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A Benefit from the Division of Labor that Adam Smith Missed

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Abstract:

This paper argues that there is an important benefit from the division of labor that Adam Smith failed to mention, that is, the division of labor shortens the time required for capital formation and makes continuous roundabout production possible. The paper presents a simple general equilibrium model to demonstrate this benefit, which is referred to as the “value of time” benefit of the division of labor.

Key words: the division of labor, time structure of production, capital formation, transaction costs

JEL classification: D92, O10

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1. Introduction

In the opening paragraph of his *Wealth of Nations* (Book 1, Chapter 1), Adam Smith (1776) postulated that the “greatest improvement in the productive powers of labour, and the greater part of the skill, dexterity, and judgment ... seem to have been the effects of the division of labour.” According to Smith, the benefit of the division of labor comes from three main sources. First, the division of labor increases the dexterity of each worker as he specializes in a single operation. Secondly, it saves time that is otherwise lost in changing from one task to another; and lastly, it facilitates the use of machines that complements with specialized labor.

There is in fact another important source of benefit from the division of labor that Smith failed to mention, namely that the division of labor shortens the time required in capital formation and makes continuous roundabout production (i.e., production with capital-using technology) possible. To illustrate, imagine a fishing village where the villagers can use two different fishing “technologies”: one is to fish with their bare hands; the alternative is to make nets first and then catch more fish with the nets. Assume for simplicity that if the villagers fish with their bare hands, their daily catch is sufficient to sustain them for two days. Assume further that it takes one day to make a net. In the absence of the division of labor, each villager catches fish with his bare hands on Day 1, consumes half of the catch on Day 1 and saves the rest for Day 2 when he makes a net. He can then use his net on Day 3. In contrast, with the division of labor, some villagers will fish with their bare hands on Day 1, others will make nets. They exchange with each other at the end of Day 1, and nets are available for use from Day 2, one day earlier than the case with no division of labor. This

simple example points to an additional mechanism through which the division of labour improves individual welfare. That is, the division of labor shortens capital formation time – the capital good, net, becomes available one day earlier and consequently the villagers enjoy the benefit of the more productive “capital-using technology” sooner. A related benefit is that the time during which some resource lies idle is shortened and the cost of storage reduced as well – fish caught on Day 1 are consumed on Day 1 instead of being stored until the end of Day 2 to be consumed. In addition, the division of labor enables continuous production with capital-using technology – a villager does not have to stop fishing to make a new net when his old net wears out.

We shall call the benefit from the division of labor illustrated in the above example the “value of time” benefit, which is very different from Adam Smith’s second claim that the division of labor saves the time that is otherwise lost in changing from one task to another. Adam Smith’s second claim is about less time spent on the productive activities. For example, if fish are caught in a river, and nets are made in a workshop, then the division of labor between fishing and net making saves the time involved in changing between the activities of fishing and making nets, which may include the travelling time between the river and the workshop and perhaps also the additional mental preparation required before an individual starts a different task. In contrast, the “value of time” benefit flows from the fact that a roundabout productivity process consists of a series of sequential tasks, and that the division of labor allows the sequential tasks to proceed continuously and in parallel, thereby shortening the time required to produce the final product. Thus in our fishing example, with the division of labor, fishing and net-making are carried out at the same time, which

means that the villagers can use nets sooner compared to the case with no division of labor.

We can also use Smith's famous example of the pin factory to illustrate the difference between Smith's time-saving benefit of the division of labor and what we call the "value of time" benefit. According to Smith (1776), the process of pin-making was divided into about 18 distinct operations. Suppose each operation took 2 minutes and changing between operations cost 4 minutes in total. In the absence of the division of labor, 18 workers could make 18 pins every 40 ($=36+4$) minutes, so each pin would cost 40 minutes labor. With the division of labor, each pin would only cost 36 minutes of labor as the 4 minutes required to change between operations were avoided. This is Smith's time-saving benefit. The "value of time" benefit refers not to the fact that each pin cost 4 minutes less labor, but that instead of having a batch of 18 pins made every 40 minutes, now pins were made continuously every 2 minutes (after the initial 36 minutes required for the first pin), so that users of pins can have them sooner.

