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Abstract

This paper takes an axiomatic approach to find rules for allocating the value of a network when the externalities generated across components are identifiable. Two new, and different, allocation rules are defined and characterized in this context. The first one is an extension of the player-based flexible network allocation rule (Jackson (2005)). The second one follows the flexible network approach from a component-wise point of view, where the notion of network flexibility is adjusted with a flavor of core stability. Furthermore, two other allocation rules are proposed by relaxing the axiom of equal treatment of vital players. These collapse into the player-based flexible network allocation rule (Jackson (2005)) for zero-normalized value functions with no externalities across components.

Keywords: Allocation rules, networks, player-based flexible network allocation rule, Myerson value.

JEL Classification: C71, C79.

1 Introduction

This paper proposes new ways for allocating the value of a network among its participants assuming that the structure of the externalities across components is known. In cases where this information is available, it is useful to define concepts that will take the "component-wise allocation" of the total value of a network into account. In this paper I first extend to this setting the flexible network approach introduced by Jackson (2005) and then define and

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characterize a new flexible network allocation rule that allows no transfers across components, an idea introduced by Myerson (Myerson (1977).)

An allocation rule is said to be component efficient (Myerson (1977)) if it distributes the total value of a connected network among its participants and is said to be fair (Myerson (1977)) if for every direct connection in the network both participants lose or gain the same amount from breaking this connection. Fairness as defined by Myerson implies an equal bargaining power when the only threat in case of disagreement between the two players participating in the link consists of breaking such a link. Therefore, this axiom implicitly assumes that the bargaining power of an agent depends only on subnetworks of the network under consideration. Jackson (2005) criticizes this since the bargaining possibilities of a player should depend on all networks, not only on the subnetworks, as agents could not only delete connections, but create new ones.

According to this viewpoint, the axiom of flexible network (Jackson (2005)) is introduced as follows. A network is flexible if, once organized into a given network structure, the group of agents is always allowed to reorganize internally. If this group of agents cooperates at the grand coalition level they will organize themselves into an efficient network. An allocation rule satisfies the axiom of flexible network if it recommends the same payoff scheme at an efficient network than it would at the complete network with flexibility.

Two comments are in order. First, the axiom of flexible network implies that the payoff scheme at any efficient network is the same. Second, the axiom remains silent about the payoff scheme at inefficient networks. As a remedy for this second point, a new axiom, proportionality, is introduced, by means of which the value of a non efficient network is distributed re-scaling proportionally the payoff scheme corresponding to any efficient network. The axioms of equal treatment of vital players and a weak definition of additivity are added to obtain uniqueness.

In this paper, I first extend the (player-based) flexible network allocation rule (Jackson (2005)) to the case where the value is available component-wise. The "intra-component allocation" of the total value of a network is written as a value function assigning a real number to every component within a network, where the numbers corresponding to each component add up to the total value of the network. The value of a component is allowed to depend on the network structure outside, and therefore the case with externalities across components is captured easily. Contrary to the (player-based) flexible network allocation rule in Jackson (2005), the allocation rule in this setting recommends different payoff schemes anytime the structure of externalities across components changes. I subsequently propose two allocation rules by relaxing the axiom of equal treatment of vital players. These allocation rules collapse into the (player-based) flexible network allocation rule for zero-normalized value functions[1] with no externalities across components.

The first allocation rule proposed follows very closely the original approach by Jackson (2005), in the sense that all efficient networks, as being the ones expected to arise distribute

[1] A value function is zero-normalized if the value of any disconnected agent is equal to 0 in any possible network structure.

the value in the same way. The difference with the original proposal by Jackson (2005) lies in the network structure that coalitions take as a reference when computing their possibilities. The second rule not only relaxes the axiom of equal treatment of vital players but reshapes the notion of flexible network. As mentioned before, the axiom of flexible network combined with proportionality forces the allocation rule to recommend the same payoff at networks that distribute the total value in the same way. In other words, the network loses its meaning in favor of a more simple coalitional structure that could be represented as a TU-game, a disappointing trait given that network relations are a different object than coalitional relations, since the former are bilateral in nature. To escape from this trap I redefine the axiom of flexible network by using a stability argument.

An allocation rule is sensitive flexible network if at an efficient network it recommends the same payoff scheme as in the situation where coalitions could reorganize themselves internally and assuming the rest of the population organize as in **this efficient network**. This axiom is consistent with the idea of participants being allowed to reorganize their network structure any time. Once they have reached an efficient network, they can not do better (as a whole, or in total value). Therefore, it still reflects flexibility in the choice of the network, but recommends different payoff schemes for different efficient networks. The sensitive flexible network allocation rule also converges to the player based flexible network allocation rule defined by Jackson (2005) in the absence of externalities across components (and for zero-normalized value functions).

In the last part of the paper, I take a component-wise approach. A network is **stable for one of its components** if the agents in this component alone cannot reorganize better, taking the network structure outside the component as given. An allocation rule is called component-wise flexible network if at a stable network for one of its components it recommends the same payoff to the participants **in this component** as in the case of network flexibility inside the component. The network structure outside the component is given by the stable network. This axiom applies the idea of flexible network looking at the choice of the network by component, since an efficient network might not be the best network for each of its components. If network flexibility is understood component-wise, so is the way we are allocating the value. So I will impose the allocation rule to be component efficient, or, in other words, no transfers across components are allowed, by changing the notion of proportionality. These two axioms together with the corresponding component-wise equal treatment of vital players and additivity define a unique allocation rule.

I finish this paper with three concluding remarks. First, I show that the Myerson value (or the fair allocation rule) is the unique allocation rule satisfying **component-wise** equal treatment of vital players (different from equal treatment of vital players) and strong additivity (additivity for any possible behavior of the total value of the network and for any graph). As shown by Jackson (2005), the Myerson value is not a flexible network allocation rule. Therefore, the accompanying axioms of a flexible network approach have to be milder versions (if not abandoned altogether) of either **component-wise** equal treatment of vital players or additivity, if considered. Second, the new flexible network allocation rules are put in perspective with respect to a transformed axiom of fairness, called fairness in relative terms. Finally, the notion of component-wise flexible network is related to problems of network formation.

The rest of this paper is organized as follows. Section 2 introduces the setting and defines the Myerson value and the player-based flexible network allocation rule. Section 3 defines and axiomatically characterizes the flexible network allocation rule in this context of identifiable externalities, and proposes the two new allocation rules explained above. Section 4 introduces some examples to show the differences across the two new allocation rules with the Myerson value and the player-based flexible network allocation rule. Section 5 defines and axiomatically characterizes the component-wise flexible network allocation rule. Section 6 concludes, while all proofs are in the Appendix.

2 Definitions

2.1 Players: coalitions and networks

Let $N = \{1, ..., n\}$ be a finite set of players. A subset S of N is called a coalition and 2^N denotes the set of all possible coalitions in N. Network relations among players in N are formally represented by an undirected graph, written as a set of unordered pairs ij, where i, $j \in N$, and $i \neq j$. Each unordered pair ij is referred to as a link.

Let $g \cup ij$ denote the graph resulting from adding the link ij to the existing graph g, while $g \backslash ij$ denotes the graph resulting from deleting the link ij from g. Let g^S be the set of all unordered pairs in $S \subseteq N$, that is, the complete graph over S. Define the restriction of g to a coalition S as $g|S = \{ij \in g : \ i \in S \text{ and } j \in S\}$. Note that $g|S \subseteq g$ and $g|N = g$. A coalition $T \subseteq S$ is called a connected component of S in g if: (1) for every two players in T, there is a path, that is, a set of consecutive links, in $g|S$ connecting them, and (2) for any player i in T and any player j not in T, there is no path in $g|S$ which connects them. For simplicity, I will call them simply *components*. Let $S|g$ be the set of components of S in g. Note that $S|g$ is a partition of S. Similarly, $N|g$ denotes the set of components in g. Finally, let $N(g)$ denote the set of agents in N participating in at least one link in g and let $n(g)$ be its cardinality.

2.2 Payoffs: values and allocation rules

Any function v assigning to each possible graph g a real number $v(g)$ is a *value function* (Jackson and Wolinsky (1996)). Let V denote the set of all possible value functions. A function w, which to every graph g and every component T in $N|g$ assigns a value $w(T, g)$, is called a *component-wise (CW) value function.* Let W be the set of all possible CW value functions.

