TOPOLOGIES OF SOCIAL INTERACTIONS

by

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Abstract

The paper extends the Brock-Durlauf model of interactive discrete choice, where individuals' decisions are influenced by the decisions of others with whom they are in contact (interact), to richer social structures. Social structure is modelled by a graph, with individuals as vertices and interaction between individuals as edges. The paper extends the mean field case to general interaction topologies and examines in detail such stylized topologies like the star, the cycle (or wheel) and the one-dimensional lattice (or path). It explores the properties of Nash equilibria when agents act on the basis of expectations over their neighbors' decisions. It links social interactions theory with the econometric theory of systems of simultaneous equations modelling discrete decisions. The paper obtains general results for the dynamics of adjustment towards steady states and shows that they combine spectral properties of the adjacency matrix with those associated with the nonlinearity of the reaction functions that lead to multiplicity of steady states. When all agents have the same number of neighbors the dynamics of adjustment exhibit relative persistence. Cyclical interaction is associated with endogenous and generally transient spatial oscillations that take the form of islands of conformity, but multiplicity of equilibria leads to permanent effects of initial conditions. The paper also analyzes stochastic dynamics for arbitrary interaction topologies, when agents acts with knowledge of their neighbors' actual decisions, which involve networked Markov chains in sample spaces of very high dimensionality.

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1 Introduction

The paper examines an economy which is populated by individuals whose discrete decisions are influenced by the decisions of other individuals. Interactions among individuals exhibit *spatial* structure that is modelled as a graph, with individuals as vertices and interaction between individuals as edges, and interpreted as *social* structure. The paper explores how patterns in the graph topology may affect the properties of the decisions of agents at equilibrium. E.g., what difference does it make if not all individuals are directly connected? What if all individuals are connected only through a common intermediary (neighbor), in which case the topology of interconnections is a *star*, or if each individual is connected to only two other neighbors, with the topology being a *cycle*? What if the topology forms an one-dimensional lattice (*path*)?

The paper extends the Brock–Durlauf model of interactive discrete choice [Brock and Durlauf (2001)] to richer social structures, which range from a number of stylized topologies, like the ones mentioned above, to arbitrary topologies. It explores the properties of Nash equilibria when agents act on the basis of expectations over their neighbors' decisions. It links social interactions theory with the econometric theory of systems of simultaneous equations modelling discrete decisions. The paper obtains general results for the dynamics of adjustment towards steady states and shows that they combine spectral properties of the adjacency matrix of the underlying graph topology with the properties of individuals' nonlinear reaction functions. Multiplicity of equilibria is possible in both static and dynamic settings. Such multiplicity of equilibria is significant because it may lead to permanent effects of initial conditions. That is, if the economy starts with different individuals making different decisions, such differences across the economy may persist, under certain conditions. In contrast, if the economy starts with individuals making identical decisions, social interactions may push the economy away towards heterogeneous outcomes. When all individuals have the same number of neighbors, the dynamics of adjustment exhibit relative persistence. Cyclical interaction is associated with endogenous and generally transient spatial oscillations that form "islands of conformity": groups of adjacent individuals are more likely to be making similar decisions. The paper also analyzes stochastic dynamics for arbitrary interaction topologies, when agents act with knowledge of their neighbors' actual decisions, which involve networked Markov chains in sample spaces of very high dimensionality.

Some results for arbitrary interaction structures are qualitatively similar to the global interaction case, but a richer class of *anisotropic* equilibria may arise, for some topologies, like the cycle and one-dimensional lattice, in static settings and provided that the interaction coefficients differ. Equilibria with social interactions when individuals react to the actual behavior of their neighbors in the previous period differ qualitatively from those when individuals use the mean of their neighbors' lagged decisions as forecasts of their current ones.

Current interest in the study of social interactions aims *inter alia* at a better understanding of conditions under which interdependence is responsible for multiplicity in conformist behavior. Ellison (1993) and Young (1998) obtain results that rest on individual agents' behavior being affected by the behavior of a *subset* of other agents, rather than of *all* other agents. Ellison (1993) and Glaeser, Sacerdote and Scheinkman (1996) exploit the intrinsic symmetry of the cyclical interaction topology in modelling, respectively, *local* matching and by relying on its tractable analytics when the number of agents is large.

The present paper brings different topologies together under the same overarching model. Its single most important result is that unless individuals located in different positions value differently their interactions with others in the social interaction topology, Nash equilibrium outcomes when individuals make decisions based on the expectation of their neighbors' decisions do not differ from the Brock–Durlauf case in static settings. In dynamic settings and regardless of the rule individuals use to forecast their neighbors' contemporaneous decisions, the dynamics of adjustment and the nature of equilibrium outcomes reflect critically the social interaction topology. Our results extend Brock and Durlauf (2001) in a number of ways and complement contributions by others, including in particular, Horst and Scheinkman (2004), who emphasize continuous decisions and also allow for random topologies in static models, and Bisin *et al.* (2004), who also emphasize continuous decisions with fixed one-sided interactions and allow for dynamics with rational expectations but exclude multiple equilibria. Bala and Goyal (2000), Haag and Lagunoff (2001) and Jackson and Wolinsky (1996) have also addressed bringing together different interaction topologies.

2 Interactive Discrete Choice

Let the elements of a set $\mathcal{I}$ represent individuals. Social interactions among individuals $\mathcal{I}$ are defined by an undirected graph $G(V, E)$, where: V is the set of vertices, $V = \{v_1, v_2, \ldots, v_I\}$, an one-to-one map of the set of individuals $\mathcal{I}$ onto itself, and $I = |V|$ is the number of vertices (nodes), (known as the *order* of the graph); E is a subset of the collection of unordered pairs of vertices and $q = |E|$ is the number of edges, (known as the *size* of the graph). We say that agent i interacts with agent j if there is an edge in $G(V, E)$ between nodes i and j. Let $\nu(i)$ define the local neighborhood of agent $i : \nu(i) = \{j \in \mathcal{I} | j \neq i, \{i, j\} \in E\}$. The number of i's neighbors is the *degree* of node i : $d_i = |\nu(i)|$. Graph $G(V, E)$ may be represented equivalently by its adjacency (a.k.a. acquaintance or sociomatrix) matrix, $\mathbf{\Gamma}$, an $I \times I$ matrix whose element (i, j) is equal to 1, if there exists an edge from agent i and to j, and is equal to 0, otherwise. For undirected graphs, matrix $\mathbf{\Gamma}$ is symmetric and its spectral properties are both well understood and are used extensively below. We use $\mathbf{N}^{-1}$ to denote the diagonal matrix with the inverse of each agent's degree, $\frac{1}{|\nu(i)|}$, as its element (i, i).

2.1 The Brock-Durlauf Interactive Discrete Choice Model with an Arbitrary Interaction Topology

This section adapts the Brock- Durlauf model of interactive discrete choice [Brock and Durlauf (2001); Durlauf (1997)] to arbitrary interaction topologies represented by an arbitrary adjacency matrix Γ. All individuals faces the binary choice set $S = \{-1, 1\}$. Let agent i choose ω_i, $\omega_i \in S$, so as to maximize her utility, which depends on the actions of her neighbors: $U_i = U(\omega_i; \widetilde{\omega}_{\nu(i)})$, where $\widetilde{\omega}_{\nu(i)}$ denotes the vector of dimension d_i containing as elements the decisions made by each of agent i's neighbors, $j \in \nu(i)$. The $I-$vector of all agents' decisions, $\widetilde{\omega}= (\omega_1, \ldots, \omega_I)$, is also known as a *configuration,* and $\widetilde{\omega}_{\nu(i)}$ is known as agent i's *environment.* We assume that an agent's utility function U_i is additively separable in a private utility component, which without loss of generality (due to the binary nature of the decision) may be written as $h\omega_i$, $h > 0$, in a social interactions component, which is written in terms of quadratic interactions between her own decision and of the expectation of the decisions of her neighbors, $\widetilde{\omega}_{\nu(i)}$, $\omega_i \mathcal{E}_i \left\{ \frac{1}{|\nu(i)|} \sum_{j \in \nu(i)} J_{ij}\omega_j \right\}$, and a random utility component, $\epsilon(\omega_i)$, which is observable only by the individual i. That is, U_i may be written as:

$$U_i(\omega_i; \mathcal{E}_i \left\{ \widetilde{\omega}_{\nu(i)} \right\}) \equiv h\omega_i + \omega_i \mathcal{E}_i \left\{ \frac{1}{|\nu(i)|} \sum_{j \in \nu(i)} J_{ij}\omega_j \right\} + \epsilon(\omega_i). \tag{1}$$

The interaction coefficients may be positive, individuals are conformist, or negative, individuals are non-conformist. We define $\mathbf{J}$ as the matrix of interaction coefficients, a $I \times I$ matrix with element J_{ij}, and $\mathbf{I}$ as the $I \times I$ identity matrix. Also, let the column $I-$vector $\widetilde{\varepsilon}$ stack the difference of $2I$ independently and identically type I extreme-value distributed random variables, $\varepsilon_i = \epsilon_i(1) - \epsilon_i(-1)$, written as a column vector, $\widetilde{\varepsilon} \equiv \widetilde{\epsilon}\ (1) - \widetilde{\epsilon}\ (-1)$, and let $\mathbf{1}[\mathcal{R}]$ is a $I-$vector indicator function of the I vector $\mathcal{R}$, with its ith element equal to 1, if the ith element of $\mathcal{R}$, $\mathcal{R}_i > 0$, and is equal to -1, otherwise.

Following Brock and Durlauf (2001)[1] and with $\epsilon(\omega_i)$ being independently and identically type I extreme-value distributed[2] across all alternatives and agents $i \in \mathcal{I}$, individual i chooses $\omega_i = 1$ with probability

$$\text{Prob}(\omega_i = 1) = \text{Prob} \left\{ 2h + 2\mathcal{E}_i \left\{ \frac{1}{|\nu(i)|} \sum_{j \in \nu(i)} J_{ij}\omega_j \right\} \geq -(\epsilon(1) - \epsilon(-1)) \right\}. \tag{2}$$

In view of the above assumptions, this may be written in terms of the logistic cumulative distribution function:

$$\text{Prob}(\omega_i = 1) = \frac{\exp\left[\beta\left(2h + 2\mathcal{E}_i \left\{ \frac{1}{|\nu(i)|} \sum_{j \in \nu(i)} J_{ij}\omega_j \right\}\right)\right]}{1 + \exp\left[\beta\left(2h + 2\mathcal{E}_i \left\{ \frac{1}{|\nu(i)|} \sum_{j \in \nu(i)} J_{ij}\omega_j \right\}\right)\right]}, \tag{3}$$

where $\beta > 0$ is a behavioral parameter that denotes the degree of precision in one response to the random component of private utility, $\epsilon(\omega_i)$ in (1). The case of $\beta = 0$ implies purely random choice, the two outcomes are equally likely, and of $\beta \to \infty$ purely deterministic choice. The extreme-value distribution assumption for the ϵ's is both convenient and links with the machinery of the Gibbs distributions theory [Blume (1997); Brock and Durlauf (2001)].

