}_1v \in Y$. In order to prove our main characterisation result, we impose the following four axioms, which carry through from the finite horizon setting.

Finite Anonymity, **FA***: $\forall {}_1u \in Y$ and $\forall$ finite permutation π of $\mathbb{N} \Rightarrow \pi({}_1u) \sim {}_1u$.

Weak Pareto Optimality, **WPO***: $\forall {}_1u,\ {}_1v \in Y$, $\exists T \geq 1\ {}_1u_T \gg\ {}_1v_T$, ${}_{T+1}u \geq\ {}_{T+1}v$, and $\exists \tau \in \{1, ..., T\}$: $u_t \geq u_\tau$ and $v_t \geq v_\tau$ $\forall t > T \Rightarrow\ {}_1u \succ\ {}_1v$.

Weak Harm Principle, **WHP****: $\forall {}_1u, {}_1v, {}_1u', {}_1v' \in Y$, ${}_1u \succ\ {}_1v$, and ${}_1u'$ and ${}_1v'$ are such that for some $i \in \mathbb{N}$,

$$\begin{aligned} u'_i &< u_i, \\ v'_i &< v_i, \\ u'_j &= u_j\ \forall j \neq i \\ v'_j &= v_j\ \forall j \neq i \end{aligned}$$

implies ${}_1u' \succcurlyeq\ {}_1v'$ whenever $u'_i > v'_i$.

Weak Continuity, **WCN**: $\forall {}_1u, {}_1v \in Y$, ${}_1u \succ\ {}_1v \Rightarrow \exists \epsilon > 0 :\ {}_1u \succ\ {}_1v +\ {}_{con}\epsilon$, $\exists \epsilon' > 0 :\ {}_1u -\ {}_{con}\epsilon' \succ\ {}_1v$.

The latter axiom is a weakening of standard continuity axioms: continuity requires that if ${}_1u$ is strictly better than ${}_1v$, then any vector sufficient close to ${}_1u$ should be strictly better than any vector sufficient close to ${}_1v$. Axiom **WCN** only requires the existence of *some* vector with the latter property.

In addition to the above requirements, we follow again Asheim and Tungodden ([1], p. 223) and impose a weak consistency property on Y.

Weak Consistency, **WCONS**: $\forall {}_1u, {}_1v \in Y$,
(a) $\pi({}_1u) \neq\ {}_1v$ $\forall$ finite permutation of $\mathbb{N}$, $\pi({}_1v) \sim\ {}_1v$ $\forall$ finite permutation of $\mathbb{N}$, $\exists \tilde{T} \geq 1$ such that ${}_1u \sim ({}_1v_T, {}_{T+1}u)$ $\forall T \geq \tilde{T} \Rightarrow\ {}_1u \sim\ {}_1v$;
(b) $\exists \tilde{T} \geq 1$ such that ${}_1u \succ ({}_1v_T, {}_{T+1}u)$ $\forall T \geq \tilde{T} \Rightarrow\ {}_1u \succ\ {}_1v$.

The latter axiom again represents a mainly technical requirement to deal with infinite dimensional vectors, which captures a continuity requirement on sequences of decisions on infinite utility streams. Axioms similar to our **WCONS** are common in the literature (e.g., see [1]) and we note that, for example, the "Partial Unit Comparability" and "Weak Consistency" axioms discussed by Basu and Mitra ([5], Axiom 3, p.354 and Axiom 5, p.359) in their analysis of utilitarianism for infinite utility streams imply **WCONS**.

Finally, we require that $\succcurlyeq$ on X is complete at least when comparing elements of Y: again, it seems obviously desirable to be able to rank as many vectors as possible.

WEAK COMPLETENESS, **WC***: $\forall {}_1u, {}_1v \in Y$, $\pi({}_1u) \neq {}_1v$ $\forall$ finite permutation of $\mathbb{N} \Rightarrow ({}_1u_T, {}_{T+1}v) \succcurlyeq {}_1v$ or ${}_1v \succcurlyeq ({}_1u_T, {}_{T+1}v)$.

