# Staff Paper Series

Staff Paper P73-8 January 1973

## A HEURISTIC FIXED-CHARGE QUADRATIC ALGORITHM

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## I. INTRODUCTION

A heuristic algorithm for bounding the true optimal value of a fixed-charge quadratic programing problem is developed in this paper. This problem is an extension of the linear fixed-charge problem whose treatment was first rigorously presented by Hirsch and Dantzig (8) and later by Hirsch and Hoffman (9) and others (1,6,17). The fixed-charge problem arises in economic and management problems where a fixed investment is required before production can take place. It is this characteristic which has made linear fixed-charge algorithms particularly useful in determining the optimal spatial location and size of production facilities (14,11,19,21). However, for many applications the linear formulation is restrictive.

Fixed-charge quadratic programing is a more general formulation of the fixed-charge problem which includes the linear formulation as a special case. The quadratic formulation differs from the linear problem in that marginal returns (costs) change (rather than remaining constant) as output changes. This formulation and some of its

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inherent difficulties was recognized in a problem of locating feed manufacturing firms (12).

Potential applications of fixed-charge quadratic programing are numerous. It could be utilized in spatial equilibrium problems of the type considered by Takayama and Judge (20) if explicit account is taken of fixed investments or demand shifts. It also has potential application in problems dealing with risk or stocastic programing where, for instance, the fixed-charge and linear portion of the objective function denote fixed investments and mean returns (costs) and where the quadratic form is a variance-covariance matrix of returns (costs). Further, this framework can be applied to the problem of determining the optimal cost-benefits of urban transportation systems (15).

## II. THE PROBLEM

Stated in integer formulation, the problem is to find values of $\{X_j, \delta_j\}$ which optimize the fixed-charge quadratic function

(II.1a) $$Z = f\,\delta' + a\,X' + X\,BX'$$

subject to the linear restraints

(II.1b) $$A\,X' \leq b', \quad X \geq 0,$$

and the integer restraints

(II.1c) $$\delta_j = \begin{cases} 0 \text{ if } X_j = 0 \\ 1 \text{ if } X > 0 \end{cases}$$

where

a is a n component row vector of constants,

X' is a n component column vector of variables,

B is a nxn symmetric definite or semi-definite matrix of constants, 1/

f is a n component row vector of constants denoting fixed charges which are assumed to be positive (negative) when (II.a) is minimized (maximized), 2/

$\delta$' is a n component column vector of integer variables,

A is a mxn matrix of constants and

b' is a m component column vector of constants.

This differs from the standard quadratic programing problem because of the existence of fixed-charges (f) in the objective function. These charges introduce origin discontinuities and lead to a violation of the concavity assumptions of quadratic programing even though B is a definite or semi-definite quadratic form (9). This can be shown intuitively for the two independant variable case of (II.1a) when B is a positive definite form. In this case

$$f_1\ s_1 = OA \text{ if } x_1 > o\ ,\ f_2\ s_2 = OB \text{ if } x_2 > o \text{ and}$$
$$f_1\ s_1 + f_2\ s_2 = OA + OB = OC \text{ if } x_1 > o\ , x_2 > o.$$ 3/

If the fixed-charges f were zero, then C becomes the origin and we have the traditional quadratic function where a constrained global

