The Annals of Probability
1988, Vol. 16, No. 1, 108–125

STRONG BOUNDS FOR WEIGHTED EMPIRICAL DISTRIBUTION FUNCTIONS BASED ON UNIFORM SPACINGS

BY JOHN H. J. EINMAHL AND MARTIEN C. A. VAN ZUIJLEN

University of Limburg and Catholic University, Nijmegen

Let $U_1, U_2, \ldots$ be a sequence of independent rv's having the uniform distribution on $(0,1)$. Let $\hat{F}_n$ be the empirical distribution function based on the transformed uniform spacings $\mathbb{D}_{i,n} := G(nD_{i,n})$, $i = 1, 2, \ldots, n$, where G is the exp(1) df and $D_{i,n}$ is the ith spacing based on $U_1, U_2, \ldots, U_{n-1}$. In this paper a complete characterization is obtained for the a.s. behaviour of $\limsup_{n\to\infty} b_n V_{n,\nu}$ and $\limsup_{n\to\infty} b_n W_{n,\nu}$, where $\nu \in [0, \frac{1}{2}]$, $\{b_n\}_{n=1}^{\infty}$ is a sequence of norming constants,

$$V_{n,\nu} = \sup_{0<t<1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\nu}} \quad \text{and} \quad W_{n,\nu} = \sup_{0<t<1} \frac{n|\hat{F}_n(t) - t|}{(1-t)^{1-\nu}}.$$

It turns out that compared with the i.i.d. case only $W_{n,\nu}$ behaves differently. The results imply, e.g., laws of the iterated logarithm for $\log(n^{\nu-1}V_{n,\nu})$ and $\log(n^{\nu-1}W_{n,\nu})$. Of independent interest is the theorem on the lower–upper class behaviour of the maximal spacing, which gives the final solution for this problem and generalizes some recent results in the literature.

1. Introduction and main results. Let $U_1, U_2, \ldots$ be a sequence of independent random variables (rv's), each having the uniform distribution on $(0,1)$. For $n = 2, 3, \ldots$ we define the transformed uniform spacings by $\mathbb{D}_{i,n} := G(nD_{i,n})$, where

$$D_{i,n} := U_{i:n-1} - U_{i-1:n-1}, \qquad i = 1, 2, \ldots, n,$$
$$0 := U_{0:n-1} \le U_{1:n-1} \le \cdots \le U_{n-1:n-1} \le U_{n:n-1} := 1$$

are the order statistics of the first $n-1$ rv's in the given sequence and G is defined by $G(x) = 1 - e^{-x}$, $x \in (0, \infty)$. Note that for $t \in (0, 1 - e^{-n}]$,

$$F_n(t) := P(\mathbb{D}_{i,n} \le t) = P(nU_{1:n-1} \le -\log(1-t)) = 1 - \left(1 + \frac{\log(1-t)}{n}\right)^{n-1},$$

and for $t \in (0,1)$,

$$F_n(t) \to t, \quad \text{as } n \to \infty.$$

Finally, we define the empirical df $\hat{F}_n$ based on the $\mathbb{D}_{i,n}$ by

$$\hat{F}_n(t) = n^{-1} \sum_{i=1}^{n} 1_{(0,t]}(\mathbb{D}_{i,n}), \qquad t \in (0,1).$$

We will establish a complete characterization of the almost sure behaviour of $\limsup_{n\to\infty} b_n V_{n,\nu}$ and $\limsup_{n\to\infty} b_n W_{n,\nu}$, for $\nu \in [0, \frac{1}{2}]$ and a sequence of

Received October 1985; revised July 1986.
AMS 1980 *subject classifications*. Primary 60F15; secondary 60G17, 62G30.
Key words and phrases. Order statistics, strong convergence, uniform spacings, weighted empirical process.

positive norming constants $\{b_n\}_{n=1}^{\infty}$, where

$$V_{n,\nu} = \sup_{0<t<1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\nu}} \tag{1.1}$$

and

$$W_{n,\nu} = \sup_{0<t<1} \frac{n|\hat{F}_n(t) - t|}{(1-t)^{1-\nu}}. \tag{1.2}$$

The study of these rv's is motivated by similar studies for the i.i.d. case. In that case for dimension one Csáki (1974, 1975, 1982) investigated the value $\nu = \frac{1}{2}$, Shorack and Wellner (1978) $\nu = 0$ and Mason (1981) $\nu \in (0, \frac{1}{2})$. Furthermore, Mason (1982, $\nu = 0$) and Einmahl and Mason (1985, $\nu \in [0, \frac{1}{2}]$) obtained the results in the multivariate i.i.d. case. Although in the i.i.d. case the results are known also for $\nu \in (\frac{1}{2}, 1]$, the behaviour for these values of the weighted empirical distribution function based on uniform spacings is still an interesting open question.

To prove the result for $W_{n,\nu}$ we need the final solution for a problem on the a.s. behaviour of the maximal uniform spacing. This result, which is of independent interest and generalizes the results of Slud (1978), Devroye (1981) and Deheuvels (1982), will be stated as our first theorem. For its presentation we denote, as usual, the ordered $D_{i,n}$, $i = 1, 2, \ldots, n$, by

$$D_{1:n} \le D_{2:n} \le \cdots \le D_{n:n}.$$

THEOREM 1.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive constants. Then we have*

$$\left[\sum a_n \log(a_n^{-1}) = \infty\right] \Rightarrow \left[P\left(nD_{n:n} \ge \log(a_n^{-1})\ i.o.\right) = 1\right]. \tag{1.3}$$

The proof of this theorem is deferred to Section 2.

Next let us present our characterizations concerning $V_{n,\nu}$ and $W_{n,\nu}$ and some corollaries. The proofs of these are deferred to Section 3. In both theorems $\{a_n\}_{n=1}^{\infty}$ is again a sequence of positive constants.

THEOREM 1.2.A. *For each $\nu \in [0, \frac{1}{2}]$ we have*

$$\left[\sum a_n = \infty\right] \Rightarrow \left[\limsup_{n\to\infty} a_n^{1-\nu} V_{n,\nu} = \infty\ a.s.\right] \tag{1.4}$$

and

$$\left[\sum a_n < \infty,\ a_n \downarrow \text{ and } na_n \log n \to 0\right] \Rightarrow \left[\lim_{n\to\infty} a_n^{1-\nu} V_{n,\nu} = 0\ a.s.\right]. \tag{1.5}$$

COROLLARY 1.1.A. *For each $\nu \in [0, \frac{1}{2}]$,*

$$\limsup_{n\to\infty} \frac{\log(n^{\nu-1} V_{n,\nu})}{\log\log n} = 1 - \nu \quad a.s. \tag{1.6}$$

THEOREM 1.2.B. *For each* $\nu \in [0, \frac{1}{2}]$ *we have*

$$\left[\sum a_n \log(a_n^{-1}) = \infty\right] \Rightarrow \left[\limsup_{n\to\infty} a_n^{1-\nu} W_{n,\nu} = \infty \ a.s.\right] \tag{1.7}$$

and

$$\left[\sum a_n \log(a_n^{-1}) < \infty \text{ and } a_n \downarrow 0\right] \Rightarrow \left[\lim_{n\to\infty} a_n^{1-\nu} W_{n,\nu} = 0 \ a.s.\right]. \tag{1.8}$$

COROLLARY 1.1.B. *For each* $\nu \in [0, \frac{1}{2}]$,

$$\limsup_{n\to\infty} \frac{\log\left(n^{\nu-1} W_{n,\nu}\right)}{\log\log n} = 2(1-\nu) \quad a.s. \tag{1.9}$$

COROLLARY 1.2.

$$\left[\sum a_n \log(a_n^{-1}) < \infty \text{ and } a_n \downarrow 0\right] \Rightarrow \left[P\left(nD_{n:n} \geq \log(a_n^{-1}) \ i.o.\right) = 0\right]. \tag{1.10}$$

Corollary 1.2 is more or less the same as Theorem 4.1 in Devroye (1981). The only difference is that our monotonicity condition on the a_n is somewhat milder than his conditions.