Theorem 3.9. *$\succcurlyeq$ is an extension of $\succcurlyeq^{M^*}$ if and only if $\succcurlyeq$ satisfies Finite Anonymity (**FA***), Weak Harm Principle (**WHP****), Weak Pareto Optimality (**WPO***), Weak Completeness (**WC***), Weak Continuity (**WCN**), and Weak Consistency (**WCONS**).*

Proof. ($\Rightarrow$) It is easy to see that $\succcurlyeq$ meets **FA***, **WHP****, **WPO***, **WC***, **WCN**, and **WCONS** whenever $\succcurlyeq$ is an extension of $\succcurlyeq^{M^*}$.

($\Leftarrow$) Suppose that $\succcurlyeq$ meets **FA***, **WHP****, **WPO***, **WC***, **WCN**, and **WCONS**. We show that $\succcurlyeq^{M^*} \subseteq \succcurlyeq$, that is, $\forall {}_1u, {}_1v \in Y$,

$$ {}_1u \succ^{M^*} {}_1v \Rightarrow {}_1u \succ {}_1v \tag{3.7}$$

and

$$ {}_1u \sim^{M^*} {}_1v \Rightarrow {}_1u \sim {}_1v. \tag{3.8}$$

We proceed by showing (3.7) first, and after we show that (3.8) holds as well.

Suppose that ${}_1u \succ^{M^*} {}_1v$. Take any $T \geq \tilde{T}$ and let ${}_1w = ({}_1v_T, {}_{T+1}u)$. We show that ${}_1u \succ {}_1w$. Observe that ${}_1u \succ^{M^*} {}_1w$, by construction. Assume, to the contrary, that ${}_1u \not\succ {}_1w$, so that ${}_1w \succcurlyeq {}_1u$ as the premises of **WC*** are met. We distinguish two cases.

Case 1: ${}_1w \succ {}_1u$

As **FA*** holds, let ${}_1\bar{w}, {}_1\bar{u}$ be such that ${}_{T+1}\bar{w} = {}_{T+1}w = {}_{T+1}u = {}_{T+1}\bar{u}$, and ${}_1\bar{w}_T$, ${}_1\bar{u}_T$ are such that $\bar{w}_1 \leq ... \leq \bar{w}_T$ and $\bar{u}_1 \leq ... \leq \bar{u}_T$. If ${}_1\bar{w}_T \ll {}_1\bar{u}_T$, **WPO*** implies ${}_1\bar{u} \succ {}_1\bar{w}$, a contradiction. Otherwise, let $\bar{w}_t \geq \bar{u}_t$ $\exists t \leq T$. We proceed in two steps.

Step 1.
Let

$$k = \min \{ l \leq T | \bar{w}_l \geq \bar{u}_l \}.$$

Let ${}_1\hat{w}$ and ${}_1\hat{u}$ be two finite permutations of $\mathbb{N}$ such that ${}_{T+1}\bar{w} = {}_{T+1}\hat{w} = {}_{T+1}\hat{u} = {}_{T+1}\bar{u}$ and, for some $i \leq T$, $\hat{w}_i = \bar{w}_k$ and $\hat{u}_i = \bar{u}_1$. By **FA***, ${}_1\hat{w} \sim {}_1\bar{w}$ and ${}_1\hat{u} \sim {}_1\bar{u}$, so that ${}_1\hat{w} \succ {}_1\hat{u}$. Then, let two real numbers $d_1, d_2 > 0$, and consider vectors ${}_1u', {}_1w^*$ formed as follows: first, $\hat{u}_i$ is lowered to $\hat{u}_i - d_1 > \bar{w}_1$;

next, $\hat{w}_i$ is lowered to $\hat{w}_i - d_2$ such that $\bar{u}_k > \hat{w}_i - d_2 > \hat{u}_i - d_1 > \bar{w}_1$; finally, all other entries of ${}_1\hat{u}$ and ${}_1\hat{w}$ are unchanged. It follows from **WHP**** that ${}_1w^* \succcurlyeq {}_1u'$. Let ${}_1\bar{w}^*$ and ${}_1\bar{u}'$ be two finite permutations of $\mathbb{N}$ such that ${}_{T+1}\bar{w}^* = {}_{T+1}w^* = {}_{T+1}u' = {}_{T+1}\bar{u}'$ and ${}_1\bar{w}^*_T, {}_1\bar{u}'_T$ are such that $\bar{w}^*_1 \leq ... \leq \bar{w}^*_T$ and $\bar{u}'_1 \leq ... \leq \bar{u}'_T$. By construction, $\bar{u}'_j > \bar{w}^*_j$ for all $j \leq k$. By **FA*** and transitivity, ${}_1\bar{w}^* \succcurlyeq {}_1\bar{u}'$.