Given a CW value function w, the total value which can be distributed among the players in N when the graph is g is given by $\sum\limits_{T \in N|g} w(S, g)$. I will then say that a CW value function $w \in W$ induces a value function v if $\sum_{S \in N|g} w(S, g) = v(g)$ for all g. Let v^w denote the value function induced by some given CW value function $w \in W$. Note that for every CW value function w there is a unique induced value function v^w, while given a value function $v \in V$ several CW value functions w could induce such a v.

A CW value function w is said to have no externalities across components if for any $T \subseteq N$ and any g such that $T \in N|g$ the value $w(T,g)$ stays constant as far as $g|T$ does not change.

A CW value function w is called *monotonic* if for any graph g and any component $T \in N|g$:

$$w(T,g) \geq \sum_{R \in T|g'} w\left(R, g' \cup g|\left(N \backslash T\right)\right),$$

for any $g' \subseteq g|T$.

Given any CW value function w its monotonic cover $\hat{w}$ is given by

$$\hat{w}\left(T,g\right) = \max_{g' \subseteq g|T} \sum_{R \in T|g'} w\left(R, g' \cup g|(N \backslash T)\right),$$

for all g and $T \in N|g$. Note that if a CW value function w is monotonic then it is equal to its monotonic cover $\hat{w}$.

A graph is *efficient* (Jackson and Wolinsky (1996)) if $v^w(g) \geq v^w\left(g'\right)$, for all g' in G. A graph g is called *stable* for T, with $T \in N|g$, if $w(T,g) = \hat{w}\left(T, g^T \cup g|\left(N \backslash T\right)\right)$. Stability for a component T means that this component has no incentive to reorganize internally (taking as given what the rest $N \backslash T$ are doing.)

It is clear that, in the absence of externalities across components, efficient graphs are stable. This is not true in the case of externalities across components. Take for example an efficient graph g inducing more than one component on the population N. It may not be optimal for a given component T to organize itself as in $g|T$ given what the others are doing in $g|\left(N \backslash T\right)$. Since g is efficient, in this best graph for T, $N \backslash T$ has to be earning less than in g. See the Concluding Remarks for a discussion on stability.

Before stating some of the axioms on allocation rules, it is convenient to define a basis for CW value functions. A basic CW value function, denoted $w_{T,g}$, is a CW value function taking values $w_{T,g}\left(R, g'\right) = 1$ if both $T \subseteq R$ and $g \subseteq g'$, and 0 otherwise. It is easily seen that a CW value function w can be written in a unique way as a linear combination of basic CW value functions, since $\{w_{T,g}\}_{g \in G, T \in N|g}$ form a basis of W.

An *allocation rule* y is a function that assigns to every CW value function $w \in W$ a payoff vector in $\Re^{N \times G}$, recommending for every player i and graph g a payoff $y_{i,g}(w)$, with

$$\sum_{i \in N} y_{i,g}(w) = v^w(g), \tag{1}$$

for all $g \in G$. Note that balance (condition (1)) is already included in the definition of an allocation rule, as in Jackson (2005).

An allocation rule $y(w)$ is called *insensitive to inter-component allocations* (IICA) if $y(w) = y(w')$ whenever $v^w = v^{w'}$.

I now introduce the definition of two well-known allocation rules, the Myerson value and the player-based flexible-network allocation rule. For a discussion on some of the axioms, see the Concluding Remarks section.

Let φ denote the Myerson value for games in partition function form (see Myerson (1977b)).

The Myerson value (Myerson (1977), Jackson and Wolinsky (1996), Feldman (1996) and Navarro (2007)) is the allocation rule $y^{MV}(w)$ assigning for every player $i \in N$ and every graph $g \in G$

$$y_{i,g}^{MV}(w) = \varphi_i\left(U_{w,g}^M\right),$$

where

$$U_{w,g}^M(S,P) = \sum_{R \in S|g} w(R, g|P),$$

with $S \in P$, P a partition on N.

Let Φ denote the Shapley value for games in characteristic function form.

The player-based flexible network allocation rule (Jackson (2005)) is the allocation rule $y^{PBFN}(w)$ assigning for every player $i \in N$ and every graph $g \in G$

$$y_{i,g}^{PBFN}(w) = \frac{v^w(g)}{\hat{w}(N, g^N)} \Phi_i\left(U_w^J\right),$$

where

$$U_w^J(S) = \max_{g \subseteq g^S} v^w(g),$$

with $S \subseteq N$. Note that, by definition, the PBFN allocation rule is insensitive to inter-component allocations.

3 A flexible network approach sensitive to inter-component allocations

3.1 An extension of the player-based flexible network allocation rule

Definition 3.1 *An allocation rule y is a flexible network rule if for all $w \in W$ and all g efficient relative to w:*

$$y_{i,g}(w) = y_{i,g^N}(\hat{w}),$$

for all players i.

An allocation rule that is flexible network distributes the value of an efficient network as if agents would start with the complete graph and could reorganize themselves internally by deleting links (therefore $\hat{w}$ is taken as a reference).

Definition 3.2 *An allocation rule y is weakly additive if for any monotonic w and w', and scalars $a \geq 0$ and $b \geq 0$,*

$$y_{i,g^N}\left(aw + bw'\right) = ay_{i,g^N}\left(w\right) + by_{i,g^N}\left(w'\right),$$

for all players i, and, if $aw - bw'$ is monotonic, then

$$y_{i,g^N}\left(aw - bw'\right) = ay_{i,g^N}\left(w\right) - by_{i,g^N}\left(w'\right),$$

for all players i.

Note that weak additivity applies "linearity" only on monotonic value functions and fixing graph structure g^N as a reference, as in Jackson (2005). If an allocation rule y is weakly additive for any graph g and with no restrictions on the monotonicity of w, w' and $aw - bw'$, I will say that y is *strongly additive*.

Definition 3.3 *An allocation rule y is said to satisfy equal treatment of vital players if for $w_{T,g}$ a basic value function for some g and some $T \in N|g$*

$$y_{i,\hat{g}}\left(w_{T,g}\right) = \begin{cases} \frac{1}{|T \cup N(g)|}, & \text{if } i \in T \cup N(g) \text{ and } g \subseteq \hat{g} \\ 0, & \text{otherwise.} \end{cases}$$

As in Jackson (2005) all players that are necessary in building the network structure to generate the value of 1 get equal payoffs. Furthermore, a player not in T or in $N(g)$ does not contribute to obtain the value, therefore obtaining zero payoff. Note that if we restrict our space of CW value functions to be zero-normalized (i.e., we fix $w(T,g) = 0$ if $|T| = 1$) then $T \cup N(g) = N(g)$. In that case, this axiom is equal to the one introduced by Jackson (2005).

Definition 3.4 *An allocation rule y is proportional if for every player i and every value function w either $y_{i,g}(w) = 0$ for all g, or for any g and g' such that $v^w(g') \neq 0$*

$$\frac{y_{i,g}\left(w\right)}{y_{i,g'}\left(w\right)} = \frac{v^w\left(g\right)}{v^w\left(g'\right)}.$$

This is equivalent to proportionality in Jackson (2005). As in his context, this axiom plays a role in balancing the allocation rule when distributing the value of a non efficient graph. Applying proportionality means that the allocation rule for any non efficient graph can be uniquely determined in terms of the allocation rule at an efficient graph (no matter which efficient graph is taken as a reference, as all efficient graphs distribute the maximal value in the same way) and that the loss of value from one graph to another (in particular, from an efficient graph to a non efficient one) is distributed proportionally across players.

Next theorem characterize the flexible network allocation rule that is not insensitive to inter-component allocations. In order to state it in a more compact way I will make use of the following definition. For any given w, let the associated TU-game $U_{w,g}$ for each possible g be defined as

$$U_{w,g}(S) = \hat{w}\left(S, g^S \cup g|(N \backslash S)\right),$$

for all coalitions $S \subseteq N$. Note that if w presents no externalities across components $U_{w,g}$ is equal for all graphs g, namely U_w and could be written as

$$U_w(S) = \hat{w}\left(S, g^S\right).$$

for all coalitions $S \subseteq N$. Furthermore, if any disconnected agent has a value of zero then

$$\hat{w}\left(S, g^S \cup g|(N \backslash S)\right) = \max_{g \subseteq g^S} v^w(g) = U_w^J(S),$$

for all coalitions $S \subseteq N$.

