A note on the β-measure for digraph competitions

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Abstract

Digraph games are cooperative TU-games associated to digraph competitions: domination structures that can be modeled by directed graphs. Examples come from sports competitions or from simple majority win digraphs corresponding to preference profiles for a group of individuals within the framework of social choice theory. Van den Brink and Gilles (2000) defined the β-measure of a digraph competition as the Shapley value of the corresponding digraph game. This paper provides a new characterization of the β-measure.

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1 Introduction

A directed graph can represent various domination structures that are based on (partial) pairwise comparisons. An obvious example is a sports competition in which several teams play matches against each other and the digraph summarizes the results of the various matches. Other examples include the results of paired comparison experiments for example within a group of alternative medicines, the results of aggregated pairwise preferences of a group of individuals, based e.g. on simple majority voting, over a certain set of alternatives, or, within a completely different framework, the hierarchical structure in economic organizations.

In the sequel we use the term digraph competition for a domination structure modeled by a digraph. The main issue under consideration is how to measure the "strength" of each node in a digraph competition. We continue on the lines set out by Van den Brink and Borm (2002) and consider an arbitrary digraph competition as a special type of allocation problem where we initially assume that each node is assigned equal weight (say equal to one). Measuring strength then can be seen as "fairly" reallocating these weights taking into account the domination structure that is represented by the digraph. For this aim a digraph game is associated to each digraph competition: in this game the players correspond to the nodes and the value of a coalition represents the maximal total weight for which there is no rightful direct claim from outside this coalition. Here a player has a rightful claim on the weights of all nodes that he dominates directly. One way to measure the strength of the nodes in a digraph competition is to consider the Shapley value of the associated digraph game (cf. Shapley (1953)). We follow Van den Brink and Gilles (2000) and Van den Brink and Borm (2002) in calling this the β-measure of the underlying digraph competition.

Van den Brink and Borm (2002) extensively analyze the class of digraph games. Among other things it turns out that these games are convex and that the Shapley value of these games, -by convexity the barycentre of the core-, is also the average of rather intuitive so-called simple score vectors (which are in the core) related to specific sub-digraphs. Borm, Van den Brink, and Slikker (2002) exploit the possibility to start out from arbitrary initial weights by investigating the limit behavior in the iterative process that takes the Shapley value of the digraph game as new weight parameters of the nodes to determine the new digraph game in the next step. The resulting limit measure is called the λ-measure. In Borm, Van den Brink, Levínsky, and Slikker (2004) the particular application area of aggregate preferences over alternatives is investigated: the β- and λ-measures form the basis for defining and analyzing two new social choice correspondences.

The current short note provides a characterization of the β-measure on the class of all digraph competitions using the properties of component efficiency, symmetry, triviality and decomposition additivity. Characterizations of the β-measure on specific subclasses of digraph competitions are provided in Van den Brink (1994).

The paper is organized as follows. Section 2 recalls the main definitions concerning digraph competitions, digraph games and the β-measure. Section 3 provides the characterization of the β-measure. The final Section 4 considers the card game "Frank's Zoo" as an illustration.

2 Digraph games

A digraph (competition) is a pair (N, D) where N is a finite set of nodes and $D \subset N \times N$ is a binary relation on N. The set of all digraphs is denoted by $\mathcal{D}$, the subclass $\mathcal{D}^N$ consists of all digraphs that have N as the set of nodes. For $D \in \mathcal{D}^N$ and $i \in N$ the set $P_D(i) = \{j \in N \mid (j,i) \in D\}$ is called the set of predecessors of i in D. The set $S_D(i) = \{j \in N \mid (i,j) \in D\}$ consists of all successors of i. The set of nodes with at least one predecessor is denoted by I_D, so $I_D = \{j \in N \mid P_D(j) \neq \emptyset\}$. The digraph game (N, v_D) corresponding to $D \in \mathcal{D}^N$ is given by (cf. Van den Brink and Borm (2002))

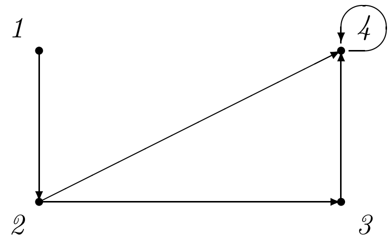
$$v_D(S) = |\{j \in I_D \mid P_D(j) \subset S\}| \quad \text{for all } S \in 2^N \backslash \{\emptyset\}.$$

As usual $v_D(\emptyset) = 0$. So a digraph game assigns to each coalition S the number of nodes in N that have all their predecessors in S, provided that this set of predecessors is non-empty. This means that the value of S is determined by the number of nodes with positive indegree on which $N \backslash S$ does not have any influence.

