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Elisa Pagani

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# On Lagrangian Duality in Vector Optimization. Applications to the linear case.

Elisa Pagani
Department of Economics, University of Verona
viale Università 3, Palazzina 32
37129 Verona, Italy

**Abstract.** The paper deals with vector constrained extremum problems. A separation scheme is recalled; starting from it, a vector Lagrangian duality theory is developed. The linear duality due to Isermann can be embedded in this separation approach. Some classical applications are extended to the multiobjective framework in the linear case, exploiting the duality theory of Isermann.

**Keywords:** Vector Optimization, Separation, Image Space Analysis, Lagrangian Duality, Set-Valued Function.

**AMS Classification:** 90C05, 90C29, 90C46, 49N15.

**JEL Classification:** C61.

## 1 Introduction and preliminaries

In the last years, there has been a growing interest addressed to the study of Vector Optimization. Since the development has been rather rapid, the research in the field of Vector Optimization has given rise to autonomous and not always well-connected investigations. In the eighties, it was evidenced the necessity of a unified approach that embraces the different developments; systematic studies in this sense can be found in [6], [7] and [12].

In the present paper, in Sect.2, we recall a general scheme for vector problems introduced in [3], where it is proposed an approach - based on separation arguments and alternative theorems - which embraces existing developments and which introduces new ones. This approach, that is based on analysis in the image space, is exploited to study Lagrangian duality in Vector Optimization. In Sect.3 it is shown that the vector linear duality of Isermann [5] can be embedded in this separation approach. In Sect.4 we propose some applications of the linear vector duality scheme introduced by Isermann in [5]. We start with a problem of minimization of costs and we extend this to a problem with two objectives; subsequently, a problem of maximization of profit is considered for two different firms. In both cases, we study the relationships between the shadow prices and we give a possible representation of the feasible region of the dual problem.

Now, we recall some notations and notions useful in what follows. $O_n$ denotes the $n$-tuple, whose entries are zero; when there is no fear of confusion the subfix is omitted; for $n = 1$, the 1-tuple is identified with its element, namely, we set $O_1 = 0$.

Let the positive integers $\ell, m, n$, the cone $C \subseteq \mathbb{R}^\ell$, the vector-valued functions $f : \mathbb{R}^n \to \mathbb{R}^\ell$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ and the subset $X \subseteq \mathbb{R}^n$ be given. In the sequel, it will be assumed that $C$ is convex, closed and pointed with apex at the origin and with $\mathrm{int}C \neq \emptyset$, namely with nonempty interior; $\mathrm{cl}A$ and $\mathrm{int}A$ will denote the closure and the interior, respectively, of the set $A \subset \mathbb{R}^n$; $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^n$.

Consider the following vector minimization problem, which is called *generalized Pareto problem*:

$$\min{}_{C\backslash\{O\}} f(x) \quad \text{subject to} \quad x \in Y = \{x \in X : g(x) \geq O\}, \tag{1}$$

where $\min_{C\backslash\{O\}}$ denotes vector minimum with respect to the cone $C\backslash\{O\}$: $y \in Y$ is a (global) vector minimum point (for short, v.m.p.) of (1), iff

$$f(y) \ngeq_{C\backslash\{O\}} f(x), \forall x \in Y, \tag{2}$$

where the inequality means $f(y) - f(x) \notin C\backslash\{O\}$. At $C = \mathbb{R}^\ell_+$ (1) becomes the classic *Pareto vector problem*. We will assume that v.m.p. exist.
Obviously, $y \in Y$ is a v.m.p. of (1), i.e (2) is fulfilled, iff the system (in the unknown $x$)

$$f(y) - f(x) \in C, \quad f(y) - f(x) \neq O, \quad g(x) \geq O, \quad x \in X \tag{3}$$

is impossible. System (3) is impossible iff $\mathcal{H} \cap \mathcal{K}(y) = \emptyset$, where $\mathcal{H} := (C\backslash\{O\}) \times \mathbb{R}^m_+$ and $\mathcal{K}(y) := \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : u = f(y) - f(x),\ v = g(x),\ x \in X\}$. $\mathcal{H}$ and $\mathcal{K}(y)$ are subsets of $\mathbb{R}^\ell \times \mathbb{R}^m$, that is called *image space*; $\mathcal{K}(y)$ is called *image* of problem (1).

In general, to prove directly $\mathcal{H} \cap \mathcal{K}(y) = \emptyset$ is a difficult task; this disjunction can be proved by means of a sufficient condition, that consists in obtaining the existence of a function, such that two of its disjoint level sets contain $\mathcal{H}$ and $\mathcal{K}(y)$, respectively.
To this end, let us consider the sets $U = C\backslash\{O\}$, $V = \mathbb{R}^m_+$ and $U^*_{C\backslash\{O\}} := \{\Theta \in \mathbb{R}^{\ell \times \ell} : \Theta u \geq_{C\backslash\{O\}} O,\ \forall u \in U\}$, $V^*_C := \{\Lambda \in \mathbb{R}^{\ell \times m} : \Lambda v \geq_C O,\ \forall v \in V\}$. Let us introduce the class of functions $w : \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^\ell$, defined by:

$$w = w(u, v, \Theta, \Lambda) = \Theta u + \Lambda v, \quad \Theta \in U^*_{C\backslash\{O\}}, \Lambda \in V^*_C, \tag{4}$$

where $\Theta, \Lambda$ are parameters; $w$ is called *separation function.* The positive and non positive level sets of a vector separation function $w$ are defined as follows:

$$W_{C\backslash\{O\}} := \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : w(u,v,\Theta,\Lambda) \geq C\backslash\{O\}\};$$

$$\overline{W}_{C\backslash\{O\}} := \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : w(u,v,\Theta,\Lambda) \ngeq C\backslash\{O\}\}.$$

