The Annals of Probability
1988, Vol. 16, No. 1, 126–141

LAWS OF THE ITERATED LOGARITHM IN THE TAILS FOR WEIGHTED UNIFORM EMPIRICAL PROCESSES

BY JOHN H. J. EINMAHL[1] AND DAVID M. MASON[2]

Katholieke Universiteit, Nijmegen and Universität München

Characterizations of laws of the iterated logarithm for the supremum of weighted uniform $[0,1]^d$ empirical processes taken over increasingly smaller regions near the origin are obtained. These results have proven to be a valuable tool in the derivation of laws of the iterated logarithm for sums of extreme values. They also constitute a further continuation of the study of the almost sure behavior of weighted uniform empirical processes, which in a certain sense was begun by Csáki.

1. Preliminaries and statements of the main results. Let $X_1, X_2, \ldots$ be a sequence of independent uniform $[0,1]^d$, $d \in \mathbb{N}$, random vectors and, for each $n \in \mathbb{N}$, let

$$F_n(t) = n^{-1}\#\{1 \le i \le n: X_i \le t\}, \qquad t \in [0,1]^d,$$

denote the empirical distribution function based on the first n of these random vectors. (Here the $\le$ sign between vectors has to be understood componentwise.) The (multivariate) uniform empirical process is defined by

$$U_n(t) = n^{1/2}(F_n(t) - |t|), \qquad t \in [0,1]^d,$$

where $|t| = \prod_{j=1}^d t_j$.

It is the purpose of this paper to study the strong limiting behavior of a certain class of weighted uniform empirical processes in the tails, i.e., for $0 \le \nu \le \frac{1}{2}$ and a sequence of numbers $\{k_n\}_{n=1}^\infty$ satisfying

$$\text{(K)} \qquad 0 < k_n \le n \quad \text{and} \quad k_n \uparrow,$$

we shall give a complete description of the almost sure behavior of the random variable

$$D_{n,\nu}(k_n) = \sup_{0<|t|\le k_n/n} (n/k_n)^\nu |U_n(t)|/|t|^{1/2-\nu}.$$

It will be seen that this behavior will often depend on the rate of growth of the sequence $\{k_n\}_{n=1}^\infty$.

In order to motivate our investigation, we shall begin by describing what is known about the asymptotic distribution of $D_{n,\nu}(k_n)$ in the one dimensional case. In Csörgő and Mason [(1985), Theorem 2.1], it is shown for $d = 1$, $0 < \nu \le \frac{1}{2}$

Received November 1985; revised August 1986.

[1]At the University of Limburg since April 1986.

[2]Research supported by the Alexander von Humboldt Foundation while the author was visiting the University of Munich on leave from the University of Delaware.

AMS 1980 *subject classifications.* Primary 60F15; secondary 60G17.

Key words and phrases. Order statistics, weighted empirical processes, laws of the iterated logarithm.

and any sequence $\{k_n\}_{n=1}^{\infty}$ satisfying (K), $k_n/n \to 0$ and $k_n \to \infty$ as $n \to \infty$, that

$$D_{n,\nu}(k_n) \to_d \sup_{0 \le t \le 1} |W(t)|/t^{1/2-\nu} \quad \text{as } n \to \infty,$$

where W denotes a standard Wiener process on $[0,1]$. Whereas, by Theorem 2.2 of that paper, for $d=1$ and $\{k_n\}_{n=1}^{\infty}$ satisfying the same conditions, one has

$$a(\log k_n)D_{n,0}(k_n) - b\big(2^{-1}\log k_n\big) \to_d E \vee E' \quad \text{as } n \to \infty,$$

where $a(x) = (2\log x)^{1/2}$, $b(x) = 2\log x + 2^{-1}\log\log x - 2^{-1}\log \pi$ and E and E' are independent random variables having the extreme value distribution function $\exp(-\exp(-t))$, $t \in \mathbb{R}$.

The corresponding limiting distributional behavior of $D_{n,\nu}(k_n)$ for higher dimensions is at present unknown.

Given the asymptotic Gaussian distribution behavior of $D_{n,\nu}(k_n)$ in the case $d=1$ and $0 < \nu \le \frac{1}{2}$, one is naturally led to conjecture that a law of the iterated logarithm should also hold for $D_{n,\nu}(k_n)$. Theorems 1 and 2 demonstrate that this is indeed the case for all dimensions, provided the sequence $\{k_n\}_{n=1}^{\infty}$ increases sufficiently rapidly.

