Hitotsubashi University

Hi-Stat Discussion Paper

Global COE Hi-Stat Discussion Paper Series 137

Research Unit for Statistical and Empirical Analysis in Social Sciences (Hi-Stat)

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May 2010

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April 2010

Abstract

In two-sided matching problems, we consider "natural" changes in preferences of agents in which only the rankings of current partners are enhanced. We introduce two desirable properties of matching rules under such rank-enhancements of partners. One property requires that an agent who becomes higher ranked by the original partner should not be punished. We show that this property cannot always be met if the matchings are required to be stable. However, if only one agent changes his preferences, the above requirement is compatible with stability, and moreover, envy-minimization in stable matchings can also be attained. The other property is a solidarity property, requiring that all of the "irrelevant" agents, whose preferences as well as whose original partners' preferences are unchanged, should be affected in the same way; either all weakly better off or all worse off. We show that when matchings are required to be stable, this property does not always hold.
Keywords: two-sided matching problem, stable matching, partnership, solidarity.
JEL Classification Numbers: C78, C71, D71, D63

1 Introduction

Consider a firm that has several factories. The firm must assign workers to these factories. Each factory manager has preferences over the workers, while each worker has preferences over which factory he works at. How should we match the workers to the factory managers?

Allocation problems such as the above example are called *two-sided matching problems*. They were first studied by Gale and Shapley (1962), who defined a matching to be *stable* if (i) no matched agent would rather be unmatched and (ii) there does not exist a pair of agents (one from each group, in our example, a worker and a manager) who would both prefer to be matched to each other than to whom they are matched. Stability is the crucial requirement for a matching to be formulated and maintained by the agents.

In this paper, we consider some "natural" changes of preferences of the agents after some matching is formulated. In our factory example, workers usually acquire factory-specific skills in the long-run, and thereby become higher ranked in the preference orders of the managers they are currently matched with. At the same time, workers also prefer working at their current places of employment to working elsewhere after obtaining such skills. Thus, it is natural to consider preference changes such that for each agent i, his current partner in a matching chosen by the rule is preferred to more of his potential partners after the preference change, holding the relative rankings of the other possible partners fixed. We call such a transformation of preferences *rank-enhancements of partners*. This kind of transformations of preferences was introduced by Tadenuma (2008).

We study how matchings should change with these "natural" changes of preferences of the agents. For that purpose, we define a *matching rule* to be a mapping that associates with each preference profile a matching, and we formulate the following two desirable properties of matching rules in relation to rank-enhancements of partners. One property is concerning the agents whose ranks are enhanced in the preference orders of their initial partners, and the other is concerning "irrelevant" agents.

First, notice that rank-enhancements of partners should be encouraged in the firm because acquisition of factory-specific skills by the workers contributes to higher productivity of labor. Thus, if some agent is ranked higher in the preference order of his current partner, he should not be punished at the new matching formulated after the preference change. He should be matched to the same partner or to another agent whose rank is higher than the current one in his own preference order. We call such a property *No Punishment for Rank-Enhanced Agents*. This is an important property of matching rules in order to provide a proper incentive for workers to acquire factory-specific skills.

Second, we consider the *solidarity principle*, a fundamental principle in normative economics. Generally, it requires that, when some data in the problem (preferences, the amount of resources, and so on) change, all the agents in the same situation with respect to the change should be affected in the same direction: They should all be better off or they should all be worse off at the new allocation chosen by the rule. One form of this principle is *solidarity under preference changes*: When the preferences of some agents change, the agents *whose preferences are fixed* should be affected in the same direction. This version of the solidarity principle was first studied by Moulin (1987) in the context of quasi-linear binary social choice.[1] In our context of rank-enhancements of partners, however, there is another

[1]Thomson (1993,1997,1998) extensively analyzed this property in classes of resource allocations problems with single-peaked preferences and with indivisible goods. Sprumont (1996) considered a class of general choice problems and formulated a solidarity property that applies when the feasibility constraints and the preferences can change simultaneously. All of these authors have considered *arbitrary changes* of preferences, and hence their versions are more demanding than the version studied in this paper which applies only to some specific kind of preference changes.

element that distinguishes the agents whose own preferences are unchanged into two types: one is the group of agents whose ranks become strictly higher in the preference orders of their current partners, and the other is the rest of the agents. As we have discussed above, the agents in the former group should not lose in the new matching after the change in preference profiles. On the other hand, the agents in the latter group are "irrelevant" agents in the sense that neither their own preferences nor their current partners' preferences change. They may gain or lose in the new matching, but there is no reason to discriminate between the agents in the same group. Thus, we require that all these "irrelevant" agents should be affected in the same direction under rank-enhancements of partners. This property is called *Solidarity under Rank-Enhancements of Partners*.

We examine the existence of matching rules that always select from the set of stable matchings and that satisfy No Punishment for Rank-Enhanced Agents or Solidarity under Rank-Enhancements of Partners. Unfortunately, neither of the two properties cannot be met with the requirement of Stability.

Then, we consider a further restricted class of preference changes: *only one* agent increases the rank of his original partner. We call such a change in preferences a *single rank-enhancement of partner*, and define weaker properties which apply only to this more restricted class of preference changes. It is shown that there exists a selection rule from stable matchings that satisfies No Punishment for Single Rank-Enhanced Agent. Moreover, any rule that selects from envy-minimizing stable matchings satisfies this property.