Step 2.
Let

$$0 < \epsilon < \min\{\bar{u}'_j - \bar{w}^*_j | \forall j \leq k\}$$

and define ${}_1\bar{w}'_T = {}_1\bar{w}^*_T + {}_1\epsilon_T$, where ${}_1\epsilon_T$ is T-head of ${}_{con}\epsilon$. Let ${}_1\bar{w}' = ({}_1\bar{w}'_T, {}_{T+1}\bar{w}^*)$. By construction, ${}_1\bar{w}^*_T \ll {}_1\bar{w}'_T$, and $\bar{w}'_j < \bar{u}'_j$ for all $j \leq k$. **WPO*** implies ${}_1\bar{w}' \succ {}_1\bar{w}^*$. As ${}_1\bar{w}^* \succcurlyeq {}_1\bar{u}'$, by *step* 1, transitivity of $\succcurlyeq$ implies ${}_1\bar{w}' \succ {}_1\bar{u}'$.

If $\bar{u}'_j > \bar{w}'_j$ for all $j \leq T$, **WPO*** implies that ${}_1\bar{u}' \succ {}_1\bar{w}'$, a contradiction. Otherwise, let $\bar{w}'_l \geq \bar{u}'_l$ for some $T \geq l > k$. Then, let

$$k' = \min\left\{l \leq T | \ \bar{w}'_l \geq \bar{u}'_l\right\}$$

where $k' > k$.
The above steps 1-2 can be applied to ${}_1\bar{u}', {}_1\bar{w}'$ to derive vectors ${}_1\bar{u}'', {}_1\bar{w}''$ such that $\bar{u}''_j > \bar{w}''_j$ for all $j \leq k' \leq T$, whereas ${}_1\bar{w}'' \succ {}_1\bar{u}''$. By **WPO***, a contradiction is obtained whenever $\bar{u}''_j > \bar{w}''_j$ for all $j \leq T$. Otherwise, let $\bar{w}''_l \geq \bar{u}''_l$ for some $T \geq l > k'$. And so on. After a finite number s of iterations, two vectors ${}_1\bar{u}^s$, ${}_1\bar{w}^s$ can be derived such that ${}_1\bar{w}^s \succ {}_1\bar{u}^s$, by steps 1-2, but ${}_1\bar{u}^s \succ {}_1\bar{w}^s$, by **WPO***, yielding a contradiction.

Case 2: ${}_1w \sim {}_1u$ Since ${}_1\underline{w}_T < {}_1\underline{u}_T$, there exists $\epsilon > 0$ such that ${}_1\underline{w}^\epsilon_T = {}_1\underline{w}_T + \epsilon < {}_1\underline{u}_T$. Then fix such $\epsilon > 0$ and let ${}_1w^\epsilon = ({}_1w_T + {}_1\epsilon_T, {}_{T+1}w)$, where ${}_1\epsilon_T$ is the T-head of ${}_{con}\epsilon$. Thus, by construction, ${}_1u \succ^{M^*} {}_1w^\epsilon$, but by **WPO*** and transitivity of $\succcurlyeq$ we have ${}_1w^\epsilon \succ {}_1u$. Hence, the above *Case 1* verifies for the vectors ${}_1w^\epsilon$ and ${}_1u$, and so the same reasoning can be carried out to obtain the desired contradiction.

Since the above reasoning holds for any $T \geq \tilde{T}$, it follows that ${}_1u \succ {}_1w = ({}_1v_T, {}_{T+1}u)$ for all $T \geq \tilde{T}$. **WCONS** implies ${}_1u \succ {}_1v$ yielding the desired result.

Next, suppose that ${}_1u \sim^{M^*} {}_1v$, so that $\exists\tilde{T} \geq 1$ such that ${}_1\underline{u}_T = {}_1\underline{v}_T$ $\forall T \geq \tilde{T}$, by definition. If ${}_1v = \pi({}_1u)$ for some finite permutation π of $\mathbb{N}$, **FA*** implies ${}_1u \sim {}_1v$. Otherwise, let ${}_1v \neq \pi({}_1u)$ for all finite permutation π of $\mathbb{N}$. Take any $T \geq \tilde{T}$ and let ${}_1w = ({}_1v_T, {}_{T+1}u)$. We show that ${}_1u \sim {}_1w$.