Next theorem presents an allocation rule collapsing into the PBFN allocation rule in the absence of externalities across components, but which is not independent of inter-component allocations.

Theorem 3.5 *Let $\Phi(U_{w,\emptyset})$ denote the Shapley value of the TU-game $[N, U_{w,g}]$ defined above when $g = \emptyset$. Then, given any CW value function w, there is a unique flexible network and proportional allocation rule y satisfying weak additivity and equal treatment of vital players, namely*

$$y_{i,g} = \frac{v^w(g)}{\hat{w}(N, g^N)} \Phi_i\left(U_{w,\emptyset}\right),$$

for all players $i \in N$ and all graphs $g \in G$.

The proof of this theorem is equivalent to the proof of Theorem 3 in Jackson (2005) and therefore omitted.

3.2 Relaxing equal treatment of vital players and reshaping the flexible network axiom

Let us take a closer look to the axiom of equal treatment of vital players. Recall that a basic value function $w_{T,g}$ for some graph g and some component $T \in N|g$ gives one unit of value only to agents in T when the graph is g. Nevertheless, agents outside T, although they do not receive any value according to $w_{T,g}$, are necessary to generate the minimal structure g in order for T to obtain such a value of 1. Therefore, for any basic value function $w_{T,g}$ we can classify the players in N into three different groups:

1. Players in T, who are assigned the value of 1 when the graph is equal to g and are necessary to generate value as far as $T \subseteq N(g)$.
2. Players in each component of $N(g)\backslash T$, who are not allocated the value of 1 but are necessary to generate it (since if g is not formed there is no value to be distributed.)
3. Players in $N\backslash (T \cup N(g))$, who are neither allocated the value of one unit nor necessary to generate it.

Equal treatment of vital players imposes the allocation rule to distribute equally the value of one unit among players that are either assigned the value or are necessary to generate the value. One could think in a milder version in which players who intervene equally generating the value obtain the same payoff. The following definition tries to capture that idea.

Definition 3.6 *An allocation rule y is said to satisfy equal treatment of equals if $w_{T,g}$ is a basic value function for some g and some $T \in N|g$, then there exists a constant c_R for any component $R \in N|g$ such that if $i \in R$ then*

$$y_{i,\hat{g}}\left(w_{T,g}\right) = \begin{cases} c_R, & \text{if } i \in T \cup N(g) \text{ and } g \subseteq \hat{g} \\ 0, & \text{otherwise} \end{cases}$$

Note that if an allocation rule satisfies equal treatment of vital players it satisfies equal treatment of equals. The axiom of equal treatment of vital players is on one of the extremes of equal treatment of equals, in which any player that is necessary to generate the value and any player belonging to the group that is assigned the value are all treated evenly. On the other extreme lies an allocation rule that would allocate positive payoff only to the players that are assigned the value. Next definition proposes a family of allocation rules satisfying equal treatment of equals.

Definition 3.7 *Let the allocation rule y satisfying*

$$y_{i,g} = \frac{v^w(g)}{\hat{w}(N, g^N)} \Phi_i\left(U_{w,g^*}\right),$$

for all players $i \in N$ and all graphs $g \in G$, be the "g^*-flexible network rule."

Intuitively, g^* is the graph structure taken as a reference when a coalition S computes its possibilities in case of disagreement and split from $N\backslash S$. This choice of g^* is constant across networks g and across agents i, meaning that the graph structure g^* taken as reference outside is common to any coalition S. Finally, note that if $g^* = \emptyset$ then we obtain the flexible network allocation rule.

Proposition 3.8 *Fix any graph $g^* \in G$. The "g^*-flexible network rule" satisfies flexible network, proportionality, weak additivity and equal treatment of equals.*

The choice of g^* will determine how the values of each of the c_R in the definition of equal vital of equals are related. From Theorem 3.5 if $g^* = \emptyset$ then all c_R take the same value. Intuitively, g^* is the graph structure taken as a reference when a coalition S computes its possibilities in case of disagreement and split with $N \backslash S$. This choice of g^* is constant across networks g and across agents i, meaning that the graph structure g^* taken as reference outside is common to any coalition S.

From an "a posteriori" point of view, a coalition S planning to deviate and stay on its own from a graph g should assume that players outside S will cooperate as in $g|\left(N \backslash S\right)$. In such a case, the graph structure outside taken as reference cannot be constant anymore, and the flexible network axiom has to be reshaped a bit.

Definition 3.9 *An allocation rule y is a sensitive flexible network rule if for all $w \in W$ and all g efficient relative to w:*

$$y_{i,g}(w) = y_{i,g}\left(\hat{w}\right),$$

for all players i.

An allocation rule that is sensitive flexible network distributes the value as if agents were allowed to reorganize internally. The difference here is in the graph that is taken as a reference. Note that at an efficient graph g the total value to be distributed $v^w\left(g\right)$ is already optimal by definition. So even in the ideal flexible world, where $\hat{w}$ is taken as a reference, the grand coalition cannot do better than they do at the efficient network. Following the argument in Jackson (2005), we should not expect reorganizations at the grand coalition level. But, if there are externalities across components and an efficient network has more than one more component, the members of one component could have incentives to reorganize internally once the efficient network structure has been formed. A sensitive flexible network allocation rule in an efficient network distributes the maximal value as if (i) players are allowed to reorganize themselves internally and (ii) they assume that the rest of agents are organized as in that efficient network. If there are no externalities across components the network structure outside does not matter and the difference between these two axioms (flexible network and sensitive flexible network) vanishes. The examples next section will illustrate how these two axioms on network flexibility differ. Next definition introduces a mild notion of proportionality that will be used in combination with the sensitive flexible network axiom.

Definition 3.10 *An allocation rule y is proportional with respect to the optimal if for every player i and every value function w either $y_{i,g}(w) = 0$ for all g, or for any g with $\hat{v^w}(g) \neq 0$*

$$\frac{y_{i,g}\left(w\right)}{y_{i,g}\left(\hat{w}\right)} = \frac{v^w(g)}{\hat{w}\left(N, g^N\right)}.$$

Recall that, in a sensitive flexible network allocation rule, whenever players arrive to an efficient graph g they obtain the same payoff as in the situation when they face the monotonic cover $\hat{w}$ instead of w. This does not necessarily mean that all efficient networks distribute the value in the same way. Proportionality with respect to the optimal means that players are treated symmetrically in the loss of value resulting from being in a non efficient graph. The difference with the notion of proportionality is that the loss of value is measured comparing w and $\hat{w}$ in the same graph, and not comparing the actual values of two different graphs.

Definition 3.11 *Let the allocation rule y satisfying*

$$y_{i,g} = \frac{v^w(g)}{\hat{w}(N, g^N)} \Phi_i \left(U_{w,g}\right),$$

for all players $i \in N$ and all graphs $g \in G$, be called the "sensitive flexible network rule."

Proposition 3.12 *The "sensitive flexible network rule" satisfies sensitive flexible network, proportionality with respect to the optimal, weak additivity and equal treatment of equals.*

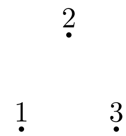
As I said above, it is easy to see that the allocation rule in Theorem 3.5, and the two alternative flexible network allocation rules collapse to the PBFN allocation rule defined before in the case of no externalities, i.e., when $w(S, g) = w(S, g')$ if $g|S = g'|S$. Nevertheless, they give different predictions when there are externalities across components, as neither of them are insensitive to inter-component allocations. The key for understanding how these externalities change the payoff scheme recommended by these allocation rules is in the definition of the game $U_{w,g}(S)$ for any given coalition S. Note that the surplus that a coalition can extract is the best network they could choose internally depends on what the others are doing. Each of the three allocation rules respectively assume (i) they are fully disconnected, driven by the axiom of equal treatment of equals defined as in Jackson (2005), (ii) they are connected as in an "a-priori" common belief g^*, or (iii) they fix it to be equal to the actual network. Next section builds on an example borrowed from Jackson (2005).

4 Examples

The following example, borrowed from Jackson (2005) will place face to face the Myerson value, the PBFN allocation rule and the new allocation rules suggested above.