It is readily seen that for any digraph game $D \in \mathcal{D}$, v_D is the sum of unanimity games[1]: $v_D = \sum_{j \in I_D} u_{P_D(j)}$. As a consequence, digraph games are convex.

Example 2.1 *Let N be $\{1,2,3,4\}$ and consider the digraph (competition)*

[1]For $T \in 2^N \backslash \{\emptyset\}$, the unanimity game u_T is given by $u_T(S) = \begin{cases} 1 & \text{if } T \subset S, \\ 0 & \text{if } T \not\subset S. \end{cases}$



The corresponding digraph game v_D is given in the table below.

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$v_D(S)$	1	1	0	0	2	1	1	1	1	0	2	2	1	2	3

The set of nodes I_D with at least one predecessor is $\{2, 3, 4\}$. The digraph game v_D can be written as $v_D = u_{\{1\}} + u_{\{2\}} + u_{\{2,3,4\}}$.

A digraph rule f assigns to each digraph $D \in \mathcal{D}$ a payoff vector $f(D) \in \mathbb{R}^N$, if $D \in \mathcal{D}^N$. One particular digraph rule is provided by the β-measure, which assigns to every $D \in \mathcal{D}$ the Shapley value of the associated digraph game v_D. Using the decomposition in terms of unanimity games one obtains that for each $D \in \mathcal{D}^N$ and $i \in N$ the β-measure is given by

$$\beta_i(D) = \sum_{j \in S_D(i)} \frac{1}{|P_D(j)|}. \tag{1}$$

So, the β-measure of the digraph competition in Example 2.1 equals $(1, 0, 0, 0) + (0, 1, 0, 0) + \frac{1}{3}(0, 1, 1, 1)$, which makes $(1, \frac{4}{3}, \frac{1}{3}, \frac{1}{3})$.

3 A characterization of the β-measure

This section provides a characterization of the β-measure as a digraph rule. To do so, we first introduce some properties. Two nodes i and j are said to be connected in a digraph D if there is a sequence of nodes $(x_1, \ldots, x_m)$ such that $x_1 = i$, $\{(x_k, x_{k+1}), (x_{k+1}, x_k)\} \cap D \neq \emptyset$ for all $k \in \{1, \ldots, m-1\}$ and $x_m = j$. A set C of nodes is maximally connected in D if each pair of nodes in C are connected and no pair (i, j) with $i \in C$ and $j \in N \backslash C$ are. Such a set is also called a component of N. The node set N is the disjoint union of its components. A digraph rule f is called component efficient if for every component C, the sum of payoffs assigned to nodes in C equals the number of nodes in C with a non-empty set of predecessors.

Property 3.1 (Component efficiency): *A digraph rule f is component efficient if for all $D \in \mathcal{D}^N$ and all maximally connected subsets $C \subset N$ we have $\sum_{i \in C} f_i(D) = |C \cap I_D|$.*

Two nodes are symmetric if they have the same sets of predecessors and successors.

Property 3.2 (Symmetry): *A digraph rule f satisfies symmetry if for all $D \in \mathcal{D}^N$ and all $i, j \in N$ such that $S_D(i) = S_D(j)$ and $P_D(i) = P_D(j)$ we have $f_i(D) = f_j(D)$.*

The following property states that a node without outgoing arcs gets nothing. We call such nodes trivial nodes.

Property 3.3 (Triviality): *A digraph rule f satisfies triviality if for all $i \in N$ with $S_D(i) = \emptyset$ we have $f_i(D) = 0$.*

The fourth property states a decomposition additivity of nodes. Consider a digraph $D \in \mathcal{D}^N$. We can decompose a node i in D in the following way: replace i by a set of $|S_D(i)| + 1$ nodes $\{i_j : j \in S_D(i) \cup \{0\}\}$.[2] The incoming arcs of i are led to node i_0, so arc (j, i) is replaced by (j, i_0) for all $j \in P_D(i)$. The node i_j becomes the starting point of the original arc (i, j), i.e. (i, j) is replaced by (i_j, j).

After decomposing node i a new digraph $d_{\{i\}}(D)$ arises with node set $(N \backslash \{i\}) \cup \{i_j : j \in S_D(i) \cup \{0\}\}$. In $d_{\{i\}}(D)$, another node of $N \backslash \{i\}$ can be decomposed, or several ones consecutively. The graph arising from D by decomposing all nodes in a subset S is denoted by $d_S(D)$. It is easily seen that the order in which the nodes in S are decomposed does not affect the resulting graph $d_S(D)$. A digraph rule satisfies decomposition additivity if for every $S \in 2^N \backslash \{\emptyset\}$, the payoff that a node $i \in S$ achieves in D equals the sum of the payoffs achieved by the nodes arising from i in $d_S(D)$.

