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A Term Structure Model with Level Factor Cannot be Realistic and Arbitrage Free

Abstract

A large part of the term structure literature interprets the first underlying factors as a level factor, a slope factor, and a curvature factor. In this paper we consider factor models interpretable as a level factor model, a level and a slope factor model, respectively. We prove that such models are compatible with no-arbitrage restrictions and the positivity of rates either under rather unrealistic conditions on the dynamic of the short term interest rate, or at the cost of explosive long-term interest rates. This introduces some doubt on the relevance of the level and slope interpretations of factors in term structure models.

Keywords : Interest Rate, Term Structure, Affine Model, No Arbitrage, Level Factor, Slope Factor.
JEL Classification : E43, E44, G12.

Un Modèle de Courbe de Taux avec Facteur Niveau ne peut être Réaliste en Absence d'Opportunités d'Arbitrage

Résumé

Une grande partie de la littérature sur les modèles de courbe de taux d'intérêt interprète les premiers facteurs comme des facteurs niveau, pente, et courbure. Dans ce papier, nous considérons des modèles de courbe de taux où les facteurs s'interprètent comme un facteur niveau (modèle à 1 facteur), ou comme des facteurs niveau et pente (modèle à 2 facteurs). Nous démontrons que, pour ces modèles, les hypothèses d'absence d'opportunités d'arbitrage et de positivité des taux d'intérêt impliquent soit une dynamique irréaliste du taux d'intérêt de court-terme, soit une explosion des taux de long-terme. Nos résultats mettent donc en question l'interprétation habituelle des facteurs dans les modèles de courbe de taux.

Mots-clés : Taux d'intérêt, Structure par Terme, Modèle Affine, Absence d'Opportunités d'Arbitrage, Facteur Niveau, Facteur Pente.
Codes JEL : E43, E44, G12.

1 Introduction

The dynamic analysis of the term structure of interest rates reveals the existence of a limited number of underlying factors. It is usual to interpret sequentially these factors as a level factor, a slope (or steepness) factor, a curvature (or butterfly) factor, and so on, even if these notions have not been precisely defined in the literature [see e.g. Litterman, Scheinkman (1991), Jones (1991)].[3] This factor interpretation has also been extended to the field of option pricing [see Rogers, Tehranchi (2008) for a study of parallel shifts in the term structure of implied volatilities].
The aim of this paper is to consider an arbitrage-free factor model of the term structure of interest rates, where one of the factors has a level interpretation (loosely speaking, any shock on the factor X will imply a parallel shift in the whole term structure). The compliance with no-arbitrage is of great importance for asset pricing modeling. In particular, it is crucial for market makers to rely on arbitrage-free asset pricing models for their quotes, to avoid the market participants to benefit from unlimited free lunches opportunities.
Let us first consider a single factor model :

$$r(t,h) = g(X_t, h), \ \forall t \geq 0, h \geq 0,$$

where $r(t,h)$ is the continuously compounded rate at date t for time-to-maturity h, and X is the single factor. A level factor is such that any shock δx, say, affecting the factor impacts the term structure by a drift independent of time-to-maturity. Thus we have :

$$g(X_t + \delta x, h) = g(X_t, h) + d(X_t, \delta x), \ \forall X_t, h, \delta x,$$

where $d(X_t, \delta x)$ denotes the drift, which can depend on the state X_t and of the magnitude of the shock. Without loss of generality, we can assume that X_t can take the value zero. Thus, we deduce that :

$$g(X_t + \delta x, h) = g(0, h) + d(0, \delta x), \ \forall h, \delta x,$$

or equivalently that g can be decomposed as :

$$g(X_t + \delta x, h) = g(0, h) + d(0, X_t).$$

[3]"Level, slope and curvature factor loadings at the core of (term structure) models have their origin in the somewhat arbitrary and atheoretical field of yield curve fitting" [Krippner (2009)].

It admits an additive decomposition into a function of the factor and a function of the time-to-maturity. Since the level factor is defined up to a nonlinear transformation, any 1-factor model with a level factor can always be written as[4] :

$$r(t,h) = X_t + c(h), \tag{1.1}$$

In Section 2, we consider a discrete time term structure model with a single factor interpretable as a level factor. Then we introduce buy and hold strategies based on two zero-coupon bonds and derive the necessary and sufficient conditions for no-arbitrage: the sequence $[c(h)]$ has to be a sequence of Cesaro means of a nonnegative increasing function. In Section 3, we discuss the implications of this result on the behavior of the long-term interest rate. Section 4 exhibits all risk-neutral dynamics compatible with parallel shifts of the yield curve. We prove that they correspond to strong random walks and we explain how the behavior of the long-term interest rate depends on the distribution of the innovation of this random walk [5]. Section 5 extend our results to 2-factor models, by adding a slope factor to the term structure model with level factor. Section 6 concludes. The history of parallel and affine shifts of the term structure in the financial literature is presented in Appendix 1, and technical derivations are gathered in the other appendix.

2 No-Arbitrage Condition for Buy-and-Hold Strategies Based on Two Zero-Coupon Bonds.

In Sections 2-4, we consider a discrete time term structure model, i.e. $t \in \mathbb{N}, h \in \mathbb{N} - \{0\}$. with a single level factor [see (1.1)]. In decomposition (1.1), the factor is defined up to an additive constant. Therefore, without loss of generality, we can always assume :

Assumption A.1 : $c(1) = 0$.

[4]Model (1.1) has been written for the continuously compounded rate. If we denote by $r^*(t,h)$ the rate which is not continuously compounded, we have : $\exp[-hr(t,h)] = [1 + r^*(t,h)]^{-h}$, or equivalently $r^*(t,h) = \exp[r(t,h)] - 1 = \exp[X_t + c(h)] - 1$. Thus the notion of level factor depends on the definition of the rate. We keep the continuously compounded definition in the rest of the paper, which is compatible with the existing literature.

[5]The main result of this Section contradicts Theorem 4 in Ingersoll, Skelton, Weil (1978). We will see later on why their result is incomplete.

Under Assumption A.1, the factor coincides with the short term interest rate : $X_t = r(t,1)$.

We also assume that model (1.1) really is a single factor model :

Assumption A.2 : The support of the conditional distribution of (X_t) given X_0 is not reduced to a single point.

Finally, we assume nonnegative rates, with a short term interest rate, which can reach value zero.

Assumption A.3 :

i) The lower bound of the support of the distribution of X_t given X_0 is zero;

ii) $c(h) \geq 0, \forall h \in \mathbb{N} - \{0\}$.

Let us consider at date t a portfolio of two zero-coupon bonds with time-to-maturity h_1 and $h_2, h_2 > h_1$, respectively. Its price at date t is :

$$\Pi_t(h_1, h_2, \alpha) = \alpha_1 B(t, h_1) + \alpha_2 B(t, h_2),$$

where $B(t,h) = \exp[-hr(t,h)]$ denotes the price of the zero-coupon bond, and (α_1, α_2) the allocations.

The value of this portfolio at date $t+k, k \leq h_1$, is :

$$\Pi_{t+k}(h_1 - k, h_2 - k, \alpha) = \alpha_1 B(t+k, h_1 - k) + \alpha_2 B(t+k, h_2 - k).$$

The no-arbitrage condition is the impossibility to ensure a positive future value with zero or negative initial endowment . This is equivalent to :

$$\begin{aligned} &\{\min_{t+k}[\Pi_{t+k}(h_1 - k, h_2 - k, \alpha] \geq 0\} \\ &\Rightarrow \{\min_t \Pi_t(h_1, h_2, \alpha)] \geq 0\}, \quad \forall \alpha, \forall k \leq h_1, h_2, \end{aligned} \tag{2.1}$$

where $\min_t$ is the minimum taken over the admissible values of the state variable of date t.