4.0.1 An example

Example 1. (Example 1, Jackson (2005)) Let $N = \{1, 2, 3\}$. In order to place the players on the possible graphs over N I will use the following convention:



Consider the following value functions v and v'. Let v take values $v(g)$ equal to 1 when g is either $\{12\}$, $\{23\}$, or $\{12, 23\}$; and 0 otherwise, and let v' take values $v'(g)$ equal to 1 for all graphs different to the empty graph. Assuming that the value of an isolated agent is normalized to 0 the CW value functions w and w' that generate v and v', respectively, are defined as follows. For every $g \in G$ and for every $S \in N|g$:

$$w(S, g) = \begin{cases} 1, & \text{if } S = \{1, 2\}, \text{ if } S = \{2, 3\}, \text{ or if } S = \{1, 2, 3\} \text{ and } g = \{12, 23\}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$w'(S, g) = \begin{cases} 0, & \text{if } |S| = 1, \\ 1, & \text{otherwise.} \end{cases}$$

Figures 1 and 2 show the payoffs recommended by the Myerson value for w and for w', respectively, while Figures 3 and 4 show the payoffs resulting from the PBFN allocation rule for w and for w'. For each graph the payoff to each of the players is written just above the corresponding node, following the convention previously defined.

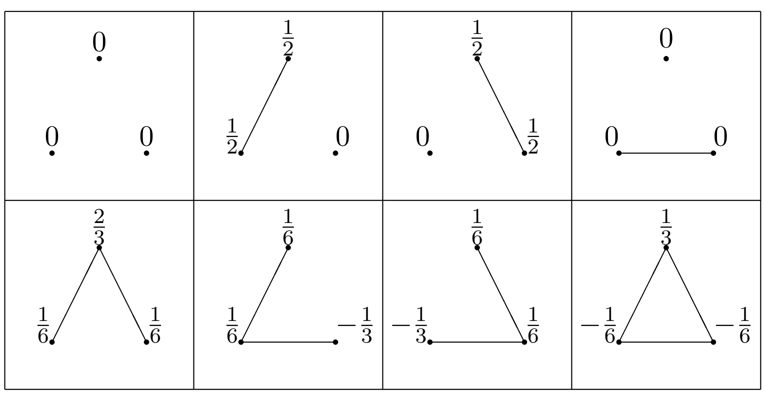


FIGURE 1: The Myerson value for w

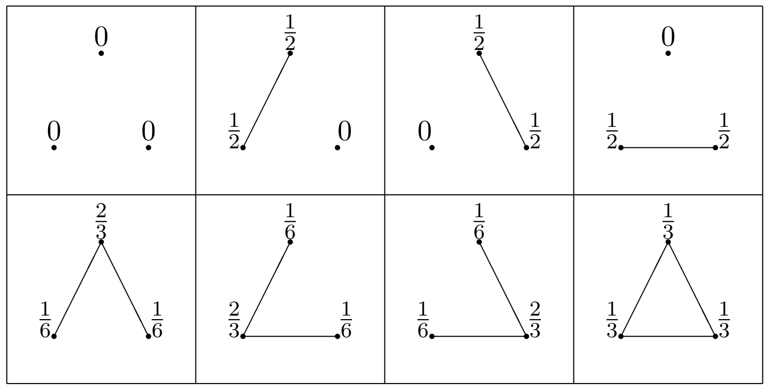


FIGURE 2: The Myerson value for w'

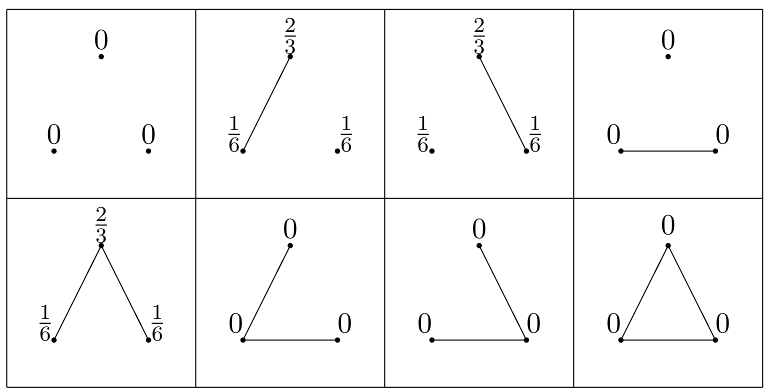


FIGURE 3: The PBFN allocation rule for w

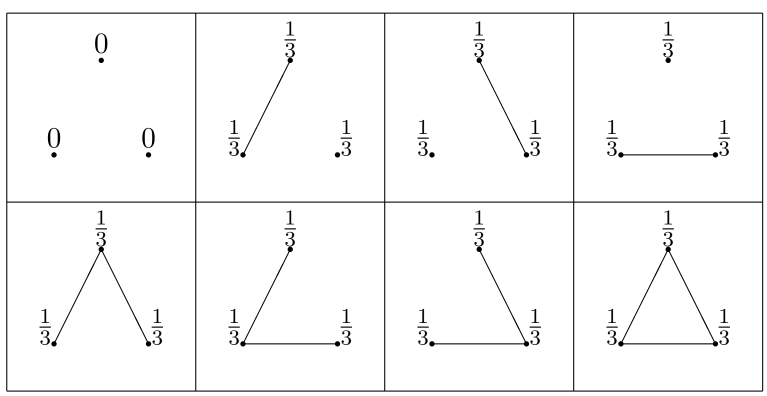


FIGURE 4: The PBFN allocation rule for w'

Jackson (2005) compares the Myerson value in w and w' to illustrate the idea that Myerson is insensitive to alternative networks. So let us compare Figures 1 and 2. The argument stated in Jackson (2005) is mainly that the fact that the Myerson value yields the same payoffs in w and w' for graph $\{12, 23\}$ shows the insensitivity of the Myerson value. The careful reader will realize that the reason why the Myerson value is equal for both value functions in that particular graph is because the values w and w' are equal for all graphs that are subsets of $\{12, 23\}$. If the value w changes for a subgraph of $\{12, 23\}$ the Myerson value will give a different recommendation. Therefore, the Myerson value is in fact sensitive to any subgraph, although it is not sensitive to *all* alternative graphs. The flexible network approach includes information coming from networks that are not necessarily subgraphs.

Let us focus now on Figures 3 and 4. As I just said, information about networks that are not necessarily subnetworks is included in the way the value is allocated: The recommendation is different for w than for w' in the network $\{12, 23\}$. On the other hand, players should be clever enough to realize the best way to organize themselves. Recall that the flexible network axiom implicitly assumes that agents will a-priori organize themselves into an efficient structure. This combined with proportionality imposes that the value should be allocated in the same way at all efficient networks, given that those are the ones that should be arising if networks are fully flexible. This payoff allocation is re-scaled proportionally at any of the other networks, which will result in the same payoff scheme for any network with the same total value. So the total value of a network is more relevant for the PBFN allocation rule than the network possibilities of coalitions that are not the grand coalition.

The flexible network allocation rule characterized in Theorem 3.5 and the one proposed in Definition 3.7 follow very closely the spirit of the PBFN allocation rule. As opposed to the PBFN allocation rule, both change when the structure of externalities given by w changes, even when the value function that is induced, v^w, stays the same. The allocation rule proposed in Definition 3.11 goes one step further and allows for different payoffs schemes in networks that distribute the same total value. These effects are captured by allowing a different assumption than the one taken in Jackson (2005) about the surplus that a coalition can achieve, as explained at the end of the previous section. In order to illustrate this point, let us take a look at a new CW value function $\tilde{w}$ that also induces v as a value function. Recall that v takes values $v(g)$ equal to 1 when g is either $\{12\}$, $\{23\}$, or $\{12, 23\}$; and 0 otherwise. It is easy to check that, in case of no externalities, as in w stated before, all allocation rules in Theorem 3.5, and in Definitions 3.7 and 3.11 are equal to the PBFN shown in Figure 2.

Let $\tilde{w}$ be another CW value function that also induces v taking the following values:

$$\tilde{w}(S, g) = \begin{cases} 1, & \text{if } |S| = 2, \text{ or } g = \{12, 23\} \\ -1, & \text{if } S = \{2\} \text{ and } g = \{13\}, \\ 0, & \text{otherwise,} \end{cases}$$

As $\tilde{w}(\{2\}, \{1, 3\}) = -1 \neq \tilde{w}(\{2\}, \{\emptyset\}) = 0$ one can say that $\tilde{w}$ presents externalities across components. Figures 5 and 6 show the payoffs recommended by the flexible network allocation rule and the g^N-flexible network allocation rule, respectively. Given the simplicity of this example, the sensitive flexible network rule recommends the same payoff scheme as the one shown in Figure 5.