Combination of Theorems 1.2.A and 1.2.B yields for

$$Z_{n,\nu} := \sup_{0<t<1} \frac{n|\hat{F}_n(t) - t|}{\left(t(1-t)\right)^{1-\nu}}:$$

COROLLARY 1.3. *Theorem* 1.2.B *and Corollary* 1.1.B *hold with* $W_{n,\nu}$ *replaced by* $Z_{n,\nu}$.

Comparison of Theorems 1.2.A and 1.2.B with the aforementioned results for the i.i.d. case shows that the result for $V_{n,\nu}$ is exactly the same as that for the one-dimensional i.i.d. case. Surprisingly, the behaviour of $W_{n,\nu}$ is different: It coincides with the result for dimension two(!) in the i.i.d. case.

2. Proof of Theorem 1.1. Before we give the actual proof we need some notation and some lemmas. For $n \in \mathbb{N}$ define the stochastic interval $I_n = (I_{n1}, I_{n2}]$, which is one of the intervals determined by two successive order statistics of $U_1, U_2, \ldots, U_{n-1}$, by

$$I_n = (U_{i-1:n-1}, U_{i:n-1}], \quad \text{if } U_{i-1:n-1} < U_n \leq U_{i:n-1}. \tag{2.1}$$

The length of I_n is denoted by S_n. Next we define

$$T_n = \begin{cases} S_n, & \text{if } I_n \not\subset I_k, \forall \ \frac{2}{3}n < k < n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Of course, we have

$$T_n \leq S_n \leq D_{n:n}. \tag{2.3}$$

For the proof of Theorem 1.1 we shall need the following extension of the Borel–Cantelli lemma.

LEMMA 2.1 [Kochen and Stone (1964)]. *If for a sequence of events $A_1, A_2, \dots$ on some probability space*

$$\sum PA_n = \infty \quad \textit{and} \quad \liminf_{N\to\infty} \frac{\sum\sum_{1\le n<m\le N} PA_nA_m}{\left(\sum_{n=1}^N PA_n\right)^2} < \infty, \tag{2.4}$$

then $P(A_n$ i.o.$) > 0$.

The next three lemmas are distributional results for S_n.

LEMMA 2.2. *Let $c \in (0,1)$. Then*

$$P(S_n \ge c) = (1-c)^{n-1}(1+(n-1)c). \tag{2.5}$$

The proof of this lemma is easy, uses only elementary probability theory and will be omitted.

LEMMA 2.3. *Let $0 < a \le b < 1$ and $m > n$. Then*

$$\begin{aligned} &P(S_m \ge a;\ S_n \ge b;\ I_m \subset I_n)\\ &\quad = (1-a)^{m-n-2}((m-n-2)a+1)n(n-1)\\ &\qquad \times\left\{\frac{(1-a^2)}{n-1}(1-b)^{n-1} - \frac{2}{n}(1-b)^n + \frac{1}{n+1}(1-b)^{n+1}\right\}\\ &\qquad -2(m-n-1)a^2(1-a)^{m-n-2}\\ &\qquad\quad \times\left\{n(1-a)(1-b)^{n-1} - (n-1)(1-b)^n\right\}. \end{aligned} \tag{2.6}$$

PROOF. Observe that, with $f_n(x) = -dP(S_n \ge x)/dx$,

$$\begin{aligned} &P(S_m \ge a;\ S_n \ge b;\ I_m \subset I_n)\\ &\quad = \int_b^1 f_n(x) P(S_m \ge a;\ I_m \subset I_n | S_n = x)\,dx\\ &\quad = \int_b^1 f_n(x) P(S_m \ge a | I_m \subset I_n;\ S_n = x) P(I_m \subset I_n | S_n = x)\,dx\\ &\quad = \int_b^1 x f_n(x) P(S_m \ge a | I_m \subset I_n;\ S_n = x)\,dx. \end{aligned} \tag{2.7}$$

For the conditional probability in this last expression we have

$$\begin{aligned} &P(S_m \ge a | I_m \subset I_n;\ S_n = x)\\ &\quad = P(S_m \ge a | I_m \subset I_n;\ S_n = x;\ U_m \le U_n)\\ &\quad = 2x^{-2}\int_a^x y P(S_m \ge a | I_m \subset I_n;\ S_n = x;\ U_m \le U_n;\ U_n - I_{n1} = y)\,dy. \end{aligned} \tag{2.8}$$

Now define the rv Z by $Z = \#\{i \in \mathbb{N}: n < i < m \text{ and } U_i \in (I_{n1}, U_n]\}$, where for a set A, $\#A$ denotes its cardinality. Using Z we see that the conditional probability in the last expression in (2.8) is equal to

$$\begin{aligned} &\sum_{k=0}^{m-n-1} P(S_m \geq a | Z = k;\ I_m \subset I_n;\ S_n = x;\ U_m \leq U_n;\ U_n - I_{n1} = y) \\ &\qquad \times P(Z = k | I_m \subset I_n;\ S_n = x;\ U_m \leq U_n;\ U_n - I_{n1} = y) \\ &\qquad = \sum_{k=0}^{m-n-1} P(S_{k+1} \geq a/y) \binom{m-n-1}{k} y^k (1-y)^{m-n-k-1} \\ &\qquad = \sum_{k=0}^{m-n-1} (1 - a/y)^k (1 + ka/y) \binom{m-n-1}{k} y^k (1-y)^{m-n-k-1}. \end{aligned} \tag{2.9}$$

Substituting (2.9) in (2.8) and in turn (2.8) in (2.7) we obtain

$$\begin{aligned} P(S_m \geq a;\ S_n \geq b;\ I_m \subset I_n) &= 2n(n-1) \int_b^1 (1-x)^{n-2} \\ &\qquad \times \int_a^x \sum_{k=0}^{m-n-1} \binom{m-n-1}{k} y^{k+1} (1-y)^{m-n-k-1} \\ &\qquad \times (1 - a/y)^k (1 + ka/y)\, dy\, dx. \end{aligned} \tag{2.10}$$