Condition (2.1) provides a restriction on zero-coupon prices, only if α_1 and α_2 have opposite sign. Thus, without loss of generality, we can choose $\alpha_1 = 1, \alpha_2 = -\alpha, \alpha > 0$, say.

Proposition 1 : Under model (1.1) and Assumptions A.1-A.3, the buy and hold strategies based on two zero-coupon bonds do not feature arbitrage opportunity if and only if the function $c^*(h) = hc(h)$ is such that : $c^*(h+1) - c^*(h)$ is a nonnegative increasing function of h.

Proof : We have :

$$\begin{aligned}
\Pi_{t+k}(h_1 - k, h_2 - k, \alpha) &= \exp[-(h_1 - k)X_{t+k} - c^*(h_1 - k)] \\
&- \alpha \exp[-(h_2 - k)X_{t+k} - c^*(h_2 - k)] \\
&\\
&= B(t+k, h_1 - k) \\
&\quad \{[1 - \alpha \exp[-(h_2 - h_1)X_{t+k}] \exp[-c^*(h_2 - k) + c^*(h_1 - k)]\}.
\end{aligned}$$

Since $\alpha \geq 0, h_2 \geq h_1$, we deduce that :

$$\min_{t+k} \Pi_{t+k}(h_1 - k, h_2 - k, \alpha) \geq 0 \text{ if and only if } 1 - \alpha \exp[-c^*(h_2 - k) + c^*(h_1 - k)] \geq 0.$$

Therefore, $\min_{t+k} \Pi_{t+k}(h_1 - k, h_2 - k, \alpha)$ is nonnegative if and only if $\alpha \leq \exp[c^*(h_2 - k) - c^*(h_1 - k)]$. Similarly, $\min_t \Pi_t(h_1, h_2, \alpha)$ is nonnegative positive if and only if $\alpha \leq \exp[c^*(h_2) - c^*(h_1)]$.

Thus the no-arbitrage condition is satisfied if and only if,

$$\{\alpha \leq \exp[c^*(h_2 - k) - c^*(h_1 - k)]\} \Rightarrow \{\alpha \leq \exp[c^*(h_2) - c^*(h_1)]\},$$

which is equivalent to :

$$c^*(h_2 - k) - c^*(h_1 - k) \leq c^*(h_2) - c^*(h_1), \forall k \leq h_1 \leq h_2. \tag{2.2}$$

i) It is easily checked that condition (2.2) above is equivalent to the fact that the function $c^*(h_2 + k) - c^*(h_1 + k)$ is increasing in k for any $h_2 \geq h_1$.

ii) Finally, by noting that :

$$\begin{aligned} c^*(h_2+k) - c^*(h_1+k) &= [c^*(h_2+k) - c^*(h_2-1+k)] + [c^*(h_2-1+k) - c^*(h_2-2+k)] \\ &+ \ldots + [c^*(h_1+1+k) - c^*(h_1+k)], \end{aligned}$$

we get the increasingness condition.

To prove the nonnegativity, we have to check that $c^*(2) - c^*(1) = 2c(2)$ is nonnegative (since $c^*(1) - c^*(0) = 0$). This is a direct consequence of Assumption A.3.

QED

Corollary 1 : The no-arbitrage condition of Proposition 1 is satisfied if and only if the sequence $[c(h)]$ is a sequence of Cesaro means of a nonnegative increasing function.

Proof : We have :

$$c(h) = c^*(h)/h = \frac{1}{h}\sum_{l=1}^{h} \Delta c^*(l),$$

with $\Delta c^*(l) = c^*(l) - c^*(l-1)$.

The result follows from Proposition 1.

QED

Corollary 2 : Under model (1.1), Assumption A1-A3 and no-arbitrage, function c^* is superadditive, that is,

$$c^*(h_1) + c^*(h_2) \leq c^*(h_1+h_2), \forall h_1, h_2 \in \mathbb{N} - \{0\}.$$

Proof : Indeed, let us consider the special case of inequality (2.2) for $k = h_1$. We get :

$$c^*(h_2 - h_1) \leq c^*(h_2) - c^*(h_1), \quad \forall h_1 \leq h_2.$$

QED

This condition was expected. Indeed, under Assumption A.3 the lower bound of the support of $r(t,h)$ is equal to $c(h)$. It has been proved in Gourieroux-Monfort (2011) that h times this lower bound, that is, $c^*(h) = hc(h)$ is necessarily superadditive under no-arbitrage condition.

No-arbitrage requires the function $\Delta^*(h) = c^*(h) - c^*(h-1)$ to be non negative increasing for all $h \in \mathbb{N} - \{0\}$. Let us for instance assume that $c^*(\bar{h}) - c^*(\bar{h}-1) > c^*(\bar{h}+1) - c^*(\bar{h}) > 0$ for a single time-to-maturity $\bar{h}$. Then, the portfolio composed at time t by $\alpha_1 = 1$ zero-coupon bond with maturity $\bar{h}$ and $\alpha_2 = -\frac{B(t,\bar{h})}{B(t,\bar{h}+1)}$ bond with maturity $\bar{h}+1$ is worthless at time t, but have a positive value Π_{t+1} at time $t+1$ with probability 1 :

$$\begin{aligned}
\Pi_{t+1} &= B(t+1,\bar{h}-1)\left[1 - \frac{\alpha_2}{\alpha_1}\exp\left(-X_t - c^*(\bar{h}) + c^*(\bar{h}-1)\right)\right] \\
&= B(t+1,\bar{h}-1)\left[1 - \exp\left(-X_{t+1} + \left[c^*(\bar{h}+1) + c^*(\bar{h})\right] - \left[c^*(\bar{h}) + c^*(\bar{h}-1)\right]\right)\right] \\
&> 0, \text{ for all } X_{t+1} \geq 0.
\end{aligned}$$

3 Behavior of the Long-Term Interest Rate

In this section, we investigate the implication of level factor for the modeling of long-term rates under no-arbitrage.

Proposition 2 : Under model (1.1) and Assumptions A.1-A.3, we get one of the two following cases :

i) $r(t,\infty) = +\infty$:

ii) $r(t,\infty) = X_t + c_\infty$, where c_∞ is a given positive constant.

Proof : Since $\Delta c^*(h)$ is nonnegative increasing, we have either $\liminf\limits_{h\to\infty} \Delta c^*(h) = \infty$, or $\liminf\limits_{h\to\infty} \Delta c^*(h) = \limsup\limits_{h\to\infty} \Delta c^*(h) = c_\infty < \infty$ say. Since $\Delta c^*(h)$ is a nonnegative increasing function, we deduce that the Cesaro mean $[c(h)]$ is such that :

$$\begin{aligned} c(h) &= \frac{1}{h}\sum_{l=1}^{h}\Delta c^*(l) \leq \Delta c^*(h), \forall h, \\ \text{and} \quad c(h) &\geq \frac{1}{h}\sum_{l=k+1}^{h}\Delta c^*(l) \geq \frac{h-k}{h}\Delta c^*(k), \forall k \leq h. \end{aligned}$$

These two inequalities explain why the sequences $[c(h)]$ and $[\Delta c^*(h)]$ have the same asymptotic behavior. For instance, let us assume that $\lim_{h\to\infty} \Delta c^*(h) = +\infty$. Then, from the second inequality, we get :

$$\liminf_{h\to\infty} c(h) \geq \Delta c^*(k), \forall k,$$

which implies $\liminf_{h\to\infty} c(h) \geq +\infty$. We deduce that $\lim_{h\to\infty} c(h) = +\infty$. When $\lim_{h\to\infty} \Delta c^*(h) = c_\infty$, the joint use of the two inequalities shows that $\liminf_{h\to\infty} c(h)$ and $\limsup_{h\to\infty} c(h)$ exist and are equal to c_∞.