In [3] there are the proofs of Proposition 1 and Theorem 1.

**Proposition 1** (see Proposition 1 of [3]) Let $w$ be given by (4); then we have $\mathcal{H} \subset W_{C\backslash\{O\}}$, $\forall \Theta \in U^*_{C\backslash\{O\}}$, $\forall \Lambda \in V^*_C$.

Proposition 1 is a first step towards a sufficient condition for the optimality of $y$. It is obvious that, if we can find one of the functions of class (4), such that $\mathcal{K}(y) \subset \overline{W}_{C\backslash\{O\}}$, then the optimality of $y$ is reached. Indeed, we have the following result.

**Theorem 1** (see Theorem 1 of [3]) Let $y \in Y$, if there exist matrices $\Theta \in U^*_{C\backslash\{O\}}, \Lambda \in V^*_C$ such that $w(f(y)-f(x), g(x), \Theta, \Lambda) = \Theta(f(y)-f(x)) + \Lambda g(x) \ngeq_{C\backslash\{O\}} O$, $\forall x \in X$, then $y$ is a (global) v.m.p. of (1).

At $\ell = 1$ and $C = \mathbb{R}_+$, the above theorem collapses to an existing one for scalar optimization (see Corollary 5.1 of [2]). Moreover, in the scalar case, there exists a known correspondence between the vector of Kuhn-Tucker multipliers and the parameters defining $w$. When $\ell > 1$ and $C = \mathbb{R}^\ell_+$, the natural extension would be to consider $\Theta$ and $\Lambda$ as the matrices obtained as marginal rate of substitution of one objective function with respect to another and of one objective function with respect to a constraint, respectively (see [9]). Unfortunately, in general, they do not satisfy the conditions $\Theta \in U^*_{C\setminus\{O\}}, \Lambda \in V^*_C$. Only if we take the matrix $\Theta$ in its absolute value, the pair $(\Theta, \Lambda)$, defined in [9], is such that $\Theta \in U^*_{C\setminus\{O\}},\ \Lambda \in V^*_C$. Observe that the identity matrix of order $\ell$, say $I_\ell$, belongs to $U^*_{C\setminus\{O\}}$ and that, when $\ell = 1$, $\Theta$ can be replaced by 1.

## 2 Lagrangian duality

In order to satisfy the sufficient optimality condition expressed by Theorem 1, it is natural to study, for each fixed $(\Theta, \Lambda) \in U^*_{C\setminus\{O\}} \times V^*_C$ the following vector optimization problem:

$$\max{}_{C\setminus\{O\}} w(u, v, \Theta, \Lambda),\ (u, v) \in \mathcal{K}(y), \tag{5}$$

where $w$ is given by (4) and $\max_{C\setminus\{O\}}$ denotes vector maximum with respect to $C\setminus\{O\}$: $(\bar{u}, \bar{v}) \in \mathcal{K}(y)$ is a vector maximum point of (5), iff

$$w(\bar{u}, \bar{v}, \Theta, \Lambda) \not\leq_{C\setminus\{O\}} w(u, v, \Theta, \Lambda),\ \forall (u, v) \in \mathcal{K}(y). \tag{6}$$

We recall some results that the interested reader can find in [3] and [10].

**Lemma 1** *If a maximum point in (5) exists, then we have*

$$z \not\leq_{C\setminus\{O\}} O,\ \forall z \in \max_{(u,v)\in\mathcal{K}}{}_{C\setminus\{O\}} w(u, v, \Theta, \Lambda),\ \forall \Theta \in U^*_{C\setminus\{O\}},\ \forall \Lambda \in V^*_C. \tag{7}$$

**Theorem 2** *For any $y \in Y$ and $\Lambda \in V^*_C$ it results:*

$$f(y) \not\leq_{C\setminus\{O\}} z,\ \forall z \in \min_{x\in X}{}_{C\setminus\{O\}} [f(x) - \Lambda g(x)]. \tag{8}$$

Let us recall the definition of *vector Maximum* of the set-valued map $\Phi : U^*_{C\setminus\{O\}} \times V^*_C \rightrightarrows \mathbb{R}^\ell$, where $\Phi(\Theta, \Lambda)$ is the set of the optimal values of (5).