Observe that ${}_1u \sim^{M^*} {}_1w$, by construction. Assume, to the contrary, that ${}_1u \not\sim {}_1w$, so that either ${}_1w \succ {}_1u$ or ${}_1u \succ {}_1w$ holds by **WC***. Without loss of generality, suppose ${}_1u \succ {}_1w$. As $\succcurlyeq$ meets **WCN** it follows that there exists $\epsilon > 0$ such that ${}_1u \succ {}_1w + {}_{con}\epsilon \equiv {}_1w^\epsilon$. However, ${}_1w^\epsilon \succ^{M^*} {}_1u$. Hence, by the implication (3.7) proved above, it follows that ${}_1w^\epsilon \succ {}_1u$ yielding a contradiction. Therefore, ${}_1u \sim {}_1w \equiv ({}_1v_{T,T+1}\,u)$. Since ${}_1u \sim {}_1w \equiv ({}_1v_{T,T+1}\,u)$ holds for any $T \geq \tilde{T}$, **WCONS** implies ${}_1u \sim {}_1v$. □

The properties in Theorem 3.9 are easily shown to be tight (see Appendix).

Theorem 3.9 provides an original characterisation of the maximin SWR in the context of societies with an infinite number of agents. This result is interesting per se, as compared to alternative characterisations of the maximin. For example, Lauwers ([12]) characterises the maximin SWO by an anonymous social welfare function (SWF) over the set of bounded utility streams, by imposing a strong version of **HE** (i.e, let ${}_1u, {}_1v$ be two bounded infinitely vectors identical up to two coordinates, that is, $u_i \geq v_i \geq v_j \geq u_j$ for some $i, j \in \mathbb{N}$ and $u_k = v_k$ $\forall k \in \mathbb{N}\backslash\{i, j\}$, then ${}_1v \succcurlyeq {}_1u$). The main focus of this paper is different and so the question of characterisation of the maximin SWO by an anonymous and liberal SWF remains open. itis worth noting, however, that we do characterise the maximin SWO on a different set of of infinite utility streams, which can be unbounded above and to this aim we do not appeal either to the continuity condition, or to the so-called "repetition approximation principle" (see [12], p.146) imposed by Lauwers.

Perhaps more importantly, Theorem 3.9 provides further support to the main theoretical arguments of this paper. For it confirms that the main intuitions concerning the role of the liberal notion of noninterference embodied in the **WHP** are extremely robust and they do not depend on the specific definition of the maximin and leximin SWR adopted to rank infinite utility streams (pionereed by Swensson, [20]).

4. Conclusions

This paper analyses Rawls's celebrated difference principle, and its lexicographic refinement, in societies with a finite and an infinite number of agents. A unified framework of analysis is set up, which allows one to characterise a family of egalitarian principles by means of a weaker version of a new axiom - the Harm Principle - recently proposed by [13]. This is quite surprising, because the Harm principle is meant to capture a liberal requirement of noninterference and it incorporates no obvious egalitarian content. A set of new characterisations of the maximin and of its lexicographic refinement are derived, including in the intergenerational context with an infinite number of agents and using different definitions of the relevant social welfare relations proposed in the literature.

The results presented in this paper have two main sets of implications from a theoretical viewpoint. First, they shed new light on the ethical foundations of the egalitarian principles stemming from Rawls's difference principle. In fact, both the leximin and the maximin are characterised by some standard axioms (such as Anonymity and the Pareto Principle) together with a liberal principle incorporating only a noninterfering view. No axiom with an explicitly egalitarian content is necessary in order to derive the main liberal egalitarian principles. Second, from the viewpoint of liberal approaches emphasising a notion of individual autonomy, or freedom, they are rather counterintuitive implication. For they prove that in a number of different contexts, liberal noninterfereing views lead straight to welfare egalitarianism.

Appendix

Let Π be the set of all finite permutations of $\mathbb{N}$.

Independence of axioms used in Theorem 3.4. In order to complete the proof of Theorem 3.4, we show that the axioms are tight.