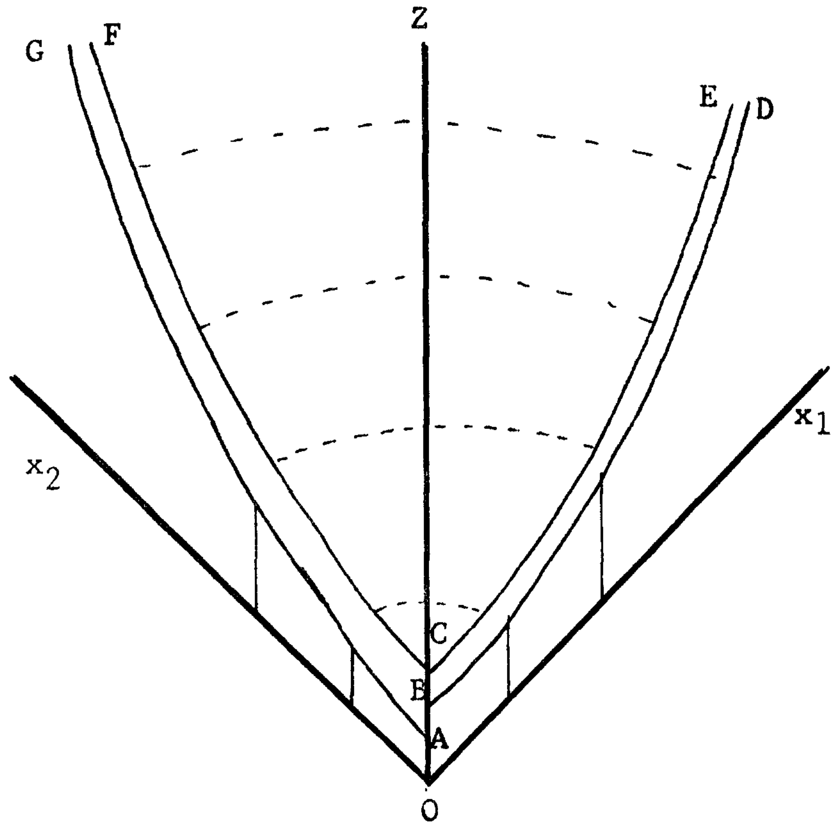


Figure I. QUADRATIC OBJECTIVE FUNCTION WITH POSITIVE FIXED-CHARGES

minimum is given by the Kuhn-Tucker conditions. For the case depicted in Figure I however, a constraint matrix (II.1b) can easily be defined such that three different local optimum solutions can be found for each of which the Kuhn-Tucker Conditions exist. These solutions would be found on BD, AG and the surface CFE, ignoring the trivial case where the origin is a solution.[4/] Thus, while any local optimum is a global optimum in the ordinary quadratic programing problem when B is a definite or semi-definite form, any local optimum of (II.1) may not be globally optimal due to the discontinuities induced by the fixed-charges (f). Therefore, to find an exact optimal solution, all extreme points and internal points as found by the Kuhn-Tucker conditions must be enumerated.

Before turning to the quadratic fixed-charge problem, briefly consider the related linear formulation which is to find values $\{X_j, \delta_j\}$ which optimize

$$W = f\ \delta' + a\ X' \tag{II.2}$$

subject to the restraints (II.1b) and (II.1c). This problem is similar to the ordinary linear programing problem where the optimal solution, is an extreme point of a closed bounded convex set of feasible solutions. Hirsch and Dantzig (8) have shown that with the existence of fixed-charges, the optimum solution lies at one of these extreme points. However, this solution can not necessarily be obtained by using the simplex method because of the existence of local optimums created by origin discontinuities. That is, the simplex procedure might lead to the derivation of a local rather than a global optimal solution. 5/

Exact methods do exist for optimizing (II.2). Among these methods is the rather computationally inefficient mixed integer and continuous variable technique described by Hadley (7). Somewhat more computationally efficient and exact methods have been developed by Steinberg (18). Jones and Soland (10) and Sa' (16) using a branch and bound approach. However, the modification of these approaches (if possible) into an efficient fixed-charge quadratic algorithm must wait further developments, especially since quadratic programing algorithms are computationally inefficient.

The literature abounds with various heuristic algorithms for obtaining approximate solutions to (II.2) which claim computational efficiency (2,3,4,5,13). A search of these algorithms does provide some insight into the development of a computationally efficient technique for obtaining good approximate solutions to (II.1). In particular, Balinski (1) developed an approximate solution procedure for deriving a solution to the fixed cost transportation problem which provide values that bound the true optimal value of the problem. It has been shown that this procedure can be adapted to obtain a solution for (II.2) (13).

In the next section the problem of minimizing and maximizing (II.1) is considered jointly. Balinski's approach is adapted to (II.1) and a method for deriving the bounds to the true optimal value of this problem is described. From these initial bounds, it is shown how superior bounds are obtained.

## III. SOLUTION PROCEDURE

Initial Bounds. Bounds to the true optimal value of the fixed-charge quadratic programing problem (II.1) can be obtained after deriving one approximate solution by a standard quadratic programing algorithm. This is achieved by defining a new problem: find $\{X_j\}$ to minimize the quadratic function

(III.1a) $$Z_L = f^* X' + X\,BX'$$

subject to the linear restraints (II.1b) where $f^*$ is a n component row vector of constants whose elements are given by the following

(III.1b) $$f_j^* = \frac{f_j}{X_j^b} + a_j,$$

where $X_j^b$ is an upper bound associated with the j-th component of X. These bounds can be obtained from (II.1b).