**Definition 1** *$(\hat{\Theta}, \hat{\Lambda}) \in U^*_{C\setminus\{O\}} \times V^*_C$ is a vector Maximum, with respect to the cone $C\setminus\{O\}$, of the set-valued map $\Phi(\Theta, \Lambda)$ iff*

$$\exists \hat{z} \in \Phi(\hat{\Theta}, \hat{\Lambda}) \ s.t.\ \hat{z} \not\leq_{C\setminus\{O\}} z,\ \forall z \in \Phi(\Theta, \Lambda),\ \forall (\Theta, \Lambda) \in U^*_{C\setminus\{O\}} \times V^*_C.$$

The definition of vector Minimum is quite analogous.

Let us now define the following vector optimization problem of set-valued functions:

$$\begin{array}{ll} \mathrm{Max}_{C\setminus\{O\}} & \min_{C\setminus\{O\}} L_V(x, \Lambda) \\ \Lambda \in V^*_C & x \in X \end{array} \tag{9}$$

where $L_V(x, \Lambda) = f(x) - \Lambda g(x)$ is the *vector Lagrangian function* and Max denotes the vector Maximum of a set-valued map. Problem (9) will be called the *vector dual problem* of (1). Observe that when $\ell = 1$ and $C = \mathbb{R}_+$, (9) collapses to the well-known Lagrangian dual. Theorem 2 states that the vector of the objectives of the primal (1), evaluated at any feasible solution $y$, is not less than or equal to the vector of the objectives of the dual (9), calculated at any $\Lambda \in V^*_C$; hence Theorem 2 is a weak Duality Theorem, in the vector case.
If in Theorem 2 all the extremes are made with respect to int$C$ instead of $C\setminus\{O\}$, we obtain Theorem 4.1 of [14]. Theorem 2 is exactly Theorem 5.2.4 of [12]; here we stress the fact that Theorem 2 is a straightforward consequence of Lemma 1; i.e. of vector separation in the image space performed by function $w(u, v, \Theta, \Lambda)$ when $\Theta = I_\ell$.

Consider the set

$$\Omega = \begin{array}{ll} \mathrm{Min}_{C\setminus\{O\}} & w(u, v, \Theta, \Lambda). \\ \Theta \in U^*_{C\setminus\{O\}} & \\ \Lambda \in V^*_C & \end{array}$$

The following result holds.

**Lemma 2** *There exist $\bar{\Theta} \in U^*_{C\setminus\{O\}}$ and $\bar{\Lambda} \in V^*_C$ such that*

$$w(u, v, \bar{\Theta}, \bar{\Lambda}) \not\geq_{C\setminus\{O\}} O,\ \forall (u, v) \in \mathcal{K}(y) \tag{10}$$

*iff $O \in \Omega$.*

Now define the following sets:

$$\Delta_1 := \begin{array}{l} \min_{C\setminus\{O\}} f(x) \\ x \in Y \end{array} \quad \text{and} \quad \Delta_2 := \begin{array}{ll} \mathrm{Max}_{C\setminus\{O\}} & [f(x) - \Lambda g(x)]. \\ \Lambda \in V^*_C & \end{array}$$

Observe that if $\Delta_1 \cap \Delta_2 \neq \emptyset$, or equivalently $O \in \Delta := \Delta_1 - \Delta_2$, then there exist an optimal solution of the primal problem and an optimal solution of the dual such that the corresponding optimal vector values are equal; i.e., the two problems possess at least a common optimal value. When $\ell = 1$ and $C = \mathbb{R}_+$, the condition $O \in \Delta$ becomes $\Delta = \{0\}$ or, equivalently,

$$\min_{x \in Y} f(x) = \max_{\lambda \in \mathbb{R}^m_+} \min_{x \in X} [f(x) - \langle \lambda, g(x) \rangle],$$

which means that the duality gap is equal to 0 in the scalar case. Hence the study of the sets $\Delta_1$ and $\Delta_2$ and of their intersection leads to consider a concept of duality gap in vector optimization; to this aim see [1].

The following lemma gives condition equivalent to $O \in \Delta$; it translates the analogous condition expressed in the image space by Lemma 2.

**Lemma 3** *There exist $y \in Y$ and $\bar{\Lambda} \in V_C^*$ such that*

$$[f(y) - f(x)] + \bar{\Lambda} g(x) \not\geq_{C \setminus \{O\}} O, \ \forall x \in X \tag{11}$$

*iff $O \in \Delta$.*

We have a necessary and sufficient condition for having an optimal solution of (1) and an optimal solution of its dual problem, such that the corresponding optimal vector values are equal. This is obviously a generalization of the duality gap equal to zero for the scalar case. It is interesting to define classes of vector problems for which (11) is fulfilled. This happens, for instance, when the problem is linear.

## 3 The linear case

In this section we want to show that a known vector linear dual, i.e. that of Isermann [5], can be enclosed in the separation scheme introduced in Sect.2.