Now elementary analysis, including Newton's binomial formula, shows that the last expression in (2.10) is equal to the last expression in (2.6). □

COROLLARY 2.1. *Let $a \in (0,1)$ and $m > n$. Then*

$$P(I_m \subset I_n | S_m \geq a) = 2/(n+1). \tag{2.11}$$

PROOF. Note that

$$P(I_m \subset I_n | S_m \geq a) = P(S_m \geq a;\ S_n \geq a;\ I_m \subset I_n)/P(S_m \geq a) \tag{2.12}$$

and apply Lemma 2.3 to the numerator and Lemma 2.2 to the denominator. □

LEMMA 2.4. *Let $a, b \in (0,1)$ and $m > n$. Then*

$$P(S_m \geq a;\ S_n \geq b;\ I_m \not\subset I_n) \leq P(S_m \geq a) P(S_n \geq b). \tag{2.13}$$

PROOF. By similar ideas as in the proof of Lemma 2.3 it can be shown that for $x \leq 1 - a$,

$$\begin{aligned} &P(S_m \geq a | S_n = x;\ I_m \not\subset I_n) \\ &\qquad = \sum_{k=0}^{m-n-1} (1-x)^k x^{m-n-k-1} \binom{m-n-1}{k} (1 - a/(1-x))^{n-2+k} \\ &\qquad \times (1 + (n-2+k)a/(1-x)). \end{aligned} \tag{2.14}$$

This last expression is decreasing in x. Hence, according to Lemma 4 in Lehmann (1966) it follows that

$$P(S_m \geq a;\ S_n \geq b | I_m \not\subset I_n) \leq P(S_m \geq a | I_m \not\subset I_n) P(S_n \geq b | I_m \not\subset I_n). \tag{2.15}$$

Using (2.15) and Corollary 2.1 we easily see

$$
\begin{aligned}
&P(S_m \geq a;\ S_n \geq b;\ I_m \not\subset I_n) \\
&\quad = P(S_m \geq a;\ S_n \geq b | I_m \not\subset I_n) P(I_m \not\subset I_n) \\
&\quad \leq P(S_m \geq a | I_m \not\subset I_n) P(S_n \geq b | I_m \not\subset I_n) P(I_m \not\subset I_n) \\
&\quad \leq P(S_m \geq a | I_m \not\subset I_n) P(S_n \geq b) \\
&\quad = P(S_m \geq a) P(S_n \geq b) P(I_m \not\subset I_n | S_m \geq a) / P(I_m \not\subset I_n) \\
&\quad = P(S_m \geq a) P(S_n \geq b).
\end{aligned} \tag{2.16}
$$

□

Now we are prepared to give the proof of Theorem 1.1. First, we restrict ourselves without loss of generality to a_n with $n^{-4/3} \leq a_n \leq e^{-1}$, which implies $1 \leq \log(a_n^{-1}) \leq \frac{4}{3}\log n$. [For $n^{-4/3} \leq a_n$ we need that $P(nD_{n:\,n} \geq \frac{4}{3}\log n \text{ i.o.}) = 0$, which follows from our Corollary 1.2 or from Theorem 2.1 in Slud (1978).] Note also that on account of (2.3) and the Hewitt–Savage 0–1 law it suffices to show that

$$
\left[\sum a_n \log(a_n^{-1}) = \infty\right] \Rightarrow \left[P\left(nT_n \geq \log(a_n^{-1}) \text{ i.o.}\right) > 0\right]. \tag{2.17}
$$

We shall prove (2.17) with the aid of Lemma 2.1. Hence we need to prove (2.4) with $A_n = [T_n \geq c_n]$, where $c_n = n^{-1}\log(a_n^{-1})$.

Let us first consider $P[S_n \geq c_n]$. By Lemma 2.2 it is easily seen that we have for large n,

$$
\begin{aligned}
P(S_n \geq c_n) &\geq (1 - c_n)^n (n-1) c_n \\
&\geq a_n\left(\log(a_n^{-1})\right)\left(\frac{n-1}{n}\right)(1 - nc_n^2) \geq \frac{1}{2} a_n \log(a_n^{-1}).
\end{aligned} \tag{2.18}
$$

Hence we have $\Sigma P(S_n \geq c_n) = \infty$. We also have

$$
\begin{aligned}
P(T_n \geq c_n) &= P(T_n \geq c_n;\ T_n = S_n) + P(T_n \geq c_n;\ T_n \neq S_n) \\
&= P(S_n \geq c_n;\ T_n = S_n) \\
&= P(T_n = S_n | S_n \geq c_n) P(S_n \geq c_n).
\end{aligned} \tag{2.19}
$$

So for establishing $\Sigma P(T_n \geq c_n) = \infty$ we only have to show that

$$
\liminf_{n \to \infty} P(T_n = S_n | S_n \geq c_n) > 0.
$$

But by Corollary 2.1 we have

$$
\begin{aligned}
P(T_n \neq S_n | S_n \geq c_n) &= P(\exists_{k:\, 2n/3 < k < n}\ I_n \subset I_k | S_n \geq c_n) \\
&\leq \sum_{k=[2n/3]+1}^{n-1} P(I_n \subset I_k | S_n \geq c_n) \\
&= \sum_{k=[2n/3]+1}^{n-1} 2/(k+1) \\
&\leq 2\int_{2n/3}^{n} x^{-1}\,dx \\
&= 2\log\tfrac{3}{2} < 1.
\end{aligned} \tag{2.20}
$$

Hence $P(T_n = S_n | S_n \geq c_n) \geq 1 - 2\log\frac{3}{2} =: \delta > 0$, which proves that the first condition in Lemma 2.1 is fulfilled.

Now it remains to show that the second condition in Lemma 2.1 is fulfilled. It is sufficient to prove that for large n and $m > n$,

$$P(T_m \geq c_m;\ T_n \geq c_n) \leq (2\delta^{-2})P(T_m \geq c_m)P(T_n \geq c_n), \tag{2.21}$$

with δ as above. In the proof of (2.21) we need that for large n and $m \geq \frac{3}{2}n$,

$$P(S_m \geq c_m;\ S_n \geq c_n;\ I_m \subset I_n) \leq P(S_m \geq c_m)P(S_n \geq c_n). \tag{2.22}$$

We shall now establish (2.22). We have to distinguish between the cases $c_m \leq c_n$ and $c_m > c_n$. The case $c_m > c_n$ is the easiest one, hence its proof will be omitted. So let us assume $c_m \leq c_n$. By an application of Lemma 2.3, using $mc_m \geq 1$, we have

$$\begin{aligned} P(S_m \geq c_m;\ S_n \geq c_n;\ I_m \subset I_n) &\leq 12 m c_m n c_n^2 (1-c)^{m-n-2}(1-c_n)^{n-1} \\ &\leq 13 m c_m a_m n c_n^2 a_n (1-c_m)^{-n}. \end{aligned} \tag{2.23}$$

Now elementary analysis shows, using $m \geq \frac{3}{2}n$ and $a_k \geq k^{-4/3}$, $k = n, m$, that for large n,

$$c_n(1-c_m)^{-n} \leq n^{-1/18} \to 0, \quad \text{as } n \to \infty. \tag{2.24}$$

But (2.23) and (2.24), combined with (2.18), imply (2.22).