Before stating our main results, which are for the case $0 < \nu \le \frac{1}{2}$, we first describe the almost sure behavior of $D_{n,0}(k_n)$. In this case, the discussion in Csáki (1975, 1982) for $d=1$ and in Einmahl and Mason (1985) for $d \ge 1$ concerning the random variable

$$\sup_{0<|t|<1} |U_n(t)|/\big(|t|(1-|t|)\big)^{1/2}$$

easily carries over to show that no matter what sequence of numbers is chosen satisfying (K), there never exists a sequence of positive numbers $\{c_n\}_{n=1}^{\infty}$ such that $nc_n \downarrow$ and

$$\limsup_{n\to\infty} (nc_n)^{1/2} D_{n,0}(k_n) = c \quad \text{a.s.},$$

for $0 < c < \infty$. To be more specific, the following result can be readily inferred from the theorem in Einmahl and Mason (1985) and its proof: Assume that the sequence $\{k_n\}_{n=1}^{\infty}$ satisfies (K) and let $\{c_n\}_{n=1}^{\infty}$ be any sequence of positive numbers. Then

$$\sum_{n=1}^{\infty} c_n(\log(1/c_n))^{d-1} = \infty \quad \text{implies} \quad \limsup_{n\to\infty} (nc_n)^{1/2} D_{n,0}(k_n) = \infty \quad \text{a.s.},$$

whereas if $nc_n \downarrow$, the finiteness of the preceding series implies that the lim sup is equal to zero almost surely.

We shall now state our main results. The proofs are postponed until the next section. In the remainder of this paper, it is assumed that $\{k_n\}_{n=1}^{\infty}$ satisfies (K). Also for any $0 < \nu < \frac{1}{2}$, we write

$$a_n = \Big(nk_n^{2\nu/(1-2\nu)}(\log\log n)^{1/(1-2\nu)}\Big)^{-1}.$$

THEOREM 1. *Let* $0 < \nu < \frac{1}{2}$.

(i) $\sum_{n=1}^{\infty} a_n(\log(1/a_n))^{d-1} = \infty$ *implies*

$$\limsup_{n \to \infty} D_{n,\nu}(k_n)/(\log\log n)^{1/2} = \infty \quad a.s. \tag{1.1}$$

(ii) $\sum_{n=1}^{\infty} a_n(\log(1/a_n))^{d-1} < \infty$ *and* $k_n/n \downarrow 0$ *implies*

$$\limsup_{n \to \infty} D_{n,\nu}(k_n)/(\log\log n)^{1/2} \le (2d)^{1/2} \quad a.s., \tag{1.2}$$

with equality almost surely for the case $d = 1$.

(iii) *If in addition to condition* (ii) *we have*

$$\lim_{n \to \infty} \log\log(n/k_n)/\log\log n = \alpha,$$

then

$$\limsup_{n \to \infty} D_{n,\nu}(k_n)/(\log\log n)^{1/2} = (2(1 + \alpha(d-1)))^{1/2} \quad a.s. \tag{1.3}$$

For any $0 < c < \infty$, define β_c by the conditions $\beta_c > 1$ and $\beta_c(\log \beta_c - 1) + 1 = c^{-1}$.

THEOREM 2.

(i) $k_n/\log\log n \to 0$ *and* $k_n/n \downarrow 0$ *implies*

$$\limsup_{n \to \infty} D_{n,1/2}(k_n)/(\log\log n)^{1/2} = \infty \quad a.s. \tag{1.4}$$

(ii) $k_n/\log\log n \to c$ *with* $0 < c < \infty$ *implies*

$$\limsup_{n \to \infty} D_{n,1/2}(k_n)/(\log\log n)^{1/2} = c^{1/2}(\beta_{c/d} - 1) \quad a.s. \tag{1.5}$$

(iii) $k_n/\log\log n \to \infty$ *and* $k_n/n \downarrow 0$ *implies*

$$\limsup_{n \to \infty} D_{n,1/2}(k_n)/(\log\log n)^{1/2} \le (2d)^{1/2} \quad a.s., \tag{1.6}$$

with equality almost surely for the case $d = 1$.