For the purpose of the following theorems, the elements of (f) are assumed positive for the minimization problem and negative for the maximization problem. Generality is not lost since the addition or subtraction of an arbiturary constant to the elements of (f) to meet this condition does not alter the gradient of the function (II.1a), i.e., the surface depicted in Figure I is merely shifted vertically. Also, let $Z^*$ be the true optimal value of the objective function in problem (II.1) having optimal solution (basis) vectors $X^*$, $\delta^*$. Let $Z_L^*$ ($Z_U^*$) denote the optimal value of the objective function in the minimization (maximization) problem (III.1), where $X^\circ$ is the corresponding optimal solution (basis) vector. Finally, let $Z_{1U}^+$ ($Z_{1L}^+$) denote the objective function in the minimization (maximization) problem (II.1) which is obtained as follows: set

(III.2a) $$X = X^\circ \text{ and } \delta_j^+ = \begin{cases} 1 \text{ if } X_j^o > 0 \\ 0 \text{ if } X_j^o = 0 \end{cases}$$

and compute the corresponding objective function value $Z_{1U}^+$ ($Z_{1L}^+$) where [6/]

(III.2b) $$Z_{1U}^+ = f\,\delta^{+'} + a\,X^{+'} + X^+\,B\,X^{+'}.$$

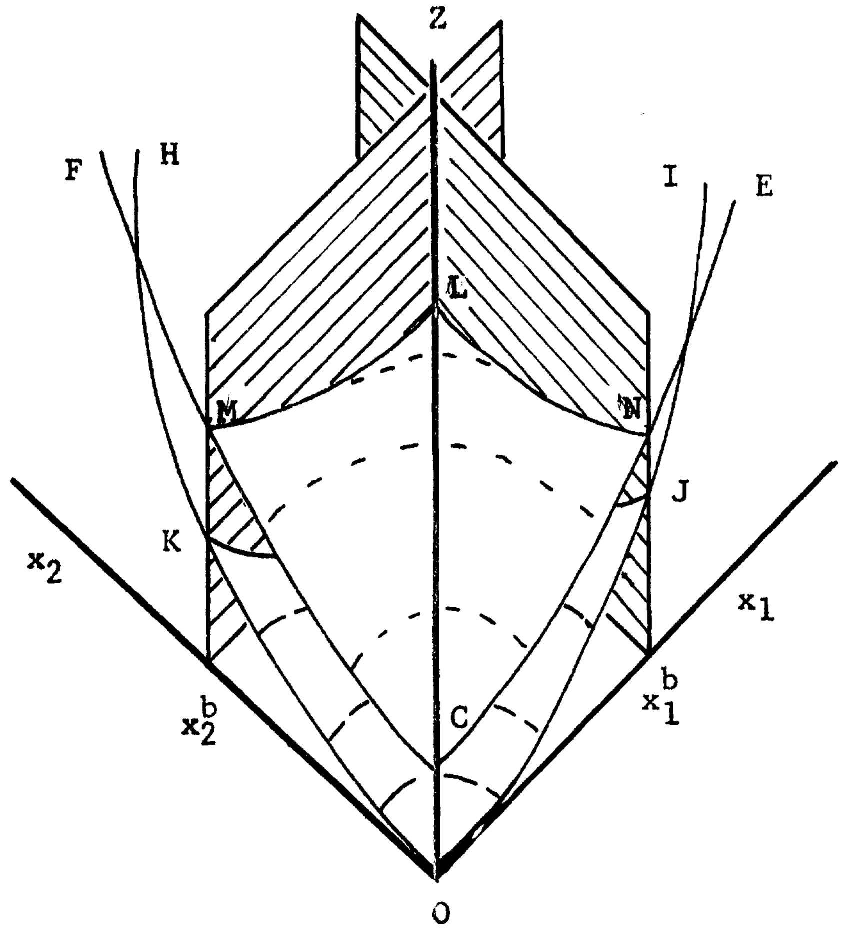


Figure II. QUADRATIC OBJECTIVE FUNCTION WITH POSITIVE FIXED CHARGES AND ITS MODIFIED FUNCTION

We will now show intuitively and then mathematically that the values $Z_L^*$ and $Z_{1U}^+$ ($Z_U^*$ and $Z_{1L}^+$) bound the true optimal value $Z^*$ of the minimization (maximization) problem (II.1).

The bounding procedure can be shown intuitively by considering the consequence of modification (III.1a) on the previous two variable examples. This modification creates a nonlinear surface which intersects the origin and lies beneath a portion of the surface CFE depicted in Figure I. This is shown in Figure II where the new nonlinear surface is OHI and the curves BD and AG of Figure I are omitted for clarification. The linear surface $x_2^b MLN x_1^b$ is the apriori bounds which are the maximum feasible values the corresponding $x_1$, $x_2$ variables can obtain. The points K and J are where the hull OH and OI intersect the curves AG and BD respectively. Within these bounds, the surface CFE intersects the surface OHI only at L. At this point, (III.2b) is identical to (III.1a) which is identical to (II.1a) since the apriori values $x_1^b$, $x_2^b$ equal the solution values of the variables $x_1$, $x_2$ in (II.1).