In the context of the pin factory example, the "value of time" benefit may not seem so important. However the pin factory example does not capture a key feature of the modern production system, which is that the completion of a final good involves many different interrelated stages of production. Take Leonard E. Read's well-known story of "I, Pencil"(Read, 2008). To make an ordinary pencil, a tree is grown, harvested and logs shipped to a mill. The logs are cut into slats, sent to a pencil factory where the slats are machined; leads are placed between two slats which are glued together. To make the leads, graphite needs to be mined and mixed with other

materials. Many other things are also required: lacquer for coating, films for labelling, brass for holding the eraser, rubber for making the eraser, and so on. Of course the making of lacquer, films, brass and rubber again involve complex chains of production stages. Thus millions of people play a part in the making of an ordinary pencil, each contributing a “tiny, infinitesimal bit of know how”. For this reason, Read claims that “not a single person on the face of this earth knows how to make” a pencil (p.8). Read is no doubt correct in suggesting that no single individual could have possessed the vast knowledge required to make a pencil the way it is routinely made today. However, recognising the fact that each of the complex production stages takes time, we may add another similarly astonishing claim: in the absence of the division of labor, even if a single person could have possessed all the know-how to make a pencil, he would not live long enough to make one from beginning to finish!

As Read’s story of the pencil illustrates, the modern production system is characterized by the time structure of production, i.e., production processes consisting of multiple, sequential, and time-consuming stages (Leijonhufvud, 1986). The “value of time” benefit of the division of labor flows logically from the time structure of production because the division of labor allows sequential production stages to proceed in parallel thereby shortening the time required for product completion. We therefore argue that the “value of time” benefit of the division of labor is a prevalent feature of the modern economy. In fact, probably because the benefit is so prevalent, it is often unrecognized just as the miracle of the pencil is taken for granted by most.

In this paper, we develop a simple general equilibrium model to demonstrate the “value of time” benefit from the division of labor as illustrated in our fishing village example. To our knowledge, the “value of time” benefit of the division of labor has not been studied formally in the literature. There is however a related literature on the relationship between the division of labor and capital formation. For example, Becker (1985) examines how increasing returns from specialised capital affects the division of labor between married men and women. Becker and Murphy (1992) investigate how the division of labor may be affected by coordination costs and knowledge which is determined by human capital investment. Yang (2002) studies the complex relationship between capital investment, the division of labor, productivity, the extent of the market and trade dependence. Cheng and Zhao (2008) incorporate capital investment and the division of labor in a general equilibrium model to explain economic performance through time. Our paper contributes to this literature and more broadly to the literature that puts the division of labor at the centre of economic analysis (see Cheng & Yang, 2004 for a survey). To the extent that our paper attempts to model the time structure of production which Leijonhufvud (1986) argues is absent in the neoclassical production function, it is related, in spirit, to the macroeconomic literature pioneered by Kydland and Prescott (1982) that highlights the importance of multiple-period construction in explaining aggregate fluctuations.

We present our model in the following section and offer some concluding remarks in section 3.

2. The model

2.1. The set up

Consider an economy with many *ex ante* identical individuals who derive utility from a single final good, Z (fish), over an infinite time horizon. The individual's utility function is:

$$U = \sum_{t=1}^{\infty} \beta^{t-1} Z_t \quad (\beta < 1) \quad (1)$$

The final good, Z , can be produced with either labor (l) alone or a combination of labor and a capital good, net (N). N is produced with labor only. Each individual is assumed to be endowed with L units of labor per period.