Taking into account the approach of Isermann, but using our notation, let us consider the following *linear Pareto vector problem*:

$$\min{}_{\mathbb{R}^\ell_+ \setminus \{O\}} Dx \text{ subject to } x \in P = \{x \in \mathbb{R}^n : Ax = b, x \geq O\}, \tag{12}$$

where $D$ is the $\ell \times n$ criterion matrix, $A$ is an $m \times n$ matrix with $rank A = m$ and $b \in \mathbb{R}^m$. Problem (12) can be put in the format of problem (1) if we consider

$$g(x) = \begin{pmatrix} Ax - b \\ -Ax + b \end{pmatrix}$$

$X = \{x \in \mathbb{R}^n : x \geq O\}$ and the cone $C = \mathbb{R}^\ell_+$. Hence, following the definition of vector dual problem given in (9), the vector dual of (12) is:

$$\begin{array}{ccc} \mathrm{Max}_{\mathbb{R}^\ell_+\setminus\{O\}} & \min_{\mathbb{R}^\ell_+\setminus\{O\}} & [Dx - \Lambda(Ax - b)], \\ \Lambda & x \geq 0 & \end{array} \tag{13}$$

where no condition is imposed on $\Lambda$ because of the equality constraints in (12).

Let us consider the following set: $T = \{\Lambda \in \mathbb{R}^{\ell \times m} : (\Lambda A - D)x \not\geq_{\mathbb{R}^\ell_+\setminus\{O\}} O,\ \forall x \geq O\}$ and its complement $\overline{T} = \{\Lambda \in \mathbb{R}^{\ell \times m} : \exists x \geq O$ such that $(\Lambda A - D)x \geq_{\mathbb{R}^\ell_+\setminus\{O\}} O\}$. For every $\Lambda \in \overline{T}$ there exist $x \geq O$ such that $(D - \Lambda A)x \in -\mathbb{R}^\ell_+\setminus\{O\}$ and hence $(D - \Lambda A)x + \Lambda b$ is not bounded from below on $x \geq O$. Therefore in (13) we can restrict the analysis to $\Lambda \in T$.

**Proposition 2**

$$\Lambda \in T \ \textit{iff}\ \Lambda b \in \min_{x \geq O}\ {}_{\mathbb{R}^\ell_+\setminus\{O\}}[Dx - \Lambda(Ax - b)].$$

Proof: Starting from the definition of set $T$, we have the following sequence of equivalences:

$$\begin{aligned} \Lambda \in T \quad & \Longleftrightarrow\ Dx - \Lambda Ax \not\leq_{\mathbb{R}^\ell_+\setminus\{O\}} O,\ \forall x \geq O \\ & \Longleftrightarrow\ Dx - \Lambda Ax + \Lambda b \not\leq_{\mathbb{R}^\ell_+\setminus\{O\}} \Lambda b,\ \forall x \geq O \\ & \Longleftrightarrow\ Dx - \Lambda(Ax - b) \not\leq_{\mathbb{R}^\ell_+\setminus\{O\}} \Lambda b,\ \forall x \geq O \\ & \Longleftrightarrow\ \Lambda b \in \min_{\substack{\mathbb{R}^\ell_+\setminus\{O\} \\ x \geq O}} [Dx - \Lambda(Ax - b)]. \end{aligned}$$

□ Proposition 2 shows that $\Lambda \in T$ iff $\Lambda b$ is one of the minimum vector values of the problem

$$\min_{x \geq O}\ {}_{\mathbb{R}^\ell_+\setminus\{O\}}[Dx - \Lambda(Ax - b)].$$

In [5], Isermann defines the vector dual problem of (12) by substituting

$$\min_{x \geq O}\ {}_{\mathbb{R}^\ell_+\setminus\{O\}}[Dx - \Lambda(Ax - b)]$$

with the single particular value $\Lambda b$:

$$\max_{\Lambda}\ {}_{\mathbb{R}^\ell_+\setminus\{O\}}\Lambda b \text{ subject to } \Lambda \in T. \tag{14}$$

The above substitution maintains the properties of the primal-dual expressed by Theorem 2.

Let us recall that, in the linear case, we have:

$$\Delta_1 := \min_{\substack{\mathbb{R}^\ell_+\setminus\{O\} \\ x \in P}} Dx \quad \text{and} \quad \Delta_2 := \mathrm{Max}_{\substack{\mathbb{R}^\ell_+\setminus\{O\} \\ \Lambda \in T}} \Lambda b.$$

**Theorem 3** (see [5] or Theorem 5.1.4 of [12]) The following statements hold:

**i)** $Dx \nleq_{\mathbb{R}^{\ell}_{+}\setminus\{O\}} \Lambda b,\ \forall x \in P,\ \forall \Lambda \in T$;

**ii)** $\Delta_1 = \Delta_2$.

# 4 Examples and applications

In the literature, there are not many works that deals with the economic description of vector dual problems. We cite, for instance, [15] and [16], that propose a non linear primal problem for minimizing the risk (i.e., the variance) and maximizing the expected return of a financial portfolio at the same time. However, in these works, there is not an economic interpretation of the dual variables obtained by the exploited duality scheme.

## 4.1 Minimization of costs

A classical example in the economic theory, that fits in an optimization problem, is the minimization of the costs of a firm. We extend this example to one with two conflicting goals: minimization of costs and minimization of pollution. We focus on the linear case.