For an example violating *only* **FA**, define $\succcurlyeq$ on X in the following way: $\forall {}_1x,\ {}_1y \in X$

$$\begin{aligned} &1)\ {}_1x = {}_1y \Rightarrow {}_1x \sim {}_1y \\ &2)\ {}_1x \neq {}_1y \text{ and } {}_1x = \pi\left({}_1y\right) \exists \pi \in \Pi :\ {}_1x \not\succcurlyeq {}_1y \text{ and } {}_1x \not\succcurlyeq {}_1y \\ &3)\ {}_1x \neq {}_1y \text{ and } {}_1x \neq \pi\left({}_1y\right) \forall \pi \in \Pi\text{: } {}_1x \succcurlyeq^{LM} {}_1y \Rightarrow {}_1x \succcurlyeq {}_1y \end{aligned}$$

The SWR $\succcurlyeq$ on X is not an extension of the leximin SWR $\succcurlyeq^{LM^*}$. The SWR $\succcurlyeq$ on X satisfies all properties except **FA**.

For an example violating *only* **SPO**, for all ${}_1x,\ {}_1y \in X$, define $\succcurlyeq$ on X in the following way: ${}_1x \sim {}_1y$. The SWR $\succcurlyeq$ on X is not an extension of the leximin SWR $\succcurlyeq^{LM^*}$. Clearly, the SWR $\succcurlyeq$ on X satisfies all properties except **SPO**.

For an example violating *only* **WC**, for all ${}_1x, {}_1y \in X$, define $\succcurlyeq$ on X in the following way: ${}_1x \succ {}_1y$ if ${}_1x > \pi\left({}_1y\right) \exists \pi \in \Pi$; ${}_1x \sim {}_1y$ if ${}_1x = \pi\left({}_1y\right) \exists \pi \in \Pi$; and ${}_1x \not\succcurlyeq {}_1y$ and ${}_1y \not\succcurlyeq {}_1x$ if ${}_1x \not> {}_1y$, ${}_1y \not> {}_1x$, and ${}_1x \neq \pi\left({}_1y\right)\ \forall \pi \in \Pi$. The SWR $\succcurlyeq$ on X is not an extension of the leximin SWR $\succcurlyeq^{LM^*}$. Clearly, the SWR $\succcurlyeq$ on X satisfies all properties except **WC**.

For an example violating *only* **WPC**, define $\succcurlyeq$ on X in the following way: $\forall {}_1x,\ {}_1y \in X$

$$\nexists T \geq 1 \text{ s.t. } {}_Tx = {}_Ty,\ {}_1x \not> {}_1y \text{ and } {}_1y \not> {}_1x \Rightarrow {}_1x \not\succcurlyeq {}_1y \text{ and } {}_1y \not\succcurlyeq {}_1x,$$

otherwise,

$${}_1x \succcurlyeq^{LM^*} {}_1y \Rightarrow {}_1x \succcurlyeq {}_1y.$$

$\succcurlyeq$ on X is a SWR. Fix $\mu, \nu, \epsilon, \delta \in \mathbb{R}$, with $\mu > \delta > \epsilon > \nu$. Let ${}_1x = (\delta,\ {}_{con}\epsilon)$ and ${}_1y = (\mu,\ {}_{con}\nu)$. Clearly, ${}_1x,\ {}_1y \in X$, and $({}_1x_T,\ {}_{T+1}y) \succ^{LM^*} {}_1y\ \forall T \geq 2$, ${}_1x \succ^{LM^*} {}_1y$, but ${}_1x \not\succcurlyeq {}_1y$ and ${}_1y \not\succcurlyeq {}_1x$. It follows that the SWR $\succcurlyeq$ on X is not an extension of the leximin SWR. The SWR $\succcurlyeq$ on X satisfies all properties except **WPC**.

For an example violating *only* **WHP***, define $\succcurlyeq$ on X in the following way: $\forall {}_1x, {}_1y \in X$

$${}_1x \sim {}_1y \Leftrightarrow \exists \tilde{T} \geq 1 \text{ s.t. } \forall T \geq \tilde{T} :\ {}_1\bar{x}_T = {}_1\bar{y}_T,$$

and

$$_1x \succ {}_1y \Leftrightarrow \exists \tilde{T} \geq 1 \text{ s.t. } \forall T \geq \tilde{T} : \exists t \in \{1, ..., T\} \; \bar{u}_s = \bar{v}_s \; (\forall t < s \leq T) \text{ and } \bar{u}_t > \bar{v}_t.$$

$\succcurlyeq$ on X is a SWR (i.e., the leximax SWR). It follows that the SWR $\succcurlyeq$ on X is not an extension of the leximin SWR. The SWR $\succcurlyeq$ on X satisfies all properties except **WHP***.