QED

Proposition 2 shows that the case, where the long-term interest rate does not exist due, for instance, to a periodic asymptotic behavior of function c, has been eliminated.

Proposition 2 concerns the limiting behavior of the long run spot interest rate when the whole term structure moves by parallel shifts. The instantaneous forward interest rate is given by :

$$f(t,h) = hr(t,h) - (h-1)r(t,h-1).$$

Under model (1.1), the instantaneous forward rate is equal to :

$$f(t,h) = X_t + c^*(h) - c^*(h-1), \forall t, h.$$

It is not a constant function of time. In particular, if $\lim_{h\to\infty} \Delta c^*(h)$ exists, the long run instantaneous forward interest rate also exists and is stochastic.

4 Risk-Neutral Factor Dynamic

Let us now analyze the factor dynamics.

Proposition 3 : Under model (1.1), no-arbitrage opportunities and Assumptions A.1-A.3, the factor process is a Markov process under the risk-neutral probability $\mathbb{Q}$ and we have :

$$\overset{\mathbb{Q}}{E_t}[\exp(-hX_{t+1})] = \exp[-hX_t + c^*(h) - c^*(h+1)].$$

Proof : Under no-arbitrage, we have :

$$B(t,h+1) = \overset{\mathbb{Q}}{E_t}\{\exp[-r(t,1)]B(t+1,h)\}, \forall h,$$

or, equivalently :

$$\exp[-(h+1)r(t,h+1)] = \exp[-r(t,1)]\ \overset{\mathbb{Q}}{E_t}\{\exp[-hr(t+1,h)]\}, \forall h.$$

By decomposition (1.1), we deduce :

$$\exp[-(h+1)X_t - c^*(h+1)] = \exp(-X_t)\ \overset{\mathbb{Q}}{E_t}\{\exp[-hX_{t+1} - c^*(h)]\},$$

or :

$$\overset{\mathbb{Q}}{E_t}[\exp(-hX_{t+1})] = \exp[-hX_t + c^*(h) - c^*(h+1)], \forall h.$$

For a nonnegative variable, the knowledge of the Laplace transform for negative integer characterizes the distribution [6]. We deduce that the conditional distribution of X_{t+1} given its past depends on the past by means of the most recent observation. This is the Markov property and Proposition 3 follows.

QED

[6] Indeed, let us denote $Z = \exp(-X)$. Variable Z takes values in $(0,1)$. Thus, for any argument u, the series $\Sigma_{h=0}^{\infty} \frac{E(Z^h)(iu)^h}{h!}$ is uniformly absolutely convergent. We deduce that the characteristic function $\psi(u) = E[\exp(iuZ)]$ exists [see Feller (1971), Vol2, p430].

The conditional log-Laplace transform is an affine function of the current value of the process. This is exactly the definition of a Compound Autoregressive (CaR) process [see Darolles, Gourieroux, Jasiak (2006)], also called Affine process in continuous time [Duffie, Kan (1996), Duffie, Filipovic, Schachermayer (2003)].

Proposition 4 [7] **:** Under model (1.1), no-arbitrage opportunities, and Assumptions A.1-A.3, the level factor process is a strong random walk under $\mathbb{Q}$:

$$X_{t+1} = X_t + \varepsilon_{t+1},$$

where (ε_t) is under $\mathbb{Q}$ a sequence of nonnegative i.i.d. variables with Laplace transform :

$$\psi_\varepsilon(h) = \overset{\mathbb{Q}}{E} [\exp(-h\varepsilon_t)] = \exp[c^*(h) - c^*(h+1)].$$

Proof : Let us denote $\varepsilon_{t+1} = X_{t+1} - X_t$. From Proposition 3, we deduce that : $\overset{\mathbb{Q}}{E_t}$ $[\exp(-h\varepsilon_{t+1})] = \exp[c^*(h) - c^*(h+1)]$. This shows that the conditional distribution of ε_{t+1} is independent of the past and provides the form of its Laplace transform. Moreover, ε is nonnegative, since by Assumption A.3, X_t can be arbitrary close to zero. In this case $\varepsilon_{t+1} = X_{t+1}$, which is nonnegative.

QED

Since ε is nonnegative, $\psi_\varepsilon(h)$ is smaller than 1 and a decreasing function of h. We deduce that $c^*(h+1) - c^*(h)$ is a nonnegative increasing function of h (which is Proposition 1). We also get the following Corollaries :

Corollary 2 : Model (1.1) is compatible with the no-arbitrage condition if and only if function c^* is such that : $\exp[c^*(h) - c^*(h+1)]$ is the Laplace transform of a positive variable.

[7] Proposition 4 contradicts Theorem 4 in Ingersoll, Skelton, Weil (1978), where it is said that any parallel shift in a term structure is not arbitrage free. The random walk models in Proposition 4 are both compatible with parallel shift and no-arbitrage. This contradiction is due to a misleading proof in ISW (1978), p635, l3, where the effect of diminishing time-to-maturity is omitted when computing the future value of the portfolio of zero-coupon bonds. In some sense, they have implicitly assumed a flat term structure $c(h) = 0$ [see the discussion in Appendix 1].

Corollary 3 : Under model (1.1), no-arbitrage opportunities, and Assumptions A.1-A.3, the factor process is a non decreasing function of time: the term structure cannot make uniform downward move.

Let us now come back to the behavior of the long-term interest rate. We have the following proposition :

Proposition 5 : For a strong random walk under $\mathbb{Q}$, the long-term interest rate exists, if and only if :

$$\lim_{h\to\infty}\{-\log E[\exp(-h\varepsilon)]\} = -\log P[\varepsilon = 0] = c_\infty < \infty;$$

then the long run interest rate is equal to :

$$r(t,\infty) = X_t + c_\infty.$$

Proof : The first condition concerning $\lim_{h\to\infty}\{-\log E[\exp(-h\varepsilon)]\} = c_\infty < \infty$ is a direct consequence of Proposition 4 and the proof of Proposition 2.

Moreover, we have

$$E[\exp(-h\varepsilon)] = P[\varepsilon = 0] + \int \mathbb{1}_{x>0}\exp(-xh)dF(x).$$

But $\lim_{h\to\infty}\exp(-hx) = 0, \forall x > 0$, and since $\exp(-hx) \in (0,1)$, we deduce by Beppo-Levi theorem that $\lim_{h\to\infty}\int \mathbb{1}_{x>0}\exp(-hx)dF(x) = 0$. The result follows.

QED

In this framework, the long-term rate exists, is stochastic and provides the same information as the underlying factor. This contradicts Lemma 3 in El Karoui, Frachot, Geman (1998), which asserts that the long-term yield (if it exists) cannot be stochastic in a one-factor model.