Example 3.1 *Consider the digraph D in Example 2.1. The graphs $d_{\{2\}}(D)$ and $d_N(D)$ are drawn in Figure 1.*

Property 3.4 (Decomposition additivity): *A digraph rule f satisfies decomposition additivity if for all $D \in \mathcal{D}^N$, for all nonempty subsets S of N and for all $i \in S$ we have*

$$f_i(D) = \sum_{j \in S_D(i) \cup \{0\}} f_{i_j}(d_S(D)).$$

The idea of decomposition additivity is as follows: a node with no predecessors can be joint with another node if they have disjoint sets of successors. The payoff to the "composite" node equals the sum of the payoffs of the original nodes.

[2] We assume that $0 \notin N$.

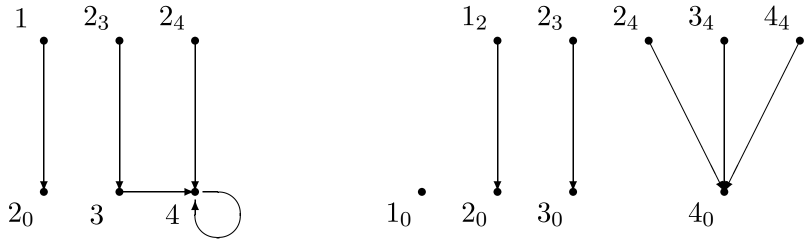


Figure 1: The graphs $d_{\{2\}}(D)$ and $d_{\{N\}}(D)$.

Theorem 3.1 *The β-measure is the unique digraph rule that satisfies component efficiency, symmetry, triviality and decomposition additivity.*

Proof: It is directly clear from formula (1) that β satisfies triviality, symmetry and decomposition additivity. The proof of component efficiency is straightforward and left to the reader.

Let $D \in \mathcal{D}^N$ and assume that the digraph rule f satisfies component efficiency, symmetry, triviality and decomposition additivity. Decomposing all nodes in D yields the graph $d_N(D)$. Clearly, the node set of $d_N(D)$ consists of $|N|$ components. Let for all $j \in N$, C_j be the component containing node j_0. Component C_j contains, besides node j_0, a node i_j for every predecessor i of j (in D). There is an arc in $d_N(D)$ from every such node to j_0, as illustrated in Figure 1. In any case, node j_0 is trivial, so $f_{j_0}(d_N(D)) = 0$. Consider a component C_j of $d_N(D)$ with $P_D(j) \neq \emptyset$. Node j_0 is the only node in C_j with predecessors, so by component efficiency f will divide one unit of payoff among the nodes in $C_j \setminus \{j_0\}$. The $|P_D(j)|$ other nodes in C_j are symmetric, so they share the unit of payoff equally: $f_{i_j}(d_N(D)) = \frac{1}{|C_j|-1} = \frac{1}{|P_D(j)|}$ for all $i \in P_D(j)$.

Decomposition additivity yields for every node $i \in N$

$$
\begin{aligned}
f_i(D) &= \sum_{j \in S_D(i) \cup \{0\}} f_{i_j}(d_N(D)) \\
&= \sum_{j \in S_D(i)} f_{i_j}(d_N(D)) \\
&= \sum_{j \in S_D(i)} \frac{1}{|P_D(j)|} \\
&= \beta_i(D).
\end{aligned}
$$

□

We conclude the section by showing the logical independence of the four properties. Firstly, let f be the digraph rule that divides all points generated within a component equally among its non-trivial nodes, i.e.

$$f_i(D) = \begin{cases} \dfrac{|\{j \in C_i \mid P_D(j) \neq \emptyset\}|}{|\{j \in C_i \mid S_D(j) \neq \emptyset\}|} & \text{if } S_D(i) \neq \emptyset, \\ 0 & \text{if } S_D(i) = \emptyset. \end{cases}$$

Here, C_i denotes the component in which node i is situated.[3] It is straightforward that this rule satisfies component efficiency, symmetry and triviality, but not decomposition additivity.

In order to be able to distinguish symmetrical nodes, we require an ordering $\prec$ on nodes. The alphabetical order applies. In the case names contain subscripts, we first compare their primary parts, so

$i_j \prec k_\ell$ if and only if $i \prec k$ or $(i = k$ and $j \prec \ell)$

and

$i \ \prec k_\ell$ if and only if $i \preceq k$.