Turning to the solidarity property, however, we show that Solidarity under Single Rank-Enhancement of Partner is still incompatible with Stability. Tadenuma (2008) considered a somewhat stronger version of Solidarity under Single Rank-Enhancement of Partner than the current one, and showed that there exists no selection rule from envy-minimizing stable matchings that satisfies the stronger version of Solidarity. The present paper strengthens this result by showing that even without any conditions on equity of matchings, Stability and the weaker version of Solidarity are incompatible.

Our properties are logically related with Maskin Monotonicity (Maskin, 1999), which is a necessary condition for a social choice correspondence to be implementable in Nash equilibria. It can be checked that in the present context of two-sided matchings, Maskin Monotonicity implies both No Punishment for Rank-Enhanced Agents and Solidarity under Rank-Enhancements of Partners, but the converse does not hold. Hence, as a corollary of our impossibility results, we can show that no (single-valued) selection rule from stable matchings is Maskin monotonic even on the domain of "pure" matching problems in which being unmatched (or unemployed) is the worst situation for every agent (Tadenuma and Toda, 1998). It should be noted that the proof of Tadenuma and Toda (1998) cannot be adapted for the theorems in the present paper because it uses transformations of preferences that are not rank-enhancements of partners. Kara and Sönmez (1996) showed that no proper subcorrespondence of the stable matchings correspondence is Maskin monotonic. But their proof relies crucially on the preference orders in which being unmatched is not necessarily worst for the agents. In fact, there exist proper (but not single-valued) subcorrespondences of the stable matchings correspondence that are Maskin monotonic on the domain of "pure" matching problems (Tadenuma and Toda, 1998).

Kojima and Manea (2009) introduced two properties related to Maskin Monotonicity, which they called Weak Maskin Monotonicity and Individually Rational (IR) Monotonicity, respectively, in the problems of assigning indivisible objects to agents. Because both No Punishment for Rank-Enhanced Agents and Solidarity under Rank-Enhancements of Partners are even weaker than these two properties, we obtain as corollaries of our theorems that no selection rule from stable matchings is weakly Maskin monotonic, nor is IR monotonic, in the two-sided matching problems. These results should be contrasted with the results by Kojima and Manea (2009) for the objects assignment problems that characterized a class of selection rules from stable assignments satisfying Individually Rational Monotonicity and Weak Maskin Monotonicity.

The organization of the rest of this paper is as follows. The next section gives basic definitions and notation. Section 3 defines rank-enhancements of partners and introduces main properties of matching rules. Section 4 presents the impossibility and possibility results on matching rules satisfying No Punishment for Single Rank-Enhanced Agent with the requirement of stability. Section 5 shows incompatibility of Solidarity under Single Rank-Enhancement of Partner with requirement of stability. Section 6 clarifies the relation of our property with Maskin Monotonicity and related properties. The final section contains some concluding remarks.

2 Basic Definitions and Notation

Let $F = \{f_1, f_2, \ldots, f_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be two fixed disjoint finite sets such that $|F| = |W| = n$. We call F *the set of factory managers* and W *the set of workers*. For each $i \in F \cup W$, let $X_i \in \{F, W\}$ be the set with $i \notin X_i$ and $Y_i \in \{F, W\}$ the set with $i \in Y_i$. We call X_i *the set of possible partners for agent* i. For each $i \in F \cup W$, a *preference relation of agent* i, denoted by R_i, is a linear order on $X_i \cup \{i\}$.[2] An alternative $j \in X_i$ indicates that agent i is matched to agent j in X_i and the alternative i that agent i is not matched to any agent in X_i (i.e., he is "matched to himself"). Let $\mathcal{R}_i$ be the set of all possible preference relations for agent i. Given $R_i \in \mathcal{R}_i$, we define the relation P_i on $X_i \cup \{i\}$ as follows: For all $x, x' \in X_i \cup \{i\}$, xP_ix' if and only if xR_ix' holds but $x'R_ix$ does not hold.[3]

To concisely express a preference relation $R_i \in \mathcal{R}_i$, we represent it, as in Roth and Sotomayor (1990), by an ordered list of the members of $X_i \cup \{i\}$.

[2] Note that we exclude indifference between any two distinct elements in $X_i \cup \{i\}$.

[3] Because R_i is a linear order, xP_ix' if and only if xR_ix' and $x' \neq x$.

For example, the list

$$R_{f_2} \quad = \quad w_3 \quad w_1 \quad f_2 \quad w_2 \quad \ldots$$

indicates that manager f_2 prefers being matched to worker w_3 to being matched to w_1, and prefers being matched to w_1 to being unmatched, and so on.

A *preference profile* is a list $R = (R_i)_{i \in F \cup W}$. Let $\mathcal{R} = \prod_{i \in F \cup W} \mathcal{R}_i$ be the class of all preference profiles. We also consider the subclass of $\mathcal{R}$ of preference profiles in which being unmatched is the worst situation for every agent:

$$\mathcal{R}^* = \{R \in \mathcal{R} \mid \forall i \in F \cup W, \forall x \in X_i, x_i \, P_i \, i\}.$$

A *matching* μ is a one-to-one function from $F \cup W$ onto itself such that for all $i \in F \cup W$, $\mu^2(i) = i$ and if $\mu(i) \notin X_i$, then $\mu(i) = i$. Let $\mathcal{M}$ be the set of all matchings.