We may imagine that a firm has to produce $m$ different outputs, i.e. $P_1, \ldots, P_m$, with $n$ available resources, i.e. $R_1, \ldots, R_n,\ x_j,\ j = 1, \ldots, n$ is the unknown quantity of $j$th resource necessary in the production process, $b_k,\ k = 1, \ldots, m$ is the demanded quantity of product $P_k$ and an element $a_{kj} \in A;\ k = 1, \ldots, m;\ j = 1, \ldots, n$ denotes the quantity of product $P_k$ obtained by the employment of a unit of resource $R_j$. This firm tries to minimize its production costs, $\langle d^1, x\rangle$, meanwhile the minimization of pollution, associated to the production process, $\langle d^2, x\rangle$, is an objective that clashes with the previous one. The vector primal problem is:

$$P_L := \begin{cases} \min(\langle d^1, x\rangle, \langle d^2, x\rangle) \\ Ax = b \\ x \geq O \end{cases}.$$

From this primal problem, we derive the vector dual due to Isermann:

$$D_L := \begin{cases} \max(\langle \lambda^1, b\rangle, \langle \lambda^2, b\rangle) \\ \Lambda \in \{\Lambda \in \mathbb{R}^{\ell \times m} s.t. (D - \Lambda A)w \notin -\mathbb{R}^2_{+}\setminus\{0\},\ \forall w \geq 0\} \end{cases}.$$

The constraints of the primal problem imply the equality between the demand and the supply of the products $P_k,\ k = 1, \ldots, m$. Suppose now that we do not start the production process, but we directly buy the final products at the price $\lambda_k^1 \geq 0$; $\sum_{k=1}^m \lambda_k^1 a_{kj}$ gives the total cost to buy the final product and $\sum_{k=1}^m \lambda_k^2 a_{kj}$ gives the total pollution implied by the production of $P_1, \ldots, P_m$. This alternative strategy is equivalent to that one of the primal problem if $\sum_{k=1}^m \lambda_k^1 a_{kj} = d_j^1$ and $\sum_{k=1}^m \lambda_k^2 a_{kj} = d_j^2$. This fact leads to the definition of the feasible region of the dual problem. In order to find a feasible matrix $\Lambda$, $\forall w \geq 0$ either $\langle d^{1T} - \lambda^{1T} A, w \rangle \geq 0$ or $\langle d^{2T} - \lambda^{2T} A, w \rangle \geq 0$ must be verified. Hence, the second strategy looks for the system of prices that satisfies the previous conditions in a way to result as less disadvantageous as possible. The term "shadow prices" referred to $\lambda_k^1$ and $\lambda_k^2$, $k = 1, \ldots, m$, is interpreted as the maximal price that the firm is willing to pay to buy directly the final product instead of to produce them by itself, or the amount of pollution that the firm is willing to take, respectively.

Let us now do some considerations based on a simple numerical example. Take, for instance,

$$P_L := \begin{cases} \min(1/4 x_1 + x_2, 4x_1 + x_2) \\ x_1 + x_2 = 4 \\ x_j \geq 0,\ j = 1, 2 \end{cases} .$$

The coefficients of both objective functions are positive, since they represent costs and pollution rates. A product is realized by means of two resources, $x_1$ and $x_2$. These resources are complementary, we mean that an increase in the use of the first one implies a decrease in the use of the other (for instance, capital and labour).

The dual problem is the following:

$$D_L := \begin{cases} \max(4\lambda_1, 4\lambda_2) \\ \begin{pmatrix} (1/4 - \lambda_1)w_1 + (1 - \lambda_1)w_2 \\ (4 - \lambda_2)w_1 + (1 - \lambda_2)w_2 \end{pmatrix} \notin -\mathbb{R}_+^2 \backslash \{0\},\ \forall w \geq 0 \end{cases} ,$$

where $\lambda_1 = \lambda_1^1$ and $\lambda_2 = \lambda_1^2$. Let us call

$$\begin{pmatrix} 1/4 - \lambda_1 & 1 - \lambda_1 \\ 4 - \lambda_2 & 1 - \lambda_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} .$$

The feasible region of the dual problem may be represented as in figure 1.

On the efficient frontier, there is a reverse relation between $\lambda_1$ and $\lambda_2$: if there is an increase of the price of a product, call it $P_k$, the demand of $P_k$ decreases, and so does the production of $P_k$

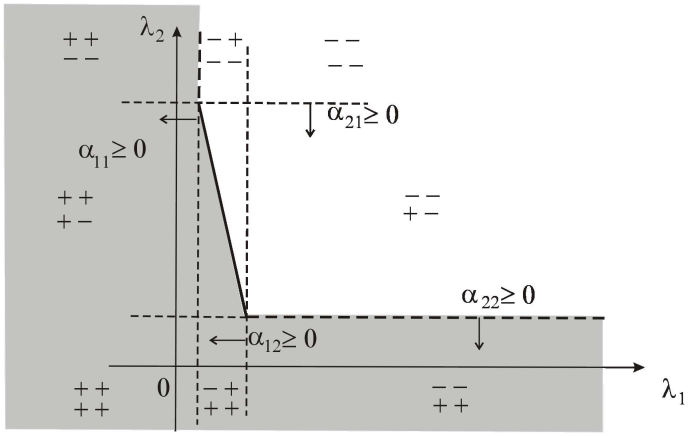


Figure 1

causing a reduction of pollution. Otherwise, we can say that a decrease in pollution, generated by the production process of $P_k$, implies an increase in the price of $P_k$, since the quality of this product is better than before.