Now we consider (2.21) for large n and $m \geq \frac{3}{2}n$. Then we have

$$\begin{aligned} P(T_m \geq c_m;\ T_n \geq c_n) &\leq P(S_m \geq c_m;\ S_n \geq c_n) \\ &\leq P(S_m \geq c_m;\ S_n \geq c_n;\ I_m \subset I_n) \\ &\quad + P(S_m \geq c_m;\ S_n \geq c_n;\ I_m \not\subset I_n) \\ &\leq 2P(S_m \geq c_m)P(S_n \geq c_n) \\ &\leq 2\delta^{-2}P(T_m \geq c_m)P(T_n \geq c_n), \end{aligned} \tag{2.25}$$

where for the third inequality also Lemma 2.4 is applied. Next, we consider (2.21) for large n and $n < m < \frac{3}{2}n$. Then we have

$$\begin{aligned} &P(T_m \geq c_m;\ T_n \geq c_n) \\ &\quad = P(T_m \geq c_m;\ T_n \geq c_n;\ I_m \subset I_n) + P(T_m \geq c_m;\ T_n \geq c_n,\ I_m \not\subset I_n) \\ &\quad \leq P(S_m \geq c_n;\ S_n \geq c_n;\ I_m \not\subset I_n) \\ &\quad \leq P(S_m \geq c_m)P(S_n \geq c_n) \\ &\quad \leq \delta^{-2}P(T_m \geq c_m)P(T_n \geq c_n), \end{aligned} \tag{2.26}$$

which completes the proof of (2.21) and hence the proof of Theorem 1.1. □

3. Proofs of Theorems 1.2.A and 1.2.B. Before giving the actual proofs we need some lemmas. Let $E_1, E_2, \ldots, E_n, \ldots$ be a sequence of independent exponentially distributed rv's with parameter 1; let H_n be the df of $\sum_{i=1}^n E_i$ [the gamma $(n,1)$ df], and let $\hat{\Gamma}_n$ be the empirical df based on $U_1, U_2, \ldots, U_n$ from the original sequence of independent uniform (0, 1) rv's. We define $\mathbb{D}_{i,n}^* := 1 - \mathbb{D}_{i,n}$ for $i = 1, 2, \ldots, n$; let $\hat{F}_n^*$ denote the empirical df based on these $\mathbb{D}_{i,n}^*$ and note that

$$W_{n,\nu} = \sup_{0<t<1} \frac{n|\hat{F}_n(t) - t|}{(1-t)^{1-\nu}} = \sup_{0<t<1} \frac{n|\hat{F}_n^*(t) - t|}{t^{1-\nu}}.$$

The first lemma is well known [see, e.g., Pyke (1965), Beirlant, van der Meulen, Ruymgaart and van Zuijlen (1982) and Devroye (1981)].

LEMMA 3.1.

(i) *The random vector* $(E_1/n\overline{E}_n, E_2/n\overline{E}_n, \dots, E_n/n\overline{E}_n)$ *is distributed as* $(D_{1,n}, D_{2,n}, \dots, D_{n,n})$ *and is independent of* $\overline{E}_n$, *where* $\overline{E}_n := n^{-1}\sum_{i=1}^n E_i$.

(ii) *Uniform spacings are exchangeable and negative lower orthant dependent (NLOD), i.e.,*

$$P(D_{1,n} \le x_1; D_{2,n} \le x_2; \cdots; D_{n,n} \le x_n) \le \prod_{i=1}^n P(D_{i,n} \le x_i).$$

(iii) *For nonnegative numbers* $c_1, c_2, \dots, c_n$ *we have*

$$P(D_{1,n} \ge c_1; D_{2,n} \ge c_2; \cdots; D_{n,n} \ge c_n) = \begin{cases} \left(1 - \sum_{i=1}^n c_i\right)^{n-1}, & \text{for } \sum_{i=1}^n c_i < 1, \\ 0, & \text{otherwise}. \end{cases}$$

LEMMA 3.2. *For each* $\lambda > 0$ *and each interval* $A \subset [0,1]$ *we have*

$$(3.1)\quad P\Big(\sup_{t\in A} n^{1/2}(\hat{F}_n(t) - t) \ge \lambda\Big) \le \frac{1}{H_n(n)} P\Big(\sup_{t\in A} n^{1/2}(\hat{\Gamma}_n(t) - t) \ge \lambda\Big),$$

$$(3.2)\quad P\Big(\sup_{t\in A} n^{1/2}(t - \hat{F}_n(t)) \ge \lambda\Big) \le \frac{1}{1 - H_n(n)} P\Big(\sup_{t\in A} n^{1/2}(t - \hat{\Gamma}_n(t)) \ge \lambda\Big).$$

PROOF. With $\tilde{t} := G^{-1}(t)$ we have with the aid of Lemma 3.1

$$\begin{aligned} &P\Big(\sup_{t\in A} n^{1/2}(\hat{F}_n(t) - t) \ge \lambda\Big) \\ &= P\left(\sup_{t\in A} n^{1/2}\left(n^{-1}\sum_{i=1}^n 1_{(0,t]}(\mathbb{D}_{i,n}) - t\right) \ge \lambda\right) \\ &= P\left(\sup_{t\in A} n^{1/2}\left(n^{-1}\sum_{i=1}^n 1_{(0,\tilde{t}]}(E_i/\overline{E}_n) - t\right) \ge \lambda | \overline{E}_n \le 1\right) \\ &= P\left(\sup_{t\in A} n^{1/2}\left(n^{-1}\sum_{i=1}^n 1_{(0,\tilde{t}]}(E_i/\overline{E}_n) - t\right) \ge \lambda; \overline{E}_n \le 1\right)\left(P(\overline{E}_n \le 1)\right)^{-1} \\ &\le P\left(\sup_{t\in A} n^{1/2}\left(n^{-1}\sum_{i=1}^n 1_{(0,\tilde{t}]}(E_i) - t\right) \ge \lambda; \overline{E}_n \le 1\right)(H_n(n))^{-1} \\ &\le (H_n(n))^{-1} P\Big(\sup_{t\in A} n^{1/2}(\hat{\Gamma}_n(t) - t) \ge \lambda\Big), \end{aligned}$$

which shows (3.1). The proof of (3.2) can be given in a similar way by conditioning on the event $[\overline{E}_n \geq 1]$. Compare with Einmahl (1985), where a similar technique is used. □

LEMMA 3.3. *There exist* $c_1, c_2, c_3 \in (0, \infty)$ *such that for every* $\lambda > 0$ *and every* $\theta \in (0,1)$,

$$\max\left(P\left(\sup_{t \geq \theta} \frac{n^{1/2}|\hat{F}_n(t) - t|}{t^{1/2}} \geq \lambda \right), P\left(\sup_{t \geq \theta} \frac{n^{1/2}|\hat{F}_n^*(t) - t|}{t^{1/2}} \geq \lambda \right) \right) \tag{3.3}$$
$$\leq c_1 \left(\log \frac{2}{\theta} \right) \exp\left(-c_2 \lambda^2 \psi\left(c_3 \lambda (n\theta)^{-1/2} \right) \right),$$

where $\psi: [0, \infty] \to (0, \infty)$ *satisfies* $\psi(\sigma) \sim 2\sigma^{-1} \log \sigma$ *as* $\sigma \to \infty$.