Following Roth and Sotomayor (1990), we represent a matching as a list of matched pairs. For example, the matching

$$\mu = \begin{cases} f_1 & f_2 & f_3 & (w_2) \\ w_3 & w_1 & (f_3) & w_2 \end{cases}$$

has two matched pairs (f_1, w_3) and (f_2, w_1) with f_3 and w_2 unmatched (i.e., self matched).

Let $R \in \mathcal{R}$ be given. A matching $\mu \in \mathcal{M}$ is *individually rational for* R if for all $i \in F \cup W$, $\mu(i) R_i i$. It is *Pareto efficient for* R if there is no $\mu' \in \mathcal{M}$ such that for all $i \in F \cup W$, $\mu'(i) R_i \mu(i)$ and for some $i \in F \cup W$, $\mu'(i) P_i \mu(i)$. It is *stable for* R if it is individually rational for R and there is no pair $(f, w) \in F \times W$ such that $w P_f \mu(f)$ and $f P_w \mu(w)$. Clearly, if a matching is stable, it is Pareto efficient, but the converse does not hold. Let $S(R)$ denote the set of stable matchings for R.

Given a class of preference profiles $\mathcal{R}_0 \subseteq \mathcal{R}$, A *matching rule* (or simply a *rule*) on $\mathcal{R}_0$ denoted by φ, is a function from $\mathcal{R}_0$ to $\mathcal{M}$. If $\varphi(R) = \mu$, we write $\varphi_i(R) = \mu(i)$ for each $i \in F \cup W$. Given a correspondence Ψ from $\mathcal{R}_0$ to $\mathcal{M}$, we say that a rule φ is a *selection rule from* Ψ if for all $R \in \mathcal{R}_0$, $\varphi(R) \in \Psi(R)$.

Let $R \in \mathcal{R}$ and a matching $\mu \in \mathcal{M}$ be given. For each agent $i \in F \cup W$, define

$$e_i(\mu, R_i) = \#\{j \in X_i \mid j = \mu(k) \text{ for some } k \in Y_i \text{ and } j \; P_i \; \mu(i)\}.^4$$

The nonnegative integer $e_i(\mu, R_i)$ is the number of instances of *envy* that agent i has in μ. Define $e(\mu, R) = (e_i(\mu, R_i))_{i \in F \cup W} \in \mathbb{Z}^{2n}_+$.[5] The vector $e(\mu, R)$ is called the *envy vector* at μ for R. We call a complete and transitive binary relation an *order*. An order $\succsim_e$ defined on $\mathbb{Z}^{2n}_+$ is said to be a *social envy order* if it satisfies the following properties:
(i) *strict monotonicity*: for all $x, y \in \mathbb{Z}^{2n}_+$, if $x \geq y$ and $x \neq y$, then $x \succ_e y$.
(ii) *order preservation under addition of a constant*: for all $x, y \in \mathbb{Z}^{2n}_+$, if $x \succsim_e y$, $a \in \mathbb{Z}^{2n}$ and $x + a, y + a \in \mathbb{Z}^{2n}_+$, then $x + a \succsim_e y + a$

The following are examples of social envy orders.

(a) The order by the weighted sum of instances of envy.
Let $\lambda \in (0, 1]^{2n}$ with $\sum_{i=1}^{2n} \lambda_i = 1$. Define $x \succsim_e y$ if and only if $\sum_{i=1}^{2n} \lambda_i x_i \geq \sum_{i=1}^{2n} \lambda_i y_i$.

(b) The lexicographic order.
Let $\theta \colon \mathbb{Z}^{2n}_+ \to \mathbb{Z}^{2n}_+$ be a function that rearranges the coordinates of each vector in $\mathbb{Z}^{2n}_+$ in nonincreasing order. Denote by $\geq_L$ the lexicographic order on $\mathbb{Z}^{2n}_+$.[6] Define $x \succsim_e y$ if and only if $\theta(x) \geq_L \theta(y)$.

Reduction of envy as a criterion of equity was first considered by Foley (1967) and Kolm (1972). The social envy order by the sum of instances of

[4] Given a set A, we denote by $\#A$ the cardinality of A.

[5] We denote by $\mathbb{Z}_+$ the set of nonnegative integers.

[6] For all $x, y \in \mathbb{Z}^{2n}_+$, $x >_L y$ if and only if there is $k \in \{1, \ldots, 2n\}$ such that for all $i < k$, $x_i = y_i$ and $x_k > y_k$. For all $x, y \in \mathbb{Z}^{2n}_+$, $x \geq_L y$ if and only if $x >_L y$ or $x = y$.

envy and the lexicographic social envy order were introduced and studied by Feldman and Kirman (1974) and Suzumura (1983), respectively.

3 Rank-Enhancements of Partners

In this section, we formulate properties of matching rules for certain "natural" changes of preferences of the agents.

