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Nr. 2

Interpolation of Discount Factors

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Abstract

This paper deals with the problem of interpolation of discount factors between time buckets. The problem occurs when price and interest rate data of a market segment are assigned to discrete time buckets. A simple criterion is developed in order to identify arbitrage-free robust interpolation methods. Methods closely examined include linear, exponential and weighted exponential interpolation. Weighted exponential interpolation, a method still preferred by some banks and also offered by commercial software vendors, creates several problems and therefore makes simple exponential interpolation a more logical choice. Linear interpolation provides a good approximation of exponential interpolation for a sufficiently dense time grid.

1 Introduction

Valuation and pricing of financial instruments generally requires knowledge of discount factors and/or zero bond prices. Fundamental to the calculation of discount factors is detailed information on interest rates, as well as on prices of fixed income securities in special market segments (Bond-, FRA-, Swap-market) at present time t_0. The procedure for calculating a discount structure *df* from this information is as follows:

– Starting with market data we define a discrete time structure $t_1, t_2, \ldots, t_N$ and calculate the implied discount factor $df(t_0, t_n)$ for every time to maturity $t_n, n = 1, \ldots, N$ e.g. by using a bootstrapping technique.

– The calculation of the present value of a cash flow *CF(t)* occurring at time t requires the conversion of the discrete structure $df(t_0, t_1), df(t_0, t_2), \ldots,$ $df(t_0, t_N)$ into a continuous discount curve $t \to df(t_0, t), t \in [t_0, t_N]$.

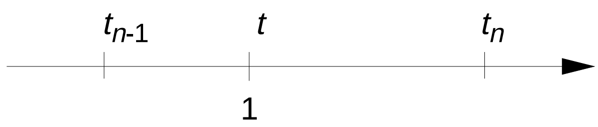
The complete set of empirical data is employed in order to derive the discrete discount structure, so that the second step of the problem is reduced to a pure interpolation problem. If the market data is incomplete then an interpolation problem may occur in the first step (e.g. this would be caused by a missing bond).

In this paper we study several widely used interpolation methods thereby confining ourselves to the study of those interpolation problems which require the knowledge of only two adjacent discount factors. McCulloch [1] has developed spline interpolation techniques by using the whole spectrum of market data. Spline interpolation offers a higher degree of smoothness, which has its price in terms of precision or even arbitrage-freeness. For a detailed discussion of this matter we refer to Breckling, Dal Dosso [2], [3] and Shea [4]. In a forthcoming paper we will investigate interpolation methods using all available market information.

Let us state the problem in more precise terms:

Problem

Let t_0 denote the present time, $t_1, t_2, \ldots, t_N$ the designated grid structure and let $df(t_0, t_1), df(t_0, t_2), \ldots, df(t_0, t_N)$ be the discount factors. The problem is the valuation of a given cash flow $CF(t) = (t,1)$, which pays an amount 1 at a time t with $t_0 \le t_{n-1} < t < t_n$, considering only the discount factors $df_{n-1} = df(t_0, t_{n-1})$ and $df_n = df(t_0, t_n)$ (without any restriction we assume $df_{n-1} > df_n$). Let the index n be fixed with $n \in \{1, \ldots, N\}$.



The problem can be looked upon from two different points of view which are somehow “dual“ to each other:

- *Interpolation*

  We calculate from discount factors df_{n-1} and df_n an interpolated value $df(t_0, t) = Ip(t, df_{n-1}, df_n)$ and determine the present value (*PV* = Present Value) of the payment $(t,1)$ to be

(1) $\quad PV(t,1) = 1 \cdot df(t_0, t)$.

- *Bucketing*

  Two functions $B_1 = B_1(t, df_{n-1}, df_n)$ and $B_2 = B_2(t, df_{n-1}, df_n)$ (bucketing functions) are to be determined in such a way that the cash flow $(t,1)$, which pays one unit in t can be replaced by the cash flows (t_{n-1}, B_1) and (t_n, B_2) (Buckets). The present value of the payment $(t,1)$ then is calculated as

(2) $\quad PV(t,1) = 1 \cdot B_1 \cdot df(t_0, t_{n-1}) + 1 \cdot B_2 \cdot df(t_0, t_n)$.

Tying together the two dual view points, i.e. equating (1) and (2) we obtain

(*) $\quad df(t_0, t) = B_1 \cdot df(t_0, t_{n-1}) + B_2 \cdot df(t_0, t_n)$;

Therefore all bucketing methods can be considered as special interpolation methods. This formula and conditions resulting from bucket hedging will be the key point in our analysis. Bucket hedging has been extensively studied by Turnbull [5].

The paper is organized as follows. First, we set some notation and state a no arbitrage condition suited for our purpose. In the second part, commonly applied interpolation techniques such as linear, exponential and weighted exponential interpolation are investigated in a qualitative manner. Their impact on zero rate structures as well as on forward rate curves is discussed in connection with some selected interest rate scenarios. It can be seen that the weighted exponential interpolation already has remarkable drawbacks. The final section contains the main results of this paper. A simple condition described by a system of differential equations is imposed on equation (*). Solutions to this system include the linear and exponential interpolation method. Interestingly, these two solutions are related by the fact that linear interpolation is the first order term of the Taylor series expansion of the exponential interpolation.