The need for an innovation with point mass at zero explains the strange behavior of the long run interest rate, in affine models with a level factor following a Gaussian random walk, even if this factor is not positive [see e.g. Christensen, Diebold, Rudebusch (2010)]. In this framework, the long run interest rate is equal to $-\infty$.

To illustrate Corollary 2, let us consider a random walk with a Poisson distributed innovation $\varepsilon_t \sim \mathcal{P}(\lambda)$. We have :

$$\begin{aligned} \psi_\varepsilon(h) &= \exp\{-\lambda[1-\exp(-h)]\}, \\ -\log P[\varepsilon = 0] &= \lambda, \end{aligned}$$

and the interest rate with time-to-maturity h is :

$$r(t,h) = X_t + \lambda\{\frac{h-1}{h} + \frac{1}{h}[1 - \frac{1-\exp(-h)}{1-\exp(-1)}]\}.$$

We check that : $\lim\limits_{h\to\infty} \psi_\varepsilon(h) = r(t,\infty) - X_t = -\log P[\varepsilon = 0] = \lambda$.

The results above concern the risk-neutral dynamics. It is known that the historical and risk-neutral dynamics are weakly linked [see Rogers (1977)]. For instance, the historical dynamic of (X_t) is not necessarily affine, and does not necessarily feature a unit root. Nevertheless, the historical and risk-neutral distributions have a same support : in particular the process (X_t) can never fall under the historical probability[8] and the probability that $X_{t+1} = X_t$ is nonzero if the long run interest rate exists. Similarly, when it exists, the long run interest rate is also an nondecreasing function of time. Therefore under model (1.1), either the long-term spot interest rate does not exist, or if it exists it can never fall. [9]

[8] The empirical literature on term structure models with level factor identify on the contrary a decreasing trend in the level factor dynamics [see for instance Diebold, Rudebush Aruoba (2006), p 312 fig. 2].

[9] Several authors argue that this property of the long-term spot rate is a consequence of no-arbitrage [Dybvig, Ingersoll, Ross (1996), El Karoui, Frachot, Geman (1998)], but prove this property under additional assumptions. These assumptions can be a predetermined long interest rate [DIR (1996)], or a long rate satisfying a diffusion equation [EFG (1998)].

5 Term Structure Model with Level and Slope Factors

At this stage, we may think that the deficiencies of the level factor modeling is specific of the single factor model and might disappear if a second factor is introduced, such as a slope factor. We will see below that this is not the case. The steps of the proof are as follows :

i) We first show that we have necessarily an affine term structure (Section 5.1).

ii) Then, we show that, except for some special baseline slope functions, the risk-neutral factor dynamics is necessarily affine (Section 5.2).

iii) In Sections 5.3-5.4, we check that an affine dynamics for the level and slope factor is not arbitrage free.

iv) Finally, we consider the cases of special baseline functions and non-affine factor dynamics in Section 5.5.

5.1 The affine term structure

To highlight the arguments, let us consider now a continuous time model with two factors. Then we can write :

$$r(t,h) = g(X_t, Z_t, h), \ \forall t, h \geq 0. \tag{5.1}$$

In order to allow for independent shocks on the level and slope factor, we need conditions on the joint support of variables X_t, Z_t. In particular, to assimilate the magnitude of the shock δx (resp. δz) with an increase in X (resp. Z), we need a property of invariance of the support. This condition is summarized in Assumption A.*1 below.

Assumption A.*1 :

i) The support of the historical (risk-neutral) conditional distribution of X_t, Z_t given its past $\underline{X_{t-h}}, \underline{Z_{t-h}}$ is $\mathcal{X}$x$\mathcal{Z}$, for any t, $h \geq 0$.

ii) The supports $\mathcal{X}$ and $\mathcal{Z}$ are additive groups.

Since the support of the historical and risk-neutral conditional distributions are the same, the condition is valid for both of them.

Proposition 6 : Under Assumption A.*1, a two factor model with a level and a slope factor can always be written as :

$$r(t,h) = X_t + Z_t\beta(h) + \gamma(h), \ \forall t, h \geq 0.$$

where $\beta(.)$ is an increasing function, $\beta(0) = 1, \gamma(0) = 1$.

Proof : X_t and Z_t are level and slope factors if and only if they can be shocked separately (under the historical distribution), with a drift and a slope effects, respectively, on the term structure.

i) Let us first consider a shock δx on X_t. By definition of the level factor, we get :

$$g(X_t + \delta x, Z_t, h) = g(X_t, Z_t, h) + d(X_t, Z_t, \delta x), \ \forall X_t, Z_t, \delta x, h,$$

where the drift effect can depend on the environment. Without loss of generality, we can assume that the level factor can take value zero. Thus we get :

$$g(\delta x, Z_t, h) = g(0, Z_t, h) + d(0, Z_t, \delta x), \ \forall Z_t, \delta x, h,$$

or equivalently, we can write :

$$g(X_t, Z_t, h) = g_1(X_t, Z_t) + g_2(Z_t, h), \text{ say.} \tag{5.2}$$

ii) Let us now apply a shock δz on Z_t. We get :

$$\begin{aligned} g(X_t, Z_t + \delta z, h) &= g_1(X_t, Z_t + \delta z) + g_2(Z_t + \delta z, h), \\ &= g_1(X_t, Z_t) + g_2(Z_t, h) + s(X_t, Z_t, \delta z)\beta(h), \ \forall X_t, Z_t, \delta z, h, \end{aligned}$$

by denoting $\beta(h)$ the baseline slope effect on the term structure.

$\beta(.)$ has to be monotonous, for instance increasing, for the slope interpretation, and the magnitude of the slope effect can depend on the environment. We can always assume that Z_t can take the value zero. Then we get :

$$g(X_t, \delta z, h) = g_1(X_t, 0) + g_2(0, h) + s(X_t, 0, \delta z)\beta(h), \ \forall X_t, \delta z, h,$$

or equivalently :

$$g(X_t, Z_t, h) = g_1(X_t) + g_2(X_t, Z_t)\beta(h) + \gamma(h), \text{ say.} \tag{5.3}$$

iii) Let us finally consider the expression (5.3) and apply a shock on the level factor. The effect of this shock equals to :

$$g_1(X_t + \delta x) - g_1(X_t) + [g_2(X_t + \delta x, Z_t) - g_2(X_t, Z_t)]\, \beta(h),$$

has to be independent of h. This implies that $g_2(X_t, Z_t)$ is independent of X_t.

To summarize we can write :

$$g(X_t, Z_t, h) = g_1(X_t) + g_2(Z_t)\beta(h) + \gamma(h),$$

or equivalently :

$$g(X_t, Z_t, h) = X_t + Z_t\beta(h) + \gamma(h),$$

since X_t (resp. Z_t) is defined up to a transformation.

Finally, if $\gamma(0) \neq 0$, $\beta(0) \neq 1$, we can always perform a drift in the definition of factor X: $X_t \to X_t + \gamma(0)$, and introduce a multiplicative scale on factor Z: $Z_t \to Z_t\beta(0)$, to satisfy the conditions $\beta(0) = 1$, $\gamma(0) = 0$.

QED

Then, the level and slope interpretations of the factors imply an affine term structure model, with a constant baseline term structure for the level factor, and an increasing baseline term structure for the slope factor.
The instantaneous interest rate $r_t = r(t, 0) = X_t + Z_t$ is the sum of the level and slope factors.