Consider, for a while, the scheme of the feasible region of the dual problem as in figure 2.

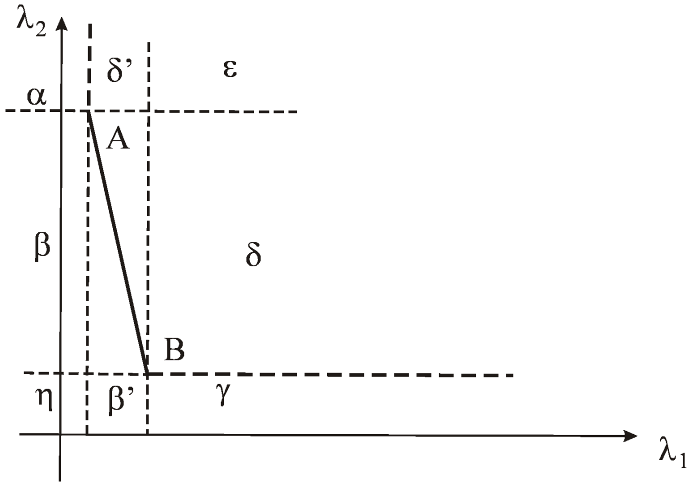


Figure 2

We want to do some remarks on the different areas that compose it. In $\alpha$ we have high rates of pollution and low prices. The production of goods of this kind should be forbid by the state till to converge to point $A$. In $\gamma$ there is also an interesting situation about pollution rates, but the production is too expensive. Maybe, for this kind of products, the state should propose some sort of incentive to maintain low prices. These two areas render the feasible region non-convex; it could be interesting to think of a dual problem that does not take them into consideration.

In $\beta$ there is a quite good level of pollution, but it is sustainable only if the selling price is that one on the frontier. In $\eta$ and $\beta'$ the situation is the best as possible (low prices, low pollution), but probably, the firms do not produce this kind of goods, since it is more convenient for them to converge to point $B$. $\delta$, $\delta'$ and $\varepsilon$ do not belong to the feasible region; in fact, the efficient frontier represent a better situation than that one described by a point of these areas.

We may also propose some considerations on the vector problem starting by the Lagrangian function. Let us define the vector Lagrangian function as: $L_V : \mathbb{R}^n \times \mathbb{R}^{\ell\times m} \to \mathbb{R}^\ell$, $L_V(x;\Lambda) = Dx - \Lambda(Ax - b)$.

In our example $L_V : \mathbb{R}^2 \times \mathbb{R}^{2\times 1} \to \mathbb{R}^2$, where

$$D = \begin{pmatrix} d^{1T} \\ d^{2T} \end{pmatrix} \quad \Lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

The two components of the Lagrangian function are:

$$\begin{aligned} (L_V)_1 &= \left\langle d^1, x \right\rangle - \langle \lambda_1, Ax - b\rangle \ \text{ and} \\ (L_V)_2 &= \left\langle d^2, x \right\rangle - \langle \lambda_2, Ax - b\rangle\,. \end{aligned}$$

If $s_k = \sum_{j=1}^n a_{kj}x_j - b_k > 0,\ k = 1, \dots, m$, we have an excess of production, then $\langle \lambda_1, s_k\rangle$ is a revenue to be subtracted to the total costs if we sell on the market this quantity; moreover, this quantity implies a decrease of the production of the other firms, so that the total pollution $\left\langle d^2, x \right\rangle$ is reduced by the quantity $\langle \lambda_2, s_k\rangle$.

### 4.2 Maximization of profit

We suppose now that a firm has to produce $n$ outputs, $P_1, \dots, P_n$, such that the unknown amount is $x = (x_1, \dots, x_n)$, using $m$ resources, $R_1, \dots, R_m$, available in the limited quantity $b = (b_1, \dots, b_m)$. An element $a_{kj} \in A$; $k = 1, \dots, m$; $j = 1, \dots, n$ denotes the quantity of resource $R_k$ necessary to the production of a unit of output $P_j$. The firm sells on the market the products in order to maximize the profit, $\langle d^1, x\rangle$. Other hypothesis are that the costs are linear, the markets are competitive and all the production may be sold. In order to render this problem a vector problem with two conflicting goals we introduce a second objective function, that is the maximization of profit by a second firm, $\langle d^2, x\rangle$. The vector primal problem is:

$$P_L := \begin{cases} \max(\langle d^1, x\rangle, \langle d^2, x\rangle) \\ Ax = b \\ x \geq O \end{cases},$$

and the Isermann vector dual problem is:

$$D_L := \begin{cases} \min(\langle \lambda^1, b\rangle, \langle \lambda^2, b\rangle) \\ \Lambda \in \{\Lambda \in \mathbb{R}^{\ell \times m} s.t. (D - \Lambda A)w \notin \mathbb{R}^2_+ \backslash \{0\},\ \forall w \geq 0\} \end{cases} .$$