The solution to problem (III.1a) given by the Kuhn-Tucker conditions yields a solution within the surface OKLJ or on the borders OK, OJ. Since these areas are on or below the surface CMLN and the curves AG, BD, $Z_L^*$ is equal to or less than $Z^*$. This is stated in Theorem A.

An upper bound to the minimization problem is obtained by modifying the solution to (III.1a) according to (III.2a). This yields a solution ($Z_{1U}^+$) on the surface CMLN or the curves AG, BG. Obviously, this solution can not be better than an optimal solution to (II.1) which also exists on these surfaces as indicated above. Thus, $Z_{1U}^+$ must be equal to or greater than $Z^*$. This is stated in Theorem B.

The following theorems state the initial bound for both the minimization and maximization problem. Their proofs appear in Appendix A.

Theorem A: The value $Z_L^*$ ($Z_U^*$) is a lower (upper) bound to the true optimal value of the objective function $Z^*$ of problem (II.1), i.e.

$$Z_L^* \leq Z^*, \text{ a min.}$$

$$Z_U^* \geq Z^*, \text{ a max.}$$

Theorem B: The value $Z_{1U}^{+}$ ($Z_{1L}^{+}$) is an upper (lower) bound to the true optimal value of the objective function $Z^{*}$ of problem (II.1), i.e.,

$$Z_{1U}^{+} \geq Z^{*}, \text{ a min.}$$

$$Z_{1L}^{+} \leq Z^{*}, \text{ a max.}$$

Thus by a single solution to the ordinary quadratic programing problem specified in (III.1) bounds can be derived about the true value of the fixed-charge quadratic programing problem specified in (II.1). It is shown below that one additional solution, obtained by an ordinary quadratic algorithm, will yield an improvement in the upper (lower) bound $Z_{1U}^{+}$ ($Z_{1L}^{+}$) obtained above.

Improved Bounds. An improvement in the upper bound $Z_{1U}^{+}$ is obtained by removing the fixed-charges f from (II.1a), and bounding all nonbasis variables associated with the optimal solution to (III.1) from consideration. Therefore, the solution surface of this new problem is continuous and an optimal solution is given by the Kuhn-Tucker conditions. The fixed charges corresponding to positive levels of the solution vector to this new problem are then added to the value of its objective function.

The intuitive implications of this procedure can be pointed out by considering the previous example. Suppose a solution to (III.1a) yields non-zero values of $x_1$ and $x_2$. Then, the fixed charges are removed from (II.1a) and C becomes the origin in Figures I and II.

The new solution is found on this surface CMLN. Finally, the fixed charges are added to the objective function value of this new solution. If the solution to (III.1a) yields $x_1$ zero and $x_2$ positive, then $x_1$ and fixed charges $f_1$ and $f_2$ are removed from (II.1). The new solution is found on curve AG where A becomes the origin. The fixed charge $f_2$ is then added to the value of the objective function.

More explicitly, the problem is to find new values for the basis variables $X^o$ of problem (III.1) which optimize the quadratic function

(III.4a) $$Z_a = a\ X' + X\ B\ X'$$

subject to the linear restraints $A\ X' \leq b'$, $X^o \geq 0$, $\tilde{X} = 0$, and where $X = \{X^o, \tilde{X}\}$.

Let the optimal solution value of this problem be denoted as $Z_a^*$ and the value of the solution vector denoted as $X^{oa}$. Also, let the corresponding zero-one vector, denoted as $Y^a$, be expressed as

$$Y^a = \{Y^{oa}, \tilde{Y}\}$$

where $Y^{oa}$ is a unit vector whose components correspond to positive basis variables $\{X_j^{oa}\}$ and $\tilde{Y}$ is a null vector corresponding to the null vector $\tilde{X}$. Finally let $Z_{2U}^*$ and $Z_{2L}^*$ denote the new bound for the minimization and maximization problem respectively where

$$Z_{2U}^* = f\ Y^{a\prime} + Z_a^*, \text{ a min.}$$

$$Z_{2L}^* = f\ Y^{a\prime} + Z_a^*, \text{ a max.}$$

The relationship of $Z^*_{2U}$ ($Z^*_{2L}$) to $Z^*$, a min. ($Z^*$, a max.) and $Z^+_{1U}$ ($Z^+_{1L}$) can now be stated.

Theorem C: The value $Z^*_{2U}$ ($Z^*_{2L}$) is a lesser upper (lower) bound to the objective function value $Z^*$ of problem (II.1) than $Z^+_{1U}$ ($Z^+_{1L}$). The proof of this statement appears in Appendix A.