PROOF. The given upper bound in the inequality of the lemma can be obtained as in Einmahl, Ruymgaart and Wellner (1984) from the well-known upper bounds for the right-hand sides of the inequalities in Lemma 3.2, taking into account that

$$P\left(\sup_{t \geq \theta} \frac{n^{1/2}|\hat{F}_n^*(t) - t|}{t^{1/2}} \geq \lambda \right) = P\left(\sup_{t \leq 1-\theta} \frac{n^{1/2}|\hat{F}_n(t) - t|}{(1-t)^{1/2}} \geq \lambda \right),$$

and that both $(H_n(n))^{-1}$ and $(1 - H_n(n))^{-1}$ are bounded above by some $M \in (0, \infty)$. □

PROOF OF THEOREM 1.2.A. Throughout the proof let $\nu \in [0, \frac{1}{2}]$ be fixed. Suppose that $\Sigma a_n = \infty$. From Theorem 3.1 in Devroye (1982) we see that $P(\mathbb{D}_{1:n} \leq a_n \text{ i.o.}) = 1$ under the extra condition that $a_n/n \downarrow$. With the aid of $[\Sigma_{n=1}^{\infty} a_n = \infty] \Rightarrow [P(U_{1:n} \leq a_n \text{ i.o.}) = 1]$ and a suitable transformation it can be shown that this extra condition is superfluous.

Hence

$$P(\mathbb{D}_{1:n} < \varepsilon a_n \text{ i.o.}) = 1, \quad \text{for every } \varepsilon > 0. \tag{3.4}$$

The inequality $V_{n,\nu} \geq (2\mathbb{D}_{1:n}^{1-\nu})^{-1}$ and (3.4) easily lead to (1.4) if we let $\varepsilon \downarrow 0$.

Next we suppose that $\Sigma a_n < \infty$, $a_n \downarrow$ and $n a_n \log n \to 0$ as $n \to \infty$. Define

$$V_n^{\pm} := \sup_{0 < t \leq \log n/n} \frac{\pm n(F_n(t) - t)}{t^{1-\nu}} \tag{3.5}$$

and let us start by showing that $\limsup_{n\to\infty} a_n^{1-\nu} V_n^+ \leq 1$ a.s., hence that $\lim_{n\to\infty} a_n^{1-\nu} V_n^+ = 0$ a.s. According to a version of the Borel–Cantelli lemma we need to show that $\Sigma P C_n < \infty$ and $P A_n \to 0$ as $n \to \infty$, where

$$A_n := \left\{ V_n^+ \geq \frac{1}{a_n^{1-\nu}} \right\} \quad \text{and} \quad C_n := A_n A_{n-1}^c.$$

Let $b_n := \log n/n$ and let $y_{k,n}$ be the solutions of the equations, $n = 1, 2, \dots,$

$$f_n(t) = k, \quad \text{for } k = 0, 1, \dots, k_n := [f_n(b_n)] + 1,$$

where

$$f_n(t) := nt + \left(\frac{t}{a_n}\right)^{1-\nu}, \qquad t \in [0, \infty). \tag{3.6}$$

Note that

$$k_n = 1 + \left[\log n + \left(\frac{\log n}{na_n}\right)^{1-\nu}\right]. \tag{3.7}$$

Clearly, since $na_n \log n \to 0$ we have

$$k_n \le 3\left(\frac{\log n}{na_n}\right)^{1-\nu}, \quad \text{for large } n. \tag{3.8}$$

Moreover, let $x_k \equiv x_{k,n} := y_{k,n} \wedge b_n$. Observe that for large n, f_n is increasing in t, $f_{n-1} \le f_n$ since $a_n \downarrow$, $b_{n-1} \ge b_n$, $x_{k_n} = b_n$ and $x_k \le k^{1/(1-\nu)} a_n$. We have

$$C_n = \left\{V_n^+ \ge \frac{1}{a_n^{1-\nu}};\ V_{n-1}^+ < \frac{1}{a_{n-1}^{1-\nu}}\right\}$$

$$= \left\{\sup_{0<t\le b_n} \frac{\hat{F}_n(t) - t}{t^{1-\nu}} \ge \frac{1}{na_n^{1-\nu}};\ \sup_{0<t\le b_{n-1}} \frac{\hat{F}_{n-1}(t) - t}{t^{1-\nu}} < \frac{1}{(n-1)a_{n-1}^{1-\nu}}\right\}$$

$$\subset \left\{\exists_{t\in(0,b_n]}\, n\hat{F}_n(t) \ge f_n(t);\ \forall_{t\in(0,b_n]}\, (n-1)\hat{F}_{n-1}(t) < f_n(t)\right\} \tag{3.9}$$

$$\subset \bigcup_{k=1}^{k_n} \left\{\exists_{t\in(x_{k-1},x_k]}\, n\hat{F}_n(t) = k;\ \forall_{t\in(x_{k-1},x_k]}\, (n-1)\hat{F}_{n-1}(t) \le k-1\right\}$$

$$\subset \bigcup_{k=1}^{n\wedge k_n} B_{n,k},$$

where

$$B_{n,k} := \{\mathbb{D}_{k:n-1} > x_k,\ \mathbb{D}_{k:n} \le x_k\} \tag{3.10}$$

and where $\mathbb{D}_{1:n} \le \mathbb{D}_{2:n} \le \cdots \le \mathbb{D}_{n:n}$ are the ordered $\mathbb{D}_{i,n}$, $i = 1, 2, \dots, n$. Define

$$A_{n,k} := \sum_{j=3}^{n} 1_{(0,z_k]}(D_{j,n}),$$

where

$$z_k \equiv z_{k,n} := G^{-1}(x_k)/n.$$

By conditioning on the event $\{U_{n-1} = U_{1:\,n-1}\}$ and using the independence of the order statistics and the ranks, we find for $n \geq 3$,

$$PB_{n,k} = P\left(\sum_{j=1}^{n-1} 1_{(0,(n/(n-1))z_k]}(D_{j,n-1}) \leq k-1;\ \sum_{j=1}^{n} 1_{(0,z_k]}(D_{j,n}) \geq k\right)$$
$$\leq P(D_{1,n} \leq z_k;\ D_{2,n} \leq z_k;\ A_{n,k} = k-2)$$
$$+P(D_{2,n} \leq z_k;\ A_{n,k} = k-1) + P(D_{1,n} \leq z_k;\ A_{n,k} = k-1).$$