2 Notation

A continuous function

$$df_t = Ip(t, df_{n-1}, df_n), \quad t \in [t_{n-1}, t_n],$$

with boundary conditions

(3) $\quad Ip(t_{n-1}, df_{n-1}, df_n) = df_{n-1}$ and $Ip(t_n, df_{n-1}, df_n) = df_n$

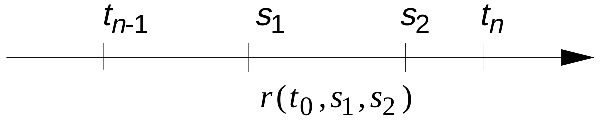
is called interpolation function. Let $df_{n-1} > df_n$ for all $n \in \{1, \dots, N\}$. An interpolation function *Ip* is called *arbitrage-free, if Ip* is strictly decreasing in t, that means

(4) $\quad Ip(s_1, df_{n-1}, df_n) > Ip(s_2, df_{n-1}, df_n) \quad \text{for} \quad t_{n-1} \le s_1 < s_2 \le t_n$.

Furthermore, we assume that the variables df_n are independent given the above restriction.

Remark 1

No arbitrage is equivalent to the fact that all forward interest rates $r(t_0, s_1, s_2)$ with $t_{n-1} \le s_1 < s_2 \le t_n$ are positive.



Proof: For the forward interest rate $r(t_0, s_1, s_2)$ one has

$$r(t_0,\ s_1,\ s_2) > 0 \qquad \Leftrightarrow \qquad \frac{df(t_0, s_2)}{df(t_0, s_1)} < 1 \qquad \Leftrightarrow \quad Ip(s_2, df_1, df_2) < Ip(s_1, df_1, df_2).$$

Since we are only interested in the relative distance of the time parameter t to the left boundary $t_{n\text{-}1}$, we will use the parameter λ instead of t where

$$\lambda = \lambda(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}.$$

We denote by df_λ the following expression

$$df_\lambda = Ip(\lambda, df_{n-1}, df_n), \quad \lambda \in [0,1]$$

Then the above boundary conditions can be restated in terms of the new parameter λ as:

(5) $\quad Ip(0, df_{n-1}, df_n) = df_{n-1}$ and $Ip(1, df_{n-1}, df_n) = df_n$.

3 Examples of interpolation functions

In the following section we look at different interpolation functions and discuss their qualitative behaviour. In analyzing the zero rate curve and the forward rate structure the following three zero rate scenarios are considered.

Maturity	Scenario 1	Scenario 2	Scenario 3
1 yr	5,0 %	8,5 %	7,0 %
2 yrs	6,5 %	7,0 %	7,0 %
3 yrs	7,5 %	6,0 %	7,0 %
4 yrs	8,2 %	5,3 %	7,0 %

3.1 Linear Interpolation

Linear interpolation is obtained by assigning the relative distances $1-\lambda$ and λ as weights to the discount factors df_{n-1} and df_n,i.e.:

(6) $df_\lambda = Ip^{lin}(\lambda, df_{n-1}, df_n) = (1-\lambda)df_{n-1} + \lambda df_n$.

The boundary conditions (5) are easily verified. The no arbitrage condition follows from

$$\frac{\partial Ip^{lin}}{\partial \lambda} = df_n - df_{n-1} < 0.$$

Discount curve

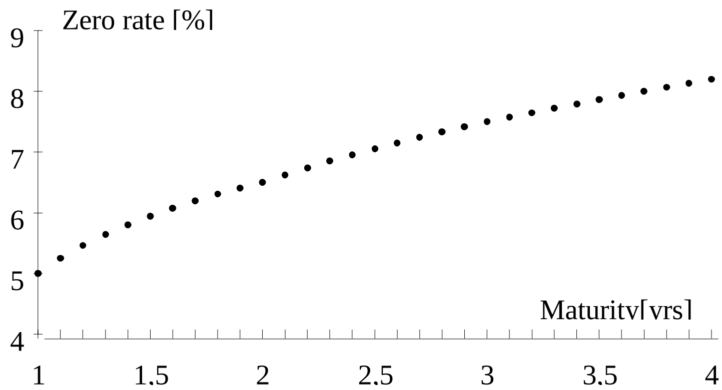
The resulting curve $t \to df_t = df_{\lambda(t)}$ is a continuous piecewise linear function which is in general not differentiable at $(t_1, df_1), (t_2, df_2), \ldots, (t_N, df_N)$.

Zero rate curve

If r_t denotes the continuously compounded zero rate of the discount factor $df_t = df_{\lambda(t)}$, then the interpolated interest rate r_t is expressed as follows:

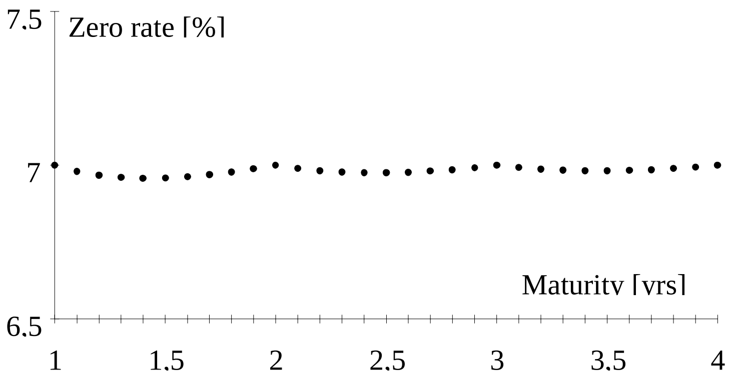
$$r_t = \frac{-\ln(df_{\lambda(t)})}{t - t_0} = \frac{-\ln((1-\lambda(t))e^{-r_{n-1}(t_{n-1}-t_0)} + \lambda(t)e^{-r_n(t_n - t_0)})}{t - t_0}$$

For the period [1 yr, 4 yrs] we obtain, using a time interval of length Δ = 0,1 yrs, and given scenario 1 the following zero rate curve.