Let an agent $i \in F \cup W$, a preference relation $R_i \in \mathcal{R}_i$, and a matching $\mu \in \mathcal{M}$ be given. Let $L_i(\mu, R_i) = \{j \in X_i \cup \{i\} \mid \mu(i) R_i j\}$ denote the lower contour set of $\mu(i)$ for R_i. We say that *a preference relation $R'_i \in \mathcal{R}_i$ is obtained from R_i by rank-enhancement of the partner in μ* if and only if:
(i) if $\mu(i) = i$, then $R'_i = R_i$, and
(ii) if $\mu(i) \neq i$, then $L_i(\mu, R_i) \subseteq L_i(\mu, R'_i)$ and for all $j, k \in X_i \cup \{i\}$ with $j, k \neq \mu(i)$, $j R'_i k$ if and only if $j R_i k$. Let $Q(R_i, \mu)$ be the set of preference relations that are obtained from R_i by rank-enhancement of the partner in μ. Given $R \in \mathcal{R}$ and $\mu \in \mathcal{M}$, let $Q(R, \mu) = \{R' \in \mathcal{R} \mid \forall i \in F \cup W,\ R'_i \in Q(R_i, \mu)\}$.

With rank-enhancements of the partners in μ, only the current partners in the matching μ can be preferred to more agents; the preferences over any other agents are unchanged. See the Introduction for some motivation for considering this kind of change in preferences.

Given $R \in \mathcal{R}$, $\mu \in \mathcal{M}$, and $R' \in Q(R, \mu)$, let $K(R, R') = \{i \in F \cup W \mid R_i = R'_i\}$ denote the set of agents whose preferences are the same in R and R'. We classified the members in $K(R, R')$ into two types. Define $N_1(R, R', \mu) = \{i \in K(R, R') \mid R'_{\mu(i)} \neq R_{\mu(i)}\}$, and $N_2(R, R', \mu) = \{i \in K(R, R') \mid R'_{\mu(i)} = R_{\mu(i)}\}$. Because $R' \in Q(R, \mu)$, $i \in N_1(R, R', \mu)$ implies that $L_{\mu(i)}(\mu, R_{\mu(i)}) \subseteq L_{\mu(i)}(\mu, R'_{\mu(i)})$ with $L_{\mu(i)}(\mu, R_{\mu(i)}) \neq L_{\mu(i)}(\mu, R'_{\mu(i)})$, that is, agent i is ranked strictly higher in the new preference order of his original partner $\mu(i)$ than initially. In contrast, if $i \in N_2(R, R', \mu)$, then neither agent i nor his partner in μ makes any changes in preferences, that is, agent i is "irrelevant" to the change in preference profiles from R to R'.

The following property of matching rules means that no agent in $N_1(R, R', \varphi(R))$ should be worse-off in the matching at R' than at R. In the following definitions of properties of matching rules, we denote by $\mathcal{R}_0$ the domain of the rules.

No Punishment for Rank-Enhanced Partners: For all $R, R' \in \mathcal{R}_0$, if $R' \in Q(R, \varphi(R))$, then for all $i \in N_1(R, R', \varphi(R))$, $\varphi_i(R') \, R_i \, \varphi_i(R)$.

The next property states solidarity among "irrelevant" agents in the change in preference profiles.

Solidarity under Rank-Enhancements of Partners: For all $R, R' \in \mathcal{R}_0$, if $R' \in Q(R, \varphi(R))$, then either $\varphi_i(R') \, R_i \, \varphi_i(R)$ for all $i \in N_2(R, R', \varphi(R))$, or $\varphi_i(R) \, R_i \, \varphi_i(R')$ for all $i \in N_2(R, R', \varphi(R))$.

4 No Punishment for Rank-Enhanced Partners

In this section, we examine whether there exist rules that always select a matching in the set of stable matchings and that satisfy No Punishment for Rank-Enhanced Partners. Our first result is an impossibility theorem when $n \geq 5$.

Theorem 1 *Suppose $n \geq 5$. No selection rule from S defined on $\mathcal{R}^*$ satisfies No Punishment for Rank-Enhanced Partners.*

Proof. Suppose, on the contrary, that there exists a selection rule φ from S that satisfies No Punishment for Rank-Enhanced Partners.

Let $R \in \mathcal{R}^*$ be a preference profile such that

$$\begin{array}{rcl} R_{f_1} & = & w_1 \quad w_2 \quad w_4 \quad \ldots \\ R_{f_2} & = & w_2 \quad w_4 \quad w_1 \quad \ldots \\ R_{f_3} & = & w_3 \quad w_1 \quad \ldots \\ R_{f_4} & = & w_4 \quad \ldots \\ R_{f_5} & = & w_5 \quad \ldots \\ R_{w_1} & = & f_2 \quad f_3 \quad f_1 \quad \ldots \\ R_{w_2} & = & f_1 \quad f_2 \quad f_5 \quad \ldots \\ R_{w_3} & = & f_3 \quad \ldots \\ R_{w_4} & = & f_4 \quad f_2 \quad \ldots \\ R_{w_5} & = & f_5 \quad \ldots \end{array}$$

and for all $i \geq 6$, $w_i \, P_{f_i} \, w$ for all $w \in [W \setminus \{w_i\}] \cup \{f_i\}$ and $f_i \, P_{w_i} \, f$ for all $f \in [F \setminus \{f_i\}] \cup \{w_i\}$. Then, $S(R) = \{\mu, \mu'\}$ where

$$\mu = \begin{cases} f_1 & f_2 & f_3 & f_4 & f_5 & \ldots & f_n \\ w_2 & w_1 & w_3 & w_4 & w_5 & \ldots & w_n \end{cases}.$$

and

$$\mu' = \begin{cases} f_1 & f_2 & f_3 & f_4 & f_5 & \ldots & f_n \\ w_1 & w_2 & w_3 & w_4 & w_5 & \ldots & w_n. \end{cases}.$$

Because φ is a selection rule from S, either $\varphi(R) = \mu$ or $\varphi(R) = \mu'$.