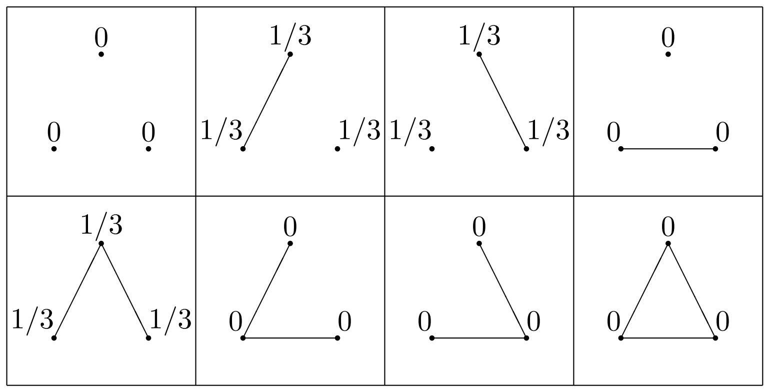


FIGURE 5: The (sensitive) flexible network allocation rule for $\tilde{w}$

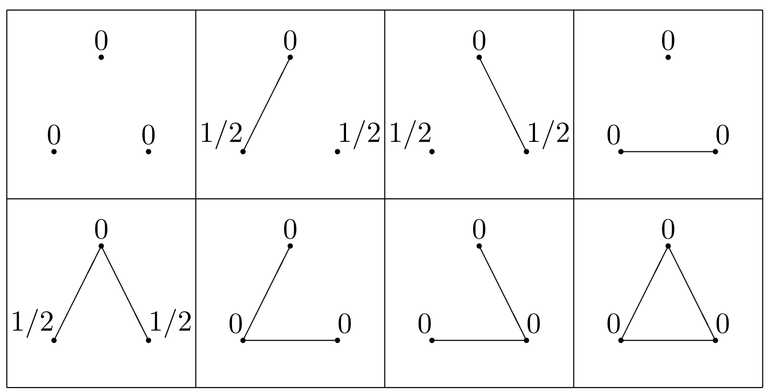


FIGURE 6: The g^N-flexible network allocation rule for $\tilde{w}$

In order to see the intuition of how these two allocation rules have readjusted their recommendations note the following features of the value functions v and $\tilde{w}$. The value function v as proposed by Jackson (2005) (with a corresponding w without externalities) captures the idea that player 2 is extremely powerful or able to generate value, as compared to agents 1 and 3, who, without player 2, would not extract any surplus. But if the structure of the externalities works according to $\tilde{w}$ the PBFN allocation rule will recommend the same payoff scheme, as it is independent of intra-component allocations. Note that in the case of $\tilde{w}$ there are two effects. First, when looking at one player coalitions, the main difference is that player 2 is weaker in the structure where she is disconnected facing players 1 and 3 being connected: She would generate a value of -1 as opposed to values of 0 for the other two players in the equivalent network structures. When we look at two-player coalitions, according to $\tilde{w}$, if players 1 and 3 get together they can also extract a value equal to 1, as do 1 and 2 together or 2 and 3. This means that all two-player coalitions can do as equally good. Finally, the grand coalition is as good in $\tilde{w}$ as in w.

If we take a look at the allocation rule in Figure 5, which is the allocation rule stated in Theorem 3.5 and in Definition 3.11, the possibilities of player 2 are measured assuming that (i) players 1 and 3 are not connected, since the empty graph is fixed as the reference outside in Theorem 3.5, or (ii) the network structure for players 1 and 3 is the same as in the graph for which the value is to be allocated, for the rule in Definition 3.11. In this latter case the weakest position for player 2, when players 1 and 3 are connected, is not considered as those graphs have zero value. Furthermore, all two-player coalitions can extract a value of 1. Therefore, player 2 has the same possibilities for extracting value as 1 and 3, and therefore receives the same payoff in the allocation rules stated in Theorem 3.5 and in Definition 3.11. The allocation rule in Figure 6, which is the one stated in Definition 3.7 for $g^* = g^N$, recommends no payoff for player 2. The reason is that the possibilities for player 2 alone when deviating from a given network are the worst, given that, for such an allocation rule, players 1 and 3 are assumed to be fully connected. Again, all two player coalitions are equivalent and the grand coalition is as good as in the previous value function w. Therefore, the allocation rule in Figure 6 recommends much less payoff for player 2 than the one in Figure 5.

Next example shows how the sensitive flexible network allocation rule changes for networks having a different total value available.

Example 2.
Let $\tilde{\tilde{w}}$ be another CW value function taking the following values:

$$\tilde{\tilde{w}}(S,g) = \begin{cases} 1, & \text{if } |S| = 2 \text{ or } \{12,23\} \subseteq g, \\ -1, & \text{if } S = \{2\} \text{ and } g = \{13\}, \\ 0, & \text{otherwise,} \end{cases}$$

The only different with respect to the previous CW value function $\tilde{w}$ is that the value the grand coalition can extract if fully connected equals 1 instead of zero. The reader can check that (i) the allocation rule in Theorem 3.5 recommends 1/3 for all players in any efficient graph and 0 in any non efficient graph and (ii) the allocation rule in Definition 3.7 for $g^* = g^N$ recommends 1/2 for players 1 and 3 and 0 for player 2 in any efficient graph and 0 in any non efficient graph. The allocation rule in Definition 3.11 allocates as shown in Figure 7.

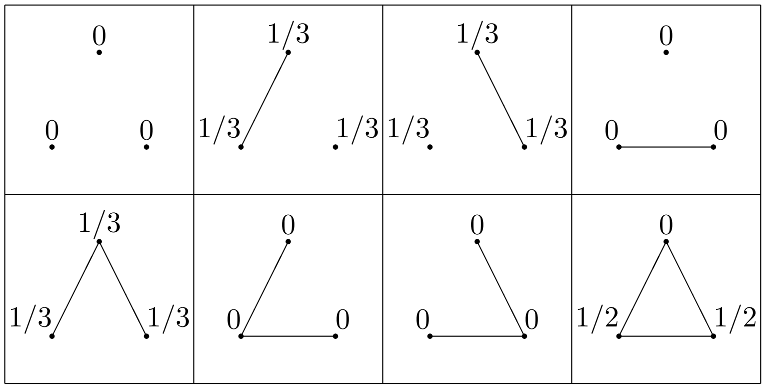


FIGURE 7: The sensitive flexible network allocation rule for $\tilde{\tilde{w}}$

As in the case of $\tilde{w}$, player 2 is weaker in the structure where she is disconnected facing players 1 and 3 being connected than players 1 and 3 in equivalent situations, and all two-player coalitions can do as equally good.

As discussed before, the allocation rule in Definition 3.7 recommends zero payoff to player 2 given that the weakest position for player 2 is relevant. If we take a look at the allocation rule in Figure 7 such weakest position for player 2 is considered only when players 1 and 3 are connected in the graph where the value is allocated. The only graph with strictly positive value and where 1 and 3 are connected is the complete graph. There, the allocation rule distributes the value in the same way as the rule proposed in Definition 3.7. For the rest of efficient networks the value is distributed evenly since the weakest position for player 2 is not taken into account. Recall that in all those graphs players 1 and 3 are disconnected already.

5 A component-wise flexible network approach

Recall that a graph g can be efficient, and nevertheless, not be stable for some of the components it induces, as far as there are externalities across components and the set of components, $N|g$, is not a singleton. Therefore, fixing the allocation rule for graphs which are efficient is not easy to sustain in the case where components could also have "flexibility" and reorganize themselves, for example taking as given what the rest of components are doing. The best possibility a given component T has in a given g depends on the value $\hat{w}(T, g)$ itself, which is in general different from what it gets in $w(T, g)$ even when g is efficient. This idea is captured in the following version for the flexible network property, called CW flexible network. Since flexibility is adapted component-wise, so are the axioms of additivity, equal treatment of vital players and proportionality.

Definition 5.1 *An allocation rule y is a CW flexible network rule if for all $w \in W$ and all g*

stable for some of its components $T \in N|g$ relative to w:

$$y_{i,g}(w) = y_{i,g}(\hat{w}),$$

for all players $i \in T$.