5.2 The risk-neutral factor dynamics

We will now use the affine term structure in Proposition 6 to restrict the specification of the risk-neutral factor dynamics. To simplify the discussion, we assume that the factor process satisfies a stochastic differential system.

Assumption A.*2 : Under the risk-neutral distribution, the bivariate process $Y_t = (X_t, Z_t)'$ satisfies the stochastic differential equation :

$$dY_t = \mu(Y_t)dt + \Sigma^{1/2}(Y_t)dW_t,$$

where (W_t) is a standard bivariate Brownian motion, $\mu(.)$, $\Sigma(.)$ are the infinitesimal drift and volatility, respectively.

By assuming a continuous time model without jumps, we avoid the limit case of the special random walk encountered in Section 4, Proposition 5.

By applying the pricing formula :

$$B(t,h) = \exp\left[-hr(t,h)\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^{t+h} r_u du\right)|Y_t\right],$$

where $\mathbb{Q}$ denotes the risk-neutral distribution, we deduce the relationship between μ, Σ and the expression of the interest rate [see e.g. Duffie (2001), Chapter 7]. Let us denote:

$$r(t,h) = g(Y_t, h),$$

where function g satisfies the partial differential equation (see Appendix 2) :

$$\begin{aligned} g(y,h) - g(y,0) + h\frac{\partial g(y,h)}{\partial h} &= h\frac{\partial g(y,h)}{\partial y'}\mu(y) + \frac{1}{2}hTr\left[\Sigma(y)\frac{\partial^2 g(y,h)}{\partial y \partial y'}\right] \\ &- \frac{h^2}{2}\frac{\partial g(y,h)}{\partial y'}\Sigma(y)\frac{\partial g(y,h)}{\partial y}, \ \forall y, h. \end{aligned} \tag{5.4}$$

In our framework, we have: $g(y,h) = x + \beta(h)z + \gamma(h)$, with $\beta(0) = 1$, $\gamma(0) = 0$.

Therefore, differential system (5.4) reduces to :

$$\begin{aligned}[\beta(h)-1]z+\gamma(h)+h\left[\frac{d\beta(h)}{dh}z+\frac{d\gamma(h)}{dh}\right] &= h\left[\begin{array}{cc}1, & \beta(h)\end{array}\right]\mu(y) \\ &- \frac{h^2}{2}\left[\begin{array}{cc}1, & \beta(h)\end{array}\right]\Sigma(y)\left[\begin{array}{c}1\\ \beta(h)\end{array}\right],\ \forall y,h.\end{aligned} \quad (5.5)$$

For given y, we get an infinite dimensional linear system of equations in $\mu_1(y)$, $\mu_2(y)$, $\sigma_{11}(y)$, $\sigma_{12}(y)$, $\sigma_{22}(y)$, that are the elements of $\mu(y)$, $\Sigma(y)$, respectively. Then, following Duffie, Kan (1996), we deduce the necessary form of the drift and volatility functions.

Proposition 7 : If the baseline slope $\beta(h)$ is not an affine function of h, and is not proportional to $\beta(h) = 1 + \frac{\sqrt{h^2+(a_2+a_3h)^2}-(a_2+a_3h)}{h}$, with $a_2 > 0$, the drift and volatility functions are necessarily affine functions of z only under Assumptions A.*1-A.*2, and no-arbitrage opportunity.

Proof : see Appendix 3.

Thus, we get an affine risk-neutral dynamics for the factor process :

$$\left(\begin{array}{c} dX_t \\ dZ_t \end{array}\right) = (\mu_0+\mu_1 Z_t)+(\Sigma_0+\Sigma_1 Z_t)^{1/2}dW_t, \text{ say,} \quad (5.6)$$

where μ_0, μ_1 are bivariate vectors and Σ_0, Σ_1 are (2,2) symmetric matrices.

As noted in Duffie, Kan (1996), p. 386, an affine term structure of interest rate implies an affine risk-neutral factor dynamics under some rank conditions. In our framework the rank condition is equivalent to the conditions on the baseline slope function in Proposition 7.

5.3 Constraints on the affine factor dynamics

Let us now discuss the constraints implied by the positivity of the factor volatility matrix and by the nonnegativity of the instantaneous rate.

Assumption A$^*_.$3 : The instantaneous interest rate $r(t,0) = X_t + Z_t$ is nonnegative, and can reach value zero.

By introducing this new assumption, we are limiting the set of possible factor values to $\{\mathcal{X}\text{x}\mathcal{Z}\} \cap \{(x,z) : x + z \geq 0\}$. In other words, we only allow for shocks on either level, or slope factor, keeping nonnegative the instantaneous rate.

Proposition 8 : Under Assumptions A$^*_.$1-A$^*_.$3 and no-arbitrage opportunities, process (Z_t) is a drifted Cox-Ingersoll-Ross process.

Proof :

i) First note that $\Sigma_1 \neq 0$. Otherwise, $Y_t = (X_t, Z_t)'$ would be a multivariate Ornstein-Uhlenbeck process, and $r(t,0)$ would be conditionally Gaussian, which contradicts Assumption A$^*_.$3.

ii) Let us now denote $\sigma_1(2,2)$ the (2,2) element of Σ_1. If $\sigma_1(2,2) = 0$, the process (Z_t) would be an Ornstein-Uhlenbeck process, and would take any value in $(-\infty, +\infty)$.
Since $\sigma_0(1,1) + \sigma_1(1,1)Z_t$ has to be nonnegative for any value of Z_t, we deduce that $\sigma_1(1,1) = 0$. Moreover the positivity of $\Sigma_0 + \Sigma_1 z_t$ for large z_t implies $\Sigma_1 >> 0$ and thus $\sigma_1(1,2) = 0$ by Cauchy-Schwarz.
We deduce that the condition $\sigma_1(2,2) = 0$ implies $\Sigma_1 = 0$, which contradicts Assumption A$^*_.$3 by i).

iii) To summarize the slope process (Z_t) satisfies the stochastic differential equation :

$$dZ_t = (\mu_0(2) + \mu_1(2)Z_t)\, dt + (\sigma_0(2,2) + \sigma_1(2,2)Z_t)^{1/2}\, dW_t^*,$$

where (W_t^*) is a one-dimensional Brownian motion and $\sigma_1(2,2) > 0$. Therefore, (Z_t) is necessarily a drifted CIR process defined on the interval $\mathcal{Z}^* = \left(-\frac{\sigma_0(2,2)}{\sigma_1(2,2)}, \infty\right)$.

QED

5.4 The absence of solution with affine level and slope factors

Proposition 9 : Under Assumptions A.*1-A.*3, a model with level and slope factors with an affine risk-neutral dynamics is not arbitrage free.

Proof :

Under Assumptions A.*1-A.*3, and the absence of arbitrage opportunity, the factor process satisfies the affine dynamics (5.6). Thus, we get an affine term structure model in which the sensitivity coefficients of the factors, that are 1 and $\beta(h)$ satisfy a Riccati equation. In our framework this equation is [see Duffie (2001), Chapter 7, eq. (31)] :

$$\begin{pmatrix} 0 \\ \frac{d\beta(h)}{dh} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \mu_1 \beta(h) - \frac{1}{2} \begin{pmatrix} 0 \\ \begin{bmatrix} 1, & \beta(h) \end{bmatrix} \Sigma_1 \begin{bmatrix} 1 \\ \beta(h) \end{bmatrix} \end{pmatrix},$$

The second equation :

$$\frac{d\beta(h)}{dh} = 1 - \mu_1(2)\beta(h) - \frac{1}{2} \begin{bmatrix} 1, & \beta(h) \end{bmatrix} \Sigma_1 \begin{bmatrix} 1 \\ \beta(h) \end{bmatrix}, \text{ with } \sigma_1(2,2) > 0,$$

is the equation corresponding to a drifted Cox-Ingersoll-Ross process, whose solution involves a rational function of exponential functions of h [see e.g. Duffie (2001), Chapter 7, eq(11)][10]. The first equation is $1 = \mu_1(1)\beta(h)$. This leads to a contradiction since function $\beta(h)$ is not constant.