The state of the economy satisfies the following condition, written in compact notation as:

$$\widetilde{\omega}_i = \mathbf{1}\left[2h\mathbf{I} + 2\mathbf{N}^{-1}\mathbf{J}\mathbf{\Gamma}\mathcal{E}\{\widetilde{\omega}_{\nu(i)}\} + \widetilde{\varepsilon}\right]. \tag{4}$$

[1] McKelvey and Palfrey (1995) and Chen, Friedman and Thisse (1997) also develop game-theoretic discrete choice models with interactive features that are based on the logit model, independently of the social interactions literature.

[2] If two independent and identically distributed random variables, $\epsilon(-1), \epsilon(1)$, obey type I extreme-value distributions, then their difference has a logistic distribution: $\text{Prob}\{\epsilon(-1) - \epsilon(1) \leq x\} = \frac{\exp[\beta x]}{1+\exp[\beta x]}$.

We assume that all agents are identical in terms of preferences but each agent holds expectations of other agents' decisions which are *contingent* on those agents' *position* in the social structure. An equilibrium confirms such expectations:

$$\mathcal{E}_i(\omega_j) = m_j, \forall i, j \in \mathcal{I}. \tag{5}$$

By writing $\mathbf{m}$ for the vector of expectations of decisions, where $m_i = \text{Prob}(\omega_i = 1) - \text{Prob}(\omega_i = -1)$, and using the hyperbolic tangent function, $\tanh(x) \equiv \frac{\exp(x)-\exp(-x)}{\exp(x)+\exp(-x)}$, $-\infty < x < \infty$, we have:

$$m_i = \tanh\left[\beta h + \beta \frac{1}{\nu(i)} \mathbf{J}\mathbf{\Gamma}_i \mathbf{m}\right] \; i = 1, \ldots, I, \tag{6}$$

where $\mathbf{\Gamma}_i$ denotes the ith row of the adjacency matrix. Succinctly, we now have:

Proposition 1. *Under the assumption of location-contingent expectations (5), the system of social interactions with an arbitrary topology (4) admits an equilibrium that satisfies (6).*

Proof. This follows readily from Brower's fixed point theorem. The mapping from $[-1, 1]^I$ into itself, defined by the RHS of (6), has at least one fixed point. □

In the mean field theory case, which is equivalent to global interaction and is considered by Brock and Durlauf (2001), each individual's subjective expectations of other agents' decisions are equal, $\mathcal{E}_i(\omega_j) = m, \forall i, j \in \mathcal{I}$, and the Nash equilibria satisfy (6), which now simplifies to

$$m = \tanh(\beta h + \beta J m). \tag{7}$$

An important implication of these results follows. Consider that all agents have the same number of neighbors, $d = \nu(i)$, that is the graph is *regular*, and the interaction coefficients are equal, $J_{ij} = J$. Then a question arises whether or not equilibria exist with agents' behavior is differentiated by their location on the graph. We call such equilibria *anisotropic* in order to distinguish them from the *isotropic* case, where individuals are not distinguished in this fashion.

For an isotropic equilibrium in the regular graph case, $\frac{1}{d}\mathcal{E}_i\left\{\sum_{\forall j \neq i} \omega_j\right\} = m$, and Equ. (7) holds. Therefore, the regular interaction case admits the same isotropic equilibria as the Brock-Durlauf mean field case.

We summarize briefly results from Durlauf (1997) and Brock and Durlauf (2001) which are critical in appreciating our own. If $\beta J > 1$, and $h = 0$, then the function $\tanh(\beta h + \beta J m)$ is centered at $m = 0$, and Equ. (7) has three roots: a positive one ("upper"), (m_+^*), zero ("middle"), and a negative one ("lower"), (m_-^*), where $m_+^* = |m_-^*|$. If $h \neq 0$ and $J > 0$, then there exists a threshold H^*, which depends on β and J, such that if $\beta h < H^*$, Equ. (7) has a unique root, which agrees with h in sign. In other words, given a private utility difference h, if the dispersion of the random utility component is sufficiently large, the random component dominates choice. If, on the other hand, $\beta h > H^*$, then Equ. (7) has three roots: one with the same sign as h, and the others of the opposite sign. That is, given a private utility difference, if the dispersion of the random utility component is small, then the social component dominates choice and is capable of producing multiplicity in conformist behavior. If $J < 0$, then there is a unique equilibrium that agrees with h in sign. In other words, as Durlauf underscores, Durlauf (1997), p. 88, economic fundamentals that drive private decisions and social norms play complementary roles. When three equilibria exist, we will refer to the middle one (m^*), as *symmetric* and to the upper and lower ones as *asymmetric* (m^*_-, m^*_+) [Figure 1]. [`INSERT FIGURE 1 HERE.`]

We note that the model exhibits nonlinear behavior with respect to parameters h and J. Conditional on a given private utility difference between the choices 1 and -1, there exists a level which the interaction effect must reach in order to produce multiple self-consistent mean choice behavior. However, as βh increases in magnitude, the importance of the conformity effect $\beta J m$ diminishes in a relative sense, and thus becomes unable to produce a self-consistent mean with the opposite sign. Even if private incentives favor a particular decision, sufficiently strong social conformity effects can bring about different social outcomes.

2.1.1 Star Interaction

The star interaction introduces a modicum of asymmetry: agent 1 is located at the center of a star, all other agents are agent 1's neighbors, $\nu(1) = \mathcal{I} - \{1\}$, and they in turn have agent 1 as their only neighbor: $\forall i,\ i \neq 1,\ \nu(i) = \{1\}$. We allow for a possibly asymmetric interaction intensity, by assuming $J_{1i} \equiv J$, $J_{i1} \equiv J_S$, $\forall i \neq 1$, and $J_{ij} = 0$, otherwise. The general expectations assumption (6) implies that individuals $2, \ldots, I$ hold common expectations of individual 1's decision, $\mathcal{E}_i\{\omega_1\} = m_1, i = 2, \ldots I$ The expected decision of a typical such agent $j, j \neq i$, depends only on m_1. Individual 1 acts with rational expectations over all other agents' decisions: $\mathcal{E}_1\{\omega_i\} = m_{-1}$, $i \neq 1$. So, we have:

$$m_{-1} = \tanh(\beta h + \beta J m_1); \tag{8}$$

$$m_1 = \tanh(\beta h + \beta J_S m_{-1}). \tag{9}$$

The equilibria in the economy with star interaction are described by the fixed points of the system of equations (8) and (9). Existence is guaranteed by Proposition 1.[3]

If $J_S = J$, then we revert back to (7), and the equilibrium is isotropic. Otherwise, the general case admits more possibilities than the mean field case. They are summarized in the following proposition, whose proof is straightforward. To visualize the equilibria, consider R^2, with axes designated by (m_1, m_{-1}) and draw the graphs of Equ. (8), (9), from which the solutions follow. See Figure 2. Without loss of generality, we assume that $h > 0$ and recall that $\beta > 0$. [`INSERT FIGURE 2 HERE.`]

Proposition 2. *The system of Equations (9)–(8) admits the following roots for* (m_1, m_{-1}).
(a) If $J, J_S > 0$, *then there exist at least one root that lies in the set* $(0,1) \times (0,1)$. *If, in addition,* $\beta J > 1$, *and* $\beta J_S > 1$, *then there exists a threshold* H^+, *such that if* $\beta h > H^+$, *then there exist two additional roots that lie in* $(-1,0) \times (-1,0)$.

[3] This is a special case of (6), with adjacency matrix $\mathbf{\Gamma}_W = \begin{pmatrix} 0 & \mathbf{1}^{\mathsf{T}}_{I-1} \\ \mathbf{1}_{I-1} & \mathbf{0}_{I-1} \end{pmatrix}$, where $\mathbf{1}_{I-1} = (1, \ldots, 1)$, the column vector of 1's of dimension $I - 1$, and $\mathbf{0}_{I-1}$ the $(I-1) \times (I-1)$ matrix of 0's.

(b) If $J > 0, J_S < 0$*, then there always exist a single root that lies in* $(-1,1) \times (0,1)$.
(c) If $J < 0, J_S > 0$*, then there always exist a single root that lies in* $(0,1) \times (-1,1)$.
(d) If $J < 0, J_S < 0$*, then there exist at least one root that lies in the set* $(0,1) \times (-1,0)$*. If, in addition,* $\beta J < -1$*, and* $\beta J_S < -1$*, then there exists a threshold* H^-*, such that if* $\beta h < H^-$*, then there exist two additional roots that lie in* $(-1,1) \times (0,1)$.

It is interesting that when both interaction coefficients have the same sign the resulting equilibria are characterized by the following properties. If both interaction coefficients are positive then there is always at least one root that is in the positive orthant of (m_1, m_{-1}) space. That is, if agents $2, \ldots, I$, are optimistic about agent 1, and agent 1 optimistic about agents $2, \ldots, I$, the upper equilibrium prevails: conformism is an equilibrium. If both groups of agents are pessimistic about the other group the lower equilibrium is also possible, provided that the interactions effect is sufficiently strong to overcome the private effect. If both interaction coefficients are negative then at least one root is associated with a positive solution for m_1 and a negative one for m_{-1} : conformism is again an equilibrium. When, on the other hand, preferences are different, in the sense that agent 1 does not wish to conform but the agents $2, \ldots, I$ do, then agents $2, \ldots, I$ choose an expected decision in the positive orthant, $m_{-1} > 0$, but agent 1 might not, m_1 may be positive or negative. If, on the other hand, agent 1 does not wish to conform but agents $2, \ldots, I$ do, agents $2, \ldots, I$ choose an expected decision in the positive orthant, $m_1 > 0$, but agent 1 might not, m_{-1} may be positive or negative.

2.1.2 Cyclical (or Wheel) Interaction

Cyclical interaction occurs when each agent interacts only with two neighboring agents, $\nu(i) = \{i-1, i+1\}$, $\nu(1) = \{I, 2\}$, $\nu(I) = \{I-1, 1\}$. From the discrete choice model generated by (1) we have: $U_i(\omega_i) \equiv h\omega_i + \omega_i \frac{1}{2}\mathcal{E}_i \left\{J_B\omega_{i-1} + J_F\omega_{i+1}\right\} + \epsilon(\omega_i)$. Let agent i's expectation of the decision by her neighbor to her left be m_{i-1} and to her right be m_{i+1}. Then, for consistency, agent i's expected decision, m_i, must be equal to that implied by her own decision, and may be expressed in terms of m_{i-1} and m_{i+1}:

$$m_i = \tanh[\beta h + \frac{1}{2}\beta(J_B m_{i-1} + J_F m_{i+1})], i = 1, \ldots, I. \quad (10)$$

This a system of I equations in the I unknowns m_i, $i = 1, \ldots, I$, with "circular symmetry" conditions $m_{I+1} = m_1$, and $m_{1-1} = m_I$.[4] By Proposition 1 an equilibrium exists. We examine next whether an anisotropic equilibrium exist.