We fix the allocation rule for graphs that are stable for a given component T, instead of for efficient graphs, to be equal to the allocation rule in the situation where $\hat{w}$, the monotonic cover of w, is distributed. As in the argument of the sensitive flexible network axiom, by doing that we assume that (i) given any graph each component is reorganizing in its best way, as in $\hat{w}$, and (ii) given that the values for T are equal in w than in $\hat{w}$ the participating players in T will obtain the same payoff in both situations.

Fix $T \subseteq N$, define $G^T \subseteq G$ as the set of graphs g on N where (i) T is a component in g, and (ii) T is fully connected inside. Formally,

$$G^T = \left\{g \in G \text{ such that } g = g^T \cup g|\left(N \backslash T\right)\right\}.$$

Recall that weak additivity takes g^N as a reference. In trying to get a component-wise version of weak additivity, G^T is going to be fix as the reference, since it is the set of graphs such that players in T are fully connected, but players outside T could be connected anyway. I introduce now a component-wise version of weak additivity.

Definition 5.2 *An allocation rule y is CW weakly additive if for any $T \subseteq N$, any monotonic w and w', and scalars $a \geq 0$ and $b \geq 0$:*

$$y_{i,g}\left(aw + bw'\right) = ay_{i,g}\left(w\right) + by_{i,g}\left(w'\right),$$

for all players $i \in T$ and all $g \in G^T$, and, if $aw - bw'$ is monotonic, then

$$y_{i,g}\left(aw - bw'\right) = ay_{i,g}\left(w\right) - by_{i,g}\left(w'\right),$$

for all players $i \in T$ and all $g \in G^T$.

Recall that weak additivity applies "linearity" only on monotonic value functions and fixing graph structure g^N as a reference. Here, we fix the set of graphs G^T given any T, as a reference. Note that for monotonic value functions the best internal structure (in terms of surplus) for a coalition T is the complete graph, independently of what players in $N \backslash T$ are doing.

Definition 5.3 *An allocation rule y is said to satisfy CW equal treatment of vital players if $w_{T,g}$ is a basic value function for some g and some $T \in N|g$ and*

$$y_{i,\hat{g}}\left(w_{T,g}\right) = \begin{cases} \frac{1}{|T|}, & \text{if } i \in T \text{ and } g \subseteq \hat{g} \\ 0, & \text{otherwise} \end{cases}$$

This axiom states that only the players in the component generating the value in the basic value function can obtain a positive payoff, and they do so equally. In other words, no transfers are allowed from one component to another. Note that an allocation rule satisfying CW equal treatment of vital players satisfies equal treatment of equals where $c_R = 0$ whenever $R \neq T$.

Definition 5.4 *An allocation rule y is CW proportional with respect to the optimal if for every player i and every value function w either $y_{i,g}(w) = 0$ for all $g \in G$ and $i \in N$, or for any g and $T \in N|g$ with $\hat{w}(T, g) \neq 0$*

$$\frac{y_{i,g}(w)}{y_{i,g}(\hat{w})} = \frac{w(T, g)}{\hat{w}(T, g^T \cup g|(N \backslash T))},$$

for $i \in T$.

An allocation rule satisfying the four properties defined just above follows the spirit of sensitive flexible network defined in the previous section. If a network g is stable for one of its components then the payoff has to correspond to what they would get even if the rest of agents are organizing their best way possible at any given network. The flexibility is therefore component-wise, as well as the analysis of the possibilities. Therefore, it is even more sensitive than the previous case. CW proportional with respect to the optimal implies that the allocation rule distributes exactly the value of a component among the participants of that component, in other words, there are no transfers of value among different components.

Given a $T \in N|g$ we define the TU-game $U_{w,g}^T$ as

$$U_{w,g}^T(S) = \hat{w}\left(S, g^S \cup g|(T \backslash S) \cup g|(N \backslash T)\right),$$

for every $S \subseteq T$.

Theorem 5.5 *There is a unique CW flexible network, CW proportional with respect to the optimal allocation rule y satisfying CW weak additivity and CW equal treatment of vital players, namely*

$$y_i(w, g) = \frac{w(T, g)}{\hat{w}(T, g^T \cup g|(N \backslash T))} \Phi_i\left(U_{w,g}^T\right),$$

for every $i \in T$, with $T \in N|g$, given any $g \in G$ and $w \in W$.

The proof of this theorem is similar to the one in Theorem 3.5 and therefore omitted.

Consider again Example 2 in the previous section. Figures 8 and 9 show, respectively, the Myerson value and the CW flexible network allocation rule for $\tilde{\tilde{w}}$.

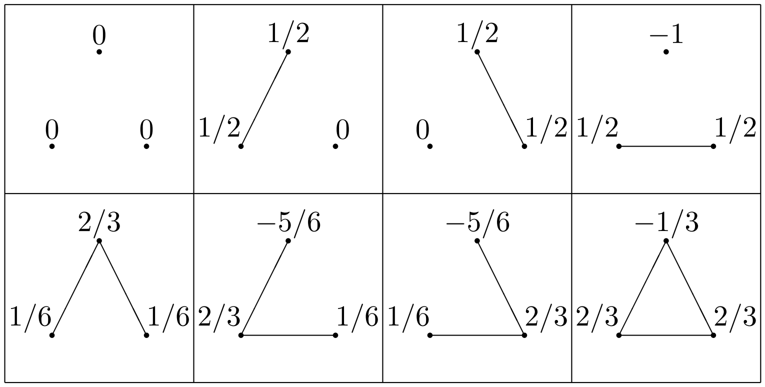


FIGURE 8: The Myerson value for $\tilde{\tilde{w}}$

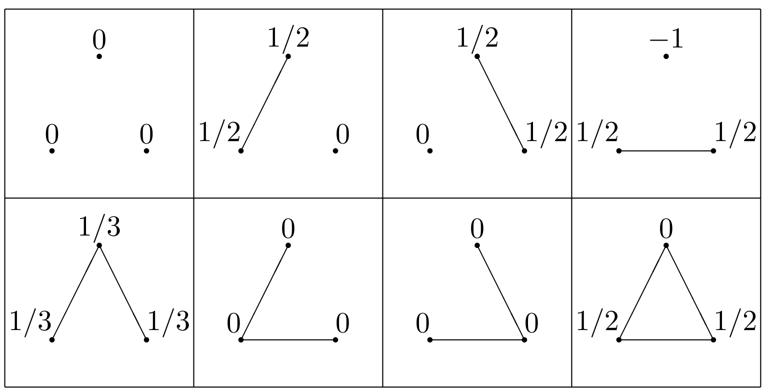


FIGURE 9: The component-wise flexible network allocation rule for $\tilde{\tilde{w}}$

For one-link networks the possibilities considered by the CW flexible network allocation rule as the same as the ones considered by the Myerson value: each component can only deviate by disconnecting the two members. Therefore, both allocation rules in Figures 8 and 9 yield the same prediction. For networks with more than one link the CW flexible network rule gives the same recommendation as the sensitive flexible network rule. Note that for all those networks, the grand coalition is the only component. The Myerson value, as mentioned before, takes into account information contained only in subnetworks when computing the possibilities for the players. Therefore, it yields more extreme recommendations than the CW or sensitive flexible network rules.

6 Concluding Remarks

The focus of this paper has been twofold. First, I suggest a flexible network approach that (i) takes inter-component allocation information into account when this information, written as a CW value function w, is available, and (ii) revises network flexibility from a component-wise point of view, as opposed to taking the total population as a reference in the definition of flexible network. If we want the flexible network approach to get a flavor of "stability" as notions of the core or VN-M stable sets do have in the context of TU-games, we can redefine flexibility as in the definition of CW flexible network. Three final comments. First, the Myerson value is characterized as the only allocation rule satisfying strong additivity and CW equal treatment of vital players, therefore showing that either additivity or CW equal treatment of vital players has to be relaxed. Second, I would like to put the flexible network approach in perspective with respect to the axiom of fairness, introduced by Myerson (1977). A small remark about the notion of stability closes this discussion.

6.1 Another characterization for the Myerson value

The following axiomatic characterization of the Myerson value is well known in the literature (See Myerson (1977), Jackson and Wolinsky (1996), Feldman (1996) and Navarro (2007).)

Definition 6.1 *(Myerson (1977)) An allocation rule y is called component efficient if given any value function $w \in W$*

$$\sum_{i \in S} y_{i,g}(w) = w(T, g),$$

for every graph $g \in G$ and every component $T \in N|g$.