The two technologies are described by the following production functions:

$$Z = \begin{cases} al & \text{if } N < 1 \\ Al & \text{if } N \geq 1 \end{cases} \quad (2)$$

and

$$N = \frac{l}{L} \quad (3)$$

Assuming that the technologies are used by individuals not groups (i.e., $l \leq L$ for each period), the above production functions imply that it takes an individual a whole period to make a net, and that only one net will be used if an individual chooses the second (net-using) technology.

We assume that the second technology is more productive than the first technology, that is, $Al > a(l + L)$, which simplifies to

$$A > 2a \quad (4)$$

if an individual devotes his entire daily labor endowment to fishing ($l = L$).

It should be noted that with this specification of the technologies, specialization has no impact on individual productivity; does not save time in switching between jobs; and does not lead to the invention of new machines. By assuming away the benefits from the division of labor emphasized by Adam Smith, we can highlight the “value of time” benefit of the division of labor that Smith missed.

Assuming the individuals have full information about their preferences and the available technologies, they need to choose what to produce (fish, net, or both) before they make any quantity decisions². The combination of all individual choices results in 3 feasible structures as illustrated in Figure 1.

[Insert Figure 1 here]

We discuss the features of each structure below.

(1) Structure A: Autarky with no capital goods.

In this structure, all individuals choose to fish with labor only; no net is produced, no trade takes place. Each individual devotes his entire labor endowment to fishing every period, and consumes all his catch at the end of that period. An individual’s total utility over the infinite time horizon amounts to:

$$U_A = \frac{aL}{1-\beta} \tag{5}$$

(2) Structure B: Autarky with capital goods

In this structure, each individual catches fish in period 1 with labor only. He consumes half of his catch in period 1, and saves the rest for consumption in period

² Indeed, the quantity decisions are trivial in this model given the simple technologies.

2.³ In period 2, he makes a net which are used together with labor to fish in period 3. For simplicity, nets are assumed to be completely depreciated after one period. So in Period 4, the individual makes a net again for use in period 5. The 2-period net-making–fishing cycle repeats itself from period 2 onwards, and the individual consumes half of his catch during the period when he fishes and leaves the other half for the following period when he makes a net. Thus the individuals’ consumption pattern over time is characterised as follows:

$$Z_1^c = Z_2^c = \frac{aL}{2}; Z_n^c = \frac{AL}{2} \quad (n \geq 3)$$

An individual’s utility over the infinite time horizon in Structure B is:

$$U_B = \frac{(1+\beta)aL}{2} + \frac{\beta^2 AL}{2(1-\beta)} \quad (6)$$

(3) Structure C: Division of labor with capital goods

The same capital-using technology is used in Structure C as in Structure B. The only difference is that in Structure C, there is a division of labor: half of the individuals specialize in fishing, the other half specialize in net making. In period 1, fishermen fish with labor only. They consume some and sell the rest to net makers in exchange of nets at the end of period 1. In period 2, fishermen use nets to fish, and exchange with net makers at the end of the period (to replace the nets that have worn out). This production and trade pattern repeats itself from period 2 onwards.

We assume that there is zero transaction cost associated with market exchange. Since all individuals are assumed to be *ex ante* identical and the choice to specialize in

³ Alternative we can assume that the individual saves the minimum amount required to sustain life for the period when he makes nets. However this assumption generates an oscillating consumption pattern and complicates computation without adding much to illuminating the main thesis of the paper.

either fishing or net-making is free, each individual has an incentive to switch profession if his current profession leads to a lower personal consumption (utility) than the other profession. Therefore in equilibrium, the utility levels of both professions should equalize, which implies that the equilibrium price ratio would be 1 net = ½ of a fisherman’s daily catch. With this price ratio, each individual, regardless of his chosen specialization will have the following consumption pattern over time:

$$Z_1^c = \frac{aL}{2}; Z_n^c = \frac{AL}{2} \quad (n \geq 2)$$

An individual’s utility over the infinite time horizon in Structure C will be:

$$U_c = \frac{aL}{2} + \frac{\beta AL}{2(1-\beta)} \quad (7)$$

Table 1 summarises individuals’ consumption patterns and utility levels for each structure.