QED

[10]The expression of $\beta(h)$ does not correspond to the expression of the sensitivity coefficient of (Z_t) in a term structure model with single factor (Z_t) satisfying the dynamics of Proposition 8, except if $\sigma_1(1,1) = \sigma_1(1,2) = 0$. This shows that factor (X_t) generally matters even if (Z_t) admits an exogenous dynamics.

5.5 Non-affine level and slope factors

Finally, let us consider the special patterns of the slope function β appearing in Proposition 7. From differential equation (5.5), we get :

$$\begin{aligned} \frac{d\left[h\gamma(h)\right]}{dh} &= -[\beta(h)-1]z - h\frac{d\beta(h)}{dh}z + h\left[\begin{array}{cc} 1, & \beta(h) \end{array}\right]\mu(y) \\ &- \frac{h^2}{2}\left[\begin{array}{cc} 1, & \beta(h) \end{array}\right]\Sigma(y)\left[\begin{array}{c} 1 \\ \beta(h) \end{array}\right], \ \forall y, h. \end{aligned} \tag{5.7}$$

i) Let us first consider the case of a slope affine function of h, i.e. $\beta(h) = b_0 + b_1 h$, with non-negative real b_1 to ensure that the slope baseline term structure is increasing in h. In this case, the linear baseline term structure $\beta(h)$ forces very long-term rates to be unboundedly large (and positive). In particular, the limiting interest rate $r(t,\infty) = \lim_{h\to\infty} r(t,h)$ is infinite, at any time t. It is interesting to discuss this special case, since the affine slope baseline was historically the first proposed specification [see Appendix 1 iv)].

ii) Let us now consider the function $\beta(h) = 1 + \frac{\sqrt{h^2+(a_2+a_3h)^2}-(a_2+a_3h)}{h}$, with $a_2 > 0$. This function tends to a finite limit when h tends to infinity. We deduce from equation (5.7), that for large h, the derivative $\frac{d[h\gamma(h)]}{dh}$ is of order h^2, whenever there exists a state y such that $\Sigma(y) \neq 0$. Under this condition, which ensures a nondegenerate 2-factor model, $\gamma(h)$ is the dominant term in the expression of $r(t,h)$. In particular, the long-term rate can be unbounded.

To summarize, let us introduce the following assumption of finiteness of the long-term rate.

Assumption A*4 : $\mathbb{P}\left[\lim_{h\to\infty}\sup|r(t,h)| < \infty\right] = 0.$

We have the Proposition below :

Proposition 10 : Under Assumptions A*1-A*4, a model with level and slope factors is not arbitrage free.

6 Concluding Remarks

A large part of the term structure literature interprets the first factors as a level factor, a slope factor, a curvature factor, respectively. Initially this interpretation relies on the pattern of the weights that each factor assigns to the different maturities: the level factor has almost equal weights, the slope factor has monotonic weights, but this interpretation has also been used to discuss the immunization of bond portfolios with respect to specific shocks on the term structure (see the discussion in Appendix 1). However, the literature does not checked if these interpretations of the first factors is coherent, that is compatible with no-arbitrage opportunity.

The aim of our paper was to show that this interpretation cannot be both realistic and arbitrage free. To discuss this point, we consider successively a single factor model with a level factor, and a 2-factor model with level and slope factors. None of these models are compatible with the positivity of interest rates, the finiteness of the long-term rate and no-arbitrage restrictions.

In particular, we show that a discrete time term structure model with a single level factor requires the short-term interest rate to be an increasing function of time. Besides, if the long run interest rate exists, the short-term interest rate has a nonzero probability to coincide with the previous rate. Both facts do not correspond to observed evolutions of short-term rates. Moreover, arbitrage-free term structure models with level and slope factors in a continuous time framework are very specific, and always feature diverging long-term rates[11].

Thus our results introduce some doubt on the relevance of the level and slope interpretations of factors underlying the term structure dynamics, but also on the practice which consists in considering basic shocks on a term structure, without checking if these shocks are compatible with both the existing term structure pattern and no-arbitrage.

[11] As noted in Andersen, Lund (1997), "We simply do not know of any theoretical rationale for explosive interest rate series".

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Appendix 1
Parallel Shift and Change of Slope in the Term Structure Literature

1. The level in the literature

i) From a coupon bond to a zero-coupon bond

Very early in the literature [Macaulay (1931), (1938), p.48] appears the idea to replace a coupon bond by an "equivalent" zero-coupon bond in order to facilitate the comparison of bonds with different maturities and seasoning. More precisely, let us consider at time t a coupon bond with nonnegative coupons $A_h, h = 0, 1 \ldots$ at the different times-to-maturity, and a current price $\Pi_t(A)$. To create the "equivalent" zero-coupon bond, we have to define the corresponding rate and time-to-maturity. They are usually defined as follows : the equivalent rate, or yield, is the solution $r_t^I(A)$ of the equation :

$$\Pi_t(A) = \sum_{h=0}^{\infty} A_h \exp[-hr_t^I(A)].$$

The equivalent time-to-maturity is the Macaulay's duration [12] defined by :

$$D_t^I(A) = \sum_{h=0}^{\infty} hA_h \exp[-hr_t^I(A)] / \sum_{h=0}^{\infty} A_h \exp[-hr_t^I(A)].$$

It is equal to the average time-to-maturity of the coupons weighted by the discounted coupons, which corresponds to a modified probability measure with elementary probabilities $\left(q_t(h) = A_h \exp[-hr_t^I(A)] / \sum_{h=0}^{\infty} A_h \exp[-hr_t^I(A)] \right)$.

In a modern terminology, these two notions are an implied rate and an implied time-to-maturity, since they are computed from a misspecified term structure model, which assumes a flat term structure, possibly varying in time :

$$r(t, h) = X_t, \forall h, \text{ say.} \tag{A.1}$$

[12]The eponym "Macaulay's duration" has been introduced in Fisher, Weil (1971), p416.

ii) Consistency with no-arbitrage

The flat term structure model (A.1) underlying the derivation and interpretation of the yield and duration hardly coincides with the true term structure. Nevertheless, this misspecified model should be consistent with no-arbitrage restrictions.

From (A.1), we note that the underlying model is a special case of model (1.1) with $c(h) = 0, \forall h$. By arguments similar to the arguments in Section 4, we deduce that, under no-arbitrage, the dynamic of (X_t) is such that :

$$X_{t+1} = X_t, \forall t. \tag{A.2}$$

Therefore, under no-arbitrage, the term structure is flat at all dates if and only if it is also time independent :

$$\begin{aligned} & X_t = X_0, \forall t, \\ \Leftrightarrow \quad & r(t,h) = X_0, \forall t, h \ \text{ say.} \end{aligned} \tag{A.3}$$

Thus the no-arbitrage restriction induces strong links between the pattern of the term structure (which is flat) and its evolution (which is constant in time).

