Type I and type II errors in the unit root determination of a fractional Brownian motion

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Introduction

Economists who deal with time-series data usually take the unit root test as the 'prerequisite' test for a Brownian motion. It is typical for any researchers to apply a battery of well-known unit root tests to their models to confirm stationarity in the model specification. Nonetheless, often times, we see a conclusion that fail to reject the null in favor of the existence of unit root even though the model specification is such that the lag coefficients of an AR(q) process do not sum up to unity. In this study, we show that having the sum of the lag coefficients equals to unity is indeed a necessary and sufficient condition for the existence of a unit root. Hence, the aforementioned incident will lead to a type II error in the unit root determination. On the other hands, type I error results when we reject the null that there exists a unit root when in fact the null is true. The fractional Brownian motion (fBm) process which has stationary but not necessarily independent increments is used to convey the findings of this study. We use Hurst exponent as a gauge for persistency in the data and show that a fBm process is a legitimate stochastic process with unit root even though it exhibits a degree of persistency in time.

Methods

To tackle the issue, we first derive a proof of two key theorems, showing sufficient and necessary condition that ties the concept of unit root down to a simple mathematical relationship for an AR(q) process. This simple law serves as a benchmark where the results from further simulation testing can be calibrated against. In the second part, we perform empirical study on a known data-generating process under controlled environment using Monte Carlo simulation. A series of fractional Brownian motions (fBm) is employed and calibrated using a scaled variance ratio test to confirm dependency among time increments. Hurst exponent is used as a gauge for persistency in the data. We then show several different cases where the test for a unit root in a fBm process with Hurst exponent significantly different from 0.5 could result in type I and type II errors.

In this study, a price series is generated from an AR(q) process

$$Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + \dots + a_q Y_{t-q} + \varepsilon_t \quad ,$$

where Y_t is the log price at time t and ε is a normally distributed, random innovation term with mean zero and variance σ^2.

To estimate the Hurst exponent on a fBm process, we use the scaled variance ratio method given by

$$\frac{E\left[Y(t+k)-Y(t)\right]^2}{E\left[Y(t+1)-Y(t)\right]^2} = \frac{\sigma_k^2}{\sigma_1^2} = (k)^{2H} \quad .$$

The estimated value of the Hurst exponent is equal to

$$H = \frac{\ln\left(\frac{\sigma_k^2}{\sigma_1^2}\right)}{2\ln(k)} \quad .$$

Results

Sufficient Condition $\sum_{i=1}^{q} a_i = 1 \rightarrow$ existence of unit root

Theorem 1: Any sequence type AR(q), $Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + \dots + a_q Y_{t-q} + \varepsilon_t$, in which the lag coefficients $\sum_{i=1}^{q} a_i = 1$ will have a unit root.

Proof: We can write $a_q = 1 - \sum_{i=1}^{q-1} a_i$. The characteristic polynomial for determining the real and complex roots of AR(q) is generally given by $v^q - \sum_{i=1}^{q-1} a_i v^{q-i} - a_q = 0$, which must be satisfied for any root v.

By definition and substitution, we can write

$$v^q - \sum_{i=1}^{q-1} a_i v^{q-i} + \sum_{i=1}^{q-1} a_i - 1 = 0 .$$

Finally,

$$v^q - \sum_{i=1}^{q-1} a_i \left(v^{q-i} - 1\right) - 1 = 0 .$$

Hence, without ambiguity, $v = 1$ is a solution and we have a unit root.

Necessary Condition existence of unit root $\rightarrow \sum_{i=1}^{q} a_i = 1$

Theorem 2: Any sequence type AR(q), $Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + \dots + a_q Y_{t-q} + \varepsilon_t$, in which the lag coefficients $\sum_{i=1}^{q} a_i \neq 1$ will NOT have a unit root.

Proof: Now, let $\sum_{i=1}^{q} a_i \neq 1$. We can write $\sum_{i=1}^{q} a_i + R = 1$, where $R \neq 0$ is a remainder. The characteristic polynomial for determining the real and complex roots of AR(q) is generally given by $v^q - \sum_{i=1}^{q-1} a_i v^{q-i} - a_q = 0$, which must be satisfied for any root v.