[Insert Table 1 here]

2.2. Pareto optimality and equilibrium

From equations (6) and (7), we derive that

$$U_C > U_B \text{ iff } A > a.$$

Given our assumption, $A > 2a$ (Equation (4)), it is clear that Structure C (with the division of labor) is Pareto superior to Structure B (without the division of labor).

The additional utility obtainable in Structure C over that obtainable in Structure B is a measure of the extent of the benefit from the division of labor. This benefit is what we call the “value of time” benefit from the division of labor because, as can be seen from Table 1, it flows from the fact that the division of labor enables the adoption of the capital-using technology from period 2 instead of period 3, thereby allowing higher consumption one period earlier than otherwise. Therefore we have

Proposition 1. *If a capital-using technology (which is superior to a labor-only technology) is adopted for the production of a final good, the division of labor between capital good and final good production leads to an earlier adoption of the capital-using technology, thereby enabling higher output to be produced and consumed sooner. This improves welfare given the existence of time preference. The improvement in welfare can thus be referred to as “the value of time” benefit of the division of labor.*

Proposition 1 suggests that if the capital-using technology is adopted, then the division of labor between capital good and final good production improves welfare. However, it is possible that there is no welfare gain in adopting the capital-using technology in the first place even though the capital-using technology is more productive. From equations (5) and (7), we have

$$U_A > U_C \text{ iff } \beta < \frac{a}{A-a}$$

This implies that Structure A (with labor-only technology) may be Pareto superior to Structure C (with capital-using technology) if individuals have sufficiently strong time preference (small β) and/or if the productivity advantage of the capital-using technology is not very pronounced (small A relative to a). For example, if $\beta=0.5$, Structure A will be Pareto superior to Structure C if $A < 3a$; and if $A=2.1a$, then $\beta < 0.91$ is sufficient to guarantee Pareto superiority for Structure A. Thus we conclude

Proposition 2. *The adoption of a more productive capital-using technology may not improve welfare if the technical advantage of the technology is relatively weak and/or if the time preference is sufficiently strong.*

The above analysis has established that if $\beta > a/(A - a)$, Structure C is Pareto optimal; and conversely if $\beta < a/(A - a)$, Structure A is Pareto optimal. However, the existence of Pareto optimality does not guarantee its attainment in equilibrium (Ng, 2004). In conventional general equilibrium analyses, the structure of production is given (i.e., we know who produces what), and the equilibrium is established when a set of prices and quantities are found which lead to utility maximisation for all individuals and clearance of all markets. In our model, we have shown that, given Structure C, a price (1 net = $\frac{1}{2}$ a fisherman's daily catch) and corresponding quantities exist that maximize individual utility and clears the market. Similarly, given the Structure A, while there is no market and therefore no prices, equilibrium can be achieved when each individual maximizes his utility by devoting all his labor endowment to fishing. However, starting from Structure A, if individuals are aware of the existence of a Pareto superior structure, Structure C, is there a mechanism that will lead the individuals to migrate from Structure A to Structure C? The answer is a qualified yes. If the condition $\beta > a/(A - a)$ holds such that Structure C is Pareto superior, there is certainly an incentive for individuals to engage in specialization and trade, and there is a price (i.e., 1 net = $\frac{1}{2}$ a fisherman's daily catch) at which mutually beneficial trade can sustain. The qualification is that, since there is no price in Structure A, the price mechanism itself cannot initiate the act of specialization, nor can it determine who is to specialize in what; thus some sort of "coordination device" is required. For example, a motivated individual who sees an opportunity for welfare gain and seizes it (i.e., an entrepreneur), may specialise and persuade another individual to participate in specialization and trade with him. Over time others will follow their lead and the migration from Autarky to the division of labor will

complete. Arguably the introduction of entrepreneurs to enable the qualitative change in production structure is not more objectionable than the use of a Walrasian Auctioneer to find the competitive equilibrium prices (which enable quantitative changes within a given production structure).

