NEW METHODS FOR SOLVING ALGEBRAIC EQUATIONS

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Abstract

Iteration methods are very useful in solving nonlinear algebraic equations. The most famous such method is Newton's method deduced by first order Taylor expansion.
In 2003, J. H. He gives a new faster convergent method, based on second order Taylor expansion, that gives a quadratic equation for the iterations difference $x_{n+1}-x_n$. However He's method is not applicable when this equation has complex roots.
In 2008, D. Wei, J. Wu and M. Mei eliminated this deficiency, obtaining from third order Taylor expansion a cubic equation, that always has a real root.
In this paper, we present the three methods and their applications to some particular equations.

1. Newton's iteration method

Given a nonlinear equation

$f(x)=0,$

Newton's method has the following from:

$f(x_n)+ f'(x_n)(x_{n+1}-x_n)=0.$

2. He's method

Using second order Taylor's expansion, He [3] developed a faster convergent iteration method:

$f(x_n)+f'(x_n)(x_{n+1}-x_n)+(1/2)f''(x_n)(x_{n+1}-x_n)^2+g(x_n)=0,$

where $g(x_n)= f(x_n)- f(x_{n-1})- f'(x_{n-1})(x_n- x_{n-1})-(1/2)f''(x_{n-1})(x_n- x_{n-1})^2$.

He's method is indeed faster convergent than Newton's method, but it does not have solutions for all initial values, for the following condition must be fulfilled at every step:

$$(f'(x_n)- f''(x_n)\, x_n)^2-f''(x_n)\, (2f(x_n)-2f'(x_n)\, x_n+f''(x_n)\, x_n^2+2g(x_n))\geq 0.$$

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3. Wei, Wu and Mei method

Following He's example, Wei, Wu and Mei, [4], proposed an even more quickly convergent method under the form of a cubic equation:

$$f(x_n)+f'(x_n)(x_{n+1}-x_n)+(1/2)f''(x_n)(x_{n+1}-x_n)^2+(1/6)f'''(x_n)(x_{n+1}-x_n)^3+g(x_n)=0,$$

where $g(x_n)= f(x_n)- f(x_{n-1})- f(x_n)- x_n- x_{n-1})-(1/2)f''(x_{n-1})(x_n- x_{n-1})^2-(1/3)f''(x_{n-1})(x_n- x_{n-1})^3$.

Being a cubic equation it will have at least one solution for any initial values, thus being more convenient than He's method.

4. Numeric examples

4.1. First, let us consider the following nonlinear algebraic equation:

$$x^3+sin(x)=1,$$

with the initial values x_0=0.5 and x_1=0.5.

It has the exact solution 0.7056936976.
Using the improved method displayed above, we will reach this solution by the 5th iteration.

x_2= 0.7057107309
x_3= 0.7056766639
x_4= 0.7056936984
x_5= 0.7056936976

4.2. Let us consider now the following equation:

$$x^3-cos(x)=0,$$

which has the exact solution 0.865474033.

We will be solving this equation with both Newton's method and the improved method.

	Newton method	Wei Wu Mei method
initial point(s)	x_1=0.5	x_0=0 , x_1=0.5
x_2	1.112141637	0.866123327
x_3	0.909672693	0.865682698
x_4	0.867263818	0.865474033
x_5	0.865477135	
x_6	0.865474033	

4.3. Maple program

We will create a maple program for the second example.

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> restart;
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The function is:

$> f := x \to x^3 - \cos(x);$

$f := x \to x^3 - \cos(x)$

The exact solution is:

$> fsolve(f = 0);$

0.865474033

The function's derivatives are:

$> fp := D(f);$

$fp := x \to 3x^2 + \sin(x)$

$> fpp := D(fp);$

$fpp := x \to 6x + \cos(x)$

$> fppp := D(fpp);$

$fppp := x \to 6 - \sin(x)$

Newton method

The initial point is:

$> x1[1] := 0.5;$

$x1_1 := 0.5$

The solution is:

$>$ **for** n **from** 1 **to** 5 **do**

$x1[n+1] := evalf\left(x1[n] - \frac{f(x1[n])}{fp(x1[n])}\right);$

end do;

$x1_2 := 1.11214163'$

$x1_3 := 0.909672693'$

$x1_4 := 0.867263818$

$x1_5 := 0.865477135$

$x1_6 := 0.865474033$

Wei, Wu and Mei method

The initial points are:

> $x2[0] := 0;$

$x2_0 := 0$

> $x2[1] := 0.5;$

$x2_1 := 0.5$

The solution is:

>

for n **from** 1 **to** 3 **do**

$$x2[n+1] := fsolve\Big(f(x2[n]) + fp(x2[n])\cdot(x2[n+1] - x2[n]) + \frac{1}{2}\cdot fpp(x2[n])\cdot(x2[n+1] - x2[n])^2 + \frac{1}{3\cdot 2}\cdot fppp(x2[n]) \cdot(x2[n+1] - x2[n])^3 + f(x2[n]) - f(x2[n-1]) - fp(x2[n-1])\cdot(x2[n] - x2[n-1]) - \frac{1}{2}\cdot fpp(x2[n-1])\cdot(x2[n] - x2[n-1])^2 - \frac{1}{3\cdot 2}\cdot fppp(x2[n-1])\cdot(x2[n] - x2[n-1])^3 = 0\Big);$$

end do;

$x2_2 := 0.866123327$:

$x2_3 := 0.865682698$‹

$x2_4 := 0.865474033$:

References.

[1] R. L. Burden, J. D. Faires, *Numerical* Analysis, 8th edition, Brook/Cole, 2004.

[2] J. H. He, *Improvement of Newton Iteration Method*, Int. J. Nonlinear Sci. Numer. Simulation, 1(2000), 239-240.

[3] J. H. He, *A New Iteration Method for Solving Algebraic Equations*, Appl. Math. Comput., 135 (2003), 81-84.

[4] D. Wei, J. Wu, M. Mei, *A More Effective Iteration Method for Solving Algebraic Equations*, Appl. Math. Sci. 2(2008), 28, 1387-1391.