Graph 1. Zero rate curve with normal term structure (scenario 1)

Similarly we obtain for an inverse term structure (scenario 2) a strictly decreasing zero rate curve with convex parts of the curve. In case of a flat term structure (scenario 3), linear interpolation yields a function which has slightly convex pieces.



Graph 2. Zero rate curve with flat interest rate structure (scenario 2)

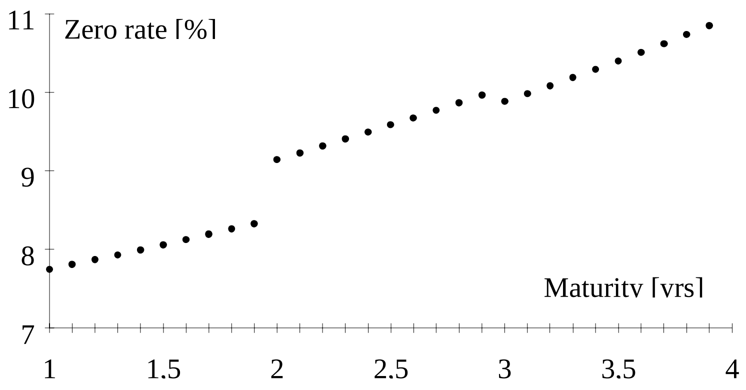
Forward rate curve

Let $df(t_0, s_1, s_2)$ denote the forward discount factor and $r(t_0, s_1, s_2)$ its exponential forward interest rate for the time interval $[s_1, s_2] \subset [t_{n-1}, t_n]$. The discount factor, respectively the forward rate, can be expressed by the following formulas

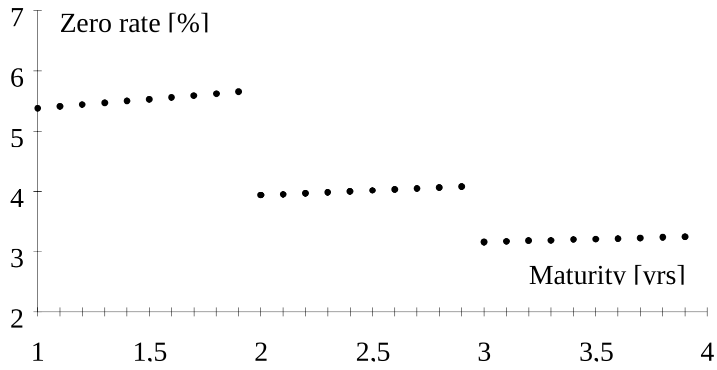
$$df(t_0, s_1, s_2) = \frac{df(t_0, s_2)}{df(t_0, s_1)} = \frac{(1-\lambda(s_2))df_{n-1} + \lambda(s_2)df_n}{(1-\lambda(s_1))df_{n-1} + \lambda(s_1)df_n} \quad \Rightarrow$$

$$r(t_0, s_1, s_2) = -\frac{\ln(df(t_0, s_1, s_2))}{s_2 - s_1}$$

For constant time intervals of length $\Delta = s_2 - s_1 = 0{,}1$ yrs and time interval [1 yr, 4 yrs] the forward curve is as follows:

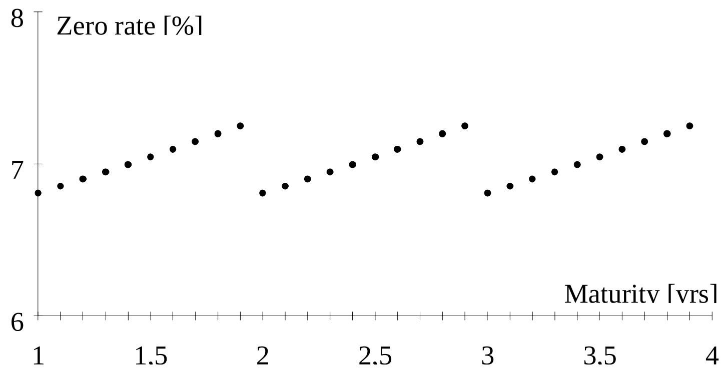


Graph 3. Forward rate curve with normal term structure (scenario 1)



Graph 4. Forward rate curve with inverse term structure (scenario 2)

In both scenarios (normal term structure as well as inverse term structure) one obtains increasing forward rates within the interpolation interval; discontinuities appear at the boundary of the time intervals. The discontinuities are due to the method of interpolation choosen, which calculates discount factors as an average of adjacent discount factors. In a flat term structure scenario (scenario 3), forward rates are not only increasing but also show a periodic behaviour.