We can think now to this alternative strategy: the firms, instead of sell their production, may sell on the market the resources necessary to the transformation process at the price $\lambda^1_k,\ k = 1, \dots, m$. The two strategies are equivalent if $\sum_{k=1}^m \lambda^1_k a_{kj} = d^1_j$ and $\sum_{k=1}^m \lambda^2_k a_{kj} = d^2_j$. This fact implies the definition of the feasible region of the dual problem. In order to find a feasible matrix $\Lambda$, $\forall w \geq 0$ either $\left\langle d^{1T} - \lambda^{1T} A, w \right\rangle \leq 0$ or $\left\langle d^{2T} - \lambda^{2T} A, w \right\rangle \leq 0$ must be verified. So, the second strategy looks for the system of prices that renders the transformation of the resources as less advantageous as possible. The term "shadow prices" referred to $\lambda^1_k$ and $\lambda^2_k$, $k = 1, \dots, m$, is interpreted as the minimal price that the firm is willing to attribute to the $k$th resource, such that the production process may not be performed.

We introduce now a simple numerical example. Take, for instance,

$$P_L := \begin{cases} \min(-1/4x_1 + x_2, 4x_1 - x_2) \\ x_1 + x_2 = 4 \\ x_j \geq 0,\ j = 1, 2 \end{cases} .$$

The coefficients of the two objective functions may be negative, since the profit may be a loss. On the efficient frontier, an increase in the production of a good implies a decrease in the production of the other, since the available resources are limited.

The dual problem is the following:

$$D_L := \begin{cases} \max(4\lambda_1, 4\lambda_2) \\ \begin{pmatrix} (-1/4 - \lambda_1)w_1 + (1 - \lambda_1)w_2 \\ (4 - \lambda_2)w_1 + (-1 - \lambda_2)w_2 \end{pmatrix} \notin \mathbb{R}^2_+ \backslash \{0\},\ \forall w \geq 0 \end{cases} ,$$

where $\lambda_1 = \lambda^1_1$ and $\lambda_2 = \lambda^2_1$. Let us call

$$\begin{pmatrix} -1/4 - \lambda_1 & 1 - \lambda_1 \\ 4 - \lambda_2 & -1 - \lambda_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} .$$

The feasible region of the dual problem may be represented by figure 3.

There exists a reverse relation between the efficient $\lambda_1$ and $\lambda_2$. We try to explain this fact as follows: if the contribution of a resource to the production process of a firm raises, then

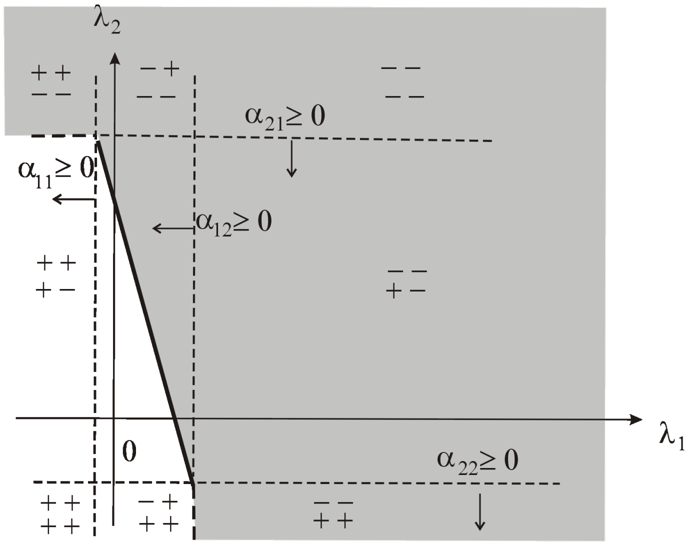


Figure 3

the demand of this resource raises and the availability of this for the second firm decreases. Consequently the growth of a firm implies necessarily the fall down of the other one.
Let us consider figure 4, that is the scheme of the feasible region of the dual problem.