(iv) *If in addition to condition* (iii) *we have* $\lim_{n \to \infty} \log\log(n/k_n)/\log\log n = \alpha$, *then*

$$\limsup_{n \to \infty} D_{n,1/2}(k_n)/(\log\log n)^{1/2} = (2(1 + \alpha(d-1)))^{1/2} \quad a.s. \tag{1.7}$$

In the one dimensional case we have the following refinements to Theorem 1 and 2: For $d = 1$ and any integer $1 \le k < \infty$, let $X_{k,n}$ denote the kth order statistic of $X_1, \ldots, X_n$ $(n \ge k)$. Write

$$D_{n,\nu}^{(k)}(k_n) = \begin{cases} \sup_{X_{k,n} \le t \le k_n/n} (n/k_n)^{\nu} |U_n(t)|/t^{1/2-\nu}, & \text{if } X_{k,n} \le k_n/n, \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 3. *Let* $0 < \nu < \frac{1}{2}$ *and* $1 \le k < \infty$.

(i) $\sum_{n=1}^{\infty} n^{k-1} a_n^k = \infty$ *implies*

$$\limsup_{n\to\infty} D_{n,\nu}^{(k)}(k_n)/(\log\log n)^{1/2} = \infty \quad a.s. \tag{1.8}$$

(ii) $\sum_{n=1}^{\infty} n^{k-1} a_n^k < \infty$ *and* $k_n/n \downarrow 0$ *implies*

$$\limsup_{n\to\infty} D_{n,\nu}^{(k)}(k_n)/(\log\log n)^{1/2} = 2^{1/2} \quad a.s. \tag{1.9}$$

THEOREM 4. *Let* $1 \le k < \infty$.

(i) $k_n/\log\log n \to 0$ *and* $k_n/n \downarrow 0$ *implies*

$$\limsup_{n\to\infty} D_{n,1/2}^{(k)}(k_n)/(\log\log n)^{1/2} = \infty \quad a.s. \tag{1.10}$$

(ii) $k_n/\log\log n \to c$ *with* $0 < c < \infty$ *implies*

$$\limsup_{n\to\infty} D_{n,1/2}^{(k)}(k_n)/(\log\log n)^{1/2} = c^{1/2}(\beta_c - 1) \quad a.s. \tag{1.11}$$

(iii) $k_n/\log\log n \to \infty$ *and* $k_n/n \downarrow 0$ *implies*

$$\limsup_{n\to\infty} D_{n,1/2}^{(k)}(k_n)/(\log\log n)^{1/2} = 2^{1/2} \quad a.s. \tag{1.12}$$

REMARK. A more delicate analysis shows that in Theorem 3(ii), the assumption that $k_n/n \downarrow 0$ can be replaced by $k_n/n \to 0$ without changing the conclusion. For the sake of brevity the details of the proof of this refinement will not be presented here.

The need to have a complete description of the almost sure behaviour of this class of weighted uniform empirical processes in the tails for the one dimensional case arose while the second named author was investigating laws of the iterated logarithm for sums of extreme values. In fact, Theorem 3 proved to be a nearly indispensable tool in the establishing of these results. The interested reader is referred to Haeusler and Mason (1987) and Deheuvels, Haeusler and Mason (1986) to see both how the random variable $D_{n,\nu}^{(k)}(k_n)$ in a sense arises naturally in the proofs and how Theorem 3 is applied.

For more about the almost sure behavior of weighted uniform empirical processes, consult Shorack and Wellner (1986) for an excellent survey in terms of results and methods of most of what is known in the one dimensional case and Einmahl (1986) for a study of the multidimensional analogues for these processes; for strong limit theorems for weighted empirical processes indexed by general classes of sets, refer to Alexander (1987).

2. Proofs of the theorems. Since the four theorems presented resemble each other, we will organize the proofs in the following way. The emphasis is laid on the proof of Theorem 1, which will be presented first. Next a short proof of Theorem 3 is given, since this theorem is a refinement of Theorem 1; especially those parts of the proof which are similar to parts of the proof of Theorem 1 are abridged. Finally Theorems 2 and 4 are proved briefly; these proofs are relatively easy compared with the preceding proofs.

PROOF OF THEOREM 1(i). Define $|X|_{1,n} = \min\{|X_i|;\ 1 \le i \le n\}$. We shall require the following fact.