2.3. Discussion

In our model we have assumed that the production of a net takes a whole period. This assumption rules out the possibility that an individual can engage in fishing and net-making during the same period. One may argue that if during each period an individual can spend some time fishing and the remaining time making a net, then the benefit from the division of labor will disappear.

Let us look at this argument closely. Suppose for simplicity that it takes half a day to make a net. In the absence of the division of labour, an individual may catch fish in the morning of Day 1, and then makes a net in the afternoon which will be available for use at the beginning of Day 2. With the division of labor, a net will be available at midday of Day 1, or half a day earlier. One may object by pointing out that the individual can make a net in the morning of Day 1! True, in this case, he will have his net ready at midday of Day 1, too. However, in the absence of the division of labor, he won't have any fish to eat until the end of Day 1. In contrast, with the division of labor, fish are available at midday of Day 1. Moreover, from Day 2 onwards, nets are available for use in the morning, instead of in the afternoon, again half a day earlier. Given time preference, that is, fish (and nets) available at midday are preferred to the same in the evening, there is still the "value of time" benefit of the division of labor.

What the foregoing example illustrates is this general point: since it takes time to make a net and since nets depreciate, in the absence of the division of labor, an individual cannot continuously fish with a net, but instead has to repeat the net making – fishing – net making cycle. This leads to two costs. One is that the use of any new net (probably after the first one) is delayed by the time required to make the net, therefore the value of time is lost. Another is that before each net is made, some fish have to be set aside for consumption during the net-making time, incurring storage costs (which are assumed to be zero in our model).

Another objection to our model may be that an individual can take credit which allows him to make a net on Day 1. This is also true. However credit has to be based on some people's savings. So in our example, someone has to have set aside food of some sort to sustain the person making a net on Day 1. Since our model looks at how saving decisions are affected by the division of labor (i.e., whether some fish are set aside so that nets can be made), it cannot assume a credit market which is an outcome of saving decisions.

Finally, the result of our model that Structure C always Pareto dominates Structure B is driven by the assumption of zero market transaction costs. Structure C is Pareto superior because of the “value of time” benefit of the division of labor. However to realize this benefit, specialization and trade must take place. It is possible that the costs associated with trade may be so large as to outweigh the benefit of the division of labor, therefore it is possible that Structure B may Pareto dominate Structure C in a world of positive transaction costs. To extend our model to include transaction cost,

we assume for simplicity that transaction cost takes the “ice-berg” form. That is, for every unit of fish traded, only k ($k \leq 1$) unit is left for consumption; $(1-k)$ is lost in transit. In equilibrium, the price of net will be such that the transaction cost is equally shared by the buyer and the seller; and that all individuals will achieve the same level of utility. This implies that the individual’s consumption pattern over time will be

$$Z_1^c = \frac{kaL}{(1+k)}; Z_n^c = \frac{kAL}{(1+k)} \quad (n \geq 2)$$

An individual’s utility over time in a world with positive transaction costs will be

$$U_C^T = \frac{k}{1+k} \left(aL + \frac{\beta}{1-\beta} AL \right) \quad (8)$$

It is straightforward to show that with positive transaction costs

$$U_B > U_C^T \quad \text{iff} \quad \beta > \frac{2k}{1+k} - \frac{1-k}{1+k} \frac{a}{(A-a)\beta}$$

$$U_B > U_A \quad \text{iff} \quad \beta > \frac{a}{(A-a)\beta}$$

$$U_C^T > U_A \quad \text{iff} \quad \beta > \frac{a}{(A-a)k}$$

Based on the above conditions, we can Pareto rank Structures A, B and C. This is presented in Table 2.