Define $R' \in \mathcal{R}^*$ as

$$\begin{array}{rcl} R'_{f_1} & = & w_1 \quad w_4 \quad w_2 \quad \ldots \\ R'_{f_3} & = & w_1 \quad w_3 \quad \ldots \end{array}$$

and

(i) for all $w_j, w_k \in X_{f_1} \setminus \{w_1, w_4, w_2\}$, $w_j \, R'_{f_1} \, w_k \Leftrightarrow w_j \, R_{f_1} \, w_k$,

(ii) for all $w_j, w_k \in X_{f_1} \setminus \{w_1, w_3\}$, $w_j \, R'_{f_3} \, w_k \Leftrightarrow w_j \, R_{f_3} \, w_k$, and

(iii) for all $i \in [F \setminus \{f_1, f_3\}] \cup W$, $R'_i = R_i$.

Then, it can be checked that $S(R') = \{\mu\}$. Because φ is a selection rule from S, we have $\varphi(R') = \mu$.

Next, define $R'' \in \mathcal{R}^*$ as

$$\begin{array}{rcl} R''_{w_2} & = & f_1 \quad f_5 \quad f_2 \quad \ldots \\ R''_{w_4} & = & f_2 \quad f_4 \quad \ldots \end{array}$$

and

(i) for all $f_j, f_k \in X_{w_2} \setminus \{f_1, f_5, f_2\}$, $f_j \; R''_{w_2} \; f_k \Leftrightarrow f_j \; R_{w_2} \; f_k$,

(ii) for all $f_j, f_k \in X_{w_4} \setminus \{f_2, f_4\}$, $f_j \; R''_{w_4} \; f_k \Leftrightarrow f_j \; R_{w_4} \; f_k$, and

(iii) for all $i \in F \cup [W \setminus \{w_2, w_4\}]$, $R''_i = R_i$.

Then, $S(R'') = \{\mu'\}$. Because φ is a selection rule from S, we have $\varphi(R'') = \mu'$.

Notice that R is obtained from R' by rank-enhancements of partners in $\varphi(R') = \mu$. Because w_2 is ranked higher by f_1 at R than at R', No Punishment for Rank-Enhanced Agents requires that he should not get worse in the matching $\varphi(R)$ than in $\varphi(R')$. Recall that either $\varphi(R) = \mu$ or $\varphi(R) = \mu'$. If we had $\varphi(R) = \mu'$, then $\varphi_{w_2}(R') = \mu(w_2) = f_1 \; P_{w_2} \; f_2 = \mu'(w_2) = \varphi_{w_2}(R)$, and w_2 would be worse off in $\varphi(R)$ than in $\varphi(R')$. Thus, we cannot have $\varphi(R) = \mu'$. Therefore, $\varphi(R) = \mu$.

However, R is also obtained from R'' by rank-enhancements of partners in $\varphi(R'') = \mu'$. Since f_2 is ranked higher by w_2 at R than at R'', No Punishment for Rank-Enhanced Agents requires that he should not get worse in $\varphi(R)$ than in $\varphi(R'')$. Then, by the same argument as above, it follows that $\varphi(R) = \mu'$. Thus, we have $\mu = \mu'$, which is a contradiction. ■

It is clear that the above result holds on the domain $\mathcal{R}$ as well.

Faced with this impossibility, we consider weakening the requirement. A weaker version of the above property is obtained if we apply the requirement only when *one* agent enhances the ranking of her original partner.

No Punishment for Single Rank-Enhanced Partner: For all $R, R' \in \mathcal{R}_0$, if $R' \in Q(R, \varphi(R))$ and $N_1(R, R', \varphi(R)) = \{i\}$ for some $i \in F \cup W$, then $\varphi_i(R') \; R_i \; \varphi_i(R)$.

We now have the following possibility result. There exist selection rules from S that satisfy No Punishment for Single Rank-Enhanced Partner. Moreover, it can be shown that any selection rule from the matchings that minimize social envy in the set of stable matchings satisfies the property on the domain $\mathcal{R}^*$ of preference profiles in which being unmatched is the worst situation for every agent.

Given a social envy order $\succsim_e$, define

$$S_{\succsim_e}(R) = \{\mu \in S(R) \mid \forall \mu' \in S(R), \mu' \succsim_e \mu\}.$$

The set $S_{\succsim_e}(R)$ is the set of matchings that minimizes social envy according to $\succsim_e$ in the set of stable matchings for R.

Theorem 2 *Let $\succsim_e$ be a social envy order. Any selection rule from $S_{\succsim_e}$ defined on $\mathcal{R}^*$ satisfies No Punishment for Single Rank-Enhanced Partner.*

Proof. Let φ be a selection rule from $S_{\succsim_e}$. Let $R \in \mathcal{R}^*$ and $\varphi(R) = \mu$. Let $i \in F \cup W$, $R'_i \in Q(R_i, \mu)$ with $R'_i \neq R_i$, $R' = (R'_i, R_{-i})$, and $\varphi(R') = \mu'$. Note that $R' \in \mathcal{R}^*$. Let $j \in F \cup W$ be such that $j = \mu(i)$. Because $R \in \mathcal{R}^*$ and $\mu \in S(R)$, $j \neq i$.