By definition and substitution, we can write

$$v^q - \sum_{i=1}^{q-1} a_i v^{q-i} + \sum_{i=1}^{q-1} a_i - 1 + R = 0 .$$

Finally,

$$v^q - \sum_{i=1}^{q-1} a_i \left(v^{q-i} - 1\right) - 1 + R = 0 .$$

For $v = 1$ as a root, we end up with R = 0 which is a contradiction. Hence, there will be no unit root if $\sum_{i=1}^{q} a_i \neq 1$.

With the two theorems, we can conclude that

$$\sum_{i=1}^{q} a_i = 1 \longleftrightarrow \text{existence of unit root.}$$

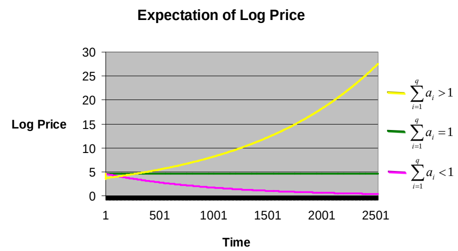

Figure 1. This plot shows comparison among the behavior of the expectation of log price paths for three different AR(q) processes. An autoregressive process with lag coefficients that do not sum up to one will either explode or goes to zero in the limit.

We run Monte Carlo simulation on the AR(q) data-generating process for 10,000 iterations with different combination of lag coefficients and estimate the mean Hurst exponent at different lag (k) for each run. The results are shown in Table 1.

Table 1. Estimated mean Hurst exponent

q	1	2	2	2	2	3	3	3	4
a1	1	0.990	0.750	0.5	0.72	0.2	0.5	1.2	1.5
a2	0	0.009	0.251	0.5	0.28	0.3	0.3	-0.5	0.3
a3	0	0	0	0	0	0.5	0.2	0.3	-1.2
a4	0	0	0	0	0	0	0	0	0.4
sum of a	1	0.999	1.001	1	1	1	1	1	1

Lag	Mean H	Mean H	Mean H	Mean H	Mean H	Mean H	Mean H	Mean H	Mean H
10	0.4986	0.4945	0.3999	0.2811	0.3845	0.1011	0.2549	0.4489	0.9127
20	0.4977	0.4940	0.4199	0.3201	0.4051	0.1535	0.2948	0.4519	0.8825
30	0.4968	0.4932	0.4290	0.3373	0.4138	0.1813	0.3133	0.4542	0.8596
40	0.4961	0.4923	0.4347	0.3475	0.4188	0.1990	0.3246	0.4556	0.8420
50	0.4953	0.4914	0.4386	0.3543	0.4220	0.2115	0.3322	0.4565	0.8280

Type II error

According to Theorem 1 and 2 derived in this study, this process has no unit root. However, the estimated mean Hurst exponents are statistically close to 0.5 for all lags and give this process the appearance of a unit-root process. In this case, a test that fail to reject the null of a unit root will lead to type II error.

Type I error

According to Theorem 1 and 2 derived in this study, this process has a unit root. In fact, the estimated mean Hurst exponent shows that it is a fractal process that is ergodic. In this case, a test that reject the null of a unit root will lead to type I error.

Conclusions

- We prove that any AR(q) process for which the sum of the lagged coefficients equals to one has a unit root, yet for any AR(q>1) process this does not suggest an independent Brownian process in increments.
- We show that a fBm process exhibits a degree of persistency in time, yet it is a legitimate stochastic process and has a unit root.
- Any sequence type AR(q) with lag coefficients that do not sum up to unity will have a limit in probability and time of either infinite or zero.
- Having the sum of the lagged coefficients equals to unity in an AR(q) process is only a necessary but not sufficient condition to ensure stability in modeling long-run economic phenomena.

This finding provides more clarity with respect to a proper model specification for work related to time-series data. An understanding of the fractal nature of a process – the coexistence between random walk and temporary dependence among time increments – is crucial when analyzing data with limited sample size. With this approach, one can make better modeling, investing, and hedging decision by tackling only the relevant risks applicable.

Literature cited

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