In the next section, this solution procedure is demonstrated for both the minimization and the maximization cases by applying it to a sample problem.

## IV. APPLICATION

Two variations on a sample problem were selected to demonstrate the solution procedure developed here for both the minimization and maximization problem. These variations were obtained by changing the fixed-charges $\{f_j\}$ and constants $\{a_j\}$ and deriving a solution for each case for a total of four cases.

The elements of the sample problem which remain unchanged are as follows:

$$B = \lambda \begin{bmatrix} 1.0 & -0.2 & -0.3 \\ -0.2 & 2.0 & -0.5 \\ -0.3 & -0.5 & 3.0 \end{bmatrix}$$

where $\lambda$ equals one for the minimization problem (referred to as problem A) and a negative one for the maximization problem (referred to as problem B).

Problem A and B contains the following constraints

$$4.0\ X_1 + 1.5\ X_2 + 5.0\ X_3 \leq 70$$

$$10.0\ X_1 + X_2 + 5.0\ X_3 \leq 100$$

$$-2.0\ X_2 + X_3 \leq 0$$

$$\delta_j = \begin{cases} 1 \text{ if } X_j > 0 \\ 0 \text{ if } X_j = 0_j \end{cases} \qquad j = 1,2,3,$$

where the following additional constraint is used in problem A,

$$X_1 + X_2 + X_3 = 14.$$

The coefficients $\{f_j,\ a_j,\ f_j^*\}$ of the sample problem which change depending on the case and problem are presented in Table 1.

TABLE 1. UPPER VARIABLE BOUNDS, FIXED-CHARGES, CONSTANTS AND UNIT VALUES FOR TWO CASES EACH OF PROBLEM A AND PROBLEM B.

| | Problem A (Min.) | | | | | | Problem B (Max.) | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Case A.i | | | Case A.ii | | | Case B.i | | | Case B.ii | | |
| | $X_1$ | $X_2$ | $X_3$ | $X_1$ | $X_2$ | $X_3$ | $X_1$ | $X_2$ | $X_3$ | $X_1$ | $X_2$ | $X_3$ |
| Variable Bounds ($X_j^b$) | 10.0 | 14.0 | 14.0 | 10.0 | 14.0 | 14.0 | 10.0 | 9.682 | 7.44 | 10.0 | 9.682 | 7.44 |
| Fixed Charges ($f_j$) | 20.0 | 50.0 | 8.0 | 60.0 | 48.0 | 330.0 | -20.0 | -50.0 | -8.0 | -10.0 | -20.0 | -600.0 |
| Constants ($a_j$) | 28.0 | 27.0 | 24.0 | 20.0 | 20.0 | 65.0 | 28.0 | 27.0 | 24.0 | 28.0 | 27.0 | 60.65 |
| Unit Values ($f_j^*$)[a/] | 30.0 | 30.57 | 24.57 | 26.0 | 23.42 | 88.8 | 26.0 | 21.84 | 22.15 | 27.0 | 24.93 | -20.0 |

[a/] $f_j^* = a_j + f_j/X_j^b$, $j = 1,2,3$.

Theorems A, B and C provide a basis for deriving a solution procedure for the quadratic fixed-charge problem. The following five steps are one such procedure:

(1) Find the least upper bounds for all $X_j$ when $f_j \neq 0$ and compute $f_j^*$ according to (III.1b). Remove all $f_j \neq 0$ and substitute the corresponding $f_j^*$ for $a_j$ in (II.1a).[7/]

(2) Using a standard quadratic algorithm, find the lower (upper) bound, $Z_L^*$ ($Z_U^*$), by solving the minimization (maximization) problem (III.1).

(3) Compute the first upper (lower) bound, $Z_{1U}^+$ ($Z_{1L}^+$), to the minimization (maximization) problem according to (III.2b).

(4) Remove all $f_j$ from (II.1a) and prevent the nonbasis variables $\{X_j = 0\}$ from appearing in the basis of the resulting problem (III.4a). This can be accomplished in the minimization (maximization) problem by setting the corresponding $\{a_j\}$ to an arbitrarily large (small) value. However, for large problems, it is computationally more efficient to remove those column equations corresponding to $\{X_j = 0\}$ from (II.1b).

(5) Find the minimum (maximum) of (III.4a) subject to (II.1b) using a standard quadratic algorithm and the compute second upper (lower) found, $Z_{2U}^*$ ($Z_{2L}^*$), according to (III.4c).