Now we can define f to be the digraph rule that assigns the point of a node j to the predecessor i of j that alphabetically precedes the other predecessors of j, i.e.

$$f_i(D) = |\{j \in S_D(i) \mid i \preceq m \text{ for all } m \in P_D(j)\}|.$$

It is easy to verify that this rule satisfies component efficiency, triviality and decomposition additivity, but not symmetry.

Thirdly, let f be the rule that assigns the point of a node to the node itself, i.e.

$$f_i(D) = \begin{cases} 1 & \text{if } i \in I_D, \\ 0 & \text{if } i \notin I_D. \end{cases}$$

This rule satisfies component efficiency, symmetry and decomposition additivity, but not triviality.

Finally, the digraph rule that assigns zero to every node in a digraph competition satisfies symmetry, triviality and decomposition additivity, but not component efficiency.

[3]Note that this notation differs from the notation used in the proof of Theorem 3.1.

4 Application: a card game

In the card game "Frank's Zoo" (originally "Zoff im Zoo"), each card represents an animal. One of the players shows a set of cards, all representing the same animal. A second player can beat this collection by either showing a larger set of cards representing the same animal, or showing a set of cards of equal cardinality, but displaying an animal that can beat the animal of choice of the first player. There are twelve types of cards, representing the animals: whale (w), elephant (e), crocodile (c), polar bear (b), lion (ℓ), seal (s), fox (f), perch (p), hedgehog (h), goldfish (g), mouse (m) and mosquito (q). The digraph representing the hierarchical structure of the animals is drawn in Figure 2. For example, the arc from node e to node ℓ indicates that elephants can beat lions. Because each set of animals can be beaten by a larger set of the same animals, the digraph is actually reflexive. In order to reduce the number of arcs to be drawn, loops are omitted. One can

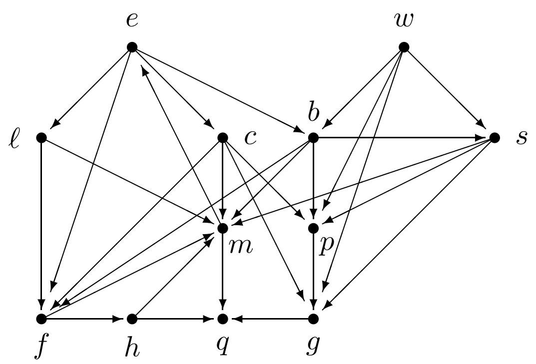


Figure 2: Graph corresponding to the game "Frank's Zoo".

use the β-measure of the digraph to determine the best card of the game. The results of this calculation can be found in Table 1. For example, the mouse beats three types of animals, i.e. elephants, mosquitos and mice. It shares the point generated by the elephant with the elephant, the point of the mosquito with four types of animals and its own point with seven types. Hence, its β-score equals $\frac{1}{2} + \frac{1}{4} + \frac{1}{7} = \frac{25}{28}$.

The game can be used to compare the β-rule with alternative rules. One could think of a score rule f_s that counts the number of successors. If one likes to take into account not only the number of successors, but also the number of predecessors, the number of types of animals that can beat a specific animal, a net score rule f_{ns} that subtracts the number of

animal	β	rank(β)	f_s	rank(f_s)	f_{ns}	rank(f_{ns})
whale	31/15	1	5	$1-4$	4	1
elephant	61/30	2	5	$1-4$	3	$2-3$
crocodile	87/70	3	5	$1-4$	3	$2-3$
polar bear	127/105	4	5	$1-4$	2	4
mouse	25/28	$5-6$	3	$6-9$	-4	12
hedgehog	25/28	$5-6$	3	$6-9$	1	$5-7$
seal	92/105	7	4	5	1	$5-7$
fox	59/70	$8-9$	3	$6-9$	-2	8
lion	59/70	$8-9$	3	$6-9$	1	$5-7$
goldfish	9/20	10	2	$10-11$	-3	$9-11$
perch	2/5	11	2	$10-11$	-3	$9-11$
mosquito	1/4	12	1	12	-3	$9-11$

Table 1: "Frank's Zoo"

predecessors from the number of successors is convenient. We have included these alternatives in the table.

The three rules all consider the whale to be the (a) strongest animal. If we look for distinctive features of the β-rule, we find that it produces much less ties than the score rule. A striking difference between the β-rule and the net score rule is the position of the mouse. This can be explained by the fact that the β-rule rewards the mouse strongly for being the only animal that can beat the elephant (apart from the elephant itself). Score rules typically only consider numbers of successors and do not explicitly consider their relative strengths.

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