Because of the exchangeability and the negative lower orthant dependence of uniform spacings we have [by convention $\binom{n}{k} = 0$ for $k < 0$ or $k > n$]

$$P(D_{1,n} \leq z_k;\ D_{2,n} \leq z_k;\ A_{n,k} = k-2) \leq \binom{n-2}{k-2}(F_n(x_k))^k.$$

Moreover,

$$P(D_{2,n} \leq z_k;\ A_{n,k} = k-1) \leq \binom{n-2}{k-1}(F_n(x_k))^k$$

and

$$P(D_{1,n} \leq z_k;\ A_{n,k} = k-1) \leq \binom{n-2}{k-1}(F_n(x_k))^k,$$

so that we obtain ($F_n(x) \leq ex$ for $x \in [0, \frac{1}{2}]$)

$$PB_{n,k} \leq 3\binom{n-1}{k-1}(F_n(x_k))^k$$
$$\leq 3\binom{n-1}{k-1}(ex_k)^k$$
$$\leq 3e^k \frac{n^{k-1}}{(k-1)!} k^{k/(1-\nu)} a_n^k \tag{3.11}$$
$$\leq 3a_n e^k (na_n)^{k-1} \frac{k^{k/(1-\nu)+1}}{k!}$$
$$\leq 3a_n e^{2k} (na_n)^{k-1} k^{k\nu/(1-\nu)+1}$$
$$\leq 3a_n e^{2k} \left(na_n k_n^{\nu/(1-\nu)}\right)^{k-1} k^{1/(1-\nu)},$$

where

$$na_n k_n^{\nu/(1-\nu)} \leq 3(na_n \log n)^{1/2}.$$

Since $na_n \log n \to 0$ as $n \to \infty$, it follows that $PC_n \leq M_\nu a_n$ for certain $M_\nu \in (0, \infty)$ (for large n) and hence $\Sigma PC_n < \infty$.

For the proof that $PA_n \to 0$ we note that for large n,

$$
\begin{aligned}
PA_n &\leq P\Big(\exists_{t \in (0, b_n]}\, n\hat{F}_n(t) \geq f_n(t)\Big) \\
&\leq P\left(\bigcup_{k=1}^{n \wedge k_n} \left\{ \sup_{t \leq x_k} n\hat{F}_n(t) \geq k \right\}\right) \\
&\leq \sum_{k=1}^{n \wedge k_n} P(\mathbb{D}_{k:n} \leq x_k) \\
&\leq \sum_{k=1}^{n \wedge k_n} \binom{n}{k} (F_n(x_k))^k \\
&\leq \sum_{k=1}^{n \wedge k_n} \frac{n}{k} \binom{n-1}{k-1} (ex_k)^k \\
&\leq \frac{1}{3} M_\nu n a_n \to 0, \quad \text{as } n \to \infty,
\end{aligned}
\tag{3.12}
$$

which completes the proof of

$$\lim_{n \to \infty} a_n^{1-\nu} V_n^+ = 0 \quad \text{a.s.} \tag{3.13}$$

Remark that also

$$a_n^{1-\nu} V_n^- \leq (n a_n \log n)^{1-\nu} (\log n)^{2\nu - 1} \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Finally, Lemma 3.3 immediately yields for large n,

$$
\begin{aligned}
&P\left(\sup_{\log n/n < t < 1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\nu}} \geq \frac{1}{a_n^{1-\nu}} \right) \\
&\qquad \leq P\left(\sup_{\log n/n < t < 1} \frac{n|\hat{F}_n(t) - t|}{t^{1/2}} \geq \frac{1}{a_n^{1/2}} \right) \leq \frac{1}{n^2},
\end{aligned}
$$

so that because of the Borel–Cantelli lemma

$$\lim_{n \to \infty} a_n^{1-\nu} \sup_{\log n/n < t < 1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\nu}} = 0 \quad \text{a.s.} \tag{3.15}$$

Combination of the obtained partial results in (3.13)–(3.15) yields

$$\lim_{n \to \infty} a_n^{1-\nu} \sup_{0 < t < 1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\nu}} = 0 \quad \text{a.s.} \tag{3.16}$$

□

PROOF OF THEOREM 1.2.B. We follow the lines of the proof of Theorem 1.2.A; however, the tools and the calculations will be different. Let $\nu \in [0, \frac{1}{2}]$ be fixed. Suppose that $\Sigma a_n \log(a_n^{-1}) = \infty$. Let $\mathbb{D}_{1,n}^* \le \mathbb{D}_{2,n}^* \le \cdots \le \mathbb{D}_{n:n}^*$ denote the ordered $\mathbb{D}_{i,n}^*$, $i = 1, 2, \dots, n$. From Theorem 1.1 we have

$$P\big(nD_{n:n} \ge \log\big(a_n^{-1}\big) \text{ i.o.}\big) = P(\mathbb{D}_{n:n} \ge 1 - a_n \text{ i.o.}) = P(\mathbb{D}_{1:n}^* \le a_n \text{ i.o.}) = 1,$$

and hence also $P(\mathbb{D}_{1:n}^* \le \varepsilon a_n \text{ i.o.}) = 1$ for every $\varepsilon > 0$. Now (1.7) follows by the argument used below (3.4) since $W_{n,\nu} \ge (2(\mathbb{D}_{1:n}^*)^{1-\nu})^{-1}$.

For the proof of (1.8) we suppose that $\Sigma a_n \log a_n^{-1} < \infty$ and $a_n \downarrow 0$ (which implies that $na_n \log n \to 0$ as $n \to \infty$ and $\Sigma a_n \log n < \infty$). Without loss of generality we restrict ourselves to sequences $\{a_n\}_{n=1}^\infty$ with $a_n \ge n^{-3/2}$. Define

$$W_n^{\pm} := \sup_{0<t<\log n/n} \frac{\pm\big(\hat{F}_n^*(t) - t\big)}{t^{1-\nu}}. \tag{3.17}$$

To obtain $\lim_{n\to\infty} a_n^{1-\nu} W_n^+ = 0$ a.s. we need to show that $\Sigma PC_n^* < \infty$ and $PA_n^* \to 0$ as $n \to \infty$, where now

$$A_n^* := \left\{W_n^+ \ge \frac{1}{a_n^{1-\nu}}\right\} \quad \text{and} \quad C_n^* := A_n^* A_{n-1}^{*c}. \tag{3.18}$$

Let b_n, $y_{k,n}$, $f_n(t)$, k_n and x_k be the quantities which are defined in the proof of Theorem 1.2.A. Note that

$$(2n^{3/2})^{-1} \le x_k \le k^{1/(1-\nu)} a_n, \tag{3.19}$$

and similarly as in (3.9) we have

$$C_n^* = \left\{W_n^+ \ge 1/a_n^{1-\nu};\ W_{n-1}^+ < \frac{1}{a_n^{1-\nu}}\right\} \subset \bigcup_{k=1}^{n\wedge k_n} B_{n,k}^*, \tag{3.20}$$

where now

$$B_{n,k}^* := \{\mathbb{D}_{k:n-1}^* > x_k;\ \mathbb{D}_{k:n}^* \le x_k\}. \tag{3.21}$$