Tables II and III present the results of utilizing this procedure for the solution of two minimization and two maximization problems (referred to as problem A and B respectively). The percent difference between the bounds on the maximization problems, case B.i and B.ii, are less than for the minimization problems, case A.i and A.ii. This is due in part to the nature of the variable $\{X_j^b\}$ bounding procedure discussed in the following section and suggests that in many applications the approximate solutions to the maximization problem will be better than those of minimization problem. In terms of the sample problems, the percent difference between the bounds for A.i is 10.6 percent in the first approximation and 10 percent in the second. This compares with 7.4 percent and 6.1 percent for B.i. Thus these solutions seem to be "good" approximations of the true optimal solutions.

In cases A.ii and B.ii, the fixed-charges $\{f_j\}$ were increased (Table I). The first solution to these cases yielded positive values of $X_1$ and $X_2$ (Tables II and III). Thus in both $X_3$ was bounded out of the basis of the second solution. The percent difference between bounds for A.ii decreased from 9.9 percent to 7.7 percent while for B.ii they decreased from 2.6 percent to 2.3 percent. Therefore, these solutions also appear to provide "good" estimates of the true optimal solutions to these problems.

TABLE II. SOLUTION VALUES AND BOUNDS TO THE TRUE MINIMUM OF PROBLEM A FOR TWO CASES

| Case | First Solution | | | Second Solution | | | Lower Bound | First Upper[a] | Second Upper[b] |
|---|---|---|---|---|---|---|---|---|---|
| | $X_1$ | $X_2$ | $X_3$ | $X_1$ | $X_2$ | $X_3$ | $(Z^*_L)$ | $(Z^+_{1U})$ | $(Z^*_{2U})$ |
| A.i | 6.51 | 3.76 | 3.73 | 6.43 | 4.06 | 3.51 | 475.97 | 529.15 | 526.11 |
| A.ii | 8.68 | 5.32 | 0.0 | 9.06 | 4.94 | 0 [c] | 463.75 | 511.42 | 500.99 |

a/ $Z^+_{1U} = \Sigma_j [f_j \delta^+_j - (f_j/X^b_j) X^o_j] + Z^*_L$

b/ $Z^*_{2U} = \Sigma_j f_j \delta^+_j + Z^*_a$, where all $f_j \geq 0$.

c/ Variable $X_3$ is not permitted to enter the basis in this solution since it is zero in the first solution.

TABLE III. SOLUTION VALUES AND BOUNDS TO THE TRUE MAXIMUM OF PROBLEM B FOR TWO CASES

| Case | First Solution | | | Second Solution | | | First Lower[a] Bound | Second Lower Bound | Upper Bound |
|---|---|---|---|---|---|---|---|---|---|
| | $X_1$ | $X_2$ | $X_3$ | $X_1$ | $X_2$ | $X_3$ | $(Z^+_{1L})$ | $(Z^*_{2L})$ | $(Z^*_U)$ |
| B.i | 7.24 | 6.81 | 4.16 | 7.0 | 8.05 | 4.39 | 277.12 | 280.61 | 298.18 |
| B.ii | 9.31 | 6.89 | 0.0 | 9.26 | 7.37 | 0.0 | 260.31 | 261.23 | 267.26 |

a/ $Z^+_{1L} = \Sigma_j [f_j \delta^+_j - (f_j/X^b_j) X^o_j] + Z^*_U$ where all $f_j \leq 0$.

For many applications the bounds for the maximization problem may be more effective than in the minimization case. This can be demonstrated by subtracting (III.1a) from (III.2b) which yields

$$Z^{+\prime} - Z^{*\prime} = \Sigma_j \; f_j \; \delta_j^+ - \Sigma_j (f_j / X_j^b) \; X_j^o$$

where all $\{f_j\}$ are zero or negative for the maximization problem, zero or positive for the minimization problem and where $Z^{+\prime}$ and $Z^{*\prime}$ are the respective upper and lower bounds in the case of maximization. The effectiveness of these bounds is determined by how closely all $\{X_j^b\}$ approximate the values $\{X_j^o\}$, i.e., the minimization of their difference.

The selection of $\{X_j^b\}$ for the maximization of (II.1a) can be obtained by searching (II.1b) and by finding the unconstrained maximum of (III.1a). Those values for $\{X_j^b\}$ are then selected which are the smallest. However, in the minimization of (II.1a) only (II.1b) is searched for the selection of $\{X_j^b\}$ since B is a positive definite or semi-definite form, i.e., the unconstrained minimization is at the origin. Therefore, it is likely that the selection of the least upper variable bounds in the maximization problem will lead to values $\{X_j^b\}$ which are closer approximations to the true optimal values $X^*$, than are these values in the minimization problem.