Definition 6.2 *(Myerson (1977)) An allocation rule y is called fair if given any value function w*

$$y_{i,g}(w) - y_{i,g\setminus ij}(w) = y_{j,g}(w) - y_{j,g\setminus ij}(w),$$

for every graph $g \in G$ and every link $ij \in g$.

The Myerson value is the unique allocation rule satisfying component efficiency and fairness.

Recall that an allocation rule y (i) is *strong additive* if for any w and w', and any scalars $a \geq 0$ and $b \geq 0$:

$$y(aw + bw') = ay(w) + by(w').$$

and (ii) satisfies *CW equal treatment of vital players* if $w_{T,g}$ is a basic value function for some g and some $T \in N|g$, then

$$y_{i,\hat{g}}(w_{T,g}) = \begin{cases} \frac{1}{|T|}, & \text{if } i \in T \text{ and } g \subseteq \hat{g} \\ 0, & \text{otherwise} \end{cases}$$

Theorem 6.3 *The unique allocation rule satisfying strong additivity and CW equal treatment of vital players is the Myerson value.*

The proof of Theorem 6.3 is in the appendix. It consists on two parts: (1) strong additivity and CW equal treatment of vital players define a unique allocation rule, and (2) any allocation rule which is component efficient and fair also satisfies strong additivity and CW equal treatment of vital players. Therefore the Myerson value is the unique allocation rule satisfying both properties.

It is indeed not new or surprising that the Myerson value embeds a notion of additivity and equal treatment[2]. What may be new (at least to me is) is the fact that both fairness and component efficient are needed in order to imply both strong additivity and CW equal treatment of vital players. Bad news are that strong additivity and CW equal treatment of vital players are both too strong and define a unique allocation rule, being this one the Myerson value.

As noted by Jackson (2005), the Myerson value cannot be "fully" flexible network. Therefore, one has to relax either the notion of additivity or the notion of CW equal treatment of vital players in order to explore the consequences of a flexible network approach. Note that in the allocation rule characterized in Theorem 3.5 additivity is relaxed and CW equal treatment of vital players is substituted by the opposite case of equal treatment of equals, namely equal treatment of vital players. In the allocation rules proposed in Definition 3.7 and Definition 3.11 both axioms (additivity and CW equal treatment of vital players) are relaxed. In the last allocation rule proposed here the choice has been to relax only the additivity constraint.

6.2 Fairness behind the flexible-network approach

Let me first introduce a new notion of fairness that will be the one satisfied by some of the flexible network allocation rules seen along this paper.

Definition 6.4 *An allocation rule y is called fair in relative terms if for every graph $g \in G$ and every link $ij \in g$ such that $v^w(g) \neq 0$ and $v^w(g \backslash ij) \neq 0$:*

$$\frac{y_{i,g}}{v^w(g)} - \frac{y_{i,g\backslash ij}}{v^w(g\backslash ij)} = \frac{y_{j,g}}{v^w(g)} - \frac{y_{j,g\backslash ij}}{v^w(g\backslash ij)}. \tag{2}$$

[2]Matthew O. Jackson has already remarked that fact in private communications and presentations at seminars and conferences. Anne van den Nouweland (1993) has a characterization of the Myerson value (in a more restricted environment) in terms of additivity, component efficiency, superfluous arc property and point anonymity, where the superfluous arc property and point anonymity could be embedded in the idea of equal treatment of vital players here.

By fairness in relative terms one restricts attention to allocation rules where, for each link in a graph, both players should lose or win the same percentage of total wealth from severing this link. Note that (2) is equivalent to

$$\frac{y_{i,g} - y_{j,g}}{v^w(g)} = \frac{y_{i,g\backslash ij} - y_{j,g\backslash ij}}{v^w\left(g\backslash ij\right)},$$

which means that the inequality between these two individuals, expressed in relative terms to the total wealth to be divided, is the same independently of the fact that this link is present or not. Note that this does not mean that the inequality between these two individuals remains constant for all graphs.

The reader may check that the PBFN allocation rule and the allocation rules introduced in Theorem 3.5, in Definition 3.7 and in Definition 3.11 are fair in relative terms, although they are not component efficient. The CW flexible network allocation rule introduced in Theorem 5.5 is not always fair in relative terms, since Figure 9 serves as a counterexample, although it is always component efficient. As said before, the flexible network approach in Theorem 3.5 is incompatible with component efficiency. The key is in the axiom of equal treatment of vital players.

As a conclusion, it seems that the Shapley value computations when taking the total population as a reference, and not component by component, triggers some sort of fairness. By making those Shapley value computations component by component, imposes some rigidity. This rigidity is not caused by fairness, but by component efficiency.

6.3 Stability of a graph vs. efficiency

As a final remark let me comment on the definition of stability of a graph (for one of its components and with respect to the CW value function w). Recall that, given a CW value function w, a graph g is stable for one of its components T if $w(T, g) = \hat{w}\left(T, g^T \cup g|\left(N\backslash T\right)\right)$. Following the argument that this notion may be better than the notion of efficiency when taking an "a-posteriori" flexible network approach, one may think of defining, for a given CW value function w, a set of graphs g with the property of being stable for all of its components (CW stability). Formally, given a CW value function w a graph g is called *CW stable* if for all T in $N|g$ if $w(T, g) = \hat{w}\left(T, g^T \cup g|\left(N\backslash T\right)\right)$. It is easy to see that the empty graph is always a CW stable graph, since no further internal reorganizations are possible.

As expected, efficiency and CW stability are in general unrelated. Consider the following examples with four agents. The value of a component is written next to it whenever this value is different from zero (in other words, all the values that are not appearing on the Figures are equal to 0).

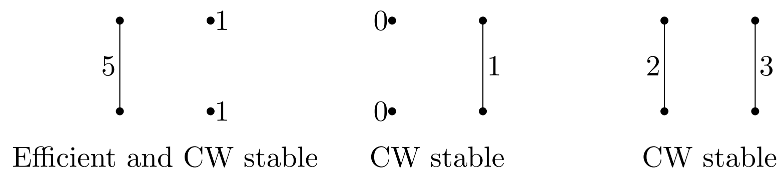


FIGURE 10

It is easy to see that (i) all the graphs containing any of the graphs shown in Figure 10 are neither efficient nor CW stable, and (ii) all the graphs containing none of the graphs shown in Figure 10 are not efficient, but they are CW stable.

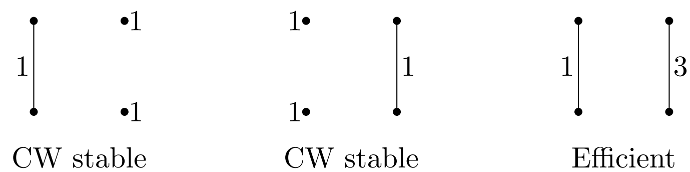


FIGURE 11

As the case in Figure 10, (i) all the graphs not shown in Figure 11 containing any structure of positive value are neither efficient nor CW stable, and (ii) all the graphs that do not contain a structure of positive value are not efficient, but they are CW stable.

As I said before, any graph g where some of its components are singletons is stable with respect to these ones. Therefore, the empty graph is always CW stable, as all its components are singletons. Although this solves the problem of existence, i.e., there always exists a graph which is CW stable, it already tells us that a CW stable graph may not efficient. But are there efficient graphs which are CW stable?

It is easy to see that, although in general they are independent notions, (i) if w presents no externalities across components the set of efficient graphs is a subset of the set of CW stable graphs, and (ii) the empty graph can be the unique CW stable graph, as Figure 12 illustrates.

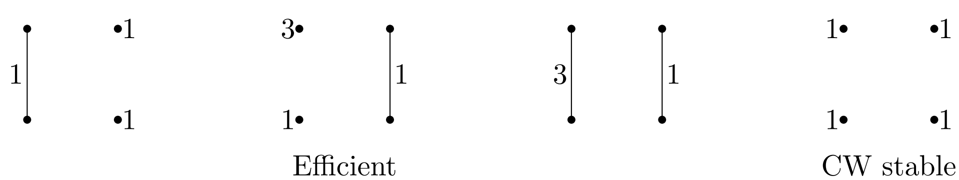


FIGURE 12

Note that, by definition of CW stability, components are allowed to reorganize internally, but not to merge to players outside the component. If we would allow components to merge then one can use a similar technique to the one developed in Pápai (2004) and write each possible network structure as a node in a directed graph. An arc joining two different network structures means that one component of the origin would like to go to the end of the arc. In the example shown in Figure 13 we could write it as follows.