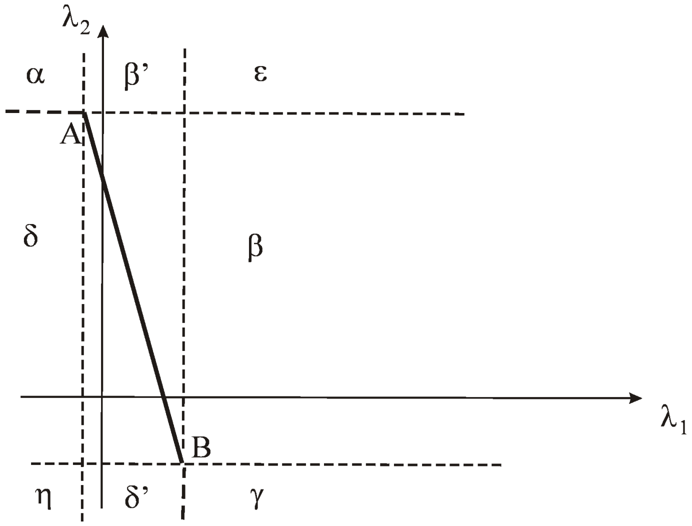


Figure 4

The different areas that compose the scheme may be interpreted in this manner: in $\alpha$ the second firm wants to attribute an high value to the resource, but it could be justified only in a monopolistic framework; thus, to avoid the exit of the first firm from the market, the best situation is that one described by the point $A$. It happens the same in $\gamma$ with the reverse order of the firms. Also in this example, $\alpha$ and $\gamma$ are the areas which give a non-convex characterization to the feasible region. In $\varepsilon$ both the firms are overestimating the resource; probably the value of this resource could be that one only in the case of a cartel of the two firms. In $\beta$ and $\beta'$ there is a firm which attributes a too high value to the resource, the first one and the second one, respectively. $\delta$, $\delta'$ and $\eta$ do not belong to the feasible region.

As in the previous example, we can now do some remarks about the vector Lagrangian function. The two components of the Lagrangian function are:

$$(L_V)_1 = \left\langle d^1, x \right\rangle + \langle \lambda_1, b - Ax \rangle \text{ and}$$
$$(L_V)_2 = \left\langle d^2, x \right\rangle + \langle \lambda_2, b - Ax \rangle .$$

If $s_k = b_k - \sum_{j=1}^{n} a_{kj} x_j > 0,\ k = 1, \ldots, m$, we have an unsold stock of the $k$th resource, then $\langle \lambda_1, s_k \rangle$ is the revenue that the first firm obtains from the sell of this quantity; moreover, this revenue may be summed to the profit of the firm, that is $\left\langle d^1, x \right\rangle$. The same argument is valid for the second firm.

## 5 Concluding remarks

This paper proposes a way to represent the vector dual problem in the linear case, in order to interpret the dual variables (a vector of variables for each objective function) and the possible relationships existing among them.

The examples are taken from an elementary context, but it is only the first step towards a theory that can be analyzed in a more general framework, for instance the non-linear or the non-convex one.

Obviously only in a setting of two objectives and one constraint we can plot easily the graph.

We observe that the feasible region of the dual problem is not convex, even in the linear case; moreover, the areas which renders this region non-convex are those of less interest, we mean that by the interpretation of the points belonging to these areas, they could be disregarded.

Matter of future studies could be to find a duality scheme that reduces the feasible region of the dual in order to make it convex, at least in the linear case.

## References


[1] Bigi G., Pappalardo M. (2001), "What is duality gap in vector optimization?", TR 01/15, Department of Computer Sciences, University of Pisa.

[2] Giannessi F. (1984), "Theorems of the alternative and optimality conditions", *JOTA,* **42**, 331–365.

[3] Giannessi F., Mastroeni G., Pellegrini L. (2000), "On the theory of vector optimization and variational inequalities. Image space analysis and separation.", In *Vector variational inequalities and vector equilibria. Mathematical theories*, edited by Giannessi F., Kluwer Academic Publisher, 153–215.

[4] Giannessi F. (2005), "Constrained optimization and image space analysis. Vol.1: Separation of sets and optimality conditions", Springer.

[5] Isermann H. (1978), "On some relations between a dual pair of multiple objective linear programs", *Zeitschrift für Operations Research,* **22**, 33–41.

[6] Jahn J. (1986) "Mathematical vector optimization in partially ordered linear spaces", *Methoden und Verahren der Mathematischen Physik,* **31**, Peter Lang, Frankfurt.

[7] Luc D.T. (1989), "Theory of vector optimization, lectures notes in economics and mathematical systems", 319 Springer-Verlag, Berlin.

[8] Mastroeni G., Rapcsak T. (2000), "On convex generalized systems", *JOTA,* **104**, 605–627.

[9] Pagani E., Pellegrini L. (2009), "Scalarization and sensitivity analysis in vector optimization. The linear case", Working Paper n. 56, Dept. of Economics, University of Verona.

[10] Pellegrini L. (2004), "On dual vector optimization and shadow prices", *RAIRO,* **38**, 305–317.

[11] Quang P.H., Yen N.D. (1991), "New proof of a theorem of F.Giannessi", *JOTA,* **68**, 385–387.

[12] Sawaragi Y., Nakayama H., Tanino T. (1985), "Theory of multiobjective optimization", Academic Press, New York.

[13] Song W. (1996), "Duality for vector optimization of set valued functions", *Journal of Mathematical Analysis and Applications,* **201**, 212–225.

[14] Wang S., Li Z. (1992), "Scalarization and lagrange duality in multiobjective optimization", *Optimization,* **26**, 315–324.

[15] Wanka G. (1999), "Multiobjective duality for the Markowitz portfolio optimization problem", *Control and Cybernetics* **28**, 691–702.

[16] Wanka G., Göhler (2001), "Duality for portfolio optimization with short sales", *Mathematical Methods of Operations Research* **53**, 247–263.