We now claim that for all $\mu'' \in S(R)$, (i) $e(\mu'', R') \succsim_e e(\mu, R')$, and (ii) $e(\mu'', R') \sim_e e(\mu, R')$ only if $\mu''(i) = \mu(i)$.

To prove this claim, let $\mu'' \in S(R)$. Since $\mu \in S_{\succsim_e}(R)$,

$$e(\mu'', R)) \succsim_e e(\mu, R). \tag{1}$$

For all $k \in F \cup W$ with $k \neq i$, $R'_k = R_k$. Hence,

$$e_k(\mu'', R'_k) - e_k(\mu'', R_k) = e_k(\mu, R'_k) - e_k(\mu, R_k). \tag{2}$$

Suppose that $\mu''(i) = \mu(i)$. Then, obviously

$$e_i(\mu'', R'_i) - e_i(\mu'', R_i) = e_i(\mu, R'_i) - e_i(\mu, R_i). \tag{3}$$

It follows from (1), (2), (3), and the order preservation propery of $\succsim_e$ that

$$e(\mu'', R)) + [e(\mu'', R') - e(\mu'', R)] \succsim_e e(\mu, R) + [e(\mu, R') - e(\mu, R)]. \quad (4)$$

Hence,

$$e(\mu'', R') \succsim_e e(\mu, R'). \quad (5)$$

Suppose next that $\mu''(i) \neq \mu(i)$. Then,

$$e_i(\mu'', R_i') - e_i(\mu'', R_i) \geq 0 > e_i(\mu, R_i') - e_i(\mu, R_i). \quad (6)$$

It follows from (1) and the order preservation propery of $\succsim_e$ that

$$e(\mu'', R)) + [e(\mu'', R') - e(\mu'', R)] \succsim_e e(\mu, R) + [e(\mu'', R') - e(\mu'', R)]. \quad (7)$$

By (2) and (6), we have $e(\mu'', R') - e(\mu'', R) \geq e(\mu, R') - e(\mu, R)$ and $e(\mu'', R') - e(\mu'', R) \neq e(\mu, R') - e(\mu, R)$. Hence, by the strict monotonicity of $\succsim_e$, we have

$$e(\mu, R) + [e(\mu'', R') - e(\mu'', R)] \succ_e e(\mu, R) + [e(\mu, R') - e(\mu, R)]. \quad (8)$$

It follows from (7), (8) and the transitivity of $\succsim_e$ that

$$e(\mu'', R)) + [e(\mu'', R') - e(\mu'', R)] \succ_e e(\mu, R) + [e(\mu, R') - e(\mu, R)]. \quad (9)$$

Therefore, we have

$$e(\mu'', R') \succ_e e(\mu, R'). \quad (10)$$

Thus, the claim has been proven.

Recall $\mu' = \varphi(R') \in S_{\succsim_e}(R')$. By the above claim, either [$\mu' \in S(R)$ and $\mu'(i) = \mu(i)$] or $\mu' \notin S(R)$. If $\mu'(i) = \mu(i)$, then obviously, $i = \mu'(j) \, R_j \, \mu(j) = i$. Suppose that $\mu' \notin S(R)$. Then, there exists a pair $(f, w) \in F \times W$ such that $w \; P_f \; \mu'(f)$ and $f \; P_w \; \mu'(w)$. However, since $\mu' \in S(R')$, $\mu'(f) \; R_f' \; w$ or $\mu'(w) \; R_w' \; f$ must hold. But between the preference profiles R and R', only agent i has different preference relations. Hence, either $i = f$ or $i = w$. By the definition of *rank-enhancement of partner*, for all $x \in X_i$, if there is $y \in X_i$ such that $y \; P_i \; x$ and yet $x \; R_i' \; y$, then $x = \mu(i)$. Thus, we have $\mu'(i) = \mu(i)$. It follows that $\mu'(j) = \mu(j)$, and hence $\mu'(j) \; R_j \; \mu(j)$, which completes the proof. ■

5 Solidarity under Rank-Enhancements of Partners

This section turns to the solidarity property. Unfortunately, whenever $n \geq 4$, no selection rule from stable matchings satisfies even the following weaker version of Solidarity, which applies only to the case when *one* agent enhances the rank of his original partner.

Solidarity under Single Rank-Enhancement of Partner: For all $R, R' \in \mathcal{R}_0$, if $R' \in Q(R, \varphi(R))$ and $N_1(R, R', \varphi(R)) = \{i\}$ for some $i \in F \cup W$, then either $\varphi_k(R')\ R_k\ \varphi_k(R)$ for all $k \in N_2(R, R', \varphi(R))$ or $\varphi_k(R)\ R_k\ \varphi_k(R')$ for all $k \in N_2(R, R')$.

Theorem 3 *Suppose $n \geq 4$. No selection rule from S defined on $\mathcal{R}^*$ satisfies Solidarity under Single Rank-Enhancement of Partner.*

Proof. Suppose, on the contrary, that there exists a selection rule φ from S that satisfies Solidarity under Single Rank-Enhancement of Partner.