If we impose that all agents' expectations of others' decisions be equal, then the Nash equilibria *isotropic* equilibria with circular interaction coincide with those of the mean field case for $J = \frac{1}{2}(J_B + J_F)$. Proposition 3, whose proof is available in the working paper version of the paper, details that *anisotropic* equilibria exist only if the backward and forward interaction coefficients are different: $J_B \neq J_F$.

[4] Alternatively, this is the special case of (6) with an adjacency matrix Γ_C an $I \times I$ circulant matrix generated by permutations of the $I-$vector $(0, 1, 0, \ldots, 1)$.

Proposition 3. *The system of Equations (10), $i = 1, \ldots, I$, that describe cyclical interaction may have, in general, two classes of equilibria, isotropic and anisotropic ones.*
(a) If $J_B = J_F$, all equilibria are isotropic. $m_i = m$, $\forall i$, and satisfy

$$m = \tanh\left[\beta h + \frac{1}{2}\beta(J_B + J_F)m\right]. \tag{11}$$

Their properties are identical to the mean field case. There may be either three distinct roots, or a single root, that all lie in $(-1, 1)$.
(b) If $J_B \neq J_F$, then anisotropic equilibria, if they exist, are given by the fixed points of $\Theta_{[I-2]}$, the $I - 2$th iterate of a mapping $\Theta = (\Theta^1, \Theta^2)$, defined as follows:

$$\begin{aligned} \Theta^1(m', m'') &\equiv \frac{1}{\frac{1}{2}\beta J_F}\left[\tanh^{-1}(m') - \beta h\right] - \frac{J_B}{J_F}m'', \\ \Theta^2(m', m'') &\equiv m', \end{aligned} \tag{12}$$

where $(m', m'') \in (-1, 1) \times (-1, 1)$, and $\tanh^{-1}(\cdot)$ denotes the inverse hyperbolic tangent function.
(c) Anisotropic solutions may exist for the case of $I = 3$ agents and are given by the fixed points of $\Theta_{[1]}$, the first iterate of Θ.
(d) Anisotropic solutions do not exist in the case of either only backward interaction, $J_F = 0$, or only forward interaction, $J_B = 0$.
(e) Anisotropic solutions do not exist for any I, if $J_B = J_F$.

The possibility of an anisotropic solution rests, in effect, on the properties of the iterates of the function $\tanh(\cdot)$. When $J_B = J_F$, the fixed points of the iterates of $\tanh^{-1}(\cdot)$ coincide with those of $\tanh^{-1}(\cdot)$ itself.

2.1.3 Path Interaction

With the set of agents being defined as $\mathcal{I} = \{-L, \ldots, 0, \ldots, L\}$, the equilibrium conditions for agents $-L + 1 \leq i \leq L - 1$, are as in (10), $-L + 1 \leq i \leq L - 1$. For agents $-L$ and L, the equilibrium conditions are:

$$m_{-L} = \tanh[\beta h + \beta J_F m_{-L+1}]; \tag{13}$$

$$m_L = \tanh[\beta h + \beta J_B m_{L-1}]. \tag{14}$$

Equ. (10), for $i = -L+1, \ldots, L-1$, and Equ. (13) and (14), form a system of $2L+1$ equations in the $2L + 1$ unknowns, the expected states of all agents. This is a special case of (6), with an adjacency matrix the $(2I + 1) \times (2I + 1)$ matrix, whose first and last row are $(0, 1, \ldots, 0)$ and $(0, 0, \ldots, 1, 0)$ respectively, and rows 2 through $2I$ are the $2I+1$ vector $(1, 0, 1, \ldots, 0)$ and its permutations. Unlike the cyclical interaction case, the presence of the end agents destroys circular symmetry.[5]

[5]To see this intuitively, consider the classic example from Schelling (1978): "If everybody needs 100 Watts to read by and a neighbor's bulb is equivalent to half one's own, and everybody has a 60-Watt bulb, everybody can read as long as he and both his neighbors have their lights on. Arranged on a circle, everybody will keep his lights on if everybody else does (and nobody will if his neighbors do not); arranged in a line, the people at the ends cannot read anyway and turn their lights off, and the whole thing unravels" [*ibid.* p. 214]. One would expect that the

We improve our intuition on path interaction by also considering the special case with just three agents, $L = 1$. For the three-agent case, $L = 1$, Equ. (13) yields m_{-1} as a function of m_0, and Equ. (14) yields m_1 as a function of m_0. By substituting back into the R.H.S. of (10) we obtain a single equation in m_0. Under the assumption that the interaction coefficients are both positive, we can easily see that the multiplicity of equilibria rests on the properties of the iterates of the function $\tanh(\cdot)$, whose fixed points coincide with its own. Our results are summarized in the following proposition, whose proof is given in the working paper version of the paper.

Proposition 4. *The system of Equations (10), $-(L-1) \leq i \leq L-1$, and Equations (13) and (14) that describe interaction on a path defines a mapping in R^I into itself:*

$$\mathcal{C} : R^I \to \underbrace{(-1,1) \times \ldots (-1,1)}_{I}.$$

It has two classes of solutions, isotropic and anisotropic ones.
(a) There exist in general isotropic equilibria, $m_i = m^*_{LI}$, $-(L-1) \leq i \leq L-1$, which are the roots of (7) with $J = \frac{1}{2}\beta(J_B + J_F)$; and m_{-L} and m_L, are given from (13) and (14), respectively, as functions of m^*_{LI}. There may be either three distinct roots, or a single root, that all lie in $(-1, 1)$.
(b) For $L = 1$, the case of three agents, m_0 satisfies

$$m_0 = \tanh\left(\beta h + \frac{1}{2}\beta J_B \tanh[\beta h + \beta J_F m_0] + \frac{1}{2}\beta J_F \tanh[\beta h + \beta J_B m_0]\right); \tag{15}$$

m_{-1}, m_1, follow from (13) and (14), as before. Equ. (15) has, depending upon parameter values, either three distinct roots, of which one has the same sign as h and the other two with the opposite sign, or a single root with the same sign as h. Furthermore, $m_{-1} = m_0 = m_1$, only if $J_B = J_F$.
(c) For $L > 1$, or for an even total number of agents, there exists an one-parameter family of solutions.

2.2 Remarks

First, we note that if we assume that all interaction coefficients in all of the above models are equal to one another, $J_P = J_B = J_F = J_S = J$, and agents hold position-independent expectations, then the equilibria of all models coincide with those of the Brock-Durlauf mean field case, Equ. (7). While not surprising, it will be useful to bear it in mind when we discuss equilibrium with interactions that depend on agents' actual environments, in static and dynamic settings, sections 2.3 and 3.1, respectively.

Second, at any of the isotropic equilibria examined above, which are associated with individuals' making decisions conditional on their expectations of their neighbors' decisions, individuals' states are described as independent Bernoulli distributed random variables, whose parameters, however,

importance of the two end agents would wane as the number of agents increases and as preferences differ from the fixed-proportions case implied by the above example. Also, the above example suggests that it might be important to allow for boundary conditions, as when the end agents, or the agent at 0, are constrained to be in a particular state.

may be in general functions of the interaction topology. These are defined by the respective choice probabilities as functions of the mean values at equilibrium. All models share the property that the social equilibrium may be characterized by aggregate uncertainty, even when individuals are subject to purely random shocks. That is, consider the case when $h = 0$, the two states equally likely in terms of fundamentals. Then, even in the mean field case, the economy has three isotropic equilibria, associated with the roots of Equ. (7), for $h = 0$. One of them is $m = 0$, and indeed implies no aggregate uncertainty: the expected outcome is equal to 0. However, this social equilibrium is unstable. If $\beta J > 1$, then the other two roots imply expected outcomes different from 0, and the corresponding equilibria are stable. Naturally, the emergence of aggregate activity is due to the synergistic effects $\beta J m$ [*c.f.* Kirman (1993)].

Third, since anisotropic equilibria will be excluded by the assumptions that we will maintain in the remainder of the paper, it is important to stress that they model the consequences of a basic lack of symmetry in the economy. The islands of conformity that would appear, which become clearer below in the analysis of circular interaction when agents make decisions based on their neighbors' actual decisions, are not completely random; they are instead skewed.

2.3 An Econometric Interpretation

The model of discrete decisions with social interactions, when utilities depend on the actual realizations of neighbors' decisions, as in (4) with the expectation operator removed from the RHS, admits an interpretation as an econometric model of simultaneous equations involving discrete decisions.[6] In fact, the earlier literature on structural models of discrete choice, such as Schmidt (1981) and others, emphasizes conditions for internal consistency, also known as "coherency" conditions. Such conditions guarantee that given the values of exogenous variables, observed and unobserved, unique values for the dependent variables are implied and the associated likelihood functions are well defined. For example, the consistency conditions proposed by Schmidt (1981), when applied to (4), reduce to the condition that the model be recursive [*ibid.,* Condition 12.6, p. 429]. To see this, consider the case of star interaction. It follows that not all principal minors of $\mathbf{N}^{-1}\mathbf{J}\boldsymbol{\Gamma}$ are equal to 0, and therefore, Condition 12.6, *ibid.,* is not satisfied.[7]

The recent literature on estimating models of discrete games allows for multiple equilibria. Tamer (2003) emphasizes that if one does not insist on coherency, then econometric models of social interactions may accommodate economic models with multiple equilibria. Imposing coherency eliminates multiplicity of equilibria. That is, one wants to know whether unique or multiple social outcomes are associated with a given set of parameter values and values of stochastic shocks. Therefore, what may be desirable of some regression models, like those examined in *ibid.,* may not always be desirable of models of social interactions. That is, recursiveness may be undesirable in general, because it would imply that a single agent's decisions would determine those of all others'. This may well be a feature of certain social interaction settings but should not be required of all.

[6] I am grateful to Chuck Manski for directing my attention to this interpretation.

[7] Neither is the system recursive for the cases of complete, cyclical, and path interaction topologies.

Multiplicity of equilibria may be interesting in their own right, but its econometric consequences are still poorly understood. Some additional observations are in order. First, enumeration of multiplicity of equilibria is appropriate for finite numbers of agents. Second, any expressions for the probabilities of particular outcomes for each agent are the equilibrium value for the *marginal* probability of agents' decisions. Therefore, such probabilities cannot be specified independently in a regression setting. They are no longer in the logit form. Third, as Soetevent (2003) underscores, the multiplicity of equilibria is entirely due to the social interactions component of individual decisions that introduces simultaneity. Fourth, it is tempting but actually not appropriate, as a referee pointed out, to interpret the model represented by (4) with the expectations in the RHS removed, as a simultaneous move game. But, as we see further below, such a model facilitates the derivation of the stationary distribution in dynamic settings.