[Insert Table 2 here]

A couple of observations can be made based on the results in Table 2. First, Structure B and Structure C are likely to Pareto dominate Structure A if individuals’ time preference is sufficiently weak (large β). Whether Structure B or Structure C is Pareto optimal depends importantly on the trade-off between the benefit of earlier consumption as indicated by $(1/\beta)$ and transaction costs. The lower the transaction cost, the more likely Structure C will be Pareto optimal. In the extreme case where

transaction cost is zero, we have shown that Structure C always Pareto dominates Structure B.

Second, there is a non-empty parameter subset, $(B/\beta) > \beta > (B/k)$, within which Structure C is Pareto superior to Structure A which in turn is Pareto superior to Structure B ($U_C^T > U_A > U_B$). This means that within this parameter subset, individuals' time preference is at such a level that it is efficient to wait for 1 period but not 2 periods to realize the gains from capital use. Thus, in the absence of the division of labor, saving will not take place at all. The introduction of the division of labor in this situation can have the effect of inducing capital formation which does not occur otherwise, thereby improving productivity and welfare.

Summarizing the above observations, we have

Proposition 3. *(1) There is a trade-off between the benefit from the division of labor and the transaction costs associated with the division of labor. The structure with the division of labor is Pareto optimal only if transaction costs are sufficiently low. (2) Under certain conditions, the introduction of the division of labor can induce capital formation which does not occur otherwise.*

3. Conclusion

In this paper we have presented a simple model to demonstrate a benefit of the division of labor that has largely been neglected. Our model illustrates that the division of labor has a “value of time” benefit, that is, it shortens capital formation time and enables continuous roundabout production. We have also shown that in

some cases, the presence of the division of labor is critically important in individual decisions on whether to engage in capital formation at all.

The “value of time” benefit of the division of labor in our model derives from three simple facts: (1) a roundabout production process involves the use of capital goods to produce final goods; (2) the production of capital goods and of final goods take time; and (3) individuals prefer to achieve their ends sooner rather than later. It may be noted that in our model, the production process is short as the production of the capital good takes only one period. In more roundabout production processes, the “value of time” benefit from the division of labor is more pronounced. For example, if the production of the capital good used in fishing involves making nets (1 period) and making a boat (1 period), then without the division of labor, an individual have to wait until period 4 before he can fish with a boat equipped with nets. In contrast, with the division of labor, 3 professions – fishermen, net-makers and boat-makers – may emerge in period 1 and the capital good may be available for use from period 2, which is 2 periods earlier than the case without the division of labor. Since a typical modern production process involves much longer and more complex stages, the “value of time” benefit of the division of labor is likely to be much more significant in the real world than that this simple model demonstrates.

We hope that this paper makes a contribution to the relatively small modern literature on the division of labor. As cited at the beginning of this paper, Adam Smith (1776) made a strong claim for the role of the division of labor in wealth creation. However, Stigler (1976, p.1209-10) contended that despite Adam Smith’s “immensely convincing presentation”, the division of labor was yet to become “an integral part of

the modern theory of production”. This paper is an attempt to study the division of labor within the modern general equilibrium framework. By combining the division of labor and the time-consuming roundabout production process, our model has led us to a better appreciation of a neglected benefit from the division of labor that is associated with the time structure of production.