FACT 1 [Geffroy (1958/1959); Kiefer (1972)]. Let $\{b_n\}_{n=1}^{\infty}$ be any sequence of positive constants.Then

$$\sum_{n=1}^{\infty} b_n(\log(1/b_n))^{d-1} = \infty \quad \text{implies} \quad P(|X|_{1,n} \le \varepsilon b_n \text{ i.o.}) = 1,$$

for all $\varepsilon > 0$.

Observe that we have, assuming that the series in (i) is equal to infinity, that for all $\varepsilon > 0$, $P(|X|_{1,n} \le \varepsilon a_n \wedge k_n/n \text{ i.o.}) = 1$, by Fact 1 and $k_n \uparrow$. Also notice that for all $n \in \mathbb{N}$, $|X|_{1,n} \le k_n/n$ implies

$$D_{n,\nu}(k_n)/(\log\log n)^{1/2} \ge (n/k_n)^{\nu} n^{1/2}/\left(2n(\log\log n)^{1/2}|X_{1,n}|^{1/2-\nu}\right)$$

[cf. Gaenssler and Stute (1979), page 208]. Combining these observations we have

$$\limsup_{n\to\infty} D_{n,\nu}(k_n)/(\log\log n)^{1/2} \ge 1/(2\varepsilon^{1/2-\nu}) \quad \text{a.s.}$$

Letting $\varepsilon \downarrow 0$ proves (1.1). □

PROOF OF THEOREM 1(ii). Notice that the series in (ii) being finite and $k_n \uparrow$ imply

$$\sum_{n=1}^{\infty} a_n(\log n)^{d-1} < \infty \tag{2.1a}$$

and

$$(\log n)^d/\left(k_n^{2\nu/(1-2\nu)}(\log\log n)^{1/(1-2\nu)}\right) \to 0 \quad \text{as } n \to \infty. \tag{2.1b}$$

Thus for all n large enough,

$$D_{n,\nu}(k_n) \le E_{n,\nu}(k_n) + \sup_{(\log\log n)^{1/(2\nu)}/n \le |t| \le k_n/n} (n/k_n)^{\nu}|U_n(t)|/|t|^{1/2-\nu},$$

where

$$E_{n,\nu}(k_n) = \sup_{0<|t|\le(\log\log n)^{1/(2\nu)}/n} (n/k_n)^{\nu}|U_n(t)|/|t|^{1/2-\nu}.$$

We shall first prove that $E_{n,\nu}(k_n)$ does not contribute to the value of the lim sup. To be more precise, we have

LEMMA 1. *$\sum_{n=1}^{\infty} a_n(\log n)^{d-1} < \infty$ implies*

$$\lim_{n\to\infty} E_{n,\nu}(k_n)/(\log\log n)^{1/2} = 0 \quad a.s. \tag{2.2}$$

PROOF. The proof of this lemma is along the lines of the proof of part (ii) of the theorem in Einmahl and Mason (1985). Write for either choice of sign

$$H_n^{\pm} = \sup_{0<|t|\le (\log\log n)^{1/(2\nu)}/n} \pm\, (F_n(t) - |t|)/|t|^{1/2-\nu}.$$

We first show

$$\limsup_{n\to\infty} \frac{n^{1/2+\nu}H_n^+}{k_n^\nu(\log\log n)^{1/2}} \le 0 \quad \text{a.s.} \tag{2.3}$$

To establish (2.3), it suffices to prove that

$$\limsup_{n\to\infty} \frac{n^{1/2+\nu}H_n^+}{k_n^\nu(\log\log n)^{1/2}} \le 1 \quad \text{a.s.} \tag{2.4}$$

for any sequence $\{k_n\}_{n=1}^\infty$ satisfying the assumption of Lemma 1. Define

$$A_n = \left\{H_n^+ \ge (\log\log n)^{1/2}k_n^\nu/n^{1/2+\nu}\right\} \quad \text{and} \quad C_n = A_nA_{n-1}^c.$$