3 Dynamic Analysis of Social Interactions

In a dynamic setting, agents employ their best responses based on their forecasts of neighbors' decisions and employ position-contingent strategies, given an *arbitrary* interaction topology. I wish to distinguish the role of expectations from the impact of the topology of interactions and of the nonlinearity of the model. I start with the general case where each agent uses as her forecast the *actual* state of her neighbors in the previous period, $\mathcal{E}_i\{\widetilde{\omega}_{\nu(i),t}\} = \widetilde{\omega}_{\nu(i),t-1}$, This *realized-action best response* dynamic is novel in the context of the social interactions literature. I provide a general description that establishes the existence of a stationary equilibrium distribution. I also provide specific results for the case of cyclical interactions and of the star. I examine the vastly simpler case when agents' forecasts of their neighbors' decisions are equal to their neighbors' *mean* choices in the previous period, $\mathcal{E}_i\left\{\omega_{j,t}\right\} = m_{j,t-1},\ j \in \nu(i)$.

3.1 Realized-action Best Response Dynamics for Arbitrary Topologies

Let the actual state of the economy at time t be denoted by $\widetilde{\omega}_t$. Under the best response dynamic I employ, each agent responds at time t, given her neighbors' *actual* decisions at time $t-1$, $\widetilde{\omega}_{\nu(i),t-1}$, defined as the subvector of $\widetilde{\omega}_{t-1}$ that pertains to agent i's neighbors. That is, adapting Equ. (3) in a dynamic setting, we have: $\text{Prob}(\omega_{i,t} = 1)|\widetilde{\omega}_{\nu(i),t-1}) = \frac{\exp\left[\beta\left(2h+2\frac{1}{|\nu(i)|}\sum_{j\in\nu(i)} J_{ij}\omega_{j,t-1}\right)\right]}{1+\exp\left[\beta\left(2h+2\frac{1}{|\nu(i)|}\sum_{j\in\nu(i)} J_{ij}\omega_{j,t-1}\right)\right]}$. By adopting the concise notation of Equ. (4), the above may be written as:

$$\widetilde{\omega}_t = \mathbf{1}\left[2h\mathbf{I} + 2\mathbf{N}^{-1}\mathbf{J}\boldsymbol{\Gamma}\,\widetilde{\omega}_{t-1} + \widetilde{\varepsilon}_t\right]. \tag{16}$$

Therefore, the state of the economy in period t is well defined in terms of time $t-1$, $\widetilde{\omega}_{t-1}$ and the contemporaneous shock $\widetilde{\varepsilon}_t$. This notation underscores that the interaction topology affects the dynamics via the properties of the adjacency matrix $\boldsymbol{\Gamma}$ and of the matrices containing the number of each agent's neighbors, $\mathbf{N}^{-1}$, and the interaction coefficients, $\mathbf{J}$.

Let us consider the state of the economy in two successive periods. For each of the 2^I possible realizations of $\widetilde{\omega}_{t-1}$, $\widetilde{\omega}_{t-1} \in \{-1,1\}^I$, Equ. (16) defines conditional choice probabilities for each agent in each of the models of social interaction, which are in effect transition probabilities for each of the 2^I possible realizations of $\widetilde{\omega}_t$, given $\widetilde{\omega}_{t-1}$. The dynamic counterparts of Equ. (7), (8)–(9) and (10) are special cases of this definition. The state of the economy evolves according to a Markov stochastic process which is defined from the finite sample space $\{-1,1\}^I$, into itself and has fixed transition probabilities. That is, the transition probability from $\widetilde{\omega}_{t-1} = \widetilde{\omega}'$ to $\widetilde{\omega}_t = \widetilde{\omega}''$ is equal to $\prod_{i=1}^I \mathrm{Prob}\left\{\omega_{i,t} = \widetilde{\omega}_i'' \mid \widetilde{\omega}_{\nu(i),t-1} = \widetilde{\omega}'_{\nu(i)}\right\}$. These probabilities sum to 1, when the sum is taken over all possible realizations of $\widetilde{\omega}''$, for any given $\widetilde{\omega}'$.

Taking cues from by Asavathiratham (2000) allows us to transform the general case (16) to a form that is amenable to analysis by means of standard linear algebra for Markov processes. To see how this may be done, consider representing the state of agent i instead of the binary set of outcomes $\{-1,1\}$, by the row vector $(1\ 0)$, if $\omega_i = 1$, and by the row vector $(0\ 1)$, if $\omega_i = -1$. In this fashion, a realization $\widetilde{\omega}$ may be represented by a row vector $(1\ 0|0\ 0|\cdots|0\ 0)$ with $2I$ elements, with each two of them representing the state of an agent. By stacking up all possible realization vectors we obtain a $2^I \times 2I$ matrix, known as the event matrix [Asavathiratham (2000), p. 109], which represents all possible states of the economy. The RHS of Equ. (16) allows us to define a $2^I \times 2^I$ transition matrix, to be denoted by $\mathcal{H}$, which is a stochastic matrix that expresses the transition probabilities for the Markov process that describes the evolution of the state of the economy from $\widetilde{\omega}_{t-1}$ to $\widetilde{\omega}_t$, given by (16). If the stochastic matrix $\mathcal{H}$ is irreducible, then its Perron-Frobenius (dominant) eigenvalue is equal to 1 with the 2^I-unit vector as the corresponding right eigenvector. Next, we define the column vector Ψ_t, with 2^I rows, and entries all zeroes, except one, which is 1 and indicates that the economy is at the corresponding state, that is associated with the respective row of the event matrix. Accordingly, we may define a probability distribution over the possible states of the economy, the rows of the event matrix. Clearly, a stationary probability distribution over the states of the economy, if it exists, is given by the positive left eigenvector of $\mathcal{H}$, $\tilde{\Psi}$, corresponding to eigenvalue 1 and suitably normalized, that satisfies $\tilde{\Psi}^{\mathsf{T}}\mathcal{H} = \tilde{\Psi}^{\mathsf{T}}$ [Seneta (1981), Theorem 4.1, p. 119]. As Asavathiratham discusses [*op. cit.,* Theorem 5.7, p. 109], the stationary distribution might not be unique in general, if the corresponding dominant eigenvalue that is equal to 1 does not have an algebraic multiplicity of 1. Because of the nature of the interaction topology, some of the states of the economy may not communicate, and the process might not have a unique recurrent class.[8]

The stationary distribution associated with an economy where agents act with knowledge of their neighbors' actual decisions coincides with the equilibrium distribution in the static case when interactions are based on agents' actual environments, discussed in section 2.3 above. The intuitive appeal of this claim follows from the definition of the agents' optimal decisions and the associated probability distributions that they imply.[9] We turn to special cases that are amenable to exact

[8]A synthetic review of the literature and some simulation results for large economies may be found in Verbrugge (2003). Of particular interest is his case of *effectively non-ergodic response*.

[9]For a proof, one may adapt the classic treatment of Preston (1974), p. 10–18.

solution.

3.1.1 Cyclical Interaction under Realized-Action Best Response Dynamic

When the interaction structure is *translation-invariant,* that is when only relative distance between agents matter and not their actual locations, and if the process is *reversible,* that is, the interaction structure is symmetric so that the process may be reversed, just like a movie being played backwards and still making sense, then an important result follows. Bigelis *et al.* (1999) establishes the existence of a stationary distribution and the closed form it actually assumes. By following the notation in *ibid.*, p. 3936, and writing the interaction effect from j to i as $J(i-j)$, we have:

Proposition 5. *Let the total number of agents I be finite and Λ be a d-dimensional torus containing L^d lattice sites (that is, Λ is a cube in Z^d containing L^d points and having periodic boundaries). If the Markov process for $\widetilde{\omega}_t$, defined on the configuration space $\{-1,1\}^{\Lambda}$, with fixed transition probabilities, given by:*

$$\text{Prob}\left\{\omega_{i,t}=\varpi|\,\widetilde{\omega}_{t-1}\right\}=\frac{1}{2}\left[1+\varpi\tanh\left(\beta\sum_{j\in\Lambda}J(i-j)\omega_{j,t-1}+\beta h\right)\right],\ \varpi\in\{-1,1\},\tag{17}$$

is reversible, for which it suffices that $J(i-j)=J(-i+j)$, then the process (17) has a unique stationary distribution given by: $\text{Prob}\{\widetilde{\omega}\}=\bar{\Pi}^{-1}\prod_{i\in\Lambda}e^{\beta h\omega_i}\cosh\left(\beta\sum_{j\in\Lambda}J(i-j)\omega_j+\beta h\right)$, *where $\bar{\Pi}$ is a normalizing constant.*

It is clear from the above expression for the stationary distribution that when individuals are conformist, neighbors' making similar decisions strengthen the likelihood that an agent would conform. The cases of global interactions and circular interactions with symmetric interaction coefficients, $J_{i-1}=J_{i+1}$, are translation-invariant, and under the assumption of reversibility, they become special cases of the above theory.

3.1.2 Star Interaction under Realized-Action Best Response

The star model, although not translation-invariant, may be handled from first principles as follows. From (16), $\text{Prob}\left\{\Omega_{-1,t}=K|\omega_{1,t-1}=\omega_1\right\}=\mathcal{B}\left(\frac{1}{2}(I+K-1),I-1;\text{Prob}\left\{\omega_{i,t}=1|\omega_{1,t-1}\right\}\right)$, $K\in\{-I+1,-I+3,\ldots,I-1\}$, where $\Omega_{-1,t}=\sum_{j=2}^{I}\omega_{j,t}$ and $\mathcal{B}(\cdot)$ denotes the value of the binomial distribution for $\frac{1}{2}(I+K-1)$ successes in $I-1$ Bernoulli trials, with each success having probability given by $\text{Prob}\left\{\omega_{i,t}=1|\omega_{1,t-1}\right\}=\frac{\exp[\beta h+\beta J\omega_{1,t-1}]}{1+\exp[\beta h+\beta J\omega_{1,t-1}]}$; $i=2,\ldots,I$. These equations define the transition probabilities of a Markov process in terms of the pair of discrete-values random variables, $(\omega_{1,t},\Omega_{-1,t})$, where defined over the sample space $\{-1,1\}\times\{-I+1,-I+3,\ldots,I-1\}$. The stationary probability distribution for the state of agent 1, $\text{Prob}\{\omega_1=1\}$ is given by:

$$\begin{aligned}&\text{Prob}\left\{\omega_1=1\right\}\\ =&\frac{\sum_k\text{Prob}\left\{\omega_1=1|\sum_{i=2}^{I}\omega_i=k\right\}\cdot\text{Prob}\left\{\sum_{i=2}^{I}\omega_i=k|\omega_1=-1\right\}}{1-\sum_k\text{Prob}\left\{\omega_1=1|\sum_{i=2}^{I}\omega_i=k\right\}\cdot\left[\text{Prob}\left\{\sum_{i=2}^{I}\omega_i=k|\omega_1=1\right\}-\text{Prob}\left\{\sum_{i=2}^{I}\omega_i=k|\omega_1=-1\right\}\right]}.\end{aligned}\tag{18}$$

The state of the economy is given by: $\text{Prob}\{\omega_1, \omega_2, \dots, \omega_I\} = \text{Prob}\{\omega_1\} \prod_{i=2}^{I} \text{Prob}\{\omega_i|\omega_1\}$, where $\text{Prob}\{\omega_1\}$ is given from (18) and $\text{Prob}\{\omega_i|\omega_1\}$ from the transition equation above.

3.2 Dynamics with Expectations Based on Lagged Mean Decisions of Neighbors

The assumption that each individual's expectation of her neighbors' choice at time t is equal to those agents' *mean* choice at time $t-1$ allows us to examine dynamics for general arbitrary interaction topologies by means of nonlinear deterministic difference equations. Working with a dynamic adaptation of (4), with each individual's expectation of her neighbors' choice at time t being equal to those agents' *mean* choice at time $t-1$, we have:

$$m_{i,t} = \tanh\left[\beta h + \beta \mathbf{N}^{-1} \mathbf{J} \mathbf{\Gamma}_i \mathbf{m}_{t-1}\right],\ i = 1, \dots, I, \tag{19}$$

where $\mathbf{\Gamma}_i$ denotes the ith row of the adjacency matrix, and $\mathbf{m}_t$ denotes the $I-$vector consisting of the $m_{i,t}$'s. Clearly, the steady states of (19) coincide with the fixed points of (6). The mean field case of Brock and Durlauf, *op. cit.*, readily follows as a special case: $m_t = \tanh(\beta h + \beta J m_{t-1})$.

It is interesting to contrast the dynamics under realized-action best response, expressed by (16), with the case when agents use their neighbors' lagged mean decisions as forecasts of their actions, expressed by (19). As a referee pointed out, the former is the probabilistic law of motion that determines the probabilities of the next state of the economy conditional on the current one, when the topology of social interaction is "drawn" once and remains fixed. The latter is a deterministic law that describes the evolution of the position-contingent mean state. The social interaction graph may be drawn from a random population *each* period and holds when all individuals adopt position-contingent strategies. Both reflect the interaction topology. It is an open question how to bridge the gap between the two, perhaps by means of techniques similar to the ones employed by Blume and Durlauf (2003), who show that the limit distribution of a dynamic Brock–Durlauf model clusters around the stable steady states of the counterpart to (19).

The stability of the system is characterized by the following proposition, which utilizes standard results based on the theory of dynamical systems and the Perron-Frobenius Theorem for symmetric positive matrices and holds for any social interactions topology.

Proposition 6.

Part I. *If the topology of interaction is represented by a regular graph, whose degree is $|\nu(i)| = d$ and otherwise arbitrary, and a constant interaction coefficient J, then the following hold.*

(a) The economy's steady states satisfy Equ. (7) and are isotropic. They are generically hyperbolic fixed points of the $I-$order dynamical system defined by Equ. (19).

(b) Small perturbations around a steady state $\widehat{\mathbf{m}}$ of (19), $\mathbf{\Delta m}_t = \mathbf{m}_t - \widehat{\mathbf{m}}$, satisfy:

$$\mathbf{\Delta m}_t = \beta \tau \mathbf{N}^{-1} \mathbf{J} \mathbf{\Gamma} \mathbf{\Delta m}_{t-1}, \tag{20}$$

where $\tau := \tanh'\left[\beta h + \beta \frac{1}{|\nu(i)|} J \mathbf{\Gamma}_i \widehat{\mathbf{m}}\right]$.

(c) If the graph is regular, $d = |\nu(i)|$, a necessary and sufficient condition for the local dynamics of (20) at a steady state $m_i = m$ to be stable (unstable) is:

$$\beta J \tanh'(\beta h + \beta J m) < (>) 1. \tag{21}$$

Part II. A necessary and sufficient condition for the local dynamics of (20) at a steady state $m_i = m$ to be stable (unstable) is (21), even for an economy with an arbitrary topology of social interactions.

Proof.

Part I.

(a) This part follows from the discussion in subsection 2.1. That they are all isotropic follows from symmetry. To see this, assume that for agent i, her neighbors are at nonisotropic steady states. By iterating forward with respect to i we reach a contradiction. Therefore, when Equ. (19) is taken at a steady state, $\mathbf{\Gamma}_i \mathbf{m} = dm_i$ and $m_i = m^*$, where m^* is a solution of Equ. (7) is a solution. It is easy to see that it is the only solution.

(b) This part is also obvious after we linearize around a uniform steady state and apply matrix notation.

(c) Since the adjacency matrix $\mathbf{\Gamma}$ is symmetric and positive, it has real eigenvalues and a non-negative maximal eigenvalue, whose magnitude absolutely exceeds all other eigenvalues. This eigenvalue is "squeezed" between the average degree of G and is maximal degree, which in our case is equal to $I - 1$ [Cvetkovic, *et al.* (1995), 381–382]. However, all agents have the same number of neighbors, therefore, the maximal eigenvalue is equal to d.

When the economy is at a steady state with all m_i's assuming either one of the asymmetric values m^*_-, m^*_+, or the symmetric value m^*, as defined in subsection 2.1 above, if they exist, then τ simplifies to become equal to $\tanh'[\beta h + \beta J m]$. Condition (21) follows as a necessary and sufficient condition for the solution of (20) to be stable (unstable). It would be stable if $m_i = m^*_-, m^*_+$, and unstable if $m_i = m^*$, $\forall i$. Brock and Durlauf, *op. cit.,* pp. 12–14, is a special case.

Part II. The proof follows from the fact that the row elements of the matrix consisting of rows $\frac{1}{|\nu(i)|}\mathbf{\Gamma}_i$ are positive and sum up to 1. Therefore, it is a stochastic matrix, whose maximal eigenvalue is equal to 1. Condition (21) holds, provided that its l.h.s. is evaluated at the appropriate steady state, at least one of which exists, by Part 2.

Q.E.D. □

A number of remarks are in order. First, by Theorem 3.33, p. 104, Cvetković *et al., op. cit.*, the eigenvector associated with the maximal eigenvalue is $(1, 1, \ldots, 1)^{\mathbf{T}}$. As a result, the role played by this eigenvector in the qualitative discussion of circular interaction pertains to *all* social interactions settings represented by *regular* graphs, that is where all agents have the same number of neighbors. In other words, the relative persistence result that we identify below with the circular interaction topology actually applies to all *regular* interaction topologies. Second, isotropic steady states for the entire economy would be either stable, as when $m = m^*_-, m^*_+$, or unstable, $m = m^*$. Third, the nature of the time map suggests that the dynamical system (19) possesses no periodic orbits. Fourth, the analysis following Equ. (19) above may be extended, in particular, to the case where the interaction topology results from a random graph, where any two agents may be connected with probability p [Erdös and Renyi (1960)], provided that we normalize by means of the largest possible degree, that is the size of the interaction graph, I. From Cvetković *et al.* (1988), p. 79, it follows that the largest eigenvalue of the adjacency matrix grows according to pI, as I tends to infinity. The above existence and stability results carry over to an economy whose interaction topology results from pure randomness. Fifth, as the properties of the spectrum of certain classes of interaction topologies are known [Cvetković *et al., op. cit.*], one could obtain specific results for such classes, as confirmed by treating stylized topologies separately. Sixth, the mean field case, Brock and Durlauf, *op. cit.,* yields: $\Delta m_t = \beta J \tanh'(\beta h + \beta J m^*)\, \Delta m_{t-1}$. Therefore, local stability depends entirely upon the magnitude and sign of $\beta J \tanh'(\beta h + \beta J m^*)$, exactly as in (21) above. The steady states exhibit "symmetry breaking": the symmetric one is unstable, $\beta J \tanh'(0) > 1$, while the asymmetric ones are stable: $\beta J \tanh'(\beta h + \beta J m_+^*) < 1$; $\beta J \tanh'(\beta h + \beta J m_-^*) < 1$. Seventh, the importance of nonlinearity is underscored by contrasting with linear models, even in the unbalanced graph case, as in Brueckner and Smirnov (2004),p. 10. In that case, the system converges to a uniform outcome, which is defined as a weighted sum of the initial positions multiplied by the left eigenvalue of $\mathbf{N}^{-1}\mathbf{J}\boldsymbol{\Gamma}$.

We see below that with symmetry, $J = J_S$, for the star interaction, and $J_F = J_B$, for the cyclical and path interactions, the above equations imply the same steady state equation, which coincides with that of the Brock-Durlauf mean field theory case, Equ. (7). To understand how the interaction topology affects the dynamics of adjustment to a steady state even when we do impose symmetry, we need to perturb the steady state equilibrium.

It should be noted that although individuals may be in either of the two realizations of the discrete state, the dynamic adjustment was defined earlier in terms of the expected choice of neighbors in the previous period. In that case, when self-consistent expectations exist in the Brock-Durlauf model, then Proposition 6, *ibid.,* guarantees that the sample average population choice converges weakly to the self-consistent expectation, the equilibrium solutions of (7). Blume and Durlauf (2003) examine the stochastic dynamics of this model. They show that when individuals revise their choices at independent random times given by a "Poisson" clock by looking at all other agents' lagged choices and if their number is large, then the mean decision obeys the continuous-time counterpart of the mean-field case.

3.2.1 Dynamics of Star Interaction

Adapting (19) to this topology yields $m_{1,t} = \tanh(\beta h + \beta J_S m_{-1,t-1})$, and $m_{-1,t} = \tanh(\beta h + \beta J m_{1,t-1})$. Linearization of the equations around a steady state (m_1^*, m_{-1}^*) yields a two-dimensional system:

$$\begin{bmatrix} \Delta m_{1,t} \\ \Delta m_{-1,t} \end{bmatrix} = \begin{bmatrix} 0 & \beta J_S \tau_1 \\ \beta J \tau_{-1} & 0 \end{bmatrix} \begin{bmatrix} \Delta m_{1,t-1} \\ \Delta m_{-1,t-1} \end{bmatrix}, \tag{22}$$

where $\tau_1 \equiv \tanh'(\beta h + \beta J m_{-1}^*)$ and $\tau_{-1} \equiv \tanh'(\beta h + \beta J m_1^*)$.

Working in the usual fashion, the eigenvalues of the matrix in the R.H.S. of (22), satisfy $\lambda^2 = \tau_1\tau_{-1}\beta^2 JJ_S$. Therefore, if $JJ_S > 0$, and since $\tau_1, \tau_{-1} > 0$, then the eigenvalues are real and have equal magnitudes but opposite signs: $|\lambda| = \beta\sqrt{\tau_1\tau_{-1}JJ_S}$. If, on the other hand, $JJ_S < 0$, then the eigenvalues are conjugate imaginary. In either case, we have that if the economy is near an asymmetric (symmetric) equilibrium for the agents outside the center and for the agent in the center, $\beta\tau_1|J_S| < 1$, and $\beta\tau_{-1}|J| < (>)1$, the eigenvalues have absolute values less (greater) than 1, and the dynamic adjustment is stable (unstable). Finally, if the economy is near an asymmetric equilibrium for the agents outside the center and the symmetric one for the agent in the center, $\beta\tau_1|J_S| < 1$, and $\beta\tau_{-1}|J| > 1$, the eigenvalues may have absolute values greater than 1, depending upon parameter values, and the dynamic adjustment would be unstable. The presence of factor $\beta\sqrt{\tau_1\tau_{-1}}$ carries the impact of the nonlinearity of the dynamics for the linearized model. Again, the instability of the symmetric steady state is more pronounced the larger are $\beta\tau_1$ and $\beta\tau_{-1}$.[10] The general solution of (22) when $JJ_S > 0$ is given by:

$$\begin{bmatrix} \Delta m_{1,t} \\ \Delta m_{-1,t} \end{bmatrix} = \lambda^t \begin{bmatrix} A_1 + A_{-1}(-1)^t \\ \sqrt{\frac{J\tau_{-1}}{J_S\tau_1}}\left(A_1 - A_{-1}(-1)^t\right) \end{bmatrix}, \tag{23}$$

where A_1, A_{-1} are constants that reflect initial conditions. For example, if $\Delta m_{1,0} = 1, \Delta m_{-1,0} = 0$, $A_1 = A_{-1} = \frac{1}{2}$. Clearly, the solution reflects dampened cob-web like oscillations. Oscillations also occur if $JJ_S < 0$.

3.2.2 Dynamics of Cyclical Interaction

Adapting (19) to this topology yields: $m_{i,t} = \tanh\left(\beta h + \frac{1}{2}\beta(J_B m_{i-1,t-1} + J_F m_{i+1,t-1})\right), i = 1, \ldots, I$. Linearization of this equation around its isotropic steady state m^*, that is, one of the solutions of (11), yields:

$$\begin{aligned} \Delta m_{i,t} = {} & \tanh'\left(\beta h + \frac{1}{2}\beta(J_B m^*_{i-1} + J_F m^*_{i+1})\right) \\ & \times \left[\frac{1}{2}\beta J_B \Delta m_{i-1,t-1} + \frac{1}{2}\beta J_F \Delta m_{i+1,t-1}\right], i = 1, \ldots, I. \end{aligned} \tag{24}$$

Let $\mathbf{\Delta m}_t := (\Delta m_{1,t}, \ldots, \Delta m_{I,t})^{\mathbf{T}}$; This above system, for the case of $J_F = J_B = J$, may be written, equivalently as:

$$\mathbf{\Delta m}_t = \beta\tau^C J \cdot \frac{1}{2}\mathbf{\Gamma}_C \cdot \mathbf{\Delta m}_{t-1}, \tag{25}$$

where $\tau^C := \tanh'\left(\beta h + \frac{1}{2}\beta(Jm^*_{i-1} + Jm^*_{i+1})\right)$. The adjacency matrix $\mathbf{\Gamma}_C$ has as rows the permutations of $(0, 1, 0, \ldots, 1)$. The general solution to equation (25) can be written, in the usual fashion

[10] We note that the dynamics of the model continue to reflect properties of both the topology of interaction and the nonlinearity. This is somewhat obscured by the fact that the product of the eigenvalues of the adjacency matrix, $-(I-1)$, cancels out because of the division by $I - 1$.

for linear systems and after a number of tedious steps[11], as a linear combination of the eigenvectors, each multiplied by its respective eigenvalue raised to the power of t, and weighted by arbitrary constants, which are determined by initial conditions. Specifically, let us assume that I is odd and therefore write:

$$[\Delta m_{1,t}, \ldots, \Delta m_{I,t}]^{\mathtt{T}} = A_1(\beta J\tau^C)^t\,[1,\ldots,1]^{\mathtt{T}} + A_2\left(\beta J\tau^C\cos\left(\frac{2\pi}{I}\right)\right)^t\left[\cos\left(\frac{2\pi}{I}\right),\cos\left(\frac{4\pi}{I}\right),\ldots,1\right]^{\mathtt{T}}$$
$$+A_3\left(\beta J\tau^C\cos\left(\frac{2\pi}{I}\right)\right)^t\left[\sin\left(\frac{2\pi}{I}\right),\sin\left(\frac{4\pi}{I}\right),\ldots,0\right]^{\mathtt{T}} + \ldots\ , \tag{26}$$

where $A_1, \ldots,$ denote constants which are computed from initial conditions.

To see the implications of this model, consider that the system is originally at a steady state equilibrium when it is shocked at time 0 by changing agent 1's decision, say $\Delta m_{1,0} = 1,\ \Delta m_{2,0} =$

[11] We have in closed form, from Brockwell and Davis (1991), p. 133–138, and Anderson (1971), Theorem 6.5.2, p. 279–281, the eigenvalues and eigenvectors of $\frac{1}{2}$ times the adjacency matrix for the cyclical interaction, $\mathbf{\Gamma}_C$. Specifically, if I is *odd*, the eigenvalues of this matrix are

$$\left\{\cos 0 = 1, \cos\left(\frac{2\pi}{I}\right), \cos\left(\frac{2\pi}{I}\right), \cos\left(\frac{4\pi}{I}\right), \cos\left(\frac{4\pi}{I}\right), \ldots, \cos\left(\frac{(I-1)\pi}{I}\right), \cos\left(\frac{(I-1)\pi}{I}\right)\right\}.$$

The eigenvalues come in pairs, so that there are $1+\frac{I-1}{2}$ distinct roots. The eigenvector corresponding to the eigenvalue 1 is $(1,1,\ldots,1)^{\mathtt{T}}$, and to the eigenvalues $\cos\frac{j2\pi}{T}$, $j \neq 0, \frac{I}{2}$, there correspond the eigenvectors

$$\left(\cos\left(j\frac{2\pi}{I}\right), \cos\left(j\frac{4\pi}{I}\right), \ldots, 1\right)^{\mathtt{T}},$$
$$\left(\sin\left(j\frac{2\pi}{I}\right), \sin\left(j\frac{4\pi}{I}\right), \ldots, 0\right)^{\mathtt{T}},$$

if I is odd. If I is *even*, the eigenvalues are

$$\left\{\cos 0 = 1, \cos\left(\frac{2\pi}{I}\right), \cos\left(\frac{2\pi}{I}\right), \cos\left(\frac{4\pi}{I}\right), \cos\left(\frac{4\pi}{I}\right), \ldots, \cos\left(\frac{(I-2)\pi}{I}\right), \cos\left(\frac{(I-2)\pi}{I}, \cos\pi = -1\right)\right\}.$$

The eigenvector corresponding to the eigenvalue 1 is $(1,1,\ldots,1)^{\mathtt{T}}$, and to the eigenvalues $\cos\frac{j2\pi}{T}$, $j \neq 0, \frac{I}{2}$, the same as above. The eigenvector corresponding to the eigenvalue -1, is $(-1,1,-1,\ldots,1)^{\mathtt{T}}$.

Instead of this approach, we may work, following Turing (1952) and Glauber (1963)[12], from first principles and seek a general solution for Equ. (24) in the form $A\zeta^i\rho^t$, where A is a constant and ρ, ζ are unknown, generally complex, numbers to be determined. Then we substitute into Equ. (24) to find its general solution for the special case of $J_B = J_F = J$. We note that for cyclical symmetry, it must be the case that $\zeta^I = 1$. In other words, the complex number ζ must assume the values of the I basic complex roots of 1, that is: $\left(1, \exp[\frac{2\pi}{I}\sqrt{-1}], \ldots, \exp[\frac{2\pi(I-1)}{I}\sqrt{-1}]\right)$. We then substitute into Equ. (24) and obtain the eigenvalues: $\rho = \tau^C J\zeta^{-1} + \tau^C J\zeta$, where $\tau^C := \tanh'\left(\beta h + \frac{1}{2}\beta(Jm^*_{i-1} + Jm^*_{i+1})\right)$. Proceeding in this fashion is rather tedious, however, because we need to transform back into real quantities.

This solution technique was employed by Turing (1952). A special case of Glauber's model, the "infinite ring" case, that is when $I \to \infty$, is solved by Ellis (1985), Theorem V. 10.4, p. 190–203. The circle model, studied in depth by Eisele and Ellis (1983), gives rise to some features which are absent from the Curie-Weiss model, namely a new kind of phase transition described in terms of random waves.

The general properties of the dynamics of differential equations of this type have received attention in the literature. Notable are the contributions of Hirsch (1982), who shows that such systems of ordinary differential equations have bounded solutions that converge to steady states or to periodic orbits. Mallet-Paret and Smith (1990) allow only backward feedback and show that the Poincaré–Bendixson theorem holds, roughly speaking, for such equations. Elkhader (1992) studies the general properties of bounded orbits of deterministic systems of differential equations roughly corresponding to our cyclical interaction case. His is the only paper that allows for backward *and* forward feedback. Our results are broadly consistent with those in *ibid.*, who shows that the omega limit set of such orbits contains a steady state or a nonconstant periodic orbit.

$\ldots, \Delta m_{I,0} = 0$. We thus have I equations (26) in the unknown constants $A_1, \ldots, A_I$, that reflect the initial conditions. These equations may be solved uniquely since the I eigenvectors span the space.

The solution (26) implies oscillatory behavior with spatial variation. If the factor $\beta J \tau^C$ were not present, as in the linear case, as t tends to ∞, the system would tend to the eigenvector corresponding to the maximal eigenvalue, which in that case would be 1. The corresponding eigenvector is $(1, 1, \ldots, 1)^{\mathrm{T}}$. The economy would exhibit persistence, in that case. However, the factor $\beta J \tau^C$ changes this. The amplitude of the oscillations has a maximum given by the maximal eigenvalue $\beta J \tau^C$. The oscillations range from $\beta J \tau^C$ to $-\beta J \tau^C$, depending upon the position of an agent on the circle. That is, because the eigenvectors form an orthonormal set, all eigenvectors other than the one consisting of ones contain negative terms. If, as we assumed, the economy starts from an isotropic equilibrium, then the eventual pattern of states would be determined by whether the starting isotropic equilibrium is a stable or unstable one. If all agents start from a stable equilibrium, then these amplitudes all have absolute values less than 1, because the eigenvectors form an orthonormal set. It follows that the amplitudes of the oscillations diminish over time, and the economy tends to return to the same isotropic steady state that it started from. If, on the other hand, the economy starts from the unstable isotropic equilibrium, $\beta J \tau^C > 1$. Because the magnitude of the oscillations varies spatially, depending upon the position of an agent on the circle, and is bounded upwards by $\beta J \tau^C$, even when the economy starts from the unstable equilibrium, some agents would not be changing their decisions.

The system (24) can be studied further, even if $J_F \neq J_B$, because the matrix is circulant and its diagonalization is accomplished by means of the Fourier matrix that diagonalizes all circulant matrices [Davis (1979), p. 73; Brockwell and Davis (1991), p. 135] The solution in the general case of $J_F \neq J_B$, involves, intuitively, two "trains" of I oscillatory terms with different amplitudes, indexed by the individuals and modulated by time-varying amplitudes, which move in opposite directions around the circle. In either case, that is, in either the general or the specific case, the spatial fluctuations can be interpreted as cluster emergence.

Our results are reminiscent of Danny Quah's findings on cluster emergence in continuous spatial settings, which occurs for reasons that are identical to ours [Quah (2000)].[13] Aside from Quah's use of continuous space and time, the present model has a key implication that is entirely due to the multiplicity of steady state equilibria. Once disturbed, all agents in the system will ultimately return to a steady state, which may be either the upper or the lower, as the symmetric one is unstable. While after the symmetric steady state equilibrium is disturbed, adjustment to a new steady state is associated, as in Quah's case, with spatial clustering, where some individuals may end up in the positive and other in the negative steady states. However, unlike Quah's case, clustering here is permanent.

[13] Technically, the similarity of these results originates in the fact that for consistency, the solution must obey cyclical symmetry, which brings us to circulant matrices, whose eigenvalues involve the complex roots of 1. In contrast, Quah works with Toeplitz operators, which are the continuous time and space counterparts of circulant matrices.

3.2.3 Small versus Large Neighborhoods

Does the speed of adjustment vary with neighborhood size? We can address this question by extending the range of local interactions in the model so that agent i is influenced by agents $\{i-L_n,\ldots,i-1,i+1,\ldots,i+L_n\}$, $L_n \leq I-1$. We set $J_B = J_F$. The counterpart of Equ. (19) implies that the cyclical interaction model with larger neighborhoods possesses the same steady states as the solutions of (11). The counterpart of Equ. (24) becomes:

$$\Delta m_{i,t} = \tanh'\left(\beta h + \frac{1}{2L_n}\beta J(\sum_{\ell=1}^{L_n}[m^*_{i-\ell}+m^*_{i+\ell}])\right)$$

$$\times\beta J\frac{1}{2L_n}\left[\sum_{\ell=1}^{L_n}[\Delta m_{i-\ell,t-1}+\Delta m_{i+\ell,t-1}]\right], i=1,\ldots,I. \tag{27}$$

The counterpart of Equ. (25) for Equ. (27) involves a real symmetric *circulant* matrix $\mathbf{\Gamma}_{CN}$, whose eigenvalues and eigenvectors are known in closed form [Proposition 4.5.1, p. 134–135, Brockwell and Davis (1991)]. [14] Let us define $\tau_S \equiv \left(\beta h + \frac{1}{2L}\beta J(\sum_{\ell=1}^{L}[m^*_{i-\ell}+m^*_{i+\ell}])\right)\beta J$. The eigenvalues and eigenvectors of matrix $\frac{1}{2}\mathbf{\Gamma}_{CN}$ again involve sine and cosine terms. The maximal eigenvalue is equal to 1, and the corresponding eigenvector is $(1,1,\ldots,1)$. The remaining eigenvalues come in pairs, if I is odd, or there are $\frac{I-2}{2}$ pairs of double eigenvalues and an additional distinct one, if I is even.

Dynamics for more general cyclical interaction cases may be studied even when the backward and forward interaction coefficients differ, as long as the pattern of dependence gives rise to a circulant matrix. The presence of both local and global interactions, that is where agent i is influenced by agents $\{i-1,i+1\}$ and by the mean state of all agents, can also be handled. It may be put in the above form and its dynamic analysis involves a circulant matrix.

3.2.4 Dynamics of Path Interaction

Adapting (19) for the path interaction model yields:

$$m_{-L,t} = \tanh\left(\beta h + \beta J_F m_{-L+1,t-1}\right); \tag{28}$$

$$m_{i,t} = \tanh\left(\beta h + \frac{1}{2}\beta(J_B m_{i-1,t-1} + J_F m_{i+1,t-1})\right), -L+1 \leq i \leq L-1. \tag{29}$$

$$m_{L,t} = \tanh\left(\beta h + \beta J_B m_{L-1,t-1}\right). \tag{30}$$

[14] From Davis (1979), p. 72–73, we have that all circulant matrices of the same order have the same set of (right) eigenvectors, the columns of F^*, *ibid.*, 32. Let a circulant matrix be defined by rows being permutations of $(c_1,c_2,\ldots,c_I)$. Its eigenvalues are complex and given by $\lambda_j = \phi\left(\frac{2\pi}{I}(j-1)\right)$, $j=1,2,\ldots,I$, where

$$\phi\left(\frac{2\pi}{I}(j-1)\right) \equiv c_1 + c_2\exp[\sqrt{-1}\frac{2\pi}{I}(j-1)] + \ldots + c_I\exp[\sqrt{-1}\frac{2\pi}{I}(j-1)(I-1)].$$

It turns out that equilibrium is still characterized by "spatial waves," similar to the cyclical interaction case. Linearization of Equ. (28) – Equ. (30), around an isotropic equilibrium, and writing the solution in the standard fashion for the deviations from an isotropic steady state[15] yields:

$$
\begin{aligned}
&[\Delta m_{-L}(t),\ldots,\Delta m_0(t),\ldots,\Delta m_L(t)]^{\mathtt{T}} = A_{-L}\left(\frac{1}{2}\beta J\tau^L\cos\left(\frac{\pi}{2(L+1)}\right)\right)^t \\
&\left[\sin\left(\frac{\pi}{2(L+1)}\right),\ldots,\sin\left(\frac{\pi 2}{2(L+1)}\right),\ldots,\sin\left(\frac{\pi(2L+1)}{2(L+1)}\right)\right]^{\mathtt{T}} \\
&+A_{-L-1}\left(\frac{1}{2}\beta J\tau^L\cos\left(\frac{\pi 2}{2(L+1)}\right)\right)^t\left[\sin\left(\frac{\pi 2}{2(L+1)}\right),\sin\left(\frac{\pi 4}{2(L+1)}\right),\ldots,\sin\left(\frac{\pi 2(2L+1)}{2(L+1)}\right)\right]^{\mathtt{T}}+\ldots \quad ,
\end{aligned}
\tag{31}
$$

where , $A_{-L},\ldots,A_L$, denote constants which are computed from initial conditions.

There is an important, though subtle, difference from the circular interaction case: the maximal eigenvalue of the dynamical system associated with path interaction is equal to $\cos\left(\frac{\pi}{2(I+1)}\right)$ and thus less than 1. The dynamics are characterized by spatial oscillations that are again transitory but there is no relative persistence. However, the importance of this fact vanishes asymptotically, as $L \to \infty$. In fact, the case of interactions along an infinite line is particularly interesting and fortunately, lends itself to explicit treatment.

When we let the number of agents tend to infinity, the role of the end agents vanishes asymptotically. When we linearize around a stable isotropic steady state and use $z-$ tranform techniques,

[15]By linearizing respectively for agents $-L$, $-(L-1),\ldots,L-1$, and L,

$$\tau^{-L} := \tanh'\left(\beta h + \beta J_F m^*_{-(L-1)})\right),$$

$$\tau^{L} := \tanh'\left(\beta h + \frac{1}{2}\beta(J_B m^*_{i-1} + J_F m^*_{i+1})\right),$$

$$\tau^{+L} := \tanh'\left(\beta h + \beta J_B m^*_{L-1})\right).$$

For the symmetric case where $J_F = J_B = J$, and given that we linearize around an isotropic equilibrium, we have that: $\tau^{-L} = \tau^{+L} = \tau^L$. We express Equ. (28) – Equ. (30) as a system of $2L+1$ equations in matrix form:

$$
\begin{bmatrix} \Delta m_{-L}(t) \\ . \\ \Delta m_0(t) \\ . \\ \Delta m_L(t) \end{bmatrix} = \beta J\tau^L\frac{1}{2}
\begin{bmatrix} 0 & 1 & 0 & . & 0 & 0 & 0 \\ 1 & 0 & 1 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & 0 & 1 \\ 0 & 0 & 0 & . & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} \Delta m_{-L}(t-1) \\ . \\ \Delta m_0(t-1) \\ . \\ \Delta m_L(t-1) \end{bmatrix}.
$$

The matrix in the RHS above is no longer a circulant. However, the eigenvalues and eigenvectors of $\frac{1}{2}$ times this adjacency matrix have been studied by Anderson, *op. cit.,* p. 290, Equ. (62). The adjacency matrix is an $(2I+1)\times(2I+1)$ matrix, whose first and last rows are $(0,1,\ldots,0)$ and $(0,0,\ldots,1,0)$ respectively and rows 2 through $2I$ are the $2I+1$ vector $(1,0,1,\ldots,0)$ and its permutations. Its eigenvalues are given by: $2\cos\left(\frac{\pi}{2(I+1)}i\right)$, $i=1,\ldots,2I+1$; the corresponding eigenvectors are given by

$$\left(\sin\left(\frac{\pi}{2}\frac{s}{L+1}\right),\sin\left(\frac{\pi}{2}\frac{2s}{L+1}\right),\ldots,\sin\left(\frac{\pi}{2}\frac{s(2L+1)}{L+1}\right)\right)^{\mathtt{T}}, s=1,\ldots,2L+1.$$

This result allows us to solve the above system in the standard fashion and obtain Equ. (31) in the main text.

[16] the term $\left(\frac{1}{2}\beta J\tau^L\right)^t$ tends to 0 as t tends to ∞. As the "disturbance" propagates through social interactions over time, it has a transient effect on each agent: the change originally increases, reaches a peak and then decreases. This response is like two blips that move in opposite directions away from agent 0. However, the symmetry and the setting and intuition from the spectral theory of random fields suggest that this infinite path case should be equivalent to the infinite circle case.

In concluding the analysis of local interaction, as represented by cyclical interaction and by path interaction, we wish to underscore, once again, important similarities and differences. Dynamics in both cases involves spatial oscillations. The cyclical interaction case involves exhibits relative persistence, which is due to the regularity of the social interactions topology.

4 Extensions

Several possible extensions come to mind. It is possible to study the evolution of the second moments of individuals' decisions. It would be interesting to examine the impact of the spatial extent of interactions upon the speed of adjustment. Ellison (1993) and Young (1998) have emphasized the importance of *local* interaction for the speed of adoption of norms. They show that when individuals interact mainly with small groups of neighbors, then "the smaller the size of the neighborhood groups, and the more close-knit they are, the faster the transition time for the whole population " [Young, *op. cit.,* 98–99].

Several ways in which the model may be extended are noteworthy. One is a production interpretation, where interactions suggest synergies between agents. Second, interactions may model

[16] It is convenient to apply the $z-$transform to the entire system of equations (29).This is the discrete-time counterpart of the treatment in continuous time by Glauber, *op. cit.*, for the linear ring case. We define $\mathcal{L}(z,t)$, the $z-$transform of the sequence of deviations $\mathcal{D}M(t) = \{\Delta m_i(t)\}_{i=-\infty}^{i=\infty}$:

$$\mathcal{L}(z,t) = \sum_{i=-\infty}^{i=\infty} z^i \Delta m_i(t).$$

It follows that by multiplying both sides of (24) by z^i and summing up for all i's, we have:

$$\mathcal{L}(z,t) = \frac{1}{2}\beta J\tau^L(z + z^{-1})\mathcal{L}(z,t-1).$$

This may be solved to yield

$$\mathcal{L}(z,t) = \mathcal{L}(z,0)\left(\frac{1}{2}\beta J\tau^L\right)^t z^{-t}\left(z^2+1\right)^t.$$

$\mathcal{L}(z,t)$ may be obtained as power series by noticing that the term $\left(z^2+1\right)^t$ may be written in terms of the binomial expansion formula.

Intuitively, $\mathcal{L}(z,0)$ carries the impact of initial conditions. We may solve for $\mathcal{L}(z,t)$ by assuming a set of initial conditions. Suppose, for example, that all deviations at 0 are equal to 0 except for $\Delta_0(0) = 1$. In that case, $\mathcal{L}(z,0) = 1$, and $\mathcal{L}(z,t) = \left(\frac{1}{2}\beta J\tau^L\right)^t \sum_{k=0}^t \frac{t!}{k!(t-k)!} z^{2k-t}$. The solutions for $\Delta_i(t)$ may be recovered from $\mathcal{L}(z,t)$ in the obvious way, as the coefficients of the powers of the z^i's. That is, the solution for $\Delta m_i(t)$ is given by the coefficient of z^i in the power expansion for $\mathcal{L}(z,t)$. Writing the terms of the summation in the r.h.s. of the above yields: $z^{-t} + tz^{2-t} + \frac{(t-1)t}{2}z^{4-t} + \frac{(t-2)(t-1)t}{6}z^{6-t} + \ldots + tz^{t-2} + z^t$. Since for every t, the binomial coefficient in the r.h.s. of the above increases with k, reaches a maximum, and then declines, the impact on the coefficients of the powers of z depends on the magnitude of $\left(\frac{1}{2}\beta J\tau^L\right)^t$, as well, which may be increasing or decreasing over time, depending upon the which particular isotropic equilibrium we start from.

trade among agents (nations). Puga and Venables (1997) explore the impact upon welfare from the creation of free trade areas, which would correspond to our complete pairwise interactions case, in contrast to "hub-and-spoke" arrangements, whereby a country liberalizes bilateral trade with several other countries, with barriers remaining among those other countries. They show that the "topology" of patterns of trade arrangements (interactions) does matter. In view of such potential applications, it would be interesting to see the impact of interactions patterns on the persistence of center vs. periphery type phenomena among countries engaging in liberalizing trade and the sequence with which they actually entered into such arrangements. Third, it would be interesting to further explore econometrics with models of interacting agents, where agents to choose whom to interact with [*c.f.,* Bala and Goyal (2000)] and to allow for preference heterogeneity [*c.f.*, Cont and Lowe (2003)].

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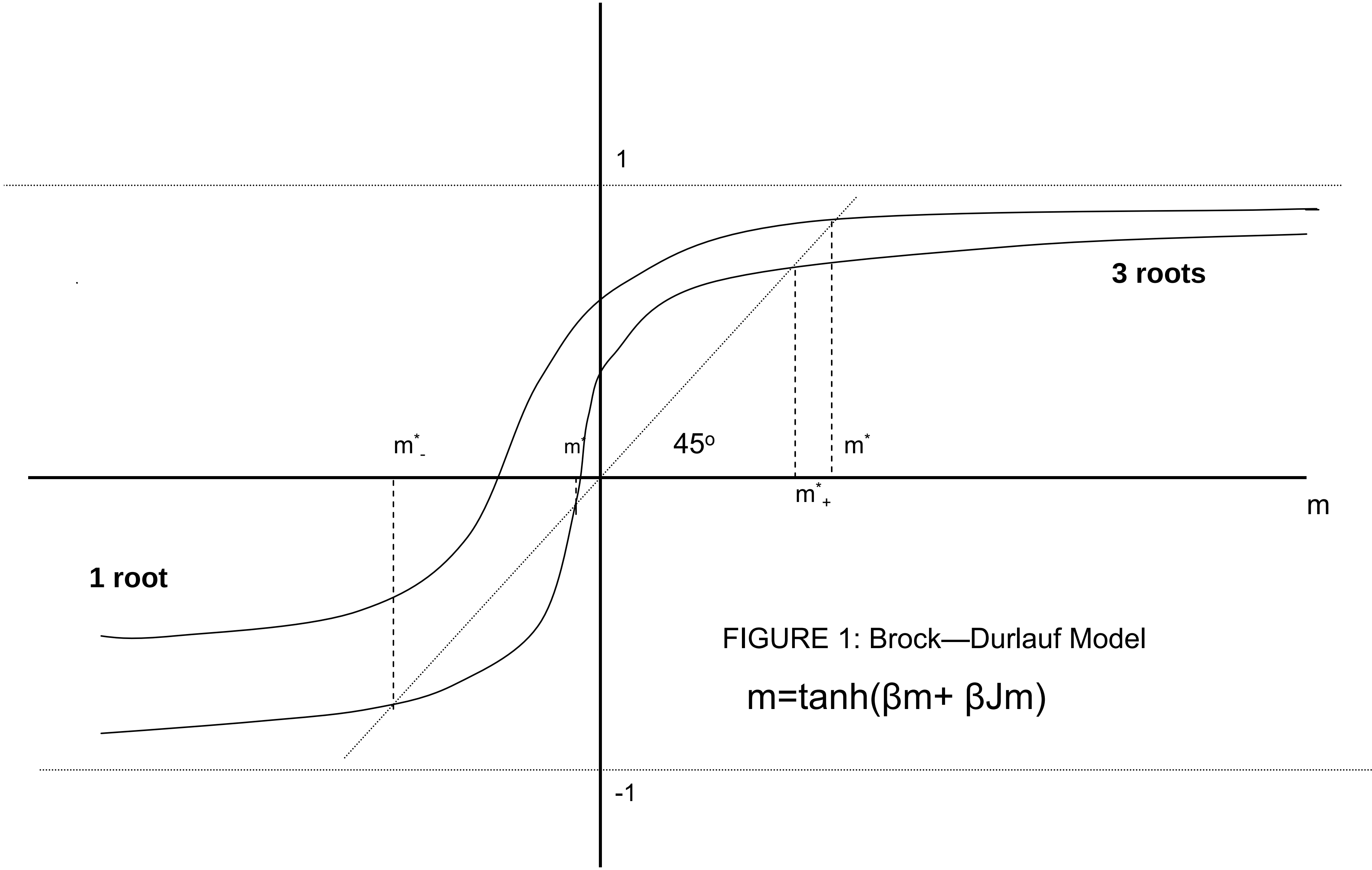

FIGURE 1: Brock—Durlauf Model

$$m=\tanh(\beta m+ \beta Jm)$$

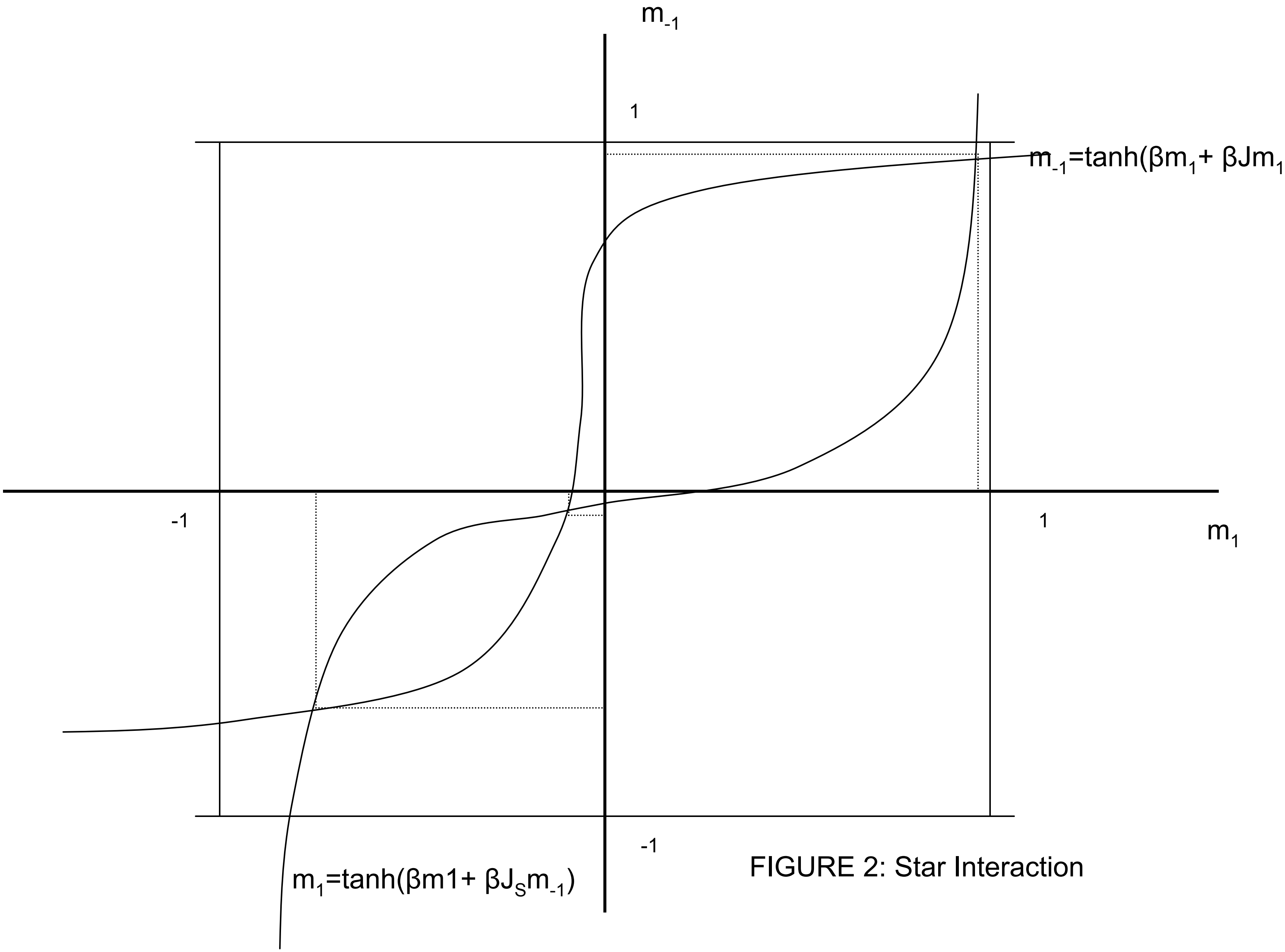

FIGURE 2: Star Interaction