Independence of axioms used in Theorem 3.6. In order to complete the proof of Theorem 3.6, we show that the axioms are tight.

As Strong Preference Continuity (**SPC**) implies Weak Preference Continuity (**WPC**), the above examples show that the axioms used in Theorem 3.6 are tight as well.

Independence of axioms used in Theorem 3.9. In order to complete the proof of Theorem 3.9, we show that the axioms are tight.

For an example violating *only* **FA***, define $\succcurlyeq$ on Y in the following way: $\forall {}_1x, {}_1y \in Y$

1) $_1x = {}_1y \Rightarrow {}_1x \sim {}_1y$

2) $_1x \neq {}_1y$ and $_1x = \pi({}_1y) \, \exists \pi \in \Pi : {}_1x \not\succcurlyeq {}_1y$ and $_1y \not\succcurlyeq {}_1x$

3) $_1x \neq {}_1y$ and $_1x \neq \pi({}_1y) \, \forall \pi \in \Pi$: $\begin{array}{c} (1) \; {}_1x \succ^{M^*} {}_1y \Rightarrow {}_1x \succ {}_1y \\ (2) \; {}_1x \sim^{M^*} {}_1y, \exists T \geq 1 : {}_Tx = {}_Ty \Rightarrow {}_1x \sim {}_1y \end{array}$.

The SWR $\succcurlyeq$ on Y is not an extension of the maximin SWR $\succcurlyeq^{M^*}$. The SWR $\succcurlyeq$ on Y satisfies all properties except **FA***.

For an example violating *only* **WPO***, for all $_1x, {}_1y \in Y$, define $\succcurlyeq$ on Y in the following way: $_1x \sim {}_1y$. The SWR $\succcurlyeq$ on Y is not an extension of the maximin SWR $\succcurlyeq^{M^*}$. Clearly, the SWR $\succcurlyeq$ on Y satisfies all properties except **WPO***.

For an example violating *only* **WC***, define $\succcurlyeq$ on Y in the following way: $\forall {}_1x, {}_1y \in Y$,

$$\begin{aligned} {}_1x &\sim^{M^*} {}_1y, \; {}_1x = \pi({}_1y) \, \exists \pi \in \Pi \Rightarrow {}_1x \sim {}_1y \\ {}_1x &\sim^{M^*} {}_1y, \; {}_1x \neq \pi({}_1y) \, \forall \pi \in \Pi \Rightarrow {}_1x \not\succcurlyeq {}_1y \; \& \; {}_1y \not\succcurlyeq {}_1x \\ {}_1x &\neq \pi({}_1y) \, \forall \pi \in \Pi, \; {}_1x \succ^{M^*} {}_1y \Rightarrow {}_1x \succ {}_1y \end{aligned}$$

The SWR $\succcurlyeq$ on Y is not an extension of the maximin SWR $\succcurlyeq^{M^*}$. Clearly, the SWR $\succcurlyeq$ on Y satisfies all properties except **WC***.

For an example violating *only* **WCN**, fix $\mathcal{T} \in \mathbb{N}$ such that $1 < \mathcal{T} < \infty$. For ${}_1x \in Y$ let ${}_1\tilde{x}$ denote a permutation of ${}_1x$ such that $\tilde{x}_1 \leq \tilde{x}_2 \leq ... \leq \tilde{x}_\mathcal{T}$ and $\tilde{x}_t \geq \tilde{x}_\mathcal{T}$ $\forall t > \mathcal{T}$. Let us define $\succcurlyeq$ on Y in the following way: $\forall {}_1x,\, {}_1y \in Y$

$$\begin{aligned} {}_1x &\succ {}_1y \text{ if } \exists t \leq \mathcal{T} :\ \tilde{x}_s = \tilde{y}_s \ \ (\forall s < t) \ \ \& \ \tilde{x}_t > \tilde{y}_t \\ {}_1x &\sim {}_1y \text{ if } {}_1\tilde{x}_\mathcal{T} = {}_1\tilde{y}_\mathcal{T} \end{aligned}$$

The SWR $\succcurlyeq$ on Y is not an extension of the maximin SWR $\succcurlyeq^{M^*}$. Clearly, the SWR $\succcurlyeq$ on Y satisfies all properties except **WCN**.