## V. CONCLUDING REMARKS

A fixed-charge quadratic problem was considered in this paper. A heuristic algorithm was developed for providing good approximate solutions which bound the true optimal value of this problem. This was done by defining two related problems which can be solved with a standard quadratic programing algorithm. Four sample problems were then solved utilizing this solution procedure. It was shown that these small problems, two maximization and two minimization, satisfied the conditions of the algorithm and demonstrated consistent results. The solutions to these problems suggested that the algorithm can derive good approximate solutions with a relatively small error, ten percent or less. It is also suggested that for many applications the procedure derives better bounds for the maximization problem than for the minimization problem.

# FOOTNOTES

1/ It is assumed throughout this paper that if the quadratic form is positive definite or positive semi-definite, the objective function is convex and that if the form is negative definite or negative semi-definite, the objective function is concave.

2/ The optimization of (II.1a) implies throughout this paper that the elements of (f) are positive in minimization problems and negative in maximization problems.

3/ Curves AG and BD can be thought of as long-run processing plant cost functions where the fixed-charges represent amortized fixed cost of plant construction. The problem is to determine the optimal size and number of plants subject the linear restraints (II.1b).

4/ Notice that if the constraint matrix requires both $x_1$, $x_2$ to be positive, the solution space is CFE where no discontinuities exist. This suggests that the Kuhn-Tucker Conditions will derive an optimal solution to (II.1) in this special case.

5/ It is possible that the application of the simplex procedure to (II.2) might arrive at an extreme point of a convex set such that no other point adjacent to it will yield an improved value of W. However, it is possible that another extreme point, which is not an adjacent extreme point, exists which can yield an improved value of W.

6/ The same procedure for deriving a lower bound to the minimization problem, yields on upper bound to the maximization problem. Therefor, the values $Z_L^*$, $Z_{1U}^+$ refer to the minimization problem and the values $Z_U^*$, $Z_{1L}^+$ refer to the maximization problem.

7/ A procedure for obtaining the <u>least</u> upper bound is presented in Appendix B.

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## APPENDIX A

Theorem A, Proof: For this theorem, it must be proved that the optimal value of the objective function of minimizing (maximizing) problem (III.1) is a lower (upper) bound to problem (II.1). For this purpose, the following modified form of problem (II.1) is defined. Find $\{X_j, Y_j\}$ to optimize the quadratic function a/

$$\bar{Z} = f\,Y' + a\,X' + X\,B\,X' \tag{III.3a}$$

subject to the linear restraints (II.1b) and

$$X_j \leq X_j^b\,Y_j \tag{III.3b}$$

where $X_j^b$ is the least upper bound of the variable $X_j$ and $Y_j$ are now permitted noninteger, $0 \leq Y_j \leq 1$. The optimal value of the objective function (III.3a) is denoted by $\bar{Z}^*$.

Notice that the true optimal solution $X^*$, $\delta^*$ to problem (II.1) is also a feasible solution to problem (III.3). Thus, the true optimal value $Z^*$ of the objective function (II.1a) corresponding to this solution $X^*$, $\delta^*$ must be equal to or greater (less) than the value $\bar{Z}^*$, a min. ($Z^*$, a max.) of the objective function (III.3), i.e.

$$\bar{Z}^* \leq Z^*, \text{ a min.}$$

$$\bar{Z}^* \geq Z^*, \text{ a max.}$$

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a/ That is, minimize (III.3a) if (II.1) is a minimization problem or maximize (III.3a) if (II.1) is a maximization problem.

The next step is to show the relationship between problem (III.3) and problem (III.1) from which it is concluded that $\bar{Z}^* = Z_L^*$ when (II.1) is a minimization problem and $\bar{Z}^* = Z_U^*$ when (II.1) is a maximization problem.

We consider two cases: (i) suppose that the optimal solution to (III.3) yields $Y_j$ equal zero for some j. Since it is required that

$$X_j \leq X_j^b\ Y_j, \text{ and } X_j \geq 0,$$

$X_j$ must also be zero in which case the above is an equality.

(ii) Suppose that the optimal solution to (III.3) yields $Y_j$ between zero and one for some j. If

$$X_j \leq X_j^b\ Y_j$$

then $Y_j$ can be arbitrarily decreased until

$$X_j = X_j^b\ Y_j$$

while maintaining a feasible solution to (III.3) and thus decreasing the value of the objective function for the minimization problem and increasing its value for the maximization problem. Hence the current solution associated with $\bar{Z}^*$ is not optimal, contradiction. Thus, in the optimal solution to (III.3).

$$X_j = X_j\ Y_j^b \ \forall j$$

which implies that this expression can be used as a constraint rather than the weaker inequality expression. This new constraint can be expressed as

$$Y_j = \frac{X_j}{X_j^b},$$

in which case the resulting problem becomes: find $\{X_j\}$ to minimize (maximize) the quadratic objective function

$$Z_+ = f^* X' + X B X'$$

subject to the linear restraints

$$A X' \leq b',$$
$$X' \geq 0$$

which is identical to problem (III.1). Hence

$$Z_L^* \leq Z^*, \text{ a min.}$$
$$Z_U^* \geq Z^*, \text{ a max.}$$

This completes the proof of theorem A.