Letting

$$\begin{aligned} z_k^* \equiv z_{k,n}^* &:= \frac{G^{-1}(1 - x_k)}{n} = \frac{\log\big(x_k^{-1}\big)}{n}, \\ u_k &:= \frac{n}{n-1} z_k^*, \\ I_n &:= \sum_{i=1}^n 1_I(D_{i,n}), \quad \text{for intervals } I, \end{aligned} \tag{3.22}$$

we obtain

$$PB^*_{n,k} = P\left(\sum_{j=1}^{n-1} 1_{[1-x_k,1]}(\mathbb{D}_{j,n-1}) \le k-1;\ \sum_{j=1}^{n} 1_{[1-x_k,1]}(\mathbb{D}_{j,n}) \ge k \right)$$

$$= P\left(\sum_{j=1}^{n-1} 1_{(0,1-x_k)}(\mathbb{D}_{j,n-1}) \ge n-k;\ \sum_{j=1}^{n} 1_{(0,1-x_k)}(\mathbb{D}_{j,n}) \le n-k \right) \tag{3.23}$$

$$= P\big((0,u_k)_{n-1} \ge n-k;\ (0,z_k^*)_n \le n-k\big)$$

$$= P_1 + P_2 + P_3,$$

where

$$P_1 := P\big((0,u_k)_{n-1} \ge n-k;\ (0,z_k^*)_n \le n-k;\ [z_k^*,u_k)_{n-1} = 0\big),$$

$$P_2 := P\big((0,u_k)_{n-1} \ge n-k;\ (0,z_k^*)_n \le n-k;\ [z_k^*,u_k)_{n-1} = 1\big),$$

$$P_3 := P\big((0,u_k)_{n-1} \ge n-k;\ (0,z_k^*)_n \le n-k;\ [z_k^*,u_k)_{n-1} \ge 2\big).$$

Next, we define the rv's

$$\begin{aligned} J_{n,k} &:= \sum_{i=3}^{n} 1_{(0,z_k^*)}(D_{i,n}), \qquad K_{n,k} := \sum_{i=4}^{n} 1_{(0,z_k^*)}(D_{i,n}), \\ S_{n,k} &:= \sum_{i=3}^{n} 1_{[u_k,1)}(D_{i,n}), \qquad T_{n,k} := \sum_{i=4}^{n} 1_{[u_k,1)}(D_{i,n}). \end{aligned} \tag{3.24}$$

We obtain, again by conditioning on the event $\{U_{n-1} = U_{1:n-1}\}$, with the aid of Lemma 3.1 for n sufficiently large,

$$P_1 = P(J_{n,k} = n-k;\ D_{1,n} \ge z_k^*;\ D_{2,n} \ge z_k^*;\ S_{n,k} = k-2)$$

$$\le \binom{n-2}{k-2}(1-kz_k^*)^{n-1}$$

$$= \binom{n-2}{k-2}\left(1 - \frac{k\log(x_k^{-1})}{n}\right)^{n-1} \tag{3.25}$$

$$\le \binom{n-2}{k-2} x_k^{k(n-1)/n}$$

$$\le \binom{n-1}{k-1}(2x_k)^k.$$

Moreover, observe that

$$P_2 = P\big((0,z_k^*)_{n-1} \ge n-k-1;\ (0,z_k^*)_n \le n-k;\ [z_k^*,u_k)_{n-1} = 1\big)$$

$$= P_{21} + P_{22} + P_{23},$$

where

$$P_{21} := P\big((0, z_k^*)_{n-1} \geq n-k-1;\ (0, z_k^*)_n \leq n-k;\ [z_k^*, u_k)_{n-1} = 1;\ D_{1,n} + D_{2,n} < z_k^*\big),$$

$$P_{22} := P\big((0, z_k^*)_{n-1} \geq n-k-1;\ (0, z_k^*)_n \leq n-k;\ [z_k^*, u_k)_{n-1} = 1;\ D_{1,n} + D_{2,n} \geq u_k\big),$$

$$P_{23} := P\big((0, z_k^*)_{n-1} \geq n-k-1;\ (0, z_k^*)_n \leq n-k;\ [z_k^*, u_k)_{n-1} = 1;\ D_{1,n} + D_{2,n} \in [z_k^*, u_k)\big).$$

We obtain, similarly as in (3.25), with the aid of the mean-value theorem for n sufficiently large,

$$\begin{aligned}
P_{21} &= (n-2)P\big(K_{n,k} = n-k-2;\ D_{3,n} \in [z_k^*, u_k);\ D_{1,n} + D_{2,n} < z_k^*;\ T_{n,k} = k-1\big)\\
&\leq (n-2)\binom{n-3}{k-1}P\big(D_{1,n} \geq u_k;\ D_{2,n} \geq u_k;\ \cdots;D_{k-1,n} \geq u_k;\ D_{k,n} \in [z_k^*, u_k)\big)\\
&= (n-2)\binom{n-3}{k-1}\big\{(1-(k-1)u_k - z_k^*)^{n-1} - (1-ku_k)^{n-1}\big\}\\
&\leq (n-2)\binom{n-3}{k-1}z_k^*(1-kz_k^*)^{n-2}\\
&\leq (n-2)\binom{n-3}{k-1}\frac{\log(x_k^{-1})}{n}x_k^{k((n-2)/n)}\\
&\leq \binom{n-1}{k-1}x_k^{k((n-2)/n)}\log(x_k^{-1})\\
&\leq \binom{n-1}{k-1}(2x_k)^k\log(x_k^{-1}),
\end{aligned}$$

$$\begin{aligned}
P_{22} &= 2(n-2)P\big(K_{n,k} = n-k-1;\ D_{3,n} \in [z_k^*, u_k);\ D_{1,n} + D_{2,n} \geq u_k;\ D_{1,n} < z_k^*;\ D_{2,n} \geq z_k^*;\ T_{n,k} = k-2\big)\\
&\quad + (n-2)P\big(K_{n,k} = n-k;\ D_{3,n} \in [z_k^*, u_k);\ D_{1,n} \geq z_k^*;\ D_{2,n} \geq z_k^*;\ T_{n,k} = k-3\big)\\
&\leq 2(n-2)\binom{n-3}{k-2}P\big(D_{1,n} \geq z_k^*;\ D_{2,n} \geq z_k^*;\ \cdots;D_{k-1,n} \geq z_k^*;\ D_{k,n} \in [z_k^*, u_k)\big)\\
&\quad + (n-2)\binom{n-3}{k-3}P\big(D_{1,n} \geq z_k^*;\ D_{2,n} \geq z_k^*;\ \cdots;D_{k-1,n} \geq z_k^*;\ D_{k,n} \in [z_k^*, u_k)\big)\\
&\leq 3(n-2)\binom{n-1}{k-1}z_k^*(1-kz_k^*)^{n-2}\\
&\leq 3\binom{n-1}{k-1}x_k^{k((n-2)/n)}\log(x_k^{-1})\\
&\leq 3\binom{n-1}{k-1}(2x_k)^k\log(x_k^{-1})
\end{aligned}$$

and

$$P_{23} = 2P(J_{n,k} = n-k-1;\ D_{1,n} + D_{2,n} \in [z_k^*, u_k);\ D_{1,n} < z_k^*;\ D_{2,n} \geq z_k^*;\ S_{n,k} = k-1)$$
$$\leq 2\binom{n-1}{k-1} P(D_{1,n} \geq u_k;\ D_{2,n} \geq u_k; \cdots; D_{k-1,n} \geq u_k;\ D_{k,n} \geq z_k^*)$$
$$\leq 2\binom{n-2}{k-1}(1 - kz_k^*)^{n-1}$$
$$\leq 2\binom{n-1}{k-1}(2x_k)^k.$$