Graph 5. Forward rate curve with flat term structure (scenario 3)

3.2 Exponential Interpolation

This form of interpolation is obtained by assigning certain exponents to the discount factors df_{n-1}, df_n:

(7) $$df_\lambda = Ip^{exp}(\lambda, df_{n-1}, df_n) = df_{n-1}^{1-\lambda} \cdot df_n^{\lambda}.$$

The boundary conditions (5) are easily verified, the no arbitrage condition (4) follows from

$$\frac{\partial Ip^{exp}}{\partial \lambda} = -(\ln df_{n-1}) df_{n-1}^{1-\lambda} df_n^{\lambda} + df_{n-1}^{1-\lambda} (\ln df_n) df_n^{\lambda} = df_\lambda (\ln df_n - \ln df_{n-1}) < 0.$$

Discount curve

Since

$$\frac{\partial^2 Ip^{exp}}{\partial \lambda^2} = df_\lambda (\ln df_n - \ln df_{n-1})^2 > 0$$

the exponential interpolation yields strongly convex pieces in the discount curve. The discount curve $t \to df_t = df_{\lambda(t)}$ is a continuous function, but in general not differentiable at the points $t_1, t_2, \ldots, t_N$.

Zero rate curve

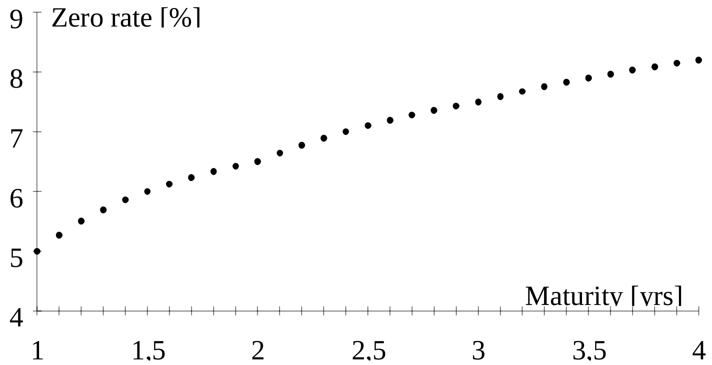
Let r_t denote the continuously compounded zero rate of the discount factor $df_t = df_{\lambda(t)}$. It is computed using the linearly interpolated value of the adjacent zero rates

$$\begin{aligned}\exp(-r_t(t-t_0)) = df_{\lambda(t)} &= df_{n-1}^{1-\lambda(t)} df_n^{\lambda(t)} \\ &= \exp(-(1-\lambda)(t_{n-1}-t_0)r_{n-1}) \exp(-\lambda(t_n - t_0)r_n) \\ &= \exp\left(-\left((1-\lambda)\frac{t_{n-1}-t_0}{t-t_0} r_{n-1} + \lambda \frac{t_n - t_0}{t-t_0} r_n\right)(t-t_0)\right)\end{aligned}$$

and

$$r_t = (1-\lambda)\frac{t_{n-1}-t_0}{t-t_0} r_{n-1} + \lambda \frac{t_n - t_0}{t-t_0} r_n .$$

Given scenario 1, the zero rate curve appears as follows, once again by using time intervals of Δ = 0,1 yrs and time periods [1 yr, 4 yrs]:

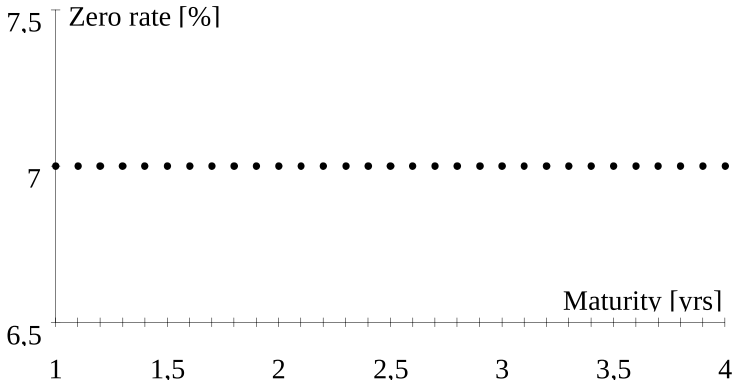


Graph 6. Zero rate curve with normal term structure (scenario 1)

Given scenario 2, the zero rate curve decreases yielding convex curve pieces. Given a flat zero rate structure (scenario 3), the exponential interpolation maintains this property, which can be derived as follows: If $r_{n-1} = r_n$ one obtains

$$r_t = (1-\lambda)\frac{t_{n-1}-t_0}{t-t_0} r_{n-1} + \lambda \frac{t_n - t_0}{t-t_0} r_n = \left((1-\lambda)\frac{t_{n-1}-t_0}{t-t_0} + \lambda \frac{t_n - t_0}{t-t_0} \right) \cdot r_n$$

$$= \frac{(t_n - t)(t_{n-1}-t_0) + (t - t_{n-1})(t_n - t_0)}{(t_n - t_{n-1})(t - t_0)} \cdot r_n = r_n \quad \text{for all } t.$$



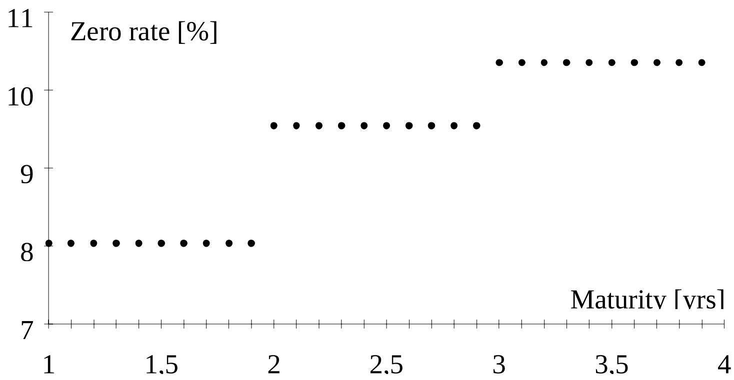
Graph 7. Zero rate curve with flat term structure (scenario 3)

Forward rate curve

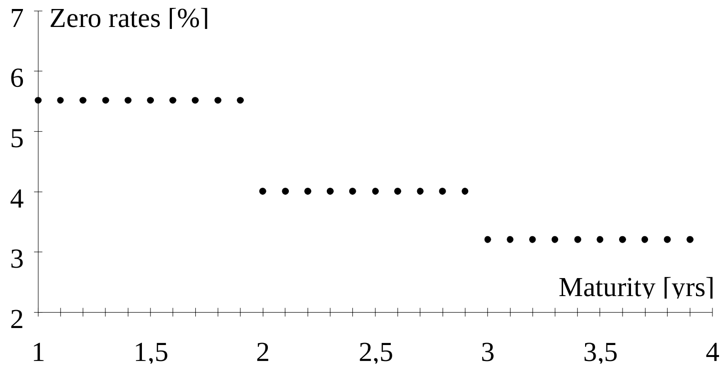
Exponential interpolation implies constant forward rates $r(t_0, s_1, s_2)$ for time intervals $[s_1, s_2]$ of equal length. Let s_1 and s_2 be such that $t_{n-1} \le s_1 < s_2 \le t_n$. Then given $\lambda_1 = \lambda(s_1)$, $\lambda_2 = \lambda(s_2)$ and a forward discount factor $df(t_0, s_1, s_2)$ it can be rewritten as

$$df(t_0, s_1, s_2) = \frac{df(t_0, s_2)}{df(t_0, s_1)} = \frac{df_{n-1}^{1-\lambda_2} df_n^{\lambda_2}}{df_{n-1}^{1-\lambda_1} df_n^{\lambda_1}} = \frac{df_n^{\lambda_2 - \lambda_1}}{df_{n-1}^{\lambda_2 - \lambda_1}} = \left(\frac{df_n}{df_{n-1}} \right)^{\frac{s_2 - s_1}{t_n - t_{n-1}}},$$

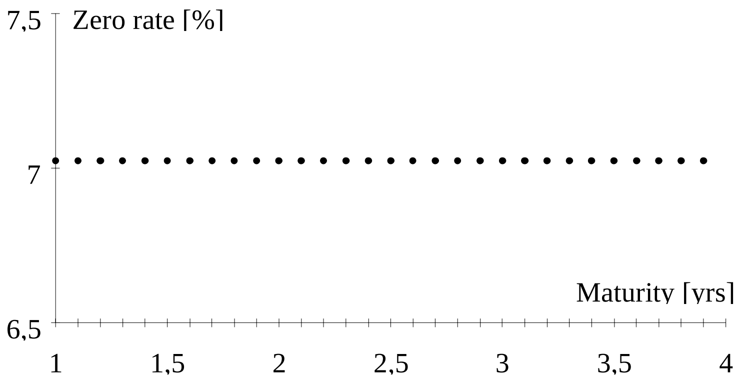
i.e. $df(t_0, s_1, s_2)$ and $r(t_0, s_1, s_2)$ as well only depend on the distance $s_2 - s_1$. For time intervals with a length of $\Delta = s_2 - s_1 = 0{,}1$ yrs and time periods [1 yr, 4 yrs] we obtain the following forward rate curve



Graph 8. Forward rate curve with normal term structure (scenario 1)



Graph 9. Forward rate curve with inverse term structure (scenario 2)



Graph 10. Forward rate curve with flat term structure (scenario 3)

3.3 Weighted Exponential Interpolation

This interpolation method is obtained by assigning additional time weights to the exponents in (7):

(8) $$df_t = Ip^{weight\ exp}(t, df_{n-1}, df_n) = df_{n-1}^{\alpha_{n-1}(t)\cdot(1-\lambda(t))} \cdot df_n^{\alpha_n(t)\cdot\lambda(t)}$$

where: $\lambda = \lambda(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}$ and $\alpha_i = \alpha_i(t) = \frac{t - t_0}{t_i - t_0}$

$Ip^{weight\ exp}$ satisfies the boundary conditions (3), however the no arbitrage condition (4) does not hold.

Counterexample

Let $df_1 = 0.91$, $df_2 = 0.89$, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$ and $t = 1.8$ then

$$df(1.8) = 0.88882386 < 0.89 = df_2 .$$

According to Remark 1 in Section I, negative or zero forward rates cannot be excluded by interpolation method (8).

Remark 2

In order to obtain a valid expression for the divisor $t_{n-1} - t_0 = t_0 - t_0 = 0$ in formula (8) for the first time interval $[t_0, t_1]$ where *n* = 1, we set: $t_1 = t_0 +$ 1 day and $r_1 =$ overnight-rate.

Discount curve

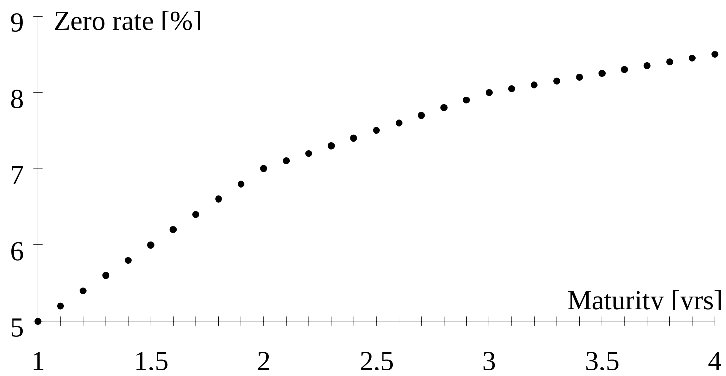
The discount curve $t \to df_t$ is a continuous function, but not necessarily differentiable at points $t_1, t_2, \ldots, t_N$.

Term structure

Let r_t be the exponential interest rate with discount factor $df_t = df_{\lambda(t)}$, then the interpolated rate r_t is given by

$$r_t = (1 - \lambda) r_{n-1} + \lambda r_n,$$

i.e. r_t is obtained by interpolating adjacent rates in a linear fashion. The term structure as defined by the previous scenarios yields the following shape:



Graph 11. Interest rate curve with normal term structure (scenario 1)

Similar graphs are obtained for inverse (scenario 2) and flat (scenario 3) term structures using piecewise linear functions.

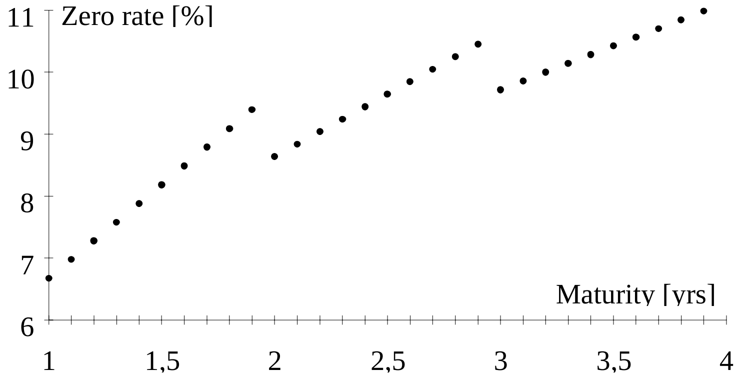
Forward curve

For the forward discount factor $df(t_0, s_1, s_2)$ and its associated weighted exponential forward rate $r(t_0, s_1, s_2)$ for the time period $[s_1, s_2] \subset [t_{n-1}, t_n]$ we have

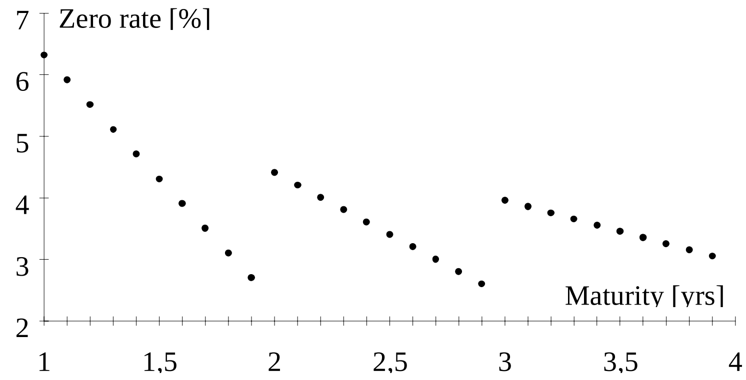
$$df(t_0, s_1, s_2) = \frac{df(t_0, s_2)}{df(t_0, s_1)}$$
$$= df_{n-1}^{\frac{(s_2-t_0)(t_n-s_2)-(s_1-t_0)(t_n-s_1)}{(t_{n-1}-t_0)(t_n-t_{n-1})}} df_n^{\frac{(s_2-t_0)(s_2-t_{n-1})-(s_1-t_0)(s_1-t_{n-1})}{(t_{n-1}-t_0)(t_n-t_{n-1})}}$$

$$r(t_0, s_1, s_2) = -\frac{\ln(df(t_0, s_1, s_2))}{s_2 - s_1}.$$

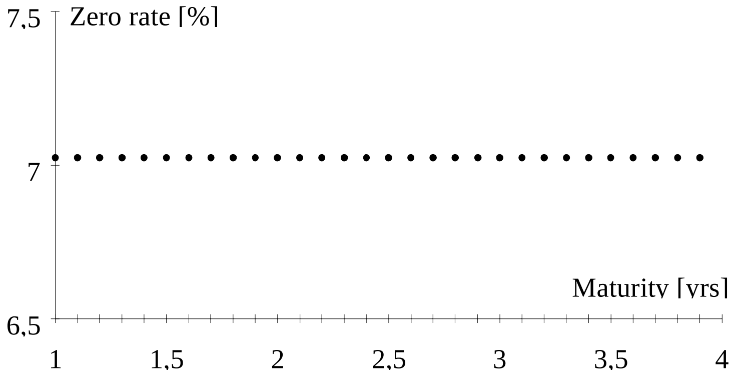
For time intervals of equal length $\Delta = s_2 - s_1 = 0{,}1$ yrs and time periods [1 yr, 4 yrs] we obtain the following forward rates, given the aforementioned scenarios:



Graph 12. Forward interest rate curve with normal term structure (scenario 1)



Graph 13. Forward rate curve with inverse term structure (scenario 2)



Graph 14. Forward rate curve with flat term structure (scenario 3)

4 Results

A large class of interpolation methods is obtained by using so called bucketing procedures. As mentioned in the introduction, »buckets« for a cash flow $(t,1)$ where $t_0 \le t_{n-1} < t < t_n$ are confined to the time period t_{n-1} and t_n. Two continuous functions

$$B_1 = B_1(t, df_{n-1}, df_n) \text{ and } B_2 = B_2(t, df_{n-1}, df_n), \quad t \in [t_{n-1}, t_n]$$

with $0 \le B_1 \le 1$ and $0 \le B_2 \le 1$ satisfying the boundary conditions

(9) $t = t_{n-1}$: $B_1(t_{n-1}, df_{n-1}, df_n) = 1$ and $B_2(t_{n-1}, df_{n-1}, df_n) = 0$

$t = t_n$: $B_1(t_n, df_{n-1}, df_n) = 0$ and $B_2(t_n, df_{n-1}, df_n) = 1$

are called *bucketing functions* or a *bucketing procedure*. As mentioned initially, every bucketing procedure defines an interpolation method. If B_1 and B_2 are bucketing functions, then

(10) $Ip(t, df_{n-1}, df_n) = B_1(t, df_{n-1}, df_n) \cdot df_{n-1} + B_2(t, df_{n-1}, df_n) \cdot df_n$

is the *associated interpolation function*. Given (9), the boundary conditions (3) are satisfied. A bucketing procedure B_1, B_2 is called *arbitrage-free*, if the associated interpolation function *Ip* is arbitrage-free, i.e. if *Ip* is strictly decreasing in *t*. A sufficient condition is

$$\frac{\partial B_1(t, df_{n-1}, df_n)}{\partial t} df_{n-1} + \frac{\partial B_2(t, df_{n-1}, df_n)}{\partial t} df_n < 0$$

provided B_1 and B_2 are differentiable in *t* . Boundary conditions and the no arbitrage property of bucketing procedures have analogue concepts for the associated interpolation function. However, the concept of robustness which is discussed below, seems to have no apparent similarities to interpolation. Robustness is the essential ingredient in deriving “reasonable“

interpolation/bucketing procedures. Further, we assume that the function *Ip* is continuously differentiable in the variables df_{n-1} and df_n.

A bucketing procedure is called *robust*, if B_1, B_2, and its associated interpolation function satisfy the following system of partial differential equations

(**) $$\frac{\partial Ip(t, df_{n-1}, df_n)}{\partial df_{n-1}} = B_1(t, df_{n-1}, df_n) \quad ,$$

$$\frac{\partial Ip(t, df_{n-1}, df_n)}{\partial df_n} = B_2(t, df_{n-1}, df_n) \quad \text{for all} \quad t \in [t_{n-1}, t_n].$$

Interpretation

The Taylor series of the associated interpolation function satisfying (**) is given for fixed $t \in [t_{n-1}, t_n]$ and (df_{n-1}^0, df_n^0) by

$$\begin{aligned} Ip(t, df_{n-1}, df_n) &= Ip(t, df_{n-1}^0, df_n^0) + \frac{\partial Ip}{\partial df_{n-1}}(t, df_{n-1}^0, df_n^0) \cdot (df_{n-1} - df_{n-1}^0) \\ &\quad + \frac{\partial Ip}{\partial df_n}(t, df_{n-1}^0, df_n^0) \cdot (df_n - df_n^0) + R_1 \\ &= B_1(t, df_{n-1}^0, df_n^0) \cdot df_{n-1} + B_2(t, df_{n-1}^0, df_n^0) \cdot df_n + R_1 \end{aligned}$$

Consequently, small changes in discount factors df_{n-1} and df_n ($\Rightarrow$ a small error term R_1) will result in invariant bucketing functions $B_1(t, df_{n-1}, df_n)$ and $B_2(t, df_{n-1}, df_n)$. Therefore, a hedge based on bucketing does not have to be adjusted for small changes in market factors.

The main conclusion of the paper is

Theorem:

Let B_1, B_2 be as stated above, and Ip the associated interpolation function. Then

(a) $B_1^{lin}(t, df_{n-1}, df_n) = 1 - \lambda(t) = \frac{t_n - t}{t_n - t_{n-1}}$ and $B_2^{lin}(t, df_{n-1}, df_n) = \lambda(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}}$

is an arbitrage-free solution to the system (**) where λ denotes the relative distance of *t* to t_{n-1}. The associated interpolation function is linear and expressed by

$$Ip^{lin}(t, df_{n-1}, df_n) = (1 - \lambda(t))df_{n-1} + \lambda(t)df_n .$$

(b) $$B_1^{exp}(t, df_{n-1}, df_n) = (1-\lambda(t))\left(\frac{df_n}{df_{n-1}}\right)^{\lambda(t)}$$ and

$$B_2^{exp}(t, df_{n-1}, df_n) = \lambda(t)\left(\frac{df_n}{df_{n-1}}\right)^{\lambda(t)-1}$$

is an arbitrage-free solution to the system (**) where λ is as above. The associated interpolation function is as follows:

$$\begin{aligned} Ip^{exp}(\lambda(t), df_{n-1}, df_n) &= B_1^{exp}(t, df_{n-1}, df_n)\cdot df_{n-1} + B_2^{exp}(t, df_{n-1}, df_n)\cdot df_n \\ &= df_{n-1}^{1-\lambda(t)}\cdot df_n^{\lambda(t)} \end{aligned}$$

(c) The two bucketing procedures are approximately the same which can be seen from the first term of the Taylor series expansion of the exponential interpolation. Let λ be between 0 and 1 and (df_{n-1}^0, df_n^0) be fixed. Then

$$\begin{aligned} Ip^{exp}(\lambda, df_{n-1}, df_n) &= Ip^{exp}(\lambda, df_{n-1}^0, df_n^0) + \frac{\partial\, Ip^{exp}}{\partial\, df_{n-1}}(df_{n-1}^0, df_n^0)(df_{n-1} - df_{n-1}^0) \\ &\quad + \frac{\partial\, Ip^{exp}}{\partial\, df_n}(df_{n-1}^0, df_n^0)(df_n - df_n^0) + R_1(df_{n-1}^0, df_n^0) \\ &= Ip^{exp}(\lambda, df_{n-1}^0, df_n^0) + (1-\lambda)(df_{n-1}^0)^{-\lambda}(df_n^0)^{\lambda}(df_{n-1} - df_{n-1}^0) \\ &\quad + \lambda\,(df_{n-1}^0)^{1-\lambda}(df_n^0)^{\lambda-1}(df_n - df_n^0) + R_1(df_{n-1}^0, df_n^0) \\ &= (1-\lambda)\left(\frac{df_n^0}{df_{n-1}^0}\right)^{\lambda}\cdot df_{n-1} + \lambda\left(\frac{df_n^0}{df_{n-1}^0}\right)^{\lambda-1}\cdot df_n + R_1(df_{n-1}^0, df_n^0) \end{aligned}$$

For small values of $t_n - t_{n-1}$ one has

$$R_1(df_{n-1}^0, df_n^0) \approx 0 \quad \text{and} \quad \frac{df_n^0}{df_{n-1}^0} \approx 1,$$

and therefore,

$$Ip^{exp}(\lambda, df_{n-1}, df_n) \approx (1-\lambda)\cdot df_{n-1} + \lambda\cdot df_n = Ip^{lin}(\lambda, df_{n-1}, df_n).$$

Remark 3

(1) The boundary conditions specified for our differential equations by no means guarantee a unique solution.

(2) The solution in (b) can be slightly generalized, if $\lambda(t)$ is replaced by a strictly increasing continuous function with values between 0 and 1.

(3) The weighted exponential interpolation does not yield a robust bucketing procedure as

$$\frac{\partial Ip^{weight\ exp}(t,df_{n-1},df_n)}{\partial df_{n-1}} = \alpha_{n-1}(1-\lambda)\left(\frac{df_n}{df_{n-1}}\right)^{\alpha_{n-1}(1-\lambda)-1} \quad \text{and}$$

$$\frac{\partial Ip^{weight\ exp}(t,df_{n-1},df_n)}{\partial df_n} = \alpha_n\lambda\left(\frac{df_n}{df_{n-1}}\right)^{\alpha_n\lambda-1}$$

produces the following expressions for the bucketing functions $B_1^{weight\ exp}$ and $B_2^{weight\ exp}$

$$B_1^{gew\ exp}(t,df_{n-1},df_n) = \alpha_{n-1}(t)\cdot(1-\lambda(t))\cdot\left(\frac{df_n}{df_{n-1}}\right)^{\alpha_{n-1}(t)(1-\lambda(t))-1} \quad \text{and}$$

$$B_2^{gew\ exp}(t,df_{n-1},df_n) = \alpha_n(t)\cdot\lambda(t)\cdot\left(\frac{df_n}{df_{n-1}}\right)^{\alpha_n(t)\lambda(t)-1}.$$

The weighted exponential interpolation method, although still often used, satisfies neither the no arbitrage nor the robustness condition.

References:

[1] Mc Culloch, H.J.: *Measuring the Term Structure of Interest Rates*, Journal of Business, XLIV (January 1971), 19 - 31

[2] Breckling, J.; L. Dal Dasso*: A Non-parametric Approach to Term Structure Estimation*, in Hrsg. G. Bol, G. Nakhaeizadeh, K.-H. Vollmer: Finanzmarktanwendungen neuronaler Netze und ökonometrischer Verfahren, Physica Verlag Heidelberg 1994

[3] Breckling, J.; L. Dal Dasso: *Modelling of Term Structure Dynamics Using Stochastic Processes*, in Hrsg. G. Bol, G. Nakhaeizadeh, K.-H. Vollmer: Finanzmarktanwendungen neuronaler Netze und ökonometrischer Verfahren, Physica Verlag Heidelberg 1994

[4] Shea, G.S.: *Pitfalls in Smoothing Interest Rate Term Structure Data: Equilibrium Models and Spline Approximations*, Journal of Financial and Quantitative Analysis, Vol 19. No 3 (September 1984), 253 - 267

[5] Turnbull, S.M.: *Evaluating and Implementing Bucket Hedging*

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1.	Moormann, Jürgen Lean Reporting und Führungsinformationssysteme bei deutschen Finanzdienstleistern	1995
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