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Figure 1. Structures

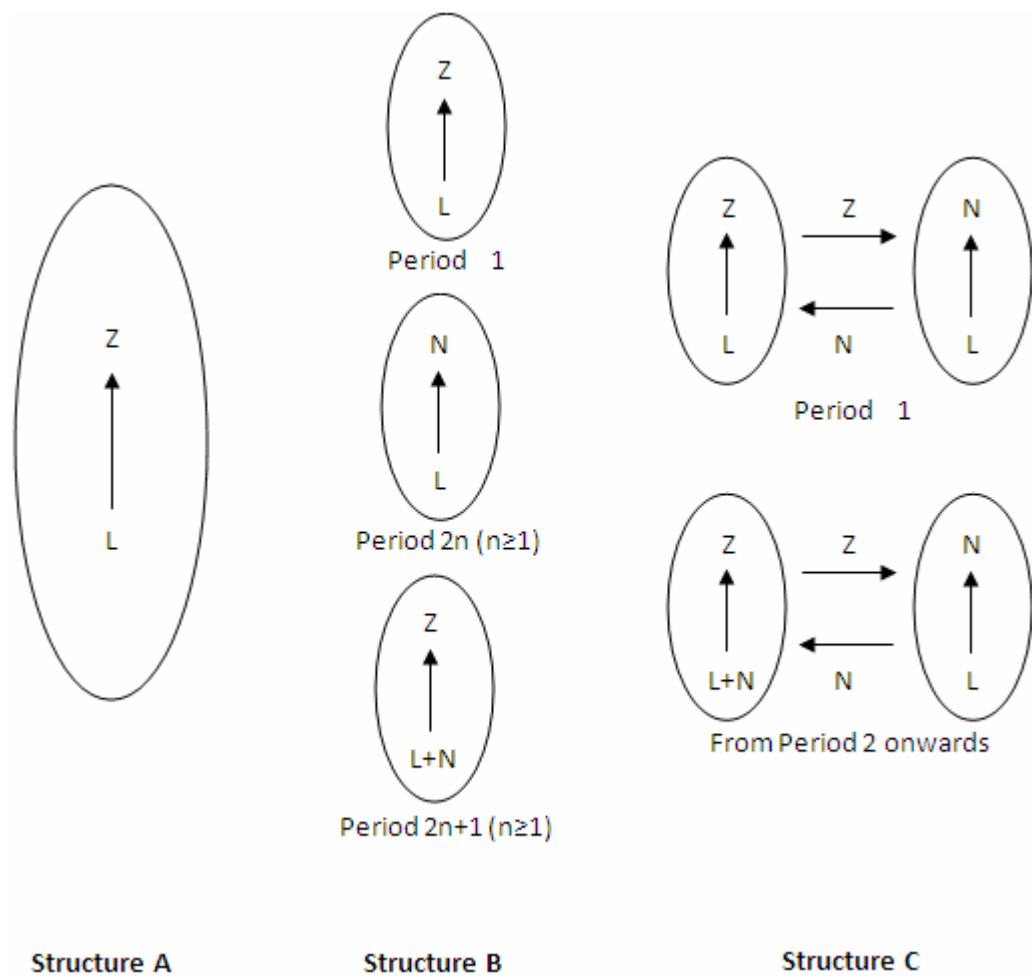


Table 1. Consumption patterns and utility levels

	Structure A	Structure B	Structure C
Period 1 consumption	aL	$\frac{aL}{2}$	$\frac{aL}{2}$
Period 2 consumption	aL	$\frac{aL}{2}$	$\frac{AL}{2}$
Consumption after period 2	aL	$\frac{AL}{2}$	$\frac{AL}{2}$
Individual utility	$U_A = \frac{aL}{1-\beta}$	$U_B = \frac{(1+\beta)aL}{2} + \frac{\beta^2 AL}{2(1-\beta)}$	$U_C = \frac{aL}{2} + \frac{\beta AL}{2(1-\beta)}$

Table 2. Pareto Optimal Structures

Pareto Optimal Structure	Conditions
Structure A	$\beta < \min\left(\frac{B}{\beta}, \frac{B}{k}\right)$
Structure B	$\beta > \max\left\{\frac{B}{\beta}, \left(\frac{2k}{1+k} - \frac{1-k}{1+k} \frac{B}{\beta}\right)\right\}$
Structure C	$\left(\frac{2k}{1+k} - \frac{1-k}{1+k} \frac{B}{\beta}\right) > \beta > \frac{B}{k}$

Where $B \equiv \frac{a}{A-a}$