Let $R \in \mathcal{R}^*$ be a preference profile such that

$$\begin{array}{rcl} R_{f_1} & = & w_1 \quad w_2 \quad \ldots \\ R_{f_2} & = & w_2 \quad w_4 \quad w_1 \quad \ldots \\ R_{f_3} & = & w_3 \quad w_1 \quad \ldots \\ R_{f_4} & = & w_4 \quad \ldots \\ R_{w_1} & = & f_2 \quad f_3 \quad f_1 \quad \ldots \\ R_{w_2} & = & f_1 \quad f_2 \quad \ldots \\ R_{w_3} & = & f_3 \quad \ldots \\ R_{w_4} & = & f_4 \quad f_2 \quad \ldots \end{array}$$

and for all $i \geq 5$, $w_i\ P_{f_i}\ w$ for all $w \in [W \setminus \{w_i\}] \cup \{f_i\}$ and $f_i\ P_{w_i}\ f$ for all $f \in [F \setminus \{f_i\}] \cup \{w_i\}$. Then, $S(R) = \{\mu, \mu'\}$ where

$$\mu = \begin{cases} f_1 & f_2 & f_3 & f_4 & \ldots & f_n \\ w_2 & w_1 & w_3 & w_4 & \ldots & w_n \end{cases}.$$

and

$$\mu' = \begin{cases} f_1 & f_2 & f_3 & f_4 & \dots & f_n \\ w_1 & w_2 & w_3 & w_4 & \dots & w_n. \end{cases}.$$

Because φ is a selection rule from S, either $\varphi(R) = \mu$ or $\varphi(R) = \mu'$.

Let $R' \in \mathcal{R}^*$ be such that

$$R'_{f_3} \quad = \quad w_1 \quad w_3 \quad \dots$$

and

(i) for all $w_j, w_k \in X_{f_3} \setminus \{w_1, w_3\}$, $w_j\ R'_{f_3}\ w_k \Leftrightarrow w_j\ R_{f_3}\ w_k$, and

(ii) for all $i \in [F \setminus \{f_3\}] \cup W$, $R'_i = R_i$.

Then, it can be checked that $S(R') = \{\mu\}$. Because φ is a selection rule from S, we have $\varphi(R') = \mu$.

Next, let $R'' \in \mathcal{R}^*$ be such that

$$R''_{w_4} \quad = \quad f_2 \quad f_4 \quad \dots$$

and

(i) for all $f_j, f_k \in X_{w_4} \setminus \{f_2, f_4\}$, $f_j\ R''_{w_4}\ f_k \Leftrightarrow f_j\ R_{w_4}\ f_k$, and

(ii) for all $i \in F \cup [W \setminus \{w_4\}]$, $R''_i = R_i$.

Then, $S(R'') = \{\mu'\}$. Because φ is a selection rule from S, we have $\varphi(R'') = \mu'$.

Notice that R is obtained from R' by single rank-enhancement of partner in $\varphi(R') = \mu$ (only f_3 enhances the rank of $w_3 = \mu(f_3) = \varphi_{f_3}(R')$), and that $f_1, w_1 \in N_2(R', R, \mu)$. If we had $\varphi(R) = \mu'$, then f_1 would be better off whereas w_1 would be worse off in $\varphi(R)$ than in $\varphi(R')$, which means that φ violates Solidarity under Single Rank-Enhancement of Partner. Hence, we must have $\varphi(R) = \mu$.

Notice that R is also obtained from R'' by single rank-enhancement of partner in $\varphi(R'') = \mu'$ (only w_4 enhances the rank of $f_4 = \mu'(w_4) = \varphi_{w_4}(R'')$), and that $f_1, w_1 \in N_2(R'', R, \mu')$. Because $\varphi(R) = \mu$, w_1 is better off whereas f_1 is worse off in $\varphi(R)$ than in $\varphi(R'')$. This means that φ violates Solidarity under Single Rank-Enhancement of Partner, which is a contradiction. ■

Clearly, the above result also holds on the domain $\mathcal{R}$.

If we impose a weaker condition than Stability as a basic requirement for matchings, does there exist a rule satisfying the solidarity property? Tadenuma (2008) showed that there exist selection rules from *individually rational* and *Pareto efficient* matchings that satisfies Solidarity under Rank-Enhancements of Partners.

6 Relations with Maskin Monotonicity

This section clarifies relations of our properties with Maskin Monotonicity (Maskin, 1999), which is a necessary condition for a social choice correspondence to be implementable in Nash equilibria. In the present context, Maskin Monotonicity is defined as follows.

Maskin Monotonicity: For all $R, R' \in \mathcal{R}_0$, if $L_i(\varphi(R), R_i) \subseteq L_i(\varphi(R), R'_i)$ for all $i \in F \cup W$, then $\varphi(R') = \varphi(R)$.

Notice that Rank-Enhancements of Partners are special cases of the transformations of preference profiles to which Maskin Monotonicity applies, and Monotonicity requires that an alternative chosen initially should also be chosen after the transformation of preference profiles. Therefore, if a matching rule is Maskin monotonic, then it satisfies both No Punishment for Rank-Enhanced Agents and Solidarity under Rank-Enhancements of Partners, but the converse does not hold. That is, both of our properties are weaker than Maskin Monotonicity. Hence, as a corollary of Theorems 1 and 3, we obtain the result that no selection rule from stable matchings is Maskin monotonic. This holds also on the domain of "pure" matching problems in which being unmatched is the worst situation for every agent.