According to a version of the Borel–Cantelli lemma, we need to verify

$$\sum_{n=1}^\infty P(C_n) < \infty \tag{2.5}$$

and

$$\lim_{n\to\infty} P(A_n) = 0. \tag{2.6}$$

For any integer $0 \le i \le n$ define $x_{i,n}$ to be $y_{i,n} \wedge b_n$ where $b_n = (\log\log n)^{1/(2\nu)}/n$ and $y_{i,n}$ is the positive solution to the equation

$$ny + k_n^\nu(\log\log n)^{1/2}n^{1/2-\nu}y^{1/2-\nu} = i. \tag{2.7}$$

Define

$$B_{n,i} = \{(n-1)F_{n-1}(t) \le i-1 \text{ for all } x_{i-1,n} < |t| \le x_{i,n}\}$$
$$\cap \{nF_n(t) = i \text{ for some } x_{i-1,n} < |t| \le x_{i,n}\}$$

and

$$B'_{n,i} = \left\{\sup_{|t|\le y_{i,n}} (n-1)F_{n-1}(t) \ge i-1;\ |X_n| \le y_{i,n}\right\}.$$

Notice that $B_{n,i} \subset B'_{n,i}$.

Define $i_n = [k_n^\nu(\log\log n)^{1/(2\nu)}]$, where $[x]$ denotes the integer part of x. Elementary analysis using (2.7) shows that $y_{i_n,n} \ge b_n$ for large n, which implies $x_{i_n,n} = b_n$. Thus we have the following inclusions for large n:

$$C_n \subset \left\{(n-1)F_{n-1}(t) < n|t| + k_n^\nu(\log\log n)^{1/2}n^{1/2-\nu}|t|^{1/2-\nu} \text{ for all } 0 < |t| \le b_n\right\}$$
$$\cap \left\{nF_n(t) \ge n|t| + k_n^\nu(\log\log n)^{1/2}n^{1/2-\nu}|t|^{1/2-\nu} \text{ for some } 0 < |t| \le b_n\right\}$$
$$\subset \bigcup_{i=1}^{i_n} B_{n,i} \subset \bigcup_{i=1}^{i_n} B'_{n,i}.$$

These inclusions yield

$$P(C_n) \le \sum_{i=1}^{i_n} P(B'_{n,i}). \tag{2.8}$$

To continue the proof of Lemma 1 we need

LEMMA 2. *For every integer m with $1 \le m \le n$ and $0 < \alpha \le 1$, we have*

$$P\Big(\sup_{|t| \le \alpha} nF_n(t) \ge m\Big) \le c_1\binom{n}{m}(c_2\alpha)^m(1 \vee \log(1/\alpha))^{d-1},$$

where $c_1 = c_1(d)$, $c_2 = c_2(d)$ and $1 \le c_1, c_2 < \infty$.

PROOF. Let $N_\alpha = [\log_2(1/\alpha)]$ and $\mathscr{I} = \{i = (i_1, \dots, i_d) \in \mathbb{Z}^d\colon i_1 \le N_\alpha, \dots, i_d \le N_\alpha\}$. For each $i \in \mathscr{I}$ let

$$R_i = (\alpha 2^{i_1}, \alpha 2^{i_1+1}] \times \cdots \times (\alpha 2^{i_d}, \alpha 2^{i_d+1}],$$

$$a_i = (\alpha 2^{i_1}, \dots, \alpha 2^{i_d}) \quad \text{and} \quad b_i = (\alpha 2^{i_1+1}, \dots, \alpha 2^{i_d+1}).$$

Notice that $|b_i| = 2^d|a_i|$.

Let $\mathscr{P}_\alpha = \{R_i\colon |a_i| < \alpha,\ i \in \mathscr{I}\}$. Observe that

$$\begin{aligned} P\Big(\sup_{|t| \le \alpha} nF_n(t) \ge m\Big) &\le P\Big(\sup_{R_i \in \mathscr{P}_\alpha} \sup_{t \in R_i} nF_n(t) \ge m\Big) \\ &\le \sum_{R_i \in \mathscr{P}_\alpha} P(nF_n(b_i) \ge m) \le \binom{n}{m} \sum_{R_i \in \mathscr{P}_\alpha} |b_i|^m \\ &\le \binom{n}{m} 2^{dm} 2^d \sum_{R_i \in \mathscr{P}_\alpha} \int_{R_i} |t|^{m-1} |dt| \le \binom{n}{m} 2^{dm} 2^d \int_{|t| \le 2^d\alpha} |t|^{m-1} |dt| \\ &\le \binom{n}{m} 2^{dm} 2^d \int_0^{2^d\alpha} (\log(1/s))^{d-1} s^{m-1}\, ds, \end{aligned}$$

where $|dt|$ denotes Lebesgue measure on $[0,1]^d$.