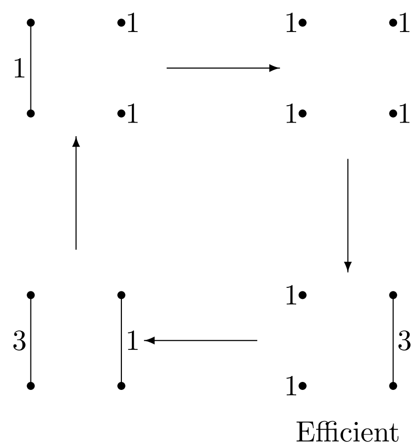


FIGURE 13

There would not be any CW stable graph if the notion of stability allows merging, since the directed graph has a cycle structure. As in Pápai (2005), having a tree-structure in the directed graphs constructed as in the example suffices to warrantee existence of this more restricted notion. It may be of interest to explore the conditions on the value function w to have at least one CW stable graph, where in this more restrictive notion it may be efficient too. This is a question related to network formation problems, and not to the allocation of the value of a network, and is therefore beyond the scope of this paper.

Appendix

Proof of Theorem 6.3

I first show that strong additivity and CW equal treatment of vital players define a unique allocation rule, and afterwards, that an allocation rule that is component efficient and fair (therefore, the Myerson value) satisfies strong additivity and CW equal treatment of vital players.

Lemma 6.3.1. There is a unique allocation rule satisfying strong additivity and CW equal treatment of vital players.

Proof of Lemma 6.3.1. Note that, by definition of basic CW value functions, any CW value function $w \in W$ can be written in a unique way as a linear combination of all possible basic value functions. By strong additivity of an allocation rule y,

$$y(w)=\sum_{g\in G,\ T\in N|g} c_{T,g}\ y\left(w_{T,g}\right),$$

where $c_{T,g}$ denotes the constant multiplying $w_{T,g}$ in the unique linear combination for w. By CW equal treatment of vital players, each of the $y\left(w_{T,g}\right)$ is uniquely determined, and therefore so is $y(w)$. This completes the proof of Lemma 6.3.1.

Lemma 6.3.2. The Myerson value, denoted $m(w)$, satisfies strong additivity.

Proof of Lemma 6.3.2. Assume not. This means that there exist games w_1 and w_2, both in W, and at least one graph $g\in G$ such that

$$m_{i,g}\left(w_1+w_2\right)\neq m_{i,g}\left(w_1\right)+m_{i,g}\left(w_2\right), \tag{3}$$

for some player $i\in N$. It is easily seen that such an i has to be connected in g, otherwise, by component efficiency, both expressions should hold equal. This implies that a g where the inequality in (3) holds has to have at least one link.

Take g to be a minimal graph such that (3) holds. This means that for all $l\in g$:

$$m_{i,g\backslash l}\left(w_1+w_2\right)=m_{i,g\backslash l}\left(w_1\right)+m_{i,g\backslash l}\left(w_2\right). \tag{4}$$

Let $w=w_1+w_2$ and let j denote a player to whom i is directly connected to in g. We know that such a j exists since i has to be connected in g. As the Myerson value is fair,

$$m_{i,g}(w)-m_{i,g\backslash ij}(w)=m_{j,g}(w)-m_{j,g\backslash ij}(w), \tag{5}$$

$$m_{i,g}(w_1)-m_{i,g\backslash ij}(w_1)=m_{j,g}(w_1)-m_{j,g\backslash ij}(w_1), \tag{6}$$

and

$$m_{i,g}(w_2)-m_{i,g\backslash ij}(w_2)=m_{j,g}(w_2)-m_{j,g\backslash ij}(w_2). \tag{7}$$

From (4), (5),(6) and (7),

$$m_{i,g}(w)-m_{i,g}(w_1)-m_{i,g}(w_2)=m_{j,g}(w)-m_{j,g}(w_1)-m_{j,g}(w_2),$$

for j directly connected to i. Note that this implies that the equality holds true even for j indirectly connected to i. Therefore, we can define $\Delta_T\left(w,w_1,w_2\right):=m_{i,g}(w)-m_{i,g}(w_1)-m_{i,g}(w_2)$, where T is the component in $N|g$ to where i belongs. But as the Myerson value is component efficient,

$$|T|\ \Delta_T\left(w,w_1,w_2\right)=\sum_{i\in T}\left(m_{i,g}(w)-m_{i,g}(w_1)-m_{i,g}(w_2)\right)=w(T,g)-w_1(T,g)-w_2(T,g)=0,$$

since $w = w_1 + w_2$. Therefore, $m_{i,g}(w) - m_{i,g}(w_1) - m_{i,g}(w_2) = 0$, a contradiction. This completes the proof of Lemma 6.3.2.

Lemma 6.3.3. The Myerson value satisfies CW equal treatment of vital players.

Proof of Lemma 6.3.3. Let $w_{T,g}$ be a basic value function. For any graph g' such that $g \nsubseteq g'$ all components are worth 0 in all subgraphs of g', so it is easily checked that in all those graphs the Myerson value gives 0 to every player. Take now g' such that $g = g'$. It is easily seen that at $g' = g$ the Myerson value recommends 0 for players out of T and equal payoffs for agents in T (as, by fairness, deleting a link in $g|T$ would give both participating players 0 payoff). Consider now any pair (R, g') with $R \in N|g'$ such that (i) $g \subseteq g'$ and $R \subseteq T$ and (ii) $g' \backslash g$ has k links. Assume by induction that, for any graph $\hat{g}$ with $g \subseteq \hat{g}$ and such that $\hat{g} \backslash g$ has $k-1$ links, all players in T get $\frac{1}{|T|}$ and players not in T get 0 payoff. I will show that the Myerson value recommends equal payoffs for agents in T and 0 payoff for agents outside T in g' too.

Note that, by fairness of the Myerson value, any two players in T directly connected in g would get equal payoffs: Deleting one link in $g|T$ gives 0 payoffs to both of them and deleting one link in $\tilde{g}$ not in g gives, by the induction assumption, $\frac{1}{|T|}$ to both of them. Any player not in T directly connected in g' to a player in T gets the payoff that would correspond to the player in T minus $\frac{1}{|T|}$: By fairness and by induction assumption, deleting their link gives zero to the player not in T and $\frac{1}{|T|}$ to the player in T. Thus, the component of g', namely R, that contains T distributes a value of 1, with all players in T obtaining equal payoff t and all players in $R \backslash T$ obtaining $t - \frac{1}{|T|}$. By component efficiency of the Myerson value,

$$|T|t + [|R| - |T|] \left[t - \frac{1}{|T|} \right] = 1,$$

which implies that $t = \frac{1}{|T|}$, and therefore all players outside T receive 0 payoffs in g'. This completes the proof of Lemma 6.3.3.

As the Myerson value satisfies both strong additivity and CW equal treatment of vital players, and these two axioms define a unique allocation rule, the Myerson value is the unique allocation rule satisfying strong additivity and CW equal treatment of vital players. This completes the proof of Theorem 6.3. □

References

[1] Feldman, B. E. (1996), *Bargaining, Coalition Formation and Value.* Ph.D. Dissertation. State University of New York at Stony Brook.

[2] Jackson, M. O. (2005), Allocation Rules for Network Games. *Games and Economic Behavior* 51: 128 – 154.

[3] Jackson, M. O. & A. Wolinsky (1996), A Strategic Model of Social and Economic Networks. *Journal of Economic Theory* 71: 44 – 74.

[4] Myerson, R. B. (1977), Graphs and Cooperation in Games. *Mathematics of Operations Research*, 2: 225 – 229.

[5] Myerson, R. B. (1977b), Myerson, R. B. (1977b), Values of Games in Partition Function Form. *International Journal of Game Theory* 6: 23 – 31

[6] Navarro, N. (2007), Fair Allocation in Networks with Externalities. *Games and Economic Behavior* 58: 354 – 364.

[7] van den Nouweland, A. (1993), *Games and Graphs in Economic Situations*. Ph.D. Dissertation. Tilburg University.

[8] Pápai, S. (2004), Unique Stability in Simple Coalition Formation Games. *Games and Economic Behavior* 48: 337 – 354.