Corollary 1 [Tadenuma and Toda, 1998] *Suppose $n \geq 3$. No selection rule from S defined on $\mathcal{R}^*$ satisfies Maskin Monotonicity.*

Kojima and Manea (2009) introduced two properties of allocation rules which are related to Maskin Monotonicity in the problem of assigning indivisible objects to agents. In the present model of two-sided matchings, these properties may be adapted as follows. First, we define for each $i \in F \cup W$, each $R_i \in \mathcal{R}_i$, and each $\mu \in \mathcal{M}$, $H_i(\mu, R_i) = \{j \in X_i \cup \{i\} \mid j \, P_i \, \mu(i)\}$ and $I_i(R_i) = \{j \in X_i \cup \{i\} \mid j \, P_i \, i\}$. The set $H_i(\mu, R_i)$ is the set of agents to whom agent i prefers being matched to being matched to $\mu(i)$. The set $I_i(R_i)$ is the set of agents to whom agent i prefers being matched to being unmatched.

Individually Rational Monotonicity: For all $R, R' \in \mathcal{R}_0$, if $H_i(\varphi(R), R'_i) \cap I_i(R'_i) \subseteq H_i(\varphi(R), R_i)$ for all $i \in F \cup W$, then $\varphi_i(R') \, R'_i \, \varphi_i(R)$ for all $i \in F \cup W$.

Individually Rational Monotonicity is logically independent of Maskin Monotonicity. Notice that if a preference relation $R'_i \in \mathcal{R}_i$ is obtained from R_i by rank-enhancement of the partner in μ, then $H_i(\varphi(R), R'_i) \subseteq H_i(\varphi(R), R'_i)$ (and hence $H_i(\varphi(R), R'_i) \cap I_i(R'_i) \subseteq H_i(\varphi(R), R'_i)$ as well). And our properties require that with rank-enhancements of partners, agents whose preferences are unchanged ($R'_i = R_i$) but who are ranked higher by their partners should not be worse off (No Punishment for Rank-Enhanced Partners), or agents whose preferences as well as whose partners' preferences are unchanged should be affected in the same way, all weakly better off or all weakly worse off (Solidarity under Rank-Enhancements of Partners). Therefore, if a matching rule satisfies Individually Rational Monotonicity, then it also satisfies No Punishment for Rank-Enhanced Partners and Solidarity under Rank-Enhancements of Partners. That is, both of our properties are weaker than Individually Rational Monotonicity.

The second axiom of Kojima and Manea (2009) is the following.

Weak Maskin Monotonicity: For all $R, R' \in \mathcal{R}_0$, if $H_i(\varphi(R), R'_i) \subseteq H_i(\varphi(R), R'_i)$ for all $i \in F \cup W$, then $\varphi_i(R') \, R'_i \, \varphi_i(R)$ for all $i \in F \cup W$.

It is clear from the above argument that both No Punishment for Rank-

Enhanced Partners and Solidarity under Rank-Enhancements of Partners are even weaker than Weak Maskin Monotonicity.

Thus, as corollaries of our theorems, we obtain the following impossibility results. Note that the results hold on the domain $\mathcal{R}$ as well.

Corollary 2 *Suppose $n \geq 3$. No selection rule from S defined on $\mathcal{R}^*$ satisfies Individually Rational Monotonicity. No selection rule from S defined on $\mathcal{R}^*$ satisfies Weak Maskin Monotonicity.*

The above result in two-sided matching problems should be contrasted with the results by Kojima and Manea (2009) in objects assignment problems that characterized a class of selection rules from stable assignments satisfying Individually Rational Monotonicity and Weak Maskin Monotonicity.

7 Conclusion

In this paper, we have considered "natural" changes in preferences of the agents in matching problems; only the rankings of current partners are enhanced. We have introduced two desirable properties of matching rules under rank-enhancements of partners. One property requires that an agent who becomes higher evaluated by the original partner should not be punished. We have shown that when two or more agents enhance the rankings of the current partners, then the above property cannot always be met if the matchings are also required to be stable. There are cases in which some agents become higher ranked by the partners and yet they get worse off in the new matching. However, if only one agent raises the ranking of the current partner, the above requirement is compatible with stability and moreover with equity as envy-minimization.

The other property under rank-enhancements of partners is the solidarity property, requiring that all of the "irrelevant" agents, whose preferences as well as whose partners' preferences are unchanged, should be affected in the same way; either all better off or all worse off. We have shown that when matchings are required to be stable, this property does not always hold. Some irrelevant agents may be better off while others may be worse off after the preference change.

Our analysis has been confined to the case of one-to-one matchings. However, our impossibility results straightforwardly extend to the more general class of many-to-one matching problems if we do not impose any constraints on the number of workers that each factory should accommodate because the class of one-to-one matching problems is a subclass of this general class.

To consider other desirable properties of matching rules under some "natural" changes of the data and examine their compatibility with basic properties such as stability, individual rationality, and Pareto efficiency may be an interesting topic for future research.

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