Theorem B, Proof: The constraints of problem (III.1) are identical to the constraints of problem (II.1) with the exception of the integer constraint (II.1c). Hence, with modification (III.2), $X^+$, $\delta^+$ is a feasible solution to (II.1). Therefore, the value $Z^+_{1U}$ ($Z^+_{1L}$) is equal to or greater (less) than the true optimal value $Z^*$ when (II.1a) is minimized (maximized).

$$Z^+_{1U} \geq Z^*, \text{ a min.}$$

$$Z^+_{1L} \leq Z^*, \text{ a max.}$$

This completes the proof of Theorem B.

Theorem C, Proof: In order to show that $Z^*_{2U} \leq Z^+_{1U}$ for the minimization problem and $Z^*_{2L} \geq Z^+_{1L}$ for the maximization problem, it is necessary to show that $X^{oa}$ is a better solution than $X^+$.

Since the vector $\tilde{X}$ is null, the vector $X^{oa}$ and the identical vectors $X^o$, $X^+$ have the same nonzero components, i.e.,

$$X^o_j = X^+_j > 0, \text{ then } X^{oa}_j > 0.$$

Therefore, the zero-one vector $\delta^+$ is identical to the $Y^a$ vector and hence

$$f\ \delta^{+\prime} = f\ Y^{a\prime}.$$

Now, the vector $X^+$ is only a feasible solution to problem (III.4) since it satisfies all the constraints, while $X^{oa}$ is an optimal solution vector to this problem as well. Thus

$$a\ (X^{oa},\tilde{X})' + (X^{oa},\tilde{X})\ B\ (X^{oa},\tilde{X})' \leq a\ (X^+,\tilde{X})' + (X^+,\tilde{X})\ B\ (X^+,\tilde{X})'$$

for the minimization problem and equal to a greater than for the maximization problem. Therefore,

$$Z^*_{2U} \leq Z^+_{1U}, \text{ a min.}$$

$$Z^*_{2L} \geq Z^+_{1L}, \text{ a max.}$$

However, $Z^*_{2U}$ ($Z^*_{2L}$) cannot be less (greater) than the true optimal value $Z^*$, since $X^{oa}$, $Y^a$ is only a feasible solution to problem (II.1) while $X^*$, $\delta^*$ is an optimal solution to this problem. Therefore, $Z^*_{2U}$ ($Z^*_{2L}$) is a lesser upper (lower) bound to $Z^*$ than is $Z^+_{1U}$ ($Z^+_{1L}$). This completes the proof of the theorem C.

# APPENDIX B

The values $x_1^b$ can be obtained by finding

(i) min. $(b_i/a_{ij})$ $\forall i$ where $b_i > 0$ and $a_{ij} > 0$,

in the minimization case and in the maximization case by finding the simultaneous solution to the set of first order conditions given by $\partial Z/\partial x_j = 0, \forall_j$ and finding (i) above. Then, select the smallest $x_j$ from among the two sets for all i where (II.1b) does not require a fixed proportion between any of the $X_{jo}$, $j_o \neq j$. This is the procedure utilized in the examples here.

However, to obtain the least upper bound for all $x_j$ where $f_j \neq 0$, the authors have found that the computationally most efficient procedure appears to be the traditional simplex algorithm, especially since most computer facilities are well equiped and experienced in the mechanics of this technique.

The procedure is to utilize the "cost ranging" routine of a traditional LP computer program where the linear objective function CX' is optimized subject to (II.1b). To find the least upper bound of $x_s$, set $c_s$ to a large value and set all other $c_j$, $j \neq s$ values to zero. The solution to this problem gives the least upper bound $x_s^b$. The least upper bound of of $x_r$ is obtained by setting $c_r$ to a large value and setting $c_s$ and all other $c_j$, $s \neq j \neq r$ to zero. The solution to this problem gives the least upper bound $x_s^b$. This process is continued for all $x_j$ where $f_j \neq 0$.

This process is not time consuming since the nature of the linear objective function induces quick convergence to an optimum solution and the use of the cost ranging routine minimizes "handing" time.