ON NEWTON-RAPHSON METHOD

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Abstract

Recent versions of the well-known Newton-Raphson method for solving algebraic equations are presented. First of these is the method given by J. H. He in 2003. He reduces the problem to solving a second degree polynomial equation. However He's method is not applicable when this equation has complex roots. In 2008, D. Wei, J. Wu and M. Mei eliminated this deficiency, obtaining a third order polynomial equation, which has always a real root.

First of the authors of present paper obtained higher order polynomial equations, which for orders 2 and 3 are reduced to equations given by He and respectively by Wei-Wu-Mei, with much improved form.

In this paper, we present these methods. An example is given.

1. Newton-Raphson method

Given a nonlinear equation

$$\mathrm{f}(\mathrm{x}) = 0,$$

the approximations x_n of an exact real root x of the equation has the following from:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2,....$$

2. He's method

Using second order Taylor's expansion, He [3] developed a faster convergent iteration method, obtaining for the variation $t_n = x_{n+1} - x_n$, the second order polynomial equation

$$\frac{1}{2} f''(x_n) t_n^2 + f'(x_n) t_n + f(x_n) + g(x_n) = 0, \ n = 1,2,\dots,$$

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where x_0 and x_1, hence $t_0 = x_1 - x_0$ are given and

$$g(x_n) = f(x_n) - f(x_{n-1}) - f'(x_{n-1})t_{n-1} - \frac{1}{2} f''(x_{n-1})t_{n-1}^2 .$$

He's method is indeed faster convergent than Newton's method, but it does not have solutions for all initial values, for the following condition must be fulfilled at every step:

$$B^2 - 4A(C + g(x_n)) \geq 0,$$

where $$\begin{cases} A = \frac{1}{2} f''(x_n) \\ B = f'(x_n) - f''(x_n)x_n \\ C = f(x_n) - f'(x_n)x_n + \frac{1}{2} f''(x_n){x_n}^2 \end{cases}$$

3. Wei, Wu and Mei method

Following He's example, Wei, Wu and Mei, [4], proposed an even more quickly convergent method under the form of a third order polynomial equation:

$$\frac{1}{3!} f'''(x_n)t_n^3 + \frac{1}{2} f''(x_n)t_n^2 + f'(x_n)t_n + f(x_n) + g(x_n) = 0, \; n = 1,2,\ldots,$$

where

$$g(x_n) = f(x_n) - f(x_{n-1}) - f'(x_{n-1})t_{n-1} - \frac{1}{2} f''(x_{n-1})t_{n-1}^2 + \frac{1}{3!} f'''(x_{n-1})t_{n-1}^3 .$$

Being a cubic equation it will have at least one real solution for any initial values, thus being more convenient than He's method.

4. Improvements of Newton-Raphson type methods

If $x_0 = x_1$, hence $t_0 = 0$, in [2] was obtained for variations t_n of the approximations x_n of an exact real solution of the algebraic equation $f(x) = 0$, the polynomial equations of order m,

$$\sum_{k=0}^{m} \frac{f^{(k)}(x_1)}{k!} t_1^k = 0,$$

$$\sum_{k=1}^{m} \frac{f^{(k)}(x_2)}{k!} t_2^k + 2f(x_2) = 0,$$

where $x_2 = x_1 + t_1$,

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$$\sum_{k=1}^{m} \frac{f^{(k)}(x_n)}{k!} t_n^k + 2f(x_n) + \sum_{j=2}^{n-1} f(x_j) = 0, \; n = 3,4,\ldots,$$

where $x_j = x_{j-1} + t_{j-1}$, $3 \le j \le n$, $n = 3,4,\ldots$.

For $m = 2$, are obtained the improved He's equations

$$\frac{f''(x_1)}{2} t_1^2 + f'(x_1)t_1 + f(x_1) = 0,$$

$$\frac{f''(x_2)}{2} t_2^2 + f'(x_2)t_2 + 2f(x_2) = 0,$$

where $x_2 = x_1 + t_1$,

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$$\frac{f''(x_n)}{2} t_n^2 + f'(x_n)t_n + 2f(x_n) + \sum_{j=2}^{n-1} f(x_j) = 0,$$

where $x_j = x_{j-1} + t_{j-1}$, $3 \le j \le n$, $n = 3,4,\ldots$.

For $m = 3$, are obtained the improved Wei-Wu-Mei equations

$$\frac{f'''(x_1)}{6} t_1^3 + \frac{f''(x_1)}{2} t_1^2 + f'(x_1)t_1 + f(x_1) = 0,$$

$$\frac{f'''(x_2)}{6} t_2^3 + \frac{f''(x_2)}{2} t_2^2 + f'(x_2)t_2 + 2f(x_2) = 0,$$

where $x_2 = x_1 + t_1$,

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$$\frac{f'''(x_n)}{6!} t_n^3 + \frac{f''(x_n)}{2} t_n^2 + f'(x_n)t_n + 2f(x_n) + \sum_{j=2}^{n-1} f(x_j) = 0,$$

where $x_j = x_{j-1} + t_{j-1}$, $3 \le j \le n$, $n = 3,4,\ldots$.

The improvement of these equations consists in replacing of $g(x_n)$ from constant term with simpler expressions.

5. Numerical example

We give an example, taken from [6], in which He's method does not apply.

Consider equation $f(x) = x^3 - e^{-x} = 0$. Newton's formula (11) gives recurrence relation $x_{n+1} = x_n - \frac{x_n^3 - e^{-x_n}}{3x_n^2 + e^{-x_n}}$. Taking $x_0 = 0$, we obtain $x_1 = 1$, $x_2 = 0.8123$, $x_3 = 0.7743$ and $x_4 = 0.7729$.

For $m = 2$, taking $x_0 = x_1 = 0$, He's method give quadratic equation $t_1^2 - 2\, t_1 + 2 = 0$, which has complex roots, therefore this method is not applicable.

For $m = 3$, taking $x_0 = x_1 = 0$, the improved Wei, Wu and Mei method give cubic equation $7\,t_1^3 - 3t_1^2 + 6t_1 - 6 = 0$, with real root $t_1 = 0.7673$, hence $x_2 = x_1 + t_1 = 0.7673$. Continuing recurrence process, we get

$$1.0774t_2^3 + 2.0698t_2^2 + 2.2305t_2 - 0.025 = 0, \quad t_2 = 0.0111, \quad x_3 = 0.7784,$$

and

$$1.0765t_3^3 + 2.1056t_3^2 + 2.2769t_3 + 0.0125 = 0, \; t_3 = -0.0055, \; x_4 = 0.7729.$$

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