For an example violating *only* **WHP****, fix $\mathcal{T} \in \mathbb{N}$ such that $1 < \mathcal{T} < \infty$. For ${}_1x \in Y$ let ${}_1\tilde{x}$ denote a permutation of ${}_1x$ such that $\tilde{x}_1 \leq \tilde{x}_2 \leq ... \leq \tilde{x}_\mathcal{T}$ and $\tilde{x}_t \geq \tilde{x}_\mathcal{T}$ $\forall t > \mathcal{T}$. Let us define $\succcurlyeq$ on Y in the following way: $\forall {}_1x,\, {}_1y \in Y$

$${}_1x \succ {}_1y \text{ if } \sum_{i=1}^{\mathcal{T}} \tilde{x}_i > \sum_{i=1}^{\mathcal{T}} \tilde{y}_i$$

and

$${}_1x \sim {}_1y \text{ if } \sum_{i=1}^{\mathcal{T}} \tilde{x}_i = \sum_{i=1}^{\mathcal{T}} \tilde{y}_i$$

The SWR $\succcurlyeq$ on Y is not an extension of the maximin SWR $\succcurlyeq^{M^*}$. Clearly, the SWR $\succcurlyeq$ on Y satisfies all properties except **WHP****.

For an example violating *only* **WCONS (a)**, define $\succcurlyeq$ on Y in the following way: $\forall {}_1x,\, {}_1y \in Y$,

$$\begin{aligned} &1)\ {}_1x \sim^{M^*} {}_1y,\ \exists T \geq 1 :\ {}_Tx = {}_Ty \Rightarrow {}_1x \sim {}_1y \\ &2)\ {}_1x \succ^{M^*} {}_1y \Rightarrow {}_1x \succ {}_1y \\ &3)\ {}_1x \sim^{M^*} {}_1y,\ \nexists T \geq 1 :\ {}_Tx = {}_Ty \Rightarrow {}_1x \not\succcurlyeq {}_1y \text{ and } {}_1y \not\succcurlyeq {}_1x. \end{aligned}$$

The SWR $\succcurlyeq$ on Y is not an extension of the maximin SWR $\succcurlyeq^{M^*}$. Clearly, the SWR $\succcurlyeq$ on Y satisfies all properties except **WCONS (a)**.

For an example violating *only* **WCONS (b)**, define $\succcurlyeq$ on Y in the following way: $\forall {}_1x,\, {}_1y \in Y$,

$$\begin{aligned} &1)\ {}_1x \sim^{M^*} {}_1y \Rightarrow {}_1x \sim {}_1y \\ &2)\ {}_1x \succ^{M^*} {}_1y,\ \exists T \geq 1 :\ {}_Tx = {}_Ty \Rightarrow {}_1x \succ {}_1y \\ &3)\ {}_1x \succ^{M^*} {}_1y,\ \nexists T \geq 1 :\ {}_Tx = {}_Ty \Rightarrow {}_1x \succ {}_1y \end{aligned}$$

The SWR $\succcurlyeq$ on Y is not an extension of the maximin SWR $\succcurlyeq^{M^*}$. Clearly, the SWR $\succcurlyeq$ on Y satisfies all properties except **WCONS (b)**.

Acknowledgments

We are grateful to José-Carlos Alcantud, Kaushik Basu, Andrés Carvajal, Bhaskar Dutta, Peter Hammond, Koichi Tadenuma, Naoki Yoshihara, and audiences at the University of Warwick (CRETA), the University of Maastricht, K.U. Leuven, Hitotsubashi University, the University of Massachusetts (Amherst), the Midwest Political Science Association conference (Chicago), and the New Directions in Welfare Conference (Oxford) for useful comments and suggestions. Special thanks go to François Maniquet and Marco Mariotti, whose comments have led to substantial improvements in the paper. The usual disclaimer applies.

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