Finally, for n sufficiently large,

$$P_3 \leq P([z_k^*, u_k)_{n-1} \geq 2)$$
$$\leq n^2 P(D_{1,n-1} \in [z_k^*, u_k);\ D_{2,n-1} \in [z_k^*, u_k))$$
$$= n^2\Big\{\big((1-2z_k^*)^{n-2} - (1 - z_k^* - u_k)^{n-2}\big) - \big((1 - u_k - z_k^*)^{n-2} - (1-2u_k)^{n-2}\big)\Big\}$$
$$\leq n^2\left\{\frac{n-2}{n-1} z_k^*(1-2z_k^*)^{n-3} - \frac{n-2}{n-1} z_k^*(1-2u_k)^{n-3}\right\}$$
$$\leq 2n^2 z_k^{*2}(1-2z_k^*)^{n-4}$$
$$\leq 2n^2\left(\frac{\log x_k^{-1}}{n}\right)^2 x_k^{2((n-4)/n)}$$
$$\leq 2(2x_k)^2(\log x_k^{-1})^2,$$

so that

$$PB_{n,k} \leq 7\binom{n-1}{k-1}(2x_k)^k \log(x_k^{-1}) + 2(2x_k)^2(\log(x_k^{-1}))^2$$
$$\leq 11\binom{n-1}{k-1}(2x_k)^k \log n + 5\left(\frac{2\log n}{n}\right)^2(\log n)^2. \tag{3.26}$$

As in (3.11) we obtain (since $a_n \geq n^{-3/2}$) for certain $M_\nu^* \in (0, \infty)$,

$$PC_n^* \leq \sum_{k=1}^{k_n} 11 \log n a_n e^{2k}\big(3(na_n \log n)^{1/2}\big)^{k-1} k^{1/(1-\nu)} + 60\left(\frac{\log n}{na_n}\right)^{1-\nu}\frac{(\log n)^4}{n^2}$$
$$\leq M_\nu^*\big(a_n \log n + (\log n)^5 n^{-3/2}\big),$$

so that $\Sigma PC_n^* < \infty$. Moreover,

$$
\begin{aligned}
PA_n^* &\le \sum_{k=1}^{n\wedge k_n} P(\mathbb{D}_{k:n}^* \le x_k) \\
&\le \sum_{k=1}^{n\wedge k_n} \binom{n}{k}(1 - kz_k^*)^{n-1} \qquad (3.27)\\
&\le \sum_{k=1}^{n\wedge k_n} \binom{n}{k}(2x_k)^k \to 0, \quad \text{as } n \to \infty,
\end{aligned}
$$

which completes the proof of

$$\lim_{n\to\infty} a_n^{1-\nu} W_n^+ = 0 \quad \text{a.s.} \tag{3.28}$$

Finally, from the reasoning in (3.13)–(3.16) with $V_n^{\pm}$ replaced by $W_n^{\pm}$ and $\hat{F}_n$ replaced by $\hat{F}_n^*$ it follows that

$$\lim_{n\to\infty} a_n^{1-\nu} \sup_{0<t<1} \frac{n|F_n^*(t) - t|}{t^{1-\nu}} = 0 \quad \text{a.s.,} \tag{3.29}$$

which completes the proof of (1.8). □

REFERENCES

BEIRLANT, J., VAN DER MEULEN, E. C., RUYMGAART, F. H. and ZUIJLEN, M. C. A. VAN (1982). On functions bounding the empirical distribution of uniform spacings. *Z. Wahrsch. verw. Gebiete* **61** 417–430.

CSÁKI, E. (1974). Studies on the empirical d.f. *MTA III. Oszt. Közl.* **23** 239–327 (in Hungarian).

CSÁKI, E. (1975). Some notes on the law of the iterated logarithm for empirical distribution function. *Colloq. Math. Soc. János Bolyai. Limit Theorems of Probability Theory* **11** 47–58.

CSÁKI, E. (1982). On the standarized empirical distribution function. *Colloq. Math. Soc. János Bolyai. Nonparametric Statistical Inference* **32** 123–138.

DEHEUVELS, P. (1982). Strong limiting bounds for maximal uniform spacings. *Ann. Probab.* **10** 1058–1065.

DEVROYE, L. (1981). Laws of the iterated logarithm for order statistics of uniform spacings. *Ann. Probab.* **9** 860–867.

DEVROYE, L. (1982). Upper and lower class sequences for minimal uniform spacings. *Z. Wahrsch. verw. Gebiete* **61** 237–254.

EINMAHL, J. H. J. (1985). On the Kolmogorov–Smirnov statistic of certain dependent random variables. *Publ. Inst. Statist. Univ. Paris* **30** 53–57.

EINMAHL, J. H. J. and MASON, D. M. (1985). Bounds for weighted multivariate empirical distribution functions. *Z. Wahrsch. verw. Gebiete* **70** 563–571.

EINMAHL, J. H. J., RUYMGAART, F. H. and WELLNER, J. A. (1984). A characterization of weak convergence of weighted multivariate empirical processes. *Acta Sci. Math. (Szeged)*. To appear.

KOCHEN, S. B. and STONE, C. J. (1964). A note on the Borel–Cantelli problem. *Illinois J. Math.* **8** 248–251.

LEHMANN, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–1153.

MASON, D. M. (1981). Bounds for weighted empirical distribution functions. *Ann. Probab.* **9** 881–884.

MASON, D. M. (1982). Some characterizations of almost sure bounds for weighted multidimensional empirical distributions and a Glivenko–Cantelli theorem for sample quantiles. *Z. Wahrsch. verw. Gebiete* **59** 505–513.

PYKE, R. (1965). Spacings (with discussion). *J. Roy. Statist. Soc. Ser. B* **27** 395–449.

SHORACK, G. R. and WELLNER, J. A. (1978). Linear bounds on the empirical distribution function. *Ann. Probab.* **6** 349–353.

SLUD, E. (1978). Entropy and maximal spacings for random partitions. *Z. Wahrsch. verw. Gebiete* **41** 341–352.

DEPARTMENT OF MEDICAL INFORMATICS
AND STATISTICS
UNIVERSITY OF LIMBURG
P.O. BOX 616
6200 MD MAASTRICHT
THE NETHERLANDS

DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
TOERNOOIVELD
6525 ED NIJMEGEN
THE NETHERLANDS