Elementary analysis shows that the integral in this last expression is less than or equal to

$$c(d)(2^d\alpha)^m(1 \vee \log(1/\alpha))^{d-1},$$

which completes the proof of Lemma 2. □

We mention that the proof of Lemma 2 was based on the method of proof of Inequality 2.1 in Einmahl, Ruymgaart and Wellner (1984).

We now resume the proof of Lemma 1. Observe that for some constant $0 < c(\nu) < \infty$, we have

$$\log(1/y_{i,n}) \le c(\nu)\log n, \quad \text{for } i = 1, \dots, i_n, \tag{2.9}$$

for all sufficiently large n. Thus for all large n

$$P(B'_{n,1}) \le P(|X_n| \le y_{1,n}) = \int_{-\log y_{1,n}}^{\infty} \frac{x^{d-1}}{(d-1)!} e^{-x}\, dx \le d(c(\nu))^{d-1} y_{1,n} (\log n)^{d-1}. \tag{2.10}$$

Notice that from (2.7) it follows that for all $1 \le i \le n$,

$$y_{i,n} \le i^{2/(1-2\nu)} a_n. \tag{2.11}$$

Now from (2.10) and (2.11), we have

$$P(B'_{n,1}) \le (c(\nu))^{d-1} d a_n (\log n)^{d-1}. \tag{2.12}$$

For $2 \le i \le i_n$, we see from Lemma 2 and (2.9) that for large n,

$$\begin{aligned} P(B'_{n,i}) &\le c_1 \binom{n-1}{i-1} (c_2 y_{i,n})^{i-1} \\ &\quad \times \left(1 \vee \log\left(\frac{1}{y_{i,n}}\right)\right)^{d-1} (c(\nu))^{d-1} d y_{1,n} (\log n)^{d-1} \\ &\le c_3 i \frac{n^{i-1}}{i!} c_2^{i-1} y_{i,n}^{i} (\log n)^{2d-2}, \end{aligned} \tag{2.13}$$

for some $0 < c_3 = c_3(d, \nu) < \infty$. By Stirling's formula

$$(1/m!) \le (e/m)^m, \quad \text{for all } m \in \mathbb{N}. \tag{2.14}$$

Combining (2.13) and (2.14) yields, for large n,

$$P(B'_{n,i}) \le c_3 \left(n^{i-1}/i^{i-1}\right) c_4^i y_{i,n}^i (\log n)^{2d-2}, \tag{2.15}$$

where $c_4 = c_2 e$.

Using (2.11) and $2 \le i \le i_n$, we obtain from (2.15) for large n,

$$\begin{aligned} P(B'_{n,i}) &\le n a_n^2 (\log n)^{2d-2} c_3 c_4^2 i^{(3+2\nu)/(1-2\nu)} \\ &\quad \times \left\{c_4 (\log\log n)^{1/[2\nu(1-2\nu)]} k_n^{-\nu}\right\}^{i-2}. \end{aligned} \tag{2.16}$$

By (2.1b), it is easily seen that the right side of (2.16) in less than or equal to (for large n)

$$c_5 \left(\tfrac{1}{2}\right)^{i-2} n a_n^2 (\log n)^{2d-2}, \tag{2.17}$$

with $0 < c_5 = c_5(d, \nu) < \infty$.

Combining (2.12), (2.16) and (2.17) with (2.8), we have for large n,

$$\begin{aligned} P(C_n) &\le (c(\nu))^{d-1} d a_n (\log n)^{d-1} + c_5 \sum_{i=2}^{i_n} \left(\tfrac{1}{2}\right)^{i-2} n a_n^2 (\log n)^{2d-2} \\ &\le c_6 a_n (\log n)^{d-1} \left(1 + n a_n (\log n)^{d-1}\right) \le 2 c_6 a_n (\log n)^{d-1}, \end{aligned}$$

where $c_6 = ((c(\nu))^{d-1} d) \vee 2c_5$. [The last inequality follows from (2.1b).] Now by the assumption of Lemma 1 we see that

$$\sum_{n=1}^{\infty} P(C_n) < \infty.$$

The proof of (2.6) is similar to the proof of (2.5). With the aid