The underlying model can be stochastic if the initial value is stochastic, but the associated notion of shock is very special. Indeed, a shock on X_0 can be introduced : this shock will have a drift effect not only on the term structure at date t, but on the term structures of all dates jointly. Under no-arbitrage on the underlying misspecified model, a transitory shock on a term structure, that is a shock specific to date t, cannot be defined. This shock is systematically permanent.[13]

The aim of Assumption A.2 was to eliminate this very special limiting case.

iii) The duration as a sensitivity parameter.

It is also well-known that the duration is a measure of the sensitivity of the bond price with respect to shock on the level of interest rate, which does not depend on time-to-maturity

[13] Theorem 4 in Ingersoll, Skelton, Weil (1978) provides an alternative proof of the result. They show that a transitory shift in a flat term structure is not arbitrage free.

due to the assumption of a flat term structure [see e.g. Hicks (1939), Redington (1952), Fisher (1966), Hopewell, Kaufman (1973)].
More precisely, let us shock the flat term structure $r(t,h) = r_t^I(A)$, $\forall h$, into $r^l(t,h) = r_t^I(A) + \delta$, $\forall h$, and consider the associated value $\Pi_t^l(A,\delta) = \sum_{h=0}^{\infty} A_h \exp[-h\left(r_t^I(A)+\delta\right)]$, where the upper index l mentions a shock on level. We have :

$$\left.\frac{\partial \log \Pi_t^l(A,\delta)}{\partial \delta}\right|_{\delta=0} = D_t^I(A). \tag{A.4}$$

2. The slope in the literature

i) The convexity

The sensitivity analysis can be extended at second-order. The convexity is the second-order derivative of the log-price[14] function with respect to shock δ :

$$\begin{aligned} C_t^I(A) &= \left.\frac{\partial^2 \log \Pi_t^l(A,\delta)}{\partial \delta^2}\right|_{\delta=0} \\ &= \frac{\sum_{h=0}^{\infty} h^2 A_h \exp[-hr_t^I(A)]}{\sum_{h=0}^{\infty} A_h \exp[-hr_t^I(A)]} - \left(\frac{\sum_{h=0}^{\infty} hA_h \exp[-hr_t^I(A)]}{\sum_{h=0}^{\infty} A_h \exp[-hr_t^I(A)]}\right)^2 \\ &= \sum_{h=0}^{\infty} h^2 q_t(h) - \left[\sum_{h=0}^{\infty} hq_t(h)\right]^2. \end{aligned}$$

Thus the convexity can be interpreted as the variance of the time-to-maturity of the bond under modified probability measure $[q_t(h)]$.

[14] Our definition differs from the usual definition of convexity [see for instance Hull (2005), p.92], according to which convexity is equal to the second-order derivative of the price function $\frac{\partial^2 \Pi_t^l(A,\delta)}{\partial \delta^2}$. The second-order Taylor expansion is often used to derive approximated prices, that is, to consider $\Pi_t^l(A,0) + \delta \frac{\partial \Pi_t^l(A,0)}{\partial \delta} + \delta^2 \frac{\partial^2 \Pi_t^l(A,0)}{\partial \delta^2}$ instead of $\Pi_t^l(A,\delta)$. Such a Taylor expansion does not ensure the positivity of the approximated prices. At the opposite, this property is satisfied when the expansion is performed on the log-price as proposed in our definition.

ii) Effect of an affine linear shock

Let us now consider an affine linear shock, whose effect on the yield curve differs with the maturity of the rates, that is :

$$r^s(t,h,\delta) = r_t^I(A) + \delta\left[h - D_t^I(A)\right], \ \forall h,$$

where the shock on the yield curve is calibrated around the bond's duration $D_t^I(A)$, and impacts the slope without affecting the level of the yield curve.

In this case, the bond price becomes $\Pi_t^s(A,\delta) = \sum_{h=0}^{\infty} A_h \exp\left[-h\left(r_t^I(A) + \delta\left[h - D_t^I(A)\right]\right)\right]$, and the first-order sensitivity of the bond price to the affine linear shock becomes :

$$\begin{aligned}
\left.\frac{\partial \log \Pi_t^s(A,\delta)}{\partial \delta}\right|_{\delta=0} &= -\frac{\sum_{h=0}^{\infty}\left\{h\left[h - D_t^I(A)\right] A_h \exp\left[-hr_t^I(A)\right]\right\}}{\sum_{h=0}^{\infty} A_h \exp[-hr_t^I(A)]} \\
&= -\sum_{h=0}^{\infty}\left\{h\left[h - D_t^I(A)\right] q_t(h)\right\} \\
&= -C_t^I(A),
\end{aligned}$$

which is the (opposite of) bond's convexity.

Appendix 2
Proof of Equation 5.4

By definition, we have :

$$B(t,h) = \exp\left[-hr(t,h)\right] = \exp\left[-hg(Y_t,h)\right] = F(Y_t,h), \text{ say.}$$

Since (Y_t) is an Ito process satisfying $dY_t = \mu(Y_t)dt + \Sigma^{1/2}(Y_t)dW_t$, the function F satisfies the following PDE [see Duffie (2001), Chapter 7, eq. (22)] :

$$g(y,0)F(y,h) = -\frac{\partial F(y,h)}{\partial h} + \frac{\partial F(y,h)}{\partial y'}\mu(y) + \frac{1}{2}Tr\left[\Sigma(y)\frac{\partial^2 F(y,h)}{\partial y \partial y'}\right], \tag{A.5}$$

with boundary conditions.

Since $F(y,h) = \exp\left[-hg(y,h)\right]$ we get:

$$\begin{aligned}
\tfrac{\partial F(y,h)}{\partial h} &= -F(y,h)\left(g(y,h) + h\tfrac{\partial g(y,h)}{\partial h}\right), \\
\tfrac{\partial F(y,h)}{\partial y} &= F(y,h)\left(-h\tfrac{\partial g(y,h)}{\partial y}\right), \\
\tfrac{\partial^2 F(y,h)}{\partial y \partial y'} &= F(y,h)\left(h^2\tfrac{\partial g(y,h)}{\partial y}\tfrac{\partial g(y,h)}{\partial y'} - h\tfrac{\partial^2 g(y,h)}{\partial y \partial y'}\right).
\end{aligned}$$

Therefore equation (A.5) becomes :

$$\begin{aligned}
g(y,0) &= \left(g(y,h) + h\tfrac{\partial g(y,h)}{\partial h}\right) - \left(h\tfrac{\partial g(y,h)}{\partial y'}\right)\mu(y) + \tfrac{1}{2}Tr\left[\Sigma(y)\left(h^2\tfrac{\partial g(y,h)}{\partial y}\tfrac{\partial g(y,h)}{\partial y'} - h\tfrac{\partial^2 g(y,h)}{\partial y \partial y'}\right)\right] \\
&= g(y,h) + h\tfrac{\partial g(y,h)}{\partial h} - h\tfrac{\partial g(y,h)}{\partial y'}\mu(y) + \tfrac{1}{2}h^2\tfrac{\partial g(y,h)}{\partial y'}\Sigma(y)\tfrac{\partial g(y,h)}{\partial y} - \tfrac{1}{2}hTr\left[\Sigma(y)\tfrac{\partial^2 g(y,h)}{\partial y \partial y'}\right],
\end{aligned}$$

which gives equation (5.4).

Appendix 3
Proof of Proposition 7

i) The coefficients of $\mu_1(y)$, $\mu_2(y)$, $\sigma_{11}(y)$, $\sigma_{12}(y)$, $\sigma_{22}(y)$ in system (5.5) are the functions :

$$h, h\beta(h), -\frac{h^2}{2}, -h^2\beta(h), -h^2\beta^2(h),\ h \in (0,\infty).$$

When these five functions are linearly independent, system (5.5) admits a unique solution $\mu(y)$, $\Sigma(y)$, (if a solution exists), which is necessarily affine in z due to the expression of the left hand side of system (5.5).

ii) Thus we have to check if these functions are linearly independent. Let us consider a linear combination :

$$a_0 h + a_1 h^2 + a_2 h\beta(h) + a_3 h^2\beta(h) + a_4 h^2\beta^2(h) = 0,\ \forall h \in (0,\infty).$$

This condition implies :

$$a_0 + a_1 h + a_2\beta(h) + a_3 h\beta(h) + a_4 h\beta^2(h) = 0,\ \forall h \in (0,\infty).$$

By setting $h = 0$, we get $a_0 + a_2 = 0$ and the condition becomes :

$$a_1 h + a_2(\beta(h) - 1) + a_3 h\beta(h) + a_4 h\beta^2(h) = 0,\ \forall h \in (0,\infty),$$

corresponding to the linear dependence between $h, \beta(h) - 1, h\beta(h), h\beta^2(h)$ values.

Let us denote : $\tilde{\beta}(h) = \beta(h) - 1$. It is equivalent to consider the linear dependence of functions $h, \tilde{\beta}(h), h\tilde{\beta}(h), h\tilde{\beta}^2(h)$. This dependence arises if :

- $\tilde{\beta}(h) = a_1 h$, where a_1 is nonnegative to ensure that the slope baseline is increasing.
- They can also exist a_1, a_2 such that :

$$h\tilde{\beta}(h) + a_2\tilde{\beta}(h) + a_1 h = 0 \Leftrightarrow \tilde{\beta}(h) = -\frac{a_1 h}{a_2 + h} = -a_1 + \frac{a_1 a_2}{a_2 + h}$$

  For $\tilde{\beta}$ to be continuous on $(0,\infty)$, we need $a_2 < 0$. For $\tilde{\beta}$ to be positive for large value of h we need $a_1 > 0$. Finally we have $\frac{d\tilde{\beta}(h)}{dh} = -\frac{a_1 a_2}{(a_2+h)^2} < 0$. We deduce that this situation cannot arise for continuous increasing function $\tilde{\beta}$ with $\tilde{\beta}(0) = 0$.

- The last possibility of linear dependence arises when there exist a_1, a_2, a_3 such that :

$$\begin{aligned} & h\tilde{\beta}^2(h) + a_3 h\tilde{\beta}(h) + a_2\tilde{\beta}(h) + a_1 h = 0 \\ \Leftrightarrow\quad & h\tilde{\beta}^2(h) + (a_2 + a_3 h)\tilde{\beta}(h) + a_1 h = 0 \end{aligned} \tag{A.6}$$

Lemma 1 : A real solution to equation (A.6) exists for any $h \in (0, \infty)$ if and only if :
i) $a_1 \le 0$,
or if
ii) $a_1 > 0$, $a_2 > 0$ and $a_3 \ge 2\sqrt{a_1}$,
or if
iii) $a_1 > 0$, $a_2 = 0$,
or if
iii) $a_1 > 0$, $a_2 < 0$ and $a_3 = -2\sqrt{a_1}$.

Proof : A solution exists if and only if

$$\Delta(h) = (a_2 + a_3 h)^2 - 4a_1 h^2 = (a_3^2 - 4a_1)h^2 + 2a_2 a_3 h + a_2^2 \ge 0,\ \forall h \in (0, \infty).$$

In particular by considering the limiting value $h = 0$ and $h \to \infty$, we see that $a_3^2 - 4a_1 \ge 0$ and $a_2^2 \ge 0$.

i) If $a_1 = 0$, we get : $\Delta(h) = (a_2 + a_3 h)^2 \ge 0$, $\forall a_2, a_3$.

ii) If $a_3^2 - 4a_1 = 0$, we get : $\Delta(h) = a_2(2a_3 h + a_2) \ge 0$, and $a_1 \ge 0$.

$a_1 = 0$ implies $a_3 = 0$ and $\Delta(h) = a_2^2 \ge 0$, $\forall h$. If $a_1 > 0$ and $a_3 = 2\sqrt{a_1} > 0$, $\Delta(h)$ is positive for all h if and only if $a_2 > 0$. Conversely, if $a_1 > 0$ and $a_3 = -2\sqrt{a_1} < 0$, $\Delta(h)$ is positive if and only if $a_2 < 0$. Finally, if $a_1 > 0$ and $a_2 = 0$, we get $\Delta(h) = 0$, $\forall h$.

iii) If $a_3^2 - 4a_1 > 0$, $\Delta(h) = 0$ is a quadratic equation, whose discriminant is equal to :

$$\Delta' = (a_2 a_3)^2 - a_2^2(a_3^2 - 4a_1) = 4a_1 a_2^2.$$

If $a_1 \leq 0$ the discriminant is negative or null and $\Delta(h) \geq 0$, $\forall h \in (-\infty, \infty)$. If $a_1 > 0$, the equation $\Delta(h) = 0$ admits the real roots :

$$\frac{-a_2 a_3 \pm 2|a_2|\sqrt{a_1}}{a_3^2 - 4a_1}.$$

$\Delta(h)$ is nonnegative for any nonnegative h, if and only if the maximal real root is nonpositive. Then the condition is : $-a_2 a_3 + 2|a_2|\sqrt{a_1} \leq 0$.
This inequality can be rewritten according to the sign of a_2. If $a_2 > 0$, we get $2\sqrt{a_1} < a_3$, and if $a_2 < 0$, we get $a_3 < -2\sqrt{a_1}$, which is not compatible with the inequality $a_3^2 - 4a_1 > 0$. Finally, if $a_2 = 0$ the maximal real root is null for all a_3, and thus $\Delta(h) \geq 0$ $\forall h \geq 0$.

QED

Then, the solution is necessarily the root :

$$\tilde{\beta}(h) = \frac{-(a_2 + a_3 h) + \sqrt{(a_2 + a_3 h)^2 - 4a_1 h^2}}{2h}, \tag{A.7}$$

due to the restriction $\tilde{\beta}(0) = 0$.

Let us now compute the derivative of this function. We get :

$$\frac{d\tilde{\beta}(h)}{dh} = \frac{a_2 \Delta^{-1/2}(h)}{2h^2}\left[\Delta^{1/2}(h) - (a_2 + a_3 h)\right].$$

This derivative is positive if and only if, either $a_2 > 0$, $a_1 < 0$, or $a_2 < 0, a_1 > 0$. By combining these restrictions with the restrictions of Lemma 1, we get the next Lemma.

Lemma 2 : A solution to equation (A.6) exists and is increasing for $h \in (0, \infty)$, if and only if $a_1 < 0$ and $a_2 > 0$. This solution is given by :

$$\tilde{\beta}(h) = \frac{\sqrt{(a_2 + a_3 h)^2 - 4a_1 h^2} - (a_2 + a_3 h)}{2h}.$$

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