# Keeping up with the technology pace: a DNS-curve and a limit cycle in a technology investment decision problem*

Josef L. Haunschmied[1]**, Gustav Feichtinger[1], Richard F. Hartl[2], Peter M. Kort[3,4]

[1]Institute of Mathematical Methods in Economics, University of Technology, Vienna, Austria

[2]Institute of Management, University of Vienna, Vienna, Austria

[3]Department of Econometrics & Operations Research and CentER, Tilburg University, Tilburg, The Netherlands,

[4]Department of Economics, University of Antwerp, Antwerp, Belgium.

**Abstract**

In order to keep up with its economic environment a firm should respond to new technological developments. In this paper we establish the optimal technology investment decision within a dynamic model, in which the baseline technology level rises over time. The problem is analyzed by designing a two state optimal control model. It turns out that in the state space a DNS-(Dechert-Nishimura-Skiba-) curve can be determined that separates different long run outcomes, viz., zero investment, constant positive investment, or a cyclical sequence of zero and positive investment.

*JEL classification:* C61, D92

*Keywords:* Investment; Multiple steady states; Threshold behavior; Optimal control; Bifurcation.

*Thanks are due to an anonymous referee for his comments and to Jon Caulkins for valuable discussions and suggestions concerning the model formulation. This research was partly financed by the Austrian National Bank (OENB) under grant No. 9414.

** Corresponding author: Vienna Univ. of Technology; Institute 105-4; Argentinierstr. 8; 1040 Vienna, Austria, Europe; Phone: +43 1 58801 11926; Fax: +43 1 58801 11999; Email: Josef.Haunschmied@tuwien.ac.at

# 1 Introduction

Technology advances quickly these days, and for firms it is important to keep up with the latest developments since customers like to buy the most modern products. However, a certain trade off may exist because of the presence of an adverse investment effect: if new versions of products appear soon after each other, customers continually have to get used to handling these new versions. Obvious examples are the new generations of software: people can be very reluctant to change the software they use. The older version still works as though it were new, while changing to a new (more powerful) version leads to an at least temporary drop in efficiency since the user has to learn how to work with it and there may also be compatibility problems. This is one of the reasons for the empirical fact that new technologies are often adopted on a large scale only after a prolonged period of time (Chari and Hopenhayn, 1991; Mansfield, 1968; Mulder et al., 2003). Further, if a firm has a reputation such that new versions of its product follow each other quickly, customers may refrain from buying a product and waiting for a more modern version to appear on the market, causing a negative effect on the volume of sales

The firm has a certain technology level that is reflected in the technological content of its product. In general it holds that the greater the technological content or the more modern the product, the more attractive it is to customers. Since an important determinant of the firm's sales is their degree of competitiveness, the absolute technological content of its products is less important than its content relative to some baseline level, so when we speak of technological content of a product in this paper, or the firm's technology level, we mean the technological content, or technology level, relative to the baseline level. Due to general technological progress, this baseline level increases over time. This implies that if the firm does nothing to increase its technology level, thus refrain from technology investments, its relative standing will decrease over time.

Firms can raise their technology level by adopting new technologies or by performing R & D themselves. It is clear that these technology investments are not very effective when there is almost no know-how present within the firm. In this case, the firm's current technological level is low and will remain low unless a very substantial effort is made to increase the technological content. Increasing the technological content is also a difficult task if the firm is already producing the most modern products. In that case very advanced R & D activities are required to increase the technological content of such a high quality product (see also Das and Van de Ven (2000)). By now it is clear that the effectiveness of technology investments depends on the present technological content of the firm's products. To increase the technology level of the firm by a given amount, more effort is needed either when the technology level is low or high compared to intermediate levels of technology.

This paper presents a model in which the above characteristics are incorporated, demonstrating that

**(i)** technology investments increase the firm's technology level which increases the number of customers and thus sales,

**(ii)** some customers are distracted by technological changes so that the sales volume decreases with technology investments, and

**(iii)** the effectiveness of technology investments depends on the firm's technology level.

The resulting model is an optimal control model with two state variables, the number of customers and the firm's technology level, and one control variable, the firm's technology investment rate.

The solution has very interesting characteristics. If the initial technology level is low, it is not optimal to increase the technology level by technology investments since such investments are not very effective due to a lack of know-how within the firm. The implication is that the firm will quit its operations in the long run. However, for a sufficiently high initial technology level it is worthwhile for the firm to converge to a long run optimal limit set, which can either be a steady state or a stable limit cycle. In a plane with the sales volume on the horizontal axis and technology level on the vertical axis a curve (called DNS-curve) separates these two regions of optimal convergence. On the DNS-curve the firm is indifferent between two long-run optimal outcomes (steady state at the origin versus interior steady state/limit cycle).

Above the DNS-curve oscillatory behavior of optimal paths is caused by model characteristic (ii). When this negative effect of technological changes on the sales volume is low, then converging to the interior steady state is optimal but happens in a less damped oscillatory way as this effect becomes larger. At a certain level of this effect, a limit point (blue sky) bifurcation occurs, which means that a semi-stable limit cycle arises ("out of the blue sky"). The implication for long-run behavior is that inside the limit cycle convergence to the steady state is still optimal, while outside the limit cycle convergence to the limit cycle itself takes place. Increasing the level of effect (ii) beyond the blue sky bifurcation level, the semi-stable limit cycle splits to a stable and an unstable limit cycle, where the unstable limit cycle is situated entirely within the stable one. Concerning long-run behavior, it now holds that within the unstable limit cycle convergence to the steady state occurs, while outside the unstable limit cycle, the system ends up at the stable limit cycle. A further increase of the effect of model characteristic (ii) results in a larger size of the stable limit cycle and a reduction in size of the unstable limit cycle; eventually the unstable limit cycle and the steady state coincide. Exactly when this happens a Hopf bifurcation arises, implying that above the DNS-curve we always have convergence to the stable limit cycle. This stable limit cycle still exists and increases in size when we increase effect (ii) beyond this point, but eventually it will turn out that the firm will refrain completely from technology investments.

The paper is organized as follows: in Section 2 the model is presented, while in Section 3 Pontryagin's maximum principle is applied to establish the necessary conditions for an optimal solution. The optimal solution is presented in Section 4 for different scenarios, while an economic interpretation is provided in Section 5.

## 2 A Model of Technology Investment and Customer Attraction

With $T(t)$ we denote the product's technological content relative to some baseline level at time $t$. In general it holds that the greater the technological content or the more modern the product, the more attractive it is to customers. The state variable $T(t)$ is also a measure for the level of know-how within the firm at time $t$.

The firm has the possibility to undertake investments, $I(t)$, in order to keep up with the technological development or even increase its own technological content relative to the baseline level. It is assumed that the effectiveness of technology investment depends on the technology level. Let us denote the rate of effectiveness of technology investment by $h(T)$. The firm will have a higher technology level when it invests (i.e. $I > 0$) than when it refrains from investment, which implies that $h(T) > 0$.

A firm lacking know-how can purchase/license standard technology and then try to adapt and improve it, but this would neither substantially increase the technological content of its products relative to the competitors nor the effectiveness of further investments. Hence, if the firm has almost no know-how, we assume that technology investments will not be very effective and, additionally, that it is hard to raise the effectiveness of technology investment in the near future. Further, it is certainly true that it is difficult to raise the technology level further when it is already large, the extreme case is a firm already producing the most modern products. Then in order to increase the technology level even further it needs to develop new technologies on its own account, which is in general quite expensive. Only top specialists can help to raise the firm's technology level, but it requires a lot of money to attract these people. Consequently, the technology increase per unit of investment is low.

Translating these observations in terms of the function $h(T)$ we conclude that the investment effectiveness function $h$ must increase for low values of $T$, while it decreases for high values of $T$. We model $h$ unimodal with a peak for a medium value of $T$. Furthermore, if it is taken into account that technology investments will be not that effective when $T$ is small (almost no know-how), and that $h(T)$ is non-negative also for very large technology levels, it can be concluded, that a convex-concave-convex-bell shape for $h$ is reasonable (cf. Figure 1). In this way we arrive at the following specification:

$$\lim_{T\to\infty} h(T) = 0, \quad h(T) > 0$$

$$h'(T)\binom{>}{<}0 \text{ for } T\binom{<}{>}T_{\max}, \quad h'(0)\geq 0, \tag{1}$$

$$\begin{aligned} h''(T) &> 0 \text{ for } T<T_1 \text{ and } T>T_2,\\ h''(T) &< 0 \text{ for } T_1<T<T_2, \end{aligned}$$

where obviously for the inflection points $T_1$ and $T_2$ it holds that

$$T_1 < T_{\max} < T_2.$$

Concerning R & D investments there is some empirical evidence for such a bell-shaped curve, in particular with respect to technological spillover effects and the concept of a firm's 'absorptive capacity' (see Cohen and Levinthal (1989)). Actually, Verspagen (1993) has empirically obtained a bell-shaped relationship between the technological distance of a firm from the technology frontier and the ability to integrate external knowledge through spillovers.

The state equation for the firm's technological content relative to the baseline level is (we omit the time argument $t$; $\dot{T}$ is the derivative of $T$ w.r.t. $t$)

$$\dot{T} = h(T)I - \delta T, \quad T(0) = T_0 > 0, \tag{2}$$

where it is assumed that in this economy technology develops with a constant rate $\delta$ ($>0$).

With $C(t)$ we denote the firm's sales volume at time $t$. Function $f(T)$ describes how much sales volume the firm would have attracted in a long run if it held its technology level at $T$ forever. We assume $f(0) = 0$, of course $f'(T) > 0$ (because the better the technology, the higher sales volume the firm would have) and non-increasing returns to technology level, $f''(T) \leq 0$. One could imagine that the rate of change in the number of sales positively depends on the difference between the current sales volume ($C$) and the number the current level of technology warrants ($f(T)$). The speed at which customers adjust to the technology level of the firm's products is denoted by the positive factor $\gamma$.

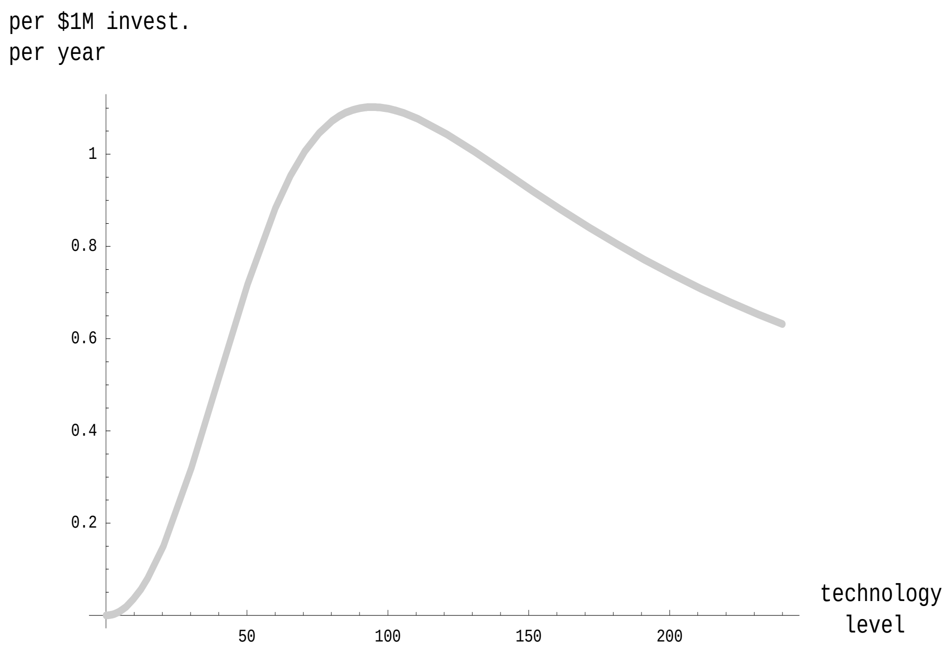



Figure 1: A generic shape of the effectiveness of technology investment $h(T)$ on the basis of the existing know-how in the firm.

Customers like products with a larger technological content, but they may get frustrated by rapid changes in the technology because it forces them to go up a learning curve, there may be compatibility problems, or purchasing a new item can happen too early, because especially in case of rapid technological developments there is always the risk that a better version appears on the market soon after the date of the present purchase. While the firm is investing a lot in improving the technology, customers will tend either to go to competitors because they want a stable product or simply wait longer with their purchase because they decide to wait for newer versions. Also the demand for low cost daily products such as shampoo, tooth paste, vanishing cream, and so on, can be affected by rapid technological change. For instance, on a spam page on the World Wide Web it can be read that "Infused with xyz's legendary age-delaying ingredients and the **best of modern technology**, xyz unite to transform the present - and future - of your skin." A firm has to improve the technology standard, for instance for a moisturizer, to stay in or even to expand its market position. Before they opt for surgery, customers demand creams with higher and higher technology level. On the other hand there are customers who are attached to a particular version of the product, and they like it for several reasons. At the moment that this version is replaced by a newer one, and thus not available in the shop anymore, such a customer may switch to a product from a competitive firm.

If we denote the factor measuring the negative effect of technology investment on the sales volume by $k$, we get the following dynamic equation for the sales volume:

$$\dot{C} = \gamma \left[ f\left(T\right) - C \right] - k\, C\, I, \quad C\left(0\right) = C_0 > 0. \tag{3}$$

Technology investments are irreversible, so that

$$I \geq 0. \tag{4}$$

It can be argued that it is not $I$ that changes technology, but $\dot{T}$. Thus the term $-k\, C\, I$ could also be $-k\, C\, (\dot{T} + \delta\, T) = -k\, C\, h(T)\, I$. (In case of $I = 0$, there is no loss of customers caused by technology change because the product remains unchanged; the change of the technology content results due to the assumed increase of the technology baseline by a constant rate $\delta$.). Besides obvious simplification in the next four paragraphs, we state additional motivation for using the first term:

In this context, we assume that customer acceptance of technology changes depends on the actual technology content of the product. If the technology content is low, the customers are attracted mainly by other features such as low cost or "easy to use" (in the sense of inability to use high technology products). Such customers act most irritated to a technology increase. As an example we want to state the strong adoption efforts in Latin American countries on Information and Communication Technology (ICT). The ICT use there was moderate, so Latin American countries have invested huge amounts of money to adopt more sophisticated ICT. The technology progress was moderate (compare the left branch of the bell-shaped curve in Figure 1), and ICT adoption resulted in an absolute loss of customers.

On the other site of the bell-shaped curve in the case of an absolute technology leader, further technology investment results in a futuristic technology where the customer may not really be able to estimate its value and advantages and may even be distracted; as an example we want to mention magnetic-levitation trains.

Additional motivation is that the firm's image plays a role in appeasing the customer. Customers are expecting that (especially) firms offering advanced product technology have to increase the technology content of their products continuously, so customers consent partly to accept the arduousness caused by a new version of the product. Being honest, was it really necessary to purchase all the stages of development of a wide-spread, well-known computer operating system in the last decade? But people did.

Modeling these three effects we scale down the loss of customers due to changes of the technology content of the product by $h(T)$,

$$\frac{-k\,C\,(\dot{T}+\delta\,T)}{h(T)} = -k\,C\,I.$$

Finally, this firm invests in such a way that the discounted profit flow is maximized. Profits decrease with investment costs while they increase with revenue where it is assumed that revenue per sale ($\alpha > 0$) is constant . Investment costs, the costs associated with carrying out technology investments can consist of purchasing new machines by which more modern products can be made, laboratory development of product improvements, and so on. Investment costs are denoted by a convexly increasing function $a(I)$ ($a(0) = 0, a' > 0, a'' > 0$). The convex shape reflects the fact that there are decreasing returns to effort at any point in time. Hence, the firm's objective functional is given by

$$\max_I \int_0^\infty e^{-rt}[\alpha C - a(I)]dt, \tag{5}$$

where $r$ is the constant discount rate.

Now the optimal control model consists of the objective (5) subject to the state equations (2) and (3) and the non-negativity constraint (4).

# 3 Analysis

In this section we present the necessary optimality conditions and analyze the possible occurrence of steady states.

### 3.1 Necessary Optimality Conditions

First we specify the current value Hamiltonian for the optimal control model (2) – (5):

$$H = \alpha\, C - a(I) + \lambda_1 \Big(\gamma\, [f\,(T) - C] - k\, C\, I\Big) + \lambda_2 \Big(h(T) I - \delta T\Big).$$

Applying Pontryagin's maximum principle (for example, Feichtinger and Hartl, 1986; Léonard and Long, 1992) (i.e. maximizing the Hamiltonian w.r.t. $I \geq 0$) we derive for the control variable $I$ that

$$\left\{ \begin{array}{rclrcl} I &=& 0 & a'(0) + \lambda_1\, k\, C &\geq& \lambda_2\, h(T) \\ a'(I) + \lambda_1\, k\, C &=& \lambda_2\, h(T) & a'(0) + \lambda_1\, k\, C &<& \lambda_2\, h(T). \end{array} \right. \tag{6}$$

Expression (6) shows that the firm invests such that marginal costs, consisting of the marginal investment costs, $a'(I)$, and the negative effect of marginal investment on the sales volume arising from the adverse investment effect, $\lambda_1\, k\, C$, equals marginal revenue. The latter consists of the increase of the technology level due to marginal investment, $h\,(T)$, which is valued by the shadow price of technology, $\lambda_2$. The firm refrains from investment when marginal costs, valued for zero investment, $I = 0$, exceeds marginal revenue.

According to the maximum principle an optimal path has to fulfill the following dynamic system (combined with the algebraic equation (6)):

$$\begin{aligned} \dot{C} &= \gamma\, [f\,(T) - C] - k\, C\, I, && (7) \\ \dot{T} &= h(T) I - \delta T, && (8) \\ \dot{\lambda}_1 &= [r + \gamma + k\, I]\ \lambda_1 - \alpha, && (9) \\ \dot{\lambda}_2 &= [r + \delta - h'(T)\, I]\ \lambda_2 - \gamma\, f'(T)\, \lambda_1. && (10) \end{aligned}$$

### 3.2 Steady States

From the algebraic-dynamic system (6) - (10) it can be obtained that there is a trivial steady state at the origin of the state-control space,

$$C = T = I = 0, \quad \lambda_1 = \frac{\alpha}{r + \gamma} \quad \lambda_2 = \frac{\gamma}{r + \delta}\, \frac{\alpha}{r + \gamma}\, f'(0), \tag{11}$$

when we assume that for zero technology level the marginal costs of investments at zero investment are greater than or equal to the marginal revenue of investment. In other words, we assume $h(0)$ small enough so that the inequality

$$a'(0) \quad \geq \quad \frac{\alpha}{r + \gamma}\, \frac{\gamma}{r + \delta}\, f'(0)\, h(0) \tag{12}$$

is fulfilled. In what follows we narrow effectiveness of investment down to an assumption that is sufficient that inequality (12) is fulfilled:

**Assumption 1** *We postulate that the average effectiveness of technology investment converges to zero when the technology level $T$ decreases to zero ( i.e.* $\lim_{T \to 0_+} \frac{h(T)}{T} = 0$*).*

Straightforward calculations show that the steady state (11) is saddle point stable in the 4-dimensional state co-state space and appears as a stable node in the state space.

Concerning the possible interior steady states (i.e. $T > 0$) it can be obtained from (7) - (10) that they have to satisfy:

$$I = \frac{\delta T}{h(T)}, \tag{13}$$

$$C = \frac{\gamma f(T)}{\gamma + k I}, \tag{14}$$

$$\lambda_1 = \frac{\alpha}{r + \gamma + k I}, \tag{15}$$

$$\lambda_2 = \frac{\gamma f'(T)}{r + \delta - h'(T) I} \lambda_1. \tag{16}$$

Furthermore, from (6) and (13) - (16) it can be derived that at a steady state it must hold that

$$G(T) = a'(I) + \frac{\alpha}{r + \gamma + k I} \left( k C - \frac{\gamma f'(T)}{r + \delta \left[ 1 - \frac{h'(T)}{h(T)} T \right]} h(T) \right) = 0. \tag{17}$$

**Assumption 2** *We postulate that the "technology investment effectiveness elasticity" $\frac{h'(T)}{h(T)} T$ decreases monotonically from a positive value for $T = 0$ to a negative value for $T \to \infty$.*

**Proposition 3** *Under the Assumption 1 and 2 we get at most three interior steady states.*

The function $G$ in (17) decreases from infinity for $T \to 0_+$ to a local minimum and increases to infinity for $T \to \infty$. Depending on the minimum value of $G$, there are zero, one or two roots of $G$. If there exists a zero in the denominator $r + \delta \left[ 1 - \frac{h'(T)}{h(T)} T \right]$, which can only happen at a $T$ larger than the minimum point, $G$ "jumps" from $+\infty$ to $-\infty$ (asymptotic pole); there exists another root right to the pole. Depending on the minimum value of $G$ and the existence of a pole the equation $G(T) = 0$ either has zero, one, two or three solutions.

# 4 Numerical Analysis

The next step is to investigate the dimensions of the stable invariant manifolds of the steady states. Due to the complexity of the analysis we have to rely on numerical tools. The numerical computations were done by MATHEMATICA (Version 4.1.0.0, Mathematica is a registered trademark by Wolfram Research) using a precision of 22 digits. The bifurcation analysis was done by CONTENT, which is an environment designed for investigating the properties of dynamical systems; its core was developed by Y.A. Kuznetsov and V.V. Levitin at the Centrum voor Wiskunde en Informatica (CWI), Amsterdam, The Netherlands, ftp.cwi.nl/pub/CONTENT.

## 4.1 Functional specifications

To start our numerical analysis we first need to specify the functions occurring in the model. For the investment costs we choose the following function:

$$a(I) = a_1 I + a_2 I^2. \tag{18}$$

For the function $f(T)$ we simply take

$$f(T) = f_1 T. \tag{19}$$

A functional form for $h(T)$ that satisfies the requirements in (1) while its elasticity decreases monotonically from a positive value for $T = 0$ to a negative value for $T \to \infty$ is

$$h(T) = \frac{h_1 T^2}{1 + \left(\frac{T}{h_2}\right)^3}. \tag{20}$$

Substitution of these specifications in the optimal control model formed by (2) – (5) leads to the conclusion that the model contains ten parameters. This model can be transformed to a model with only five parameters, facilitating numerical analysis. Moreover, this makes the results more general since a particular specification of these five parameter values coincides with many combinations of parameter values of the original model. (The transformation is put into the Appendix. Note that the Appendix will not appear in hard copy but it will be available on the JEBO website.)

The physical dimension of $C, T$ and $I$ are sales in million, technology units, and investment in million \$, respectively; the unit of time is one year. With the parameter values

$$r = 5\%, \quad \delta \approx 4.17\%, \quad \alpha = \$2, \tag{21}$$

$$\gamma = 0.0625, \quad f_1 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{3} \tag{22}$$

$$h_1 = \frac{1}{2700}, \quad h_2 = 75, \quad k = 0.00852, \tag{23}$$

the function $G$ in expression (17) has the shape depicted in Figure 2.

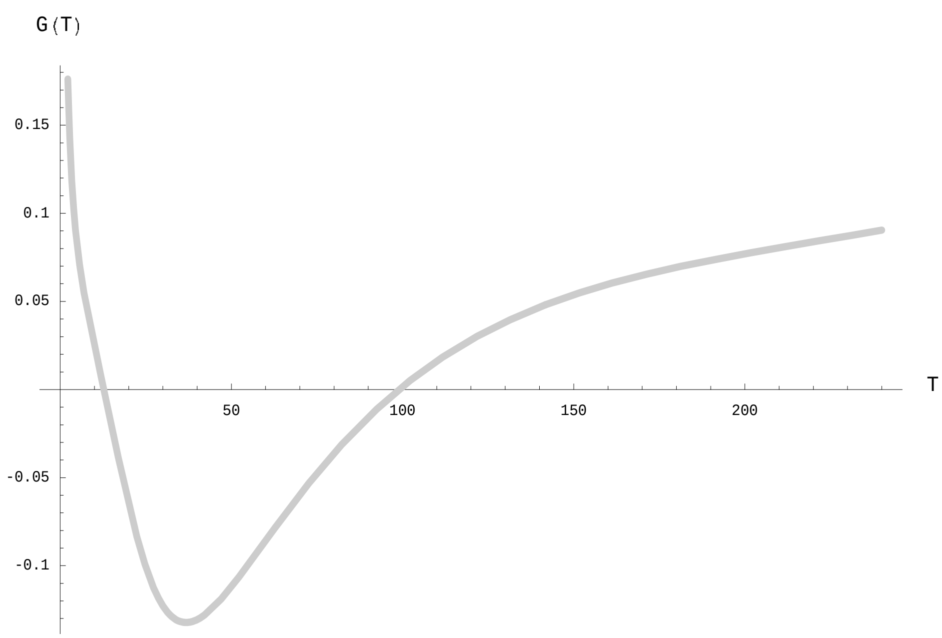



Figure 2: For the chosen parameter values the equation $G(T) = 0$ has two roots.

From Figure 2 it can be concluded that two steady states with positive technology level exist. Varying parameters, numerical calculations show that the smaller one is always unstable while the larger one can be saddle point stable or occurs simultaneously with a saddle point stable limit cycle. Investigating long-run outcomes we are interested in the stable invariant manifold of the saddle point stable steady state, or in the stable invariant manifold of the limit cycle. In the projection of these stable manifolds into the state space, saddle point stability changes to stability. Therefore, we refer to these long run outcomes as “stable steady state” and “stable limit cycle.” Numerical experiments suggest that this result seems to be robust in the sense that it also holds for other parameter sets.

Besides one non-optimal unstable steady state, we face two optimal steady states (including the origin). The question arises from what starting values convergence to which steady state is optimal. Furthermore, it must be investigated whether there exists a DNS set (i.e. whether there exist initial state values from which the optimal controlled system is indifferent to converge to one or the other steady state), and where this DNS set is located. As we will see, the situation is even more interesting since around the positive stable steady state a (saddle point) stable and an unstable limit cycle can occur simultaneously. For a particular parameter set the unstable limit cycle and the stable steady state merge and result in an unstable steady state.

In the next section we disregard the stable steady state at the origin for the moment and focus our attention on the analysis of the dynamic behavior around the interior stable steady state.

## 4.2 Bifurcation Analysis

One of the key parameters of our model is $k$ since it measures the adverse effect of investment on the sales volume. To be more precise, $k$ equals the loss of sales volume due to each \$1M investment per year (cf. $-k\,C\,I$ in (3)). The system exhibits different types of optimal long run outcomes: a steady state solution at the origin, a steady state solution with a positive technology level, and an undamped oscillation (limit cycle). Varying $k$ the model shows all possible combinations of these different long run outcomes. Therefore, we choose $k$ to be the bifurcation parameter. Depending on the value of this parameter four regimes can be distinguished; they are described below:

### 4.2.1 Regime I

If \$1M investment per year causes a loss of approximately 0.844% sales volume (i.e. 844 of 100K sales or $k = 0.00844$), investment, sales volume, and technology level converge to their steady state values. If the loss is much less than 0.844%, convergence is monotonic, otherwise it is damped oscillatory.

### 4.2.2 Regime II

We obtain that limit cycle behavior occurs for a loss greater than approximately 0.844%.

Figure 3 shows how the shape and the amplitude of stable and unstable limit cycles projected onto the state plane $(C, T)$ depend on the bifurcation parameter $k$. Parameter $k$ varies between 0.84% and 0.88%. For a given value of $k$ the intersection of the big ”basket” with the state plane $(C, T)$ gives the stable limit cycle while the intersection of the small interior ”basket” (which is placed upside down in the big basket) gives the unstable limit cycle.

**Regime IIa** At $k \approx 0.00844$ there occurs a “ limit point ” (“ blue sky ”) bifurcation (cf. e.g. Guckenheimer and Holmes (1983), Kuznetsov (1995)). The stable and the unstable limit cycles coincide and result in just one semi-stable limit cycle (being attracting from outside and repelling inside). Inside this semi-stable cycle convergence to the steady state occurs (inside and outside refers to the projection onto the state space).

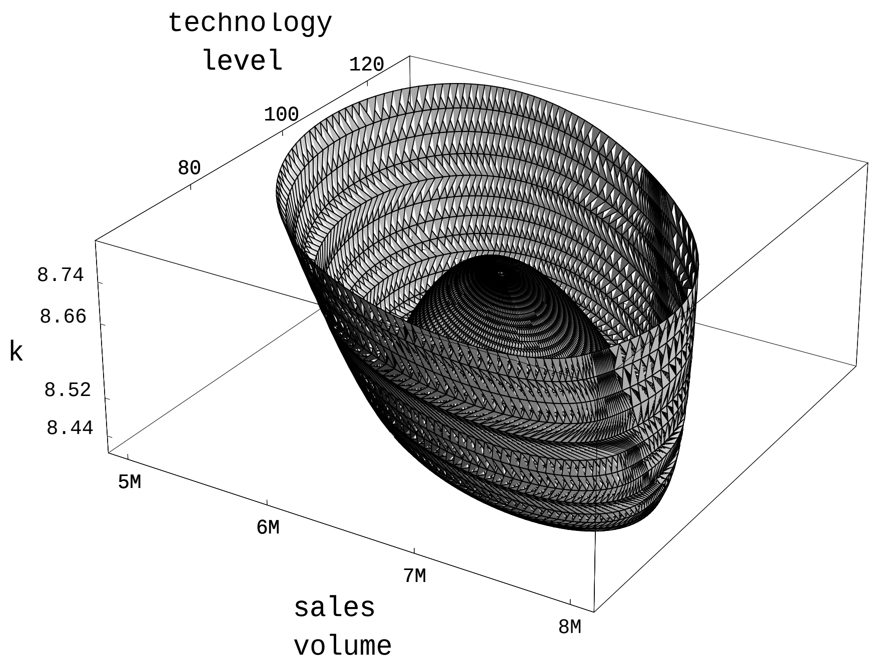



Figure 3: Shape and amplitude of stable and unstable limit cycles (parameter k in tenth percentage points).

For a loss of sales volume between $\approx 0.844\%$ and $0.866\%$ there are two limit cycles around the stable steady state: the inner one being unstable and the outer one being (saddle point) stable. Inside the inner cycle convergence to the stable steady state occurs, whereas outside the outer cycle and between the inner and outer cycle, convergence to the outer (stable) limit cycle takes place. In Figure 4 the (damped) oscillatory behavior around the stable steady state is shown. The movement of all trajectories is clockwise, which will be economically interpreted in Section 5.2.

The amplitude of the unstable (repelling) limit cycle decreases in size when $k$ increases, while the amplitude of the stable (attracting) limit cycle increases in size. At $k \approx 0.00866$ a sub-critical Hopf bifurcation (cf. e.g. Guckenheimer and Holmes (1983), Kuznetsov (1995)) arises. Then the smaller (unstable) limit cycle collapses to the steady state.

**Regime IIb** For a loss between 0.866% and 1.302% only the larger (stable) limit cycle exists and the steady state is now unstable. Increasing $k$ results in a growing amplitude of the limit cycle. For initial levels of the state variables located in- or outside the limit cycle, it is optimal to converge to the limit cycle.

#### 4.2.3 Regime III

If the loss of sales volume due to \$1M investment per year is greater than $\approx 1.3\%$, in the long run it is not optimal to sustain a positive stock of customers. For a more detailed description we refer to the next section.

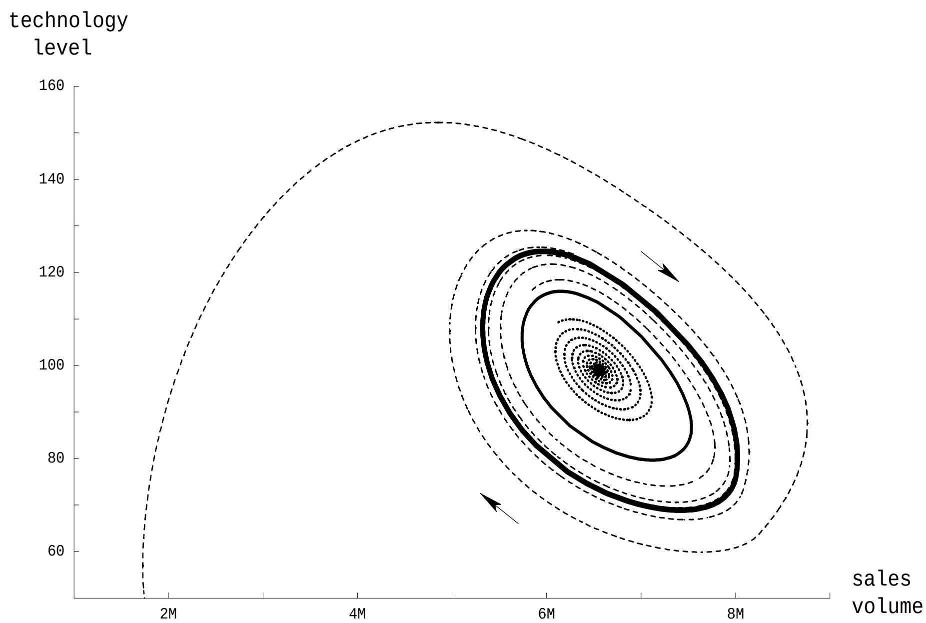



Figure 4: The oscillatory behavior around the stable steady state for parameter value $k = 0.00852$.

### 4.3 DNS–curves

So far we only have considered initial values in the neighborhood of the interior (stable) steady state and the limit cycles. Another candidate for an optimal long run outcome is the solution at the origin, as with, zero sales volume and zero technology level; see (11). (Note that zero investment is not only limited to the origin, but for instance along the limit cycles there are periods of time with zero investment; cf (6).)

Expanding Figure 4, Figure 5 shows candidate trajectories converging to the origin represented as dotted curves. Additionally, the candidates converging to the stable limit cycle are represented as solid curves. Apparently there is a region where it is possible to converge to both the origin and the stable limit cycle. In this region the objective function has to be evaluated for both of these solution candidates in order to find out to which region of optimal convergence this initial point belongs:

**Definition 4** *The region of optimal convergence of an optimal steady state/limit cycle is the set of all initial states from which it is optimal to choose a trajectory converging to the steady state/limit cycle.*

**Definition 5** *A DNS (Dechert-Nishimura-Skiba)-point is a point in the state space that belongs to at least two different regions of optimal convergence. The* DNS-set *is the set of all DNS-points.*

Figure 5 illustrates three optimal long run outcomes and their regions of optimal convergence for a loss of sales of 0.852%:

1. dotted lines in the gray-shaded area show paths of optimal long run zero investment policies (steady state at the origin),

2. solid lines in the white region show paths with an optimal long run cyclical sequence of zero and positive investment periods (limit cycle depicted by the thicker line), and

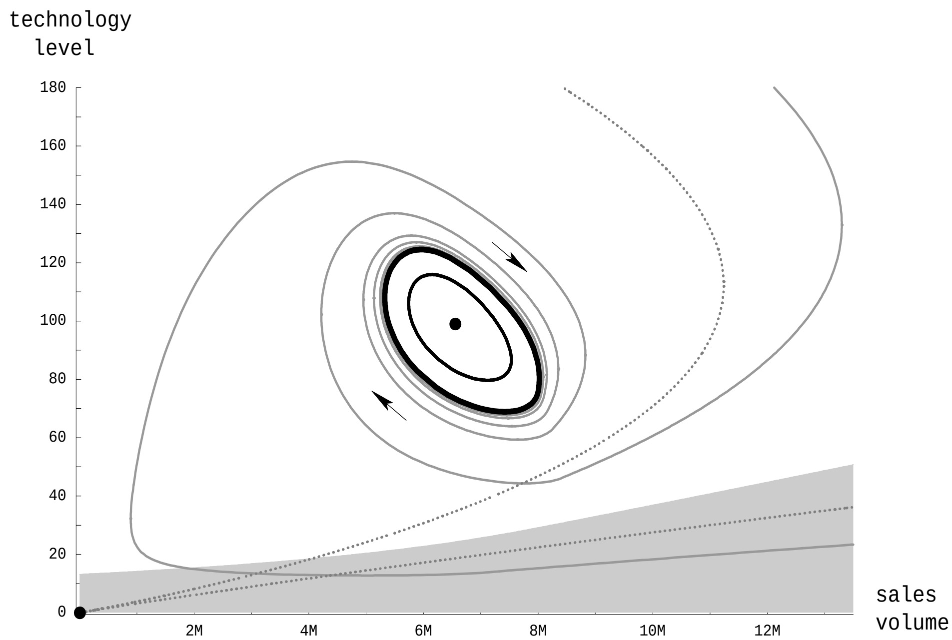



Figure 5: Three optimal long run outcomes and their regions of optimal convergence for $k = 0.00852$ : (1) gray-shaded area for the origin, (3) interior area of the unstable limit cycle (limit cycle depicted with the thinner line) for the interior steady state and (2) remaining white area for the stable limit cycle (limit cycle depicted with the thicker line).

3. in the interior of the limit cycle, which is depicted by the thinner line, in the long run a constant investment policy is optimal (steady state with approx. 6.2M sales per year and technology level of approx. 100).

In Figures 5, 6 and 7, the DNS-set is the borderline between the gray-shaded and the white region. There is no doubt that there the DNS-sets are curves. The procedure chosen to determine the DNS-curve (and the regions of optimal convergence) numerically is extensively explained in Haunschmied et al. (2003).

Increasing $k$ reduces the amplitude of the unstable limit cycle which finally disappears completely when $k$ is equal or higher than approximately 0.00866 (cf. Regime IIa and IIb). Figure 6 is computed for a loss of sales volume of 1.1%. For every initial point located on and above the DNS-curve, convergence to the stable limit cycle is optimal, whereas for every initial point located on and below the DNS-curve convergence to the origin is optimal.

Increasing $k$ further increases the amplitude of the stable limit cycle. As we can see in Figure 7 the DNS–curve folds back for larger losses (i.e. where $k = 0.012$). Also, as we can see in Figures 5, 6 and 7, the DNS-curves move upwards as parameter $k$ rises, and for larger $k$ they get closer to the stable limit cycle. For parameter values of $k$ near 0.013 we faced noticeable numerical instabilities when we tried to compute the regions of optimal convergence and DNS curves. These numerical instabilities are not astonishing knowing that for (a numerically computed) parameter value $k = 0.0130194$ the DNS-curve and the limit cycle coincide. In other words, starting from any initial value inside the limit cycle, convergence to the limit cycle is optimal, whereas for any initial value outside the limit cycle, convergence to the origin is optimal.

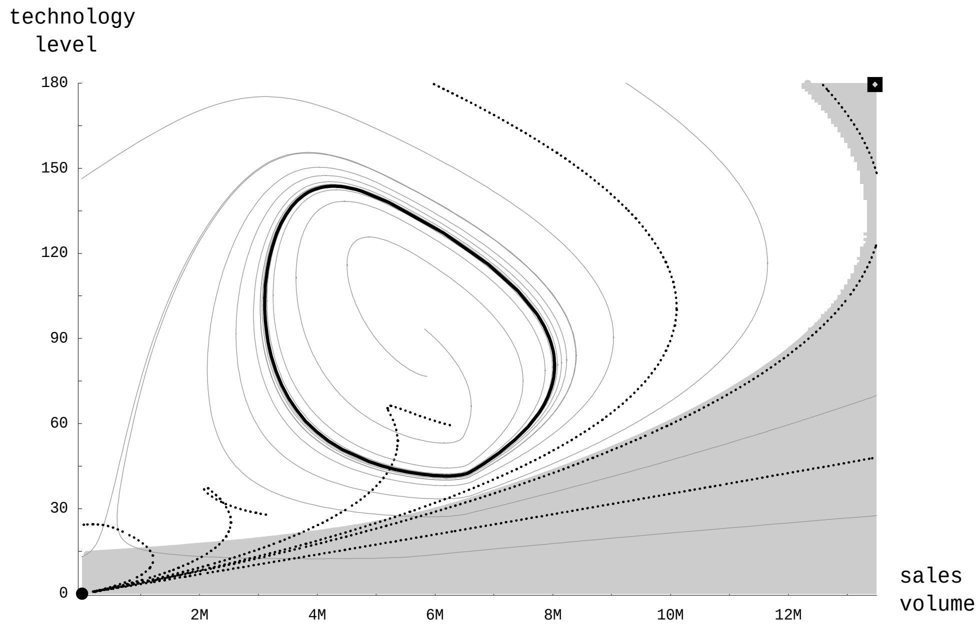



Figure 6: This figure illustrates two optimal long run outcomes and their regions of optimal convergence for $k = 0.011$: dotted lines in the gray-shaded area show paths of optimal long run zero investment policies (origin), solid lines in the white region show paths with an optimal long run cyclical sequence of zero and positive investment periods (limit cycle).

In case of a loss higher than approximately 1.3%, it is always optimal to converge to the origin. (Trajectories that converge to the limit cycle still fulfill the necessary optimality conditions, but a numerical analysis shows that this limit cycle is no longer path efficient. )

# 5 Economic Interpretation

In this section we provide the economic intuition for our results. As mentioned in the introduction, the main model characteristics are the following:

**(i)** technology investments increase the firm's technology level, increasing the number of customers and thus sales,

**(ii)** customers are distracted by technological changes, thus sales volume decreases with technology investments, and

**(iii)** the effectiveness of technology investments depends on the firm's technology level: it is difficult to build up technology when the technology level is small (no know-how makes building up the technology difficult) or large (when the firm is already producing the most modern products, advanced R&D is needed to increase the technology level even further). Hence technology investments are most effective for intermediate values of the technology level.

There are two different kinds of optimal long run behavior: on the one hand moving out of the market (see Section 5.1), and on the other hand staying in the

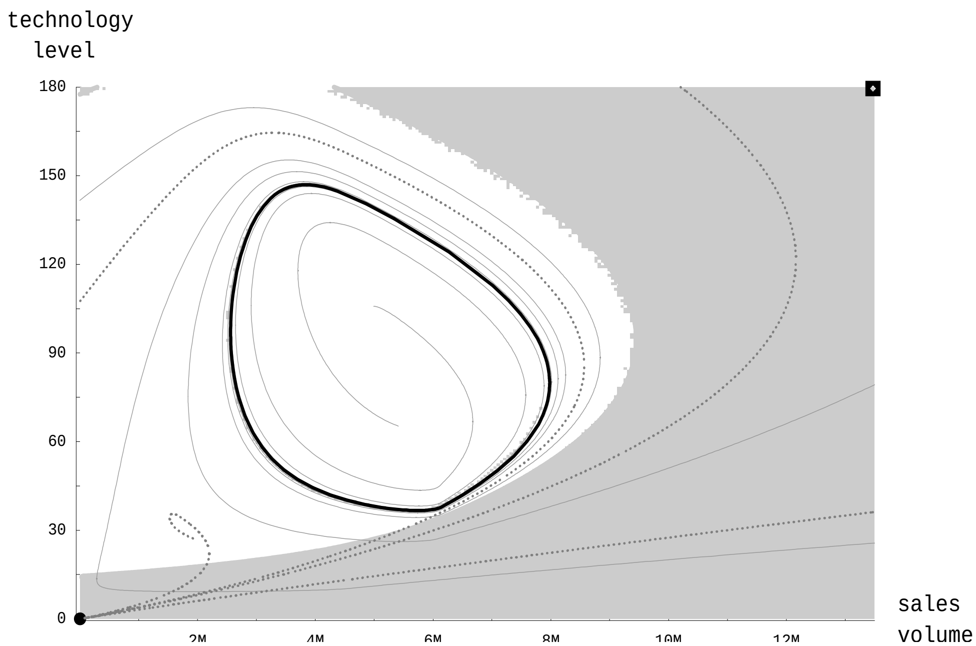



Figure 7: Similar to Figure 6 but now computed for parameter value $k = 0.012$. With the chosen precision of 22 digits the accuracy of the computed value function for $k = 0.012$ was enlarged to 6 digits. As we can see here the borderline between the gray-shaded and white region is fringed. The reason is that in case of parameter values $k \approx 0.013$, the value functions of the two different optimal long run outcomes differ, even at considerable distances to the DNS-curve (where the value functions intersect) of about values less than $10^{-6}$.

market either with constant sales volume or persistent oscillation with periods of positive and periods of zero technology investment (see Section 5.2). Consequently, the state plane (sales volume is on the abscissa and technology level on the ordinate) partitions to regions where in the long run a path efficient firm

- follows a zero investment policy (=moves out), or
- follows either a constant positive investment policy (=constant sales volume), or a cyclical sequence of zero - positive investment policy (=persistent oscillation).

Generically these two regions overlap and we call this overlapping area DNS (Dechert-Nishimura-Skiba) – curve (see Section 5.3). Numerical results show that this area is indeed a curve.

## 5.1 Moving out

For a firm with a lack of know-how, investing in technology is not effective. Hence, if the technology level is low (i.e. below the DNS-curve), an efficient firm does not invest sufficiently to prevent that sales volume and technology level from eroding to zero.

Moreover, if the loss of sales due to investment is high and if the firm's sales are booming, investment causes a serious absolute loss of customers, making it is less attractive to invest, but low or even zero investment results in a decline in technology level, which in turn results in a decline in sales volume. This pattern continues until sales volume is so small that the absolute loss of sales due to investment is reasonably negligible. Further, it could be the case that the firm's know-how is not high enough for investments to be sufficiently effective. This would imply that the firm refrains from investing. Consequently, in the long run the firm moves out of the market, in spite of having been in business with booming sales and a relatively high technology level.

## 5.2 Staying in the market

When the initial technology level is large (above the DNS-curve), in the long run we can have limit cycles on which the movement is clockwise. Hence, persistent oscillations arise for which sales volume lags behind technology level. Let us have a closer look at this undamped oscillation:

- Let us assume that the current sales volume is low. This implies that not too many customers are lost when investment is increased dramatically. The difference from the case of being below the DNS-curve is that now there is sufficient know-how within the firm for technology investment to be effective. Intensifying investment causes a boost in the technology level, and the sales volume starts to rise.
- At the end of this technology boost further technology growth is too expensive mainly due to the diminishing effectiveness of investment. Hence investment is lowered, which implies that the technology level starts to decline. As the technology level is high, sales volume still increases.
- The firm does not invest to stop declining technology because it does not want to antagonize a large stock of customers. However, due to zero investment the technology level erodes.
- When the technology level becomes dramatically low, the firm starts to invest moderately (cf. the "kink" in the right lower part of the limit cycles in Figure 5, 6, and 7). As both the low technology level and investment distracts customers, in a short period (compared with the other three phases of the oscillation) the sales volume becomes that low that it is now time to boost the technology level again.

Apparently, oscillation is induced mainly by the adverse investment effect (ii). Thus it can be expected that oscillatory behavior is especially present when the loss of sales volume due to investment (the parameter $k$ in the term $k\,C\,I$ in the state equation for $C$) is high. In order to investigate this fact we analyze the sensitivity of the solution with respect to the parameter $k$ (see additionally Figure 8):

- In case of a relatively small loss (i.e. 0.844%), the firm converges after a transition period to a constant (positive) investment policy. For a negligible small loss the convergence is monotonic, and for a slightly larger but still small loss the convergence is damped oscillatory. In the latter case we have higher investment for smaller sales volume and smaller investment for higher sales volume (cf. Regime I).
- Increasing $k$, at a certain value ("out of the blue sky") an optimal semi-stable limit cycle comes into existence. This implies that convergence to the steady state only takes place from within the limit cycle. From the outside, convergence to the limit cycle is optimal, thus preserving oscillating behavior forever.

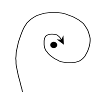
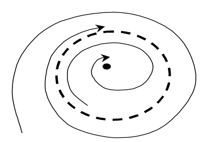
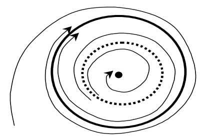
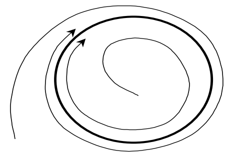
| k | | Figure |
|---|---|---|
| < 0.844% |  | - |
| 0.844% |  | - |
| 0.844% – 0.866% |  | 3.4 and 3.5 |
| 0.866% – 1.302% |  | 3.6 and 3.7 |
| > 1.302% | Stagnation to the origin | - |

Figure 8: Oscillating behavior for different values of the parameter $k$.

- If $k$ is increased further, the semi-stable limit cycle splits into a stable and an unstable limit cycle. Convergence to the (larger) stable limit cycle is optimal when starting outside the (smaller) unstable limit cycle, whereas inside the unstable limit cycle convergence to the steady state takes place.

- For intermediate values of $k$, the unstable limit cycle vanishes, and convergence to the steady state never occurs. This implies that all trajectories eventually end up at the stable limit cycle; in other words, the firm applies eventually an undamped oscillatory investment policy. This undamped oscillatory investment policy is characterized by technology boost during periods of dull sale and by no investment during periods of booming sale. With this behavior the firm is able to reduce the negative impact due to (ii).

- In case of a relatively high loss even oscillatory investment policy does not suffice to reduce the negative impact due to (ii). Hence, the firm refrains from investing and moves out of the market in the long run.

The result of oscillating behavior suggests that over time the firm switches between being (close to) the technological leader in the industry and falling behind. Admittedly, in many industries changes in technological leadership are not common. However, there are still examples where such a change can easily occur, such as, for example, the camera business (see, e.g., the Economist (March 13th-19th, 2004)). For this industry it holds that in America the point has been reached that digital cameras now outsell conventional film-based cameras. The switch to digital provides an opportunity for new players like pc-makers such as Hewlett-Packard, Dell, and Gateway to enter the market. Meanwhile, digital technology speeds up technological progress. For instance, the quality of the images produced by camera-phones is fast improving. This will further intensify competition as handset-makers such as Nokia muscle in, which can easily cause incumbents like Canon, Sony, Olympus, Nikon and Kodak fall behind.

In another example, for a long time the Swiss companies (Omega, Breitling, Tag-Heuer, etc.) were leading the watch industry with classic technology (mechanical or automatic movement). Then a new disruptive technology, the quartz movement, appeared and the market leaders did not go into this new direction (or at least not early enough). The Japanese (Seiko, Casio, ...) took over the market, and the classical watch companies had severe problems. Most of the American watch companies (Hamilton, Bulova, Elgin, Waltham, Illinois, etc.) disappeared, and the Swiss companies almost suffered the same fate. Only when Swatch came up with the idea of making watches a fashion product and a fairly cheap collector's item could they come back in the market with quartz watches. Recently, there has again been a boom of expensive and complicated automatic movement (Tourbillon, etc.) lead by Swiss companies and the revived old Saxonian companies in Glashuette, Germany (IWC, Lange, etc.). The Japanese watch companies were not able or willing to go into this direction.

### 5.3 DNS–curve

A necessary condition for the existence of the DNS-curve is the occurrence of effect (iii). As already stated the reason is that for a low technology level the effectiveness of technology investment is too low for building up the firm.

The shape of the DNS-curve is in general upward sloping. This is caused by effect (ii), implying that the number of customers has a negative effect on the profitability of investment.

For a large sales volume (and when effect (ii) is pronounced) the DNS-curve is not only upward sloping, but in fact it folds back to north-west; see for example Figure 7. The reason for the latter is that if the technology level is very high (and, or although, the sales volume is moderate), one need not invest in technology to increase the sales volume. Then, eventually the sales volume becomes so large that (because of the absolute loss of sales volume) it is also not reasonable to invest because this would mainly erode the current sales. Refraining from undertaking technology investments implies that the technology level decreases. In the end the firm's technology level is so small that investment in technology is not sufficiently effective to build up the firm.

## 6 Conclusions

In this paper, we present an investment control model where investment in technology is controlled by a firm in order to maximize profit. A higher technology level of the product has a positive effect on sales. The effectiveness of investment, however, depends on the existing know-how in the firm, and, additionally, investment in technology has a negative side effect on sales volume (adverse investment effect).

The investigated model class exhibits different types of optimal long run behavior. If the initial technology level is too low, it is not optimal to increase the technology level by investments. The reason is that technology investments are not very effective due to a lack of know-how within the firm. The implication is that in the long run the firm will cease its operations. For a sufficiently high initial technology level, in the long run, it is worthwhile for the firm to converge to either a constant or undamped oscillatory investment policy. Depending on the initial endowment the firm applies one of these types.

For certain initial endowments in the long run, the firm is indifferent between zero investment on the one hand and constant positive or a least a cyclical zero and positive investment policy on the other hand. In the plane with sales volume on the horizontal axis and technology level on the vertical axis an upward sloping curve separates these two regions of optimal convergence (different long run behavior). Deriving the terminology from one-state-optimal control models (Skiba, 1978; Dechert and Nishimura, 1983), we denote this curve by DNS-(Dechert-Nishimura-Skiba) curve. The adverse investment effect leads to an upward sloping DNS-curve since now the profitability of investment depends on the number of customers: the more customers this firm has, the more will be distracted by technology changes.

Optimal oscillatory investment policy has its seeds in the model characteristic adverse investment effect: if this effect is negligible, convergence to a constant investment policy is optimal. Otherwise, in the long run a cyclical sequence of zero and positive investment periods is applied to reduce the negative impact of the adverse investment effect. If the adverse investment effect is relatively high, the firm only remains in the market when its initial sales volume and technology level allow in to establish the optimal long run sequence of zero - positive investment policy without a long transition phase; otherwise the firm will finally quit business. In case of a pronounced adverse investment effect the firm ceases business in any case.

Recapitulating we have three main features:

- Coexistence of an optimal steady state (origin) and an optimal limit cycle separated by a DNS-curve.
- For a particular level of the adverse investment effect, the DNS-curve coincides with a limit cycle. For larger values of this effect only the steady state at the origin (zero sales volume) is optimal.
- An unstable limit cycle separating regions of optimal convergence of an optimal limit cycle and an interior steady state.

# Appendix

The parameter reduction is achieved by the following linear time scaling transformation:

$$\tau = \delta t. \tag{24}$$

Then we reformulate the state and control variables as follows:

$$\bar{C}(\tau) = \frac{\delta}{\gamma f_1 h_2} C\left(\frac{\tau}{\delta}\right), \tag{25}$$

$$\bar{T}(\tau) = \frac{1}{h_2} T\left(\frac{\tau}{\delta}\right), \tag{26}$$

and

$$\bar{I}(\tau) = \frac{h_1 h_2}{\delta} I\left(\frac{\tau}{\delta}\right). \tag{27}$$

Furthermore, we define the parameters:

$$\begin{aligned} \rho &= \frac{r}{\delta}, \\ \beta_1 &= \frac{\gamma}{\delta}, \\ \beta_2 &= \frac{a_1\delta}{\alpha f_1 h_1 h_2^2}, \\ \beta_3 &= \frac{a_2\delta^2}{\alpha f_1 h_1^2 h_2^3}, \\ \beta_4 &= \frac{k}{h_1 h_2}. \end{aligned} \tag{28}$$

Now, substituting f (24)-(28) into the original optimal control model (5), (2), (3), and (4), including the functional forms (18)-(20), gives the following model:

$$\max_I \frac{\alpha f_1 h_2}{\delta} \int_0^\infty e^{-\rho\tau} \left[\beta_1 \bar{C} - \beta_2 \bar{I} - \frac{\beta_3}{2}\bar{I}^2\right] d\tau, \tag{29}$$

subject to

$$\dot{\bar{C}} = \bar{T} - \beta_1 \bar{C} - \beta_4 \bar{C}\bar{I}, \tag{30}$$

$$\dot{\bar{T}} = \tilde{h}\left(\bar{T}\right)\bar{I} - \bar{T}, \tag{31}$$

$$\bar{I} \geq 0, \tag{32}$$

in which

$$\tilde{h}\left(T\right) = \frac{T^2}{1+T^3}. \tag{33}$$

This model provides the basis for our numerical analysis. For the parameters (21) - (23) we get

$$\rho = 1.2, \quad \beta_1 = 1.5, \quad \beta_2 = 0.01, \quad \beta_3 = 0.005, \quad \beta_4 = 0.36 \tag{34}$$

# References


Chari, V.V. , Hopenhayn, H., 1991. Vintage human capital, growth, and the diffusion of new technology. Journal of Political Economy 99, 1142-1165.

Cohen, W.M.,Levinthal, D.A., 1989. Innovation and Learning: the two faces of R & D. Economic Journal 99, 569-596.

Das, S.S., Van de Ven, A.H., 2000. Competing with new product technologies: a process model of strategy. Management Science 46, 1300-1316.

Dechert, W.D., Nishimura, K., 1983. A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. Journal of Economic Theory 31, 332-354.

Feichtinger, G., Hartl, R., 1986. Optimale Kontrolle Ökonomischer Prozesse - Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften. (Berlin: de-Gruyter).

Guckenheimer, J., Holmes, P., 1983. Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields. (New York: Springer-Verlag).

Haunschmied, J.L., Kort, P.M., Hartl, R.F., Feichtinger, G., 2003. A DNS–curve in a two state capital accumulation model: a numerical analysis. Journal of Economic Dynamics and Control 27, 701-716.

Kuznetsov, Y.A., 1995. Elements of Applied Bifurcation Theory. (New York: Springer-Verlag).

Léonard, D., Long, N.V., 1992. Optimal Control Theory and Static Optimization in Economics. (Cambridge: Cambridge University Press).

Mansfield, E., 1968. Industrial Research and Technological Innovation: An Econometric Analysis. (London: Longmans).

Mulder, P., Groot, H.L.F., de Hofkes, M.W., 2003. Explaining slow diffusion of energy-saving technologies; a vintage model with returns to diversity and learning-by-using. Resource and Energy Economics 25, 105-126.

Skiba, A.K., 1978. Optimal growth with a convex-concave production function. Econometrica 46, 527-540.

Verspagen, B., 1993. Uneven growth between interdependent economies: an evolutionary view on technology gaps, trade and growth (published form of PhD dissertation). (Aldershot: Avebury).