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The Non-Linear Cournot Model as a Best-Response Potential Game*

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Abstract

We show that the Cournot oligopoly game with non-linear market demand can be reformulated as a best-response potential game where the best-response potential function is linear-quadratic.

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1 Introduction

A recent stream of literature has focussed upon the so-called potential games (Monderer and Shapley, 1996), using the concept of potential function borrowed from physics. The essential idea behind these concepts is that the information about Nash equilibria can be nested into a single real-valued function on the strategy space. In order to clarify the aim of our paper, we briefly summarize the definition and properties of potential games.

Consider a game $\Gamma \triangleq \left\langle N, (S_i)_{i\in N}, (\pi_i(s))_{i\in N} \right\rangle$, where $N \triangleq (1, 2, 3, ...n)$ is the set of players, S_i is the strategy space for player i and $\pi_i(s)$ is the payoff function attributing a payoff to i in correspondence of any given admissible outcome s. In differentiable games, where S_i is an interval of real numbers for all $i \in N$, the function $P(s)$ is an *exact* potential function for the game Γ iff $P(s)$ is continuously differentiable and $\partial\pi_i(s)/\partial s_i = \partial P(s)/\partial s_i$ for all $i \in N$ (Monderer and Shapley, 1996, Lemma 4.4, p. 134). Additionally, if the payoff functions are twice continuously differentiable, Γ is a potential game iff $\partial^2\pi_i(s)/\partial s_i\partial s_j = \partial^2\pi_j(s)/\partial s_j\partial s_i$ for all $i, j \in N$ (Monderer and Shapley, 1996, Theorem 4.5, p. 135).

The latter property holds, e.g., in the Cournot oligopoly game with linear market demand, which is therefore an exact potential game (see Slade, 1994, p. 52; and Monderer and Shapley, 1996, pp. 124-125), while it is not met in oligopoly models with non-linear demand (see Anderson and Engers, 1992, *inter alia*). However, even if $\partial^2\pi_i(s)/\partial s_i\partial s_j \neq \partial^2\pi_j(s)/\partial s_j\partial s_i$, one can still show that the game has a *best-response* potential function $\widehat{P}(s)$ under a milder requirement, namely that

$$\arg\max_{s_i\in S_i} \pi_i(s) = \arg\max_{s_i\in S_i} \widehat{P}(s) \quad (1)$$

for all $i \in N$ and for all admissible strategy profiles s. If $\widehat{P}(s)$ takes a maximum over $S \triangleq S_1 \times S_2 \times ...S_n$ in correspondence of $s^* \triangleq (s_1^*, s_2^*, ...s_n^*)$, then the game Γ has a Nash equilibrium in s^*. (Voorneveld, 2000, Definition 2.1 and Proposition 2.2, p. 290).

In the next section, we outline the Cournot oligopoly game with non-linear market demand (Anderson and Engers, 1992) as a best-response potential game, showing that it has a linear-quadratic form. This result, which here we present as an example based on a static game, has appealing applications to dynamic games whose strategic form is not linear-quadratic and therefore, in principle, cannot be solved analytically to yield feedback equilibria.

2 The game

As in Anderson and Engers (1992), we consider a Cournot oligopoly where n firms sell a homogeneous good whose market demand function is:

$$Q = 1 - p^{\alpha} \ , \ \alpha > 0 \ . \tag{2}$$

The above function is always downward sloping, and can be either convex ($\alpha \in (0,1)$) or concave ($\alpha > 1$). Hence, the inverse demand function is:

$$p = (1 - Q)^{\frac{1}{\alpha}} \ , \tag{3}$$

where $Q = \sum_{i=1}^{n} q_i$, and q_i is the individual output of firm i. Production costs are normalised to zero. The individual profit function is therefore equal to $\pi_i = pq_i$.

It is trivial to check that, for all $j \neq i$:

$$\frac{\partial^2 \pi_i}{\partial q_i \partial q_j} \neq \frac{\partial^2 \pi_j}{\partial q_j \partial q_i} \forall \alpha \neq 1. \tag{4}$$

Consequently, for all $\alpha \neq 1$, the Cournot game with non linear demand is not a potential game. However, we can prove the following:

Proposition 1 *For all $\alpha \neq 1$, the Cournot game with non linear demand is a best-response potential game with a linear-quadratic best-response potential function:*

$$\widehat{P} = \sum_{i=1}^{n} \frac{q_i}{2}\left[2\alpha\left(1-\sum_{i\neq j} q_j\right) - q_i\left(1+\alpha\right)\right] + \sum_{i\neq j} \alpha q_i q_j .$$

Proof. Essentially, the proof consists in showing that the present Cournot game meets the conditions formulated by Voorneveld (2000, p. 290). To this aim, we proceed as follows. As a first step, observe that the Cournot-Nash first order condition

$$\frac{\partial \pi_i}{\partial q_i} = \frac{\left(1-Q\right)^{\frac{1-\alpha}{\alpha}}\left[\alpha\left(1-q_i-\sum_{i\neq j} q_j\right)-q_i\right]}{\alpha} = 0 \tag{5}$$

is met when $\alpha\left(1-q_i-\sum_{i\neq j} q_j\right) - q_i = 0$, which yields the equilibrium quantity $q^N = \alpha/\left(1+\alpha N\right)$. The other solution to (5), $q = 1/N$, can be dismissed as it implies $p = \pi = 0$.

Accordingly, define

$$\frac{\partial \widehat{\pi}_i}{\partial q_i} \triangleq \alpha\left(1-q_i-\sum_{i\neq j} q_j\right) - q_i \tag{6}$$

as the fictitious first order condition of firm i. Let

$$\widehat{\pi}_i\left(q_i, Q_{-i}\right) \triangleq \int \frac{\partial \widehat{\pi}_i}{\partial q_i} dq_i = \frac{q_i}{2}\left[2\alpha\left(1-\sum_{i\neq j} q_j\right) - q_i\left(1+\alpha\right)\right] \tag{7}$$

be the corresponding fictitious profit function of i.

If we take the summation $\widehat{\Pi} = \sum_{i=1}^{n} \widehat{\pi}_i\left(q_i, Q_{-i}\right)$ and differentiate it w.r.t. q_i, we obtain

$$\frac{\partial \widehat{\Pi}}{\partial q_i} = \alpha\left(1-q_i-\sum_{i\neq j} q_j\right) - q_i - \sum_{j\neq i} \alpha q_j = \frac{\partial \widehat{\pi}_i}{\partial q_i} - \sum_{j\neq i} \alpha q_j . \tag{8}$$

Hence, the function

$$\widehat{P} = \widehat{\Pi} + \sum_{j \neq i} \alpha q_i q_j = \sum_{i=1}^{n} \frac{q_i}{2} \left[2\alpha \left(1 - \sum_{i \neq j} q_j \right) - q_i (1 + \alpha) \right] + \sum_{i \neq j} \alpha q_i q_j \quad (9)$$

whose first derivative w.r.t. q_i is

$$\frac{\partial \widehat{P}}{\partial q_i} = \alpha \left(1 - q_i - \sum_{i \neq j} q_j \right) - q_i = \frac{\partial \widehat{\pi}_i}{\partial q_i} \quad (10)$$

is the best-response potential function of the game under examination. Hence, Cournot oligopolists facing market demand (3) act *as if* they were jointly maximising (9). ■

Therefore, as in the Cournot examples treated by Slade (1994) and Monderer and Shapley (1996), $\widehat{P}$ is a fictitious objective function whereby the Cournot model could be reinterpreted as a coordination game where every player aims at maximising $\widehat{P}$. The additional implication of the present non-linear example is that it opens the way to interesting applications in the field of dynamic games where the strategic form, being not linear-quadratic, prevents from characterising the analytical solution of the feedback equilibrium, which is instead attainable with the corresponding best-response potential game in linear-quadratic form.[1] This is left for future research.

[1]The open-loop solution of the differential Cournot game using (3) and the Ramsey dynamics is in Cellini and Lambertini (2007).

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