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Non-Common Beliefs**

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Multidimensional Political Competition with Non-Common Beliefs

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Abstract

This paper extends a probabilistic voting model with a multidimensional policy space, allowing candidates to have different prior probability distributions of the distribution of voters' ideal policies. In this model, we show that a platform pair is a Nash equilibrium if and only if both candidates choose a common generalized median of expected ideal policies. Thus, the existence of a Nash equilibrium requires not only that each candidate's belief have an expected generalized median, which is already a knife-edge condition, but also that the two medians coincide. We also study limits of ϵ -equilibria of Radner (1980) as $\epsilon \rightarrow 0$, which we call "limit equilibria." Limit equilibria are policy pairs that approximate choices by the candidates who almost perfectly optimize. We show that a policy pair is a limit equilibrium if and only if both candidates choose the same policy around which they form "opposite expectations" in a certain sense. For a limit equilibrium to exist (equivalently, for ϵ -equilibria to exist for all $\epsilon > 0$), it is sufficient, though not necessary, that either candidate has an expected generalized median.

1 Introduction

In real elections, candidates often choose their electoral platforms without knowing the exact preference distribution of the electorate. What policy choices constitute Nash equilibria in such situations? To describe and analyze this case, we need probabilistic voting models in which candidates have beliefs (i.e., prior probability distributions) of the preference distribution. Most of the probabilistic voting models assume that candidates have a common belief. Under this assumption, when the policy space is multidimensional, the existence of a (pure strategy) Nash equilibrium often requires

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a quite strong condition on the belief. It is natural to ask under what condition an equilibrium exists if candidates have different beliefs. In this paper, we try to answer this question.

We extend a standard two-candidate probabilistic voting model with a multidimensional policy space, allowing the candidates to have different beliefs. In general, differences between beliefs may arise from differences in information, experiences, and other personal characteristics. In electoral competition, the large degree of freedom in assigning a probability to every possible profile of voters' preferences may add scope for belief differences. Candidates may form beliefs from their private polls. Duggan, Bernhardt, and Squintani (2007) study this case in a Bayesian-game setting. Candidates also may draw different subjective inferences from the same public data on voters' preferences.

We consider a probabilistic voting model with a multidimensional compact, convex policy space, two vote-maximizing candidates, and a continuum of voters. The candidates believe that each voter i has some ideal policy x_i such that he prefers policies closer to x_i . They have beliefs on the distribution of ideal policies. We assume that each candidate's belief induces a distribution of expected ideal policies which has a density function on the policy space with full support.

In the common-belief case of this model, results shown by Plott (1967) and Calvert (1980) can be applied to fully characterize Nash equilibria: platforms of the candidates constitute a Nash equilibrium if and only if both candidates locate at an "expected generalized median" with respect to the belief. An expected generalized median is defined as a policy x such that any hyperplane containing x divides the policy space into halfspaces with equal proportions of expected ideal policies. Thus, a Nash equilibrium exists if and only if the common belief has an expected generalized median. Under our assumptions, any belief has at most one expected generalized median. If the policy space has only one dimension, all beliefs have an expected generalized median, as it is a median in the usual sense of expected ideal policies. If the policy space has multiple dimensions, then the existence of an expected generalized median is extremely restrictive.

We show that even in the case with non-common beliefs, the characterization of Nash equilibria, together with the condition for the existence of them, remains essentially the same. More precisely, in the model with non-common beliefs, a platform pair is a Nash equilibrium if and only if both candidates locate at a common expected generalized

median. Thus, a Nash equilibrium exists if and only if both candidates' beliefs have expected generalized medians, and the two medians coincide. For example, even if the policy space is unidimensional, no Nash equilibrium exists whenever the candidates' expected medians differ. Hence, allowing for non-common beliefs does not substantially expand the set of pairs of the candidates' beliefs for which Nash equilibria exist. We are thus led to consider a solution weaker than Nash equilibria.

We study “epsilon-equilibria” of Radner (1980). For $\epsilon \geq 0$, a platform pair is called an ϵ -equilibrium if each candidate's platform gives him an expected vote share within ϵ of the maximum that he can get given the opponent's platform. In previous papers on political competition, only Nash equilibria (ϵ -equilibria for $\epsilon = 0$) have been studied. In our model, the candidates' best response correspondences have empty values almost everywhere: while each candidate can gain by getting closer to the opponent's position y , locating exactly at y is never optimal for him unless it is an expected generalized median with respect his belief. Weakening the strict maximization behavior of the candidates allows us to deal with nonempty-valued approximate best response correspondences.

We characterize limits of ϵ -equilibria as $\epsilon \rightarrow 0$, which we simply call “limit equilibria.” Limit equilibria are policy pairs that approximate choices by the candidates who almost perfectly optimize. We show that a platform pair is a limit equilibrium if and only if both candidates choose the same policy with “symmetric maximal normals.” We say a policy z has symmetric maximal normals if the candidates can divide all votes by cutting the policy space with some hyperplane containing x (one candidate receives all votes from voters with ideal policies on one side of the hyperplane), so that no other division with a hyperplane containing x makes either candidate better off according to his own belief. In the common-belief case, a policy has symmetric maximal normals if and only if it is an expected generalized median; hence limit equilibria coincide with Nash equilibria. The characterization of limit equilibria implies that an ϵ -equilibrium exists for every $\epsilon > 0$ if and only if a policy with symmetric maximal normals exists.

In contrast to the condition for the existence of a Nash equilibrium (i.e., the existence of a common expected generalized median), the existence of a policy with symmetric maximal normals only requires that there be a relation between the candidates' beliefs, and imposes no independent constraint on each candidate's belief. In particular, each candidate's expected generalized median has symmetric maximal normals. Thus, for an ϵ -equilibrium to exist for every $\epsilon > 0$, it is sufficient, though not necessary, that either

candidate has an expected generalized median. If the policy space is unidimensional, then, since the candidates have expected medians, ϵ -equilibria exist for all $\epsilon > 0$. In fact, in this case, all policies between the candidates' expected medians have symmetric maximal normals.

The result on limit equilibria implies that only very close platforms can constitute an ϵ -equilibrium for very small $\epsilon > 0$. We thus find that differences between the candidates' beliefs alone do not generate policy divergence in Nash equilibrium, and this is approximately true even if the candidates' behavior deviates slightly from perfect optimization. In contrast, Wittman (1983) and Roemer (2001) show that in probabilistic voting models in which the candidates have different policy preferences, at any Nash equilibrium (if it exists), the candidates must choose different platforms. This comparison indicates that candidates' beliefs on the electorate and their preferences over policy outcomes have qualitatively different effects on their policy choices.

There are two recent papers that take new approaches to candidates' behavior under uncertainty. Bade (2011) constructs a multidimensional probabilistic voting model in which candidates are uncertainty averse. Each candidate has multiple beliefs on voters' preferences and seeks to maximize the minimum expected vote share, where the minimum is taken over the possible beliefs. Under some restrictions on the sets of beliefs held by the candidates and on the dimensionality of the policy space, Bade shows the existence of an equilibrium. As her focus is on uncertainty aversion, it does not fully cover the case where each candidate has a single belief, but the beliefs are different between the candidates. The present paper focuses on this case.

Another approach is taken by Bernhardt, Duggan, and Squintani (2007). In a unidimensional setting, they assume that candidates receive noisy private signals of the median ideal policy. While the candidates have a common prior belief of the median ideal policy, different signals lead them to different posterior beliefs. They show that while a pure strategy Bayesian equilibrium often does not exist, a mixed strategy Bayesian equilibrium exists generally. As the equilibrium platforms are contingent on private signals, each candidate forms a probabilistic conjecture about the opponent's platform. We do not explicitly model such an informational aspect. Our model fits the case where it is relevant to assume that each candidate predicts the opponent's platform with certainty.

The rest of this paper is organized as follows. In Section 2, we construct a probabilistic voting with non-common beliefs, and define Nash equilibria and epsilon-equilibria

of Radner (1980) in the model. In Section 3, we characterize Nash equilibria, and point out that they often do not exist. In Section 4, we define limit equilibria and points with symmetric maximal normals, characterize limit equilibria, and show conditions for ϵ -equilibria to exist for any $\epsilon > 0$. Finally, in Section 5, we conclude.

2 The Model

Our model consists of two candidates (candidates 1 and 2), a continuum of voters, the policy space (i.e., the set of all policies) X , and candidate 1's belief μ and candidate 2's belief ν about the distribution of voters' preferences.

The *policy space* X is a compact, convex subset of \mathbb{R}^n with $\text{int } X \neq \emptyset$.¹

The *belief* μ of candidate 1 is a probability measure on \mathbb{R}^n with the following interpretation: candidate 1 believes that each voter has an ideal policy $z \in X$ such that he prefers policy x to policy y if and only if $\|x - z\| < \|y - z\|$; the candidate does not observe voters' ideal policies, and has a prior joint probability distribution of voters' ideal policies in the population; for each measurable $A \subset \mathbb{R}^n$, the measure $\mu(A)$ represents the expected proportion (with respect to his prior) of voters with ideal policies in A . Similarly, the belief ν of candidate 2 is defined and interpreted. We assume that μ and ν are absolutely continuous with respect to the Lebesgue measure and have support X .

Each candidate seeks to maximize the expected vote share with respect his belief by choosing his electoral platform from the policy space X . Let $\pi_\mu(x, y)$ denote the expected vote share with respect to μ for policy x over policy y . We assume that $\pi_\mu(x, y) = \frac{1}{2}$ if $x = y$. Our assumptions on μ imply

$$\pi_\mu(x, y) = \begin{cases} \mu\{z \in \mathbb{R}^n \mid \|x - z\| < \|y - z\|\} & \text{if } x \neq y \\ \frac{1}{2} & \text{if } x = y. \end{cases}$$

Replacing μ with ν , the function π_ν is similarly defined.

Our model is a game between candidates 1 and 2 in which if candidate 1 chooses platform $x \in X$ and candidate 2 chooses platform $y \in X$, then they get payoffs $\pi_\mu(x, y)$ and $\pi_\nu(y, x)$, respectively. A *Nash equilibrium* is a (pure strategy) Nash equilibrium of this game.

¹Throughout the paper, we endow \mathbb{R}^n with the Euclidean inner product “ \cdot ”, norm $\|\cdot\|$, and topology (with $\text{int } A$ and $\text{cl } A$ denoting the interior and the closure of A).

For each $x \in \mathbb{R}^n$ and $a \in \mathbb{R}^n \setminus \{0\}$, denote by $H_{x,a}$ the open halfspace

$$H_{x,a} = \{z \in \mathbb{R}^n \mid a \cdot z > a \cdot x\}.$$

It is easy to check that $\mu(H_{x,a})$ is continuous in $(x, a) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. For each (x, y) with $x \neq y$, we have

$$\pi_\mu(x, y) = \mu(H_{(x+y)/2, x-y}). \quad (1)$$

The normal of any halfspace can be represented by a point in the $(n-1)$ -dimensional unit sphere

$$S = \{s \in \mathbb{R}^n \mid \|s\| = 1\}.$$

A policy x is called a *generalized median (in all directions)* of μ if $\mu(H_{x,s}) = \frac{1}{2}$ for every $s \in S$. A generalized median of μ is interpreted as a generalized median of expected ideal policies with respect to candidate 1's prior. If μ has a generalized median, then it is unique. If $n = 1$, then a generalized median is a median in the usual sense.

Given $\epsilon \geq 0$, the ϵ -best response correspondence of candidate 1, $F_\mu(\cdot, \epsilon) : X \rightarrow X$, is defined by

$$F_\mu(y, \epsilon) = \{x \in X \mid \pi_\mu(x, y) \geq \sup_{z \in X} \pi_\mu(z, y) - \epsilon\}.$$

A policy pair (x, y) is an ϵ -equilibrium if $x \in F_\mu(y, \epsilon)$ and $y \in F_\nu(x, \epsilon)$. Nash equilibria are exactly 0-equilibria.

3 Nash Equilibrium

To characterize Nash equilibria, we provide some notation and terminology. For each $y \in X$, let

$$\Pi_\mu(y) = \max_{s \in S} \mu(H_{y,s}).$$

As $\mu(H_{y,s})$ is continuous, the function Π_μ is continuous. A μ -maximal normal at policy y is a vector $s \in S$ such that $\mu(H_{y,s}) = \Pi_\mu(y)$. That is, among all halfspaces whose boundaries contain y , the halfspace with a maximal normal has the maximum μ -measure.

The following lemma is due to Caplin and Nalebuff (1988).²

²Caplin and Nalebuff (1988), Proposition 2. Their statement includes the assumption that μ has a concave density, but their proof only uses the assumption that μ has a density function with convex support.

Lemma 1. (Caplin and Nalebuff, 1988) $\sup_{z \in X} \pi_\mu(z, y) = \Pi_\mu(y)$ for every $y \in X$.

By (1), Lemma 1 implies that given candidate 2's platform y , any sufficiently high expected vote share that candidate 1 can get is attained by locating close to y in the direction of any μ -maximal normal at y .

Proposition 1. (x, y) is a Nash equilibrium if and only if $x = y = z$, where z is a generalized median of both μ and ν .

Proof. It suffices to show that $F_\mu(y, 0) \neq \emptyset$ if and only if y is a generalized median of μ , in which case $F_\mu(y, 0) = \{y\}$ (because then the same statement with respect to belief ν also holds). The “if” part directly follows from Lemma 1. To show the “only if” part, suppose y is not a generalized median of μ . Fix any maximal normal s at y with respect to μ . By choosing platform $y + \lambda s$ for sufficiently small $\lambda > 0$, candidate 1 gets an expected vote share greater than $\frac{1}{2}$. Thus y is not a best response of candidate 1 to y . His expected vote share becomes arbitrarily close to the supremum as λ approaches 0, while y is not optimal. Therefore $F_\mu(y, 0) = \emptyset$. \square

The corollary below follows directly from Proposition 2.

Corollary 1. A Nash equilibrium exists if and only if both candidates' beliefs have generalized medians, and the two generalized medians coincide.

When $n \geq 2$, it is already quite restrictive that a candidate's belief has a generalized median. Corollary 1 says that the existence of a Nash equilibrium requires not only that both candidates' beliefs have generalized medians, but also that the two medians coincide. Even if $n = 1$, no Nash equilibrium exists if the medians of μ and ν differ. Thus, allowing for different beliefs does not substantially expand the possibility of the existence of Nash equilibria.

4 Epsilon Equilibrium

In this section, we characterize “limit equilibria,” which are defined as the limits of ϵ -equilibria as $\epsilon \rightarrow 0$. Formally, a platform pair (x, y) is called a *limit equilibrium* if a sequence (x^k, y^k, ϵ^k) in $X \times X \times (0, \infty)$ exists such that (x^k, y^k) is an ϵ^k -equilibrium for every k and $(x^k, y^k, \epsilon^k) \rightarrow (x, y, 0)$ as $k \rightarrow \infty$. A limit equilibrium is said to have a *direction* $s \in S$ if a sequence (x^k, y^k, ϵ^k) exists which, in addition to the above

condition, satisfies that $x^k \neq y^k$ for every k and $(x^k - y^k)/\|x^k - y^k\| \rightarrow s$ as $k \rightarrow \infty$. A limit equilibrium is not necessarily an equilibrium.

We say beliefs μ and ν have *symmetric maximal normals* at policy z if there exists a normal $s \in S$ at z which is μ -maximal and ν -minimal (i.e., $-s$ is a ν -maximal normal at z). Let S_z be the set of all s which are μ -maximal and ν -minimal at z . Let Z be the set of all policies z at which the beliefs have symmetric maximal normals.

$$S_z = \arg \max_{s \in S} \mu(H_{z,s}) \cap \arg \min_{s \in S} \nu(H_{z,s}), \quad z \in X,$$

$$Z = \{z \in X \mid S_z \neq \emptyset\}.$$

An example of a policy with symmetric maximal normals is a generalized median of each candidate's belief. To see this, suppose z is a generalized median of μ . Then all $s \in S$ are μ -maximal normals at z . As at least one ν -minimal normal at z exists, we have $S_z \neq \emptyset$.

If $n = 1$, under our assumptions, both candidates' beliefs have a median. Hence $Z \neq \emptyset$. Moreover, if $n = 1$, Z is the interval between the medians of the candidates' beliefs. To see this, let $X = [0, 1]$, and z_μ and z_ν be the medians of μ and ν , with $0 < z_\mu \leq z_\nu < 1$. Then it is immediate that $S_z = \{-1\}$ if $z \in [z_\mu, z_\nu]$, and $S_z = \emptyset$ otherwise.

Proposition 2. *(x, y) is a limit equilibrium if and only if $x = y \in Z$. If (z, z) is a limit equilibrium, then the set of its directions is S_z .*

Proof. See Appendix. □

We sketch the proof of Proposition 2. In the proof, we first show that the distance between any policy y and any ϵ -best response to y converges uniformly to 0 as $\epsilon \rightarrow 0$ (Lemma 2). This is due to the absolute continuity and the compact, convex supports of beliefs. Thus any limit equilibrium must be a symmetric policy pair. This then implies that for very small ϵ , the candidates must locate close to some policy z and almost best respond to z . This occurs only if the beliefs have almost symmetric maximal normals at z . Therefore, the limit of any sequence of ϵ -equilibria as $\epsilon \rightarrow 0$ must be (z, z) with $z \in Z$. To show the converse, we note that the convexity of X ensures that given $z \in Z$ and $s \in S_z$, there is a policy pair of the form $(y + \lambda s, y)$ with $\lambda > 0$ which is arbitrarily close to (z, z) . By the absolute continuity, there is a sequence of ϵ -equilibria of this form which converges to (z, z) as $\epsilon \rightarrow 0$.

As a corollary of Proposition 2, we state a necessary and sufficient condition for an ϵ -equilibrium to exist for every $\epsilon > 0$.³

Corollary 2. *An ϵ -equilibrium exists for every $\epsilon > 0$ if and only if the candidates' beliefs have symmetric maximal normals at some policy.*

Proof. Suppose that an ϵ -equilibrium exists for every $\epsilon > 0$. Let (ϵ^k) be a sequence such that $\epsilon^k > 0$ for every k and $\epsilon^k \rightarrow 0$. For each k , let (x^k, y^k) be an ϵ^k -equilibrium. Since $X \times X$ is compact, (x^k, y^k) has a subsequence converging to some $(x, y) \in X \times X$. By Proposition 2, $x = y \in Z$. The converse follows directly from Proposition 2. \square

In contrast with the condition for the existence of Nash equilibria (Corollary 1), which requires that the candidates beliefs have the same generalized median, the candidates' beliefs may have symmetric maximal normals at some policy even if neither belief has a generalized median. On the other hand, since a generalized median of each candidate's belief is in Z , for an ϵ -equilibrium to exist for any $\epsilon > 0$, it suffices that either candidate's belief has a generalized median. This establishes the following Corollary 3. However, we do not know a meaningful sufficient condition for the existence of ϵ -equilibria which holds without assuming that either candidate's belief has a generalized median.

Corollary 3. *If at least one candidate's belief has a generalized median, an ϵ -equilibrium exists for every $\epsilon > 0$. If $n = 1$, an ϵ -equilibrium exists for every $\epsilon > 0$.*

As Corollary 4 below shows, if the candidates have a common belief, then the condition for an ϵ -equilibrium to exist for every $\epsilon > 0$ is the same as the condition for a Nash equilibrium to exist. Thus, it is not until we introduce belief differences between the candidates that we have non-equivalence between the existence of Nash equilibria and the existence of ϵ -equilibria for all $\epsilon > 0$.

Corollary 4. *If $\mu = \nu$, then an ϵ -equilibrium exists for every $\epsilon > 0$ if and only if μ ($= \nu$) has a generalized median.*

Proof. Clearly, $S_z \neq \emptyset$ if and only if $\Pi_\mu(z) = \frac{1}{2}$, which is equivalent to that z is a generalized median of μ . \square

³There are papers that derive sufficient conditions for an ϵ -equilibrium to exist for every $\epsilon > 0$ in more general games with discontinuous payoff functions (Radzik, 1991; Ziad, 1997; Carmona, 2010). Their conditions include upper semicontinuity and (a strengthening of) quasiconcavity of payoff functions, which often fail in our model.

5 Concluding Remarks

In a probabilistic voting model with non-common beliefs, we have shown that: (i) a Nash equilibrium exists if and only if the candidates' beliefs have the same expected generalized median; (ii) an ϵ -equilibrium exists for every $\epsilon > 0$ if and only if the candidates' beliefs have symmetric maximal normals at some policy. The condition for the existence of a Nash equilibrium is essentially the same as in the model with a common belief, and is quite restrictive in a multidimensional setting. The condition for the existence of ϵ -equilibria is weaker than that at least one candidate's belief has an expected generalized median. Allowing for different beliefs extends the possibility of the existence of ϵ -equilibria, but not that of Nash equilibria.

We conjecture that when no candidate has an expected generalized median, the candidates' beliefs have symmetric maximal normals only if the beliefs are sufficiently distant (in terms of some relevant distance measurement). Thus a small deviation from a common belief may not suffice for an ϵ -equilibrium to exist for arbitrarily small $\epsilon > 0$. It is interesting to clarify how large deviation from a common belief is necessary. Furthermore, while we have focused on the limit behavior of ϵ -equilibria as $\epsilon \rightarrow 0$, it is also appealing to study ϵ -equilibria for a fixed ϵ . Further analysis of the model may reveal the lower bound of ϵ above which an ϵ -equilibrium exists.

We have shown that any limit of ϵ -equilibria as $\epsilon \rightarrow 0$ is a symmetric policy pair (z, z) such that the candidates' beliefs have symmetric maximal normals at z . Around such policy z , unless it is a common expected generalized median, the candidates have opposing expectations (in some sense) on the voting outcome. Equilibria supported by such conflicting expectations may not emerge as a steady state in a long period involving many elections: if the candidates repeat the play of such an equilibrium in a large number of elections (with some fixed true preference distribution in the electorate), then the resulting polls will force at least one candidate to revise his belief and change his platform. Thus our prediction may be more appropriate for a short period between one election and the next. Yet, the candidates' entire beliefs need not converge in the long run, since polls of past elections only reveal voters' preferences over a small subset of policies. The question of how the true preference distribution, experiences in past elections, and candidates' beliefs are related is a potential subject of future research.

6 Appendix

6.1 Lemma 2

The following lemma shows that the distance between any policy y and any ϵ -best response to y converges uniformly to zero. For every $\epsilon > 0$, define a function $\Delta_\mu(\cdot, \epsilon) : X \rightarrow \mathbb{R}$ by

$$\Delta_\mu(y, \epsilon) = \sup_{x \in F_\mu(y, \epsilon)} \|x - y\|.$$

Lemma 2. *For any sequence (ϵ^k) of positive numbers with $\epsilon^k \rightarrow 0$, the sequence $(\Delta(\cdot, \epsilon^k))$ converges uniformly to the constantly zero-valued function.*

We divide the proof into two steps.

Step 1. *For every $\epsilon > 0$ and $y \in X$, $\text{cl } F_\mu(y, \epsilon) = F_\mu(y, \epsilon) \cup \{y\}$. For every $\epsilon > 0$, the correspondence $\text{cl } F_\mu(\cdot, \epsilon) : X \rightarrow X$ is upper semicontinuous.*

Proof. Let

$$\phi(x, y) = \sup_{z \in X} \pi_\mu(z, y) - \pi_\mu(x, y).$$

Then

$$F_\mu(y, \epsilon) = \{x \in X \mid \phi(x, y) \leq \epsilon\}.$$

By Lemma 1,

$$\phi(x, y) = \begin{cases} \Pi_\mu(y) - \mu(H_{(x+y)/2, x-y}) & \text{if } x \neq y, \\ \Pi_\mu(y) - \frac{1}{2} & \text{if } x = y. \end{cases}$$

The function ϕ is continuous at every (x, y) with $x \neq y$.

To show that $y \in \text{cl } F_\mu(y, \epsilon)$ for every $y \in X$ and $\epsilon > 0$, fix $y \in X$ and $s \in S$ with $\mu(H_{y,s}) = \Pi_\mu(y)$. There is $\bar{\lambda} > 0$ such that $y + \lambda s \in X$ for all $\lambda \in (0, \bar{\lambda})$. If λ^k is a sequence in $(0, \bar{\lambda})$ with $\lambda^k \rightarrow 0$, then

$$\lim_{k \rightarrow \infty} \phi(y + \lambda^k s/2, y) = \Pi_\mu(y) - \mu(H_{y,s}) = 0.$$

Thus for any $\epsilon > 0$, there exists K such that for all $k > K$, $\phi(y + \lambda^k s, y) \leq \epsilon$, and $y + \lambda^k s \rightarrow y$. Therefore $y \in \text{cl } F_\mu(y, \epsilon)$.

Now, since ϕ is continuous at every (x, y) with $x \neq y$, if $x^k \in F_\mu(y, \epsilon)$ for all k and $x^k \rightarrow x \neq y$, then $\phi(x, y) \leq \epsilon$, and hence $x \in F_\mu(y, \epsilon)$. Therefore $\text{cl } F_\mu(y, \epsilon) = F_\mu(y, \epsilon) \cup \{y\}$.

To prove upper semicontinuity, consider sequences $x^k \rightarrow x$ and $y^k \rightarrow y$ with $x^k \in \text{cl } F_\mu(y^k, \epsilon)$ for all k . If $x = y$, the first part of the lemma directly implies $x \in \text{cl } F_\mu(y, \epsilon)$. If $x \neq y$, then the continuity of ϕ at (x, y) implies $x \in F_\mu(y, \epsilon)$. \square

Step 2. For every $\epsilon > 0$, the function $\Delta_\mu(\cdot, \epsilon) : X \rightarrow X$ is upper semicontinuous. For every $y \in X$, $\lim_{\epsilon \rightarrow 0} \Delta_\mu(y, \epsilon) = 0$.

Proof. It is easy to see that $F_\mu(y, \epsilon) \setminus \{y\} \neq \emptyset$ for any $y \in X$ and any $\epsilon > 0$. Thus by Step 1,

$$\Delta_\mu(y, \epsilon) = \max_{x \in \text{cl } F_\mu(y, \epsilon)} \|x - y\|.$$

By Step 1, $\text{cl } F_\mu(\cdot, \epsilon)$ is upper semicontinuous, and has compact, nonempty values. Thus by the maximum theorem for upper semicontinuous domain correspondences (Berge (1963), Theorem 2, p.116), $\Delta_\mu(\cdot, \epsilon)$ is upper semicontinuous.

To show the last part, suppose the contrary. Then for some $\epsilon^k \rightarrow 0$ and some $\delta > 0$, there exists a sequence $x^k \in \text{cl } F_\mu(y, \epsilon^k)$ with $\|x^k - y\| > \delta$. Since X is compact, (x^k) has a subsequence converging to some $x \in X$. Then $\|x - y\| \geq \delta$ and hence $x \neq y$. By the continuity of ϕ except on the diagonal, $\phi(x, y) = 0$. For any $\lambda \in (0, 1)$, the convexity of X implies that $\lambda x + (1 - \lambda)y \in X$ and $\phi(\lambda x + (1 - \lambda)y, y) < 0$, a contradiction. \square

Finally, Lemma 2 follows from Step 2 and Dini's theorem on decreasing sequences of upper semicontinuous functions (See Royden (1988), p.195).

6.2 Proof of Proposition 2

We divide the proof into two steps.

Step 1. If (x, y) is a limit equilibrium, then $x = y = z$ for some $z \in Z$. If (z, z) is a limit equilibrium and z is not a generalized median of both μ and ν , then its direction is in S_z .

Proof. By Lemma 2, if (x, y) is a limit equilibrium, then $x = y = z$ for some $z \in X$. If z is a generalized median of both μ and ν , then $S_z = S$, and hence $z \in Z$.

Now suppose that z is not a generalized median of μ . (The proof is the same when z is not a generalized median of ν .) Let (x^k, y^k) be a sequence of ϵ^k -equilibria converging to (z, z) . By Lemma 1,

$$\sup_{z \in X} \pi_\mu(z, y^k) = \Pi_\mu(y^k) \rightarrow \Pi_\mu(z) > \frac{1}{2}.$$

Thus for K with $\epsilon^K < \Pi_\mu(z) - \frac{1}{2}$ and for every $k > K$, $y^k \notin F_\mu(y^k, \epsilon^k)$, and hence $x^k \neq y^k$. For each $k > K$, let $s^k = (x^k - y^k)/\|x^k - y^k\| \in S$ and suppose $s^k \rightarrow s$. By the definition of ϵ -equilibrium, for all $k > K$,

$$\begin{aligned} 0 &\leq \Pi_\mu(y^k) - \mu(H_{(x^k+y^k)/2, s^k}) \leq \epsilon^k, \\ 0 &\leq \nu(H_{(x^k+y^k)/2, s^k}) - (1 - \Pi_\nu(x^k)) \leq \epsilon^k. \end{aligned} \tag{2}$$

Therefore, $\Pi_\mu(z) = \mu(H_{z, s})$ and $1 - \Pi_\nu(z) = \nu(H_{z, s})$. This implies $s \in S_z$ and hence $z \in Z$. \square

Step 2. For any $z \in Z$ and $s \in S_z$, (z, z) is a limit equilibrium with direction s .

Proof. Let $z \in Z$ and $s \in S_z$. Since X is convex and $\text{int } X \neq \emptyset$, for any neighborhood U of z there exist $y \in U \cap \text{int } X$ and $\lambda > 0$ with $y + \lambda s \in U \cap X$. Thus a sequence $(x^i, y^i) = (y^i + \lambda^i s, y^i)$ in $X \times X$ exists such that $\lambda^i > 0$ for every i and $(x^i, y^i) \rightarrow (z, z)$ as $i \rightarrow \infty$. We have $\mu(H_{(x^i+y^i)/2, s}) \rightarrow \mu(H_{z, s})$ and $\Pi_\mu(y^i) \rightarrow \mu(H_{z, s})$. Similarly, $\nu(H_{(x^i+y^i)/2, s}) \rightarrow \nu(H_{z, s})$ and $1 - \Pi_\nu(y^i) \rightarrow \nu(H_{z, s})$. Thus for every k , there exists i such that (2) holds if we substitute (x^i, y^i) for (x^k, y^k) and s for s^k . Thus we obtain a subsequence (x^{i_k}, y^{i_k}) of (x^i, y^i) such that (x^{i_k}, y^{i_k}) is an ϵ^k -equilibrium for every k . \square

If z is a generalized median of both μ and ν , then $S_z = S$. Thus by Step 2, any direction of the limit equilibrium (z, z) is in S_z , finishing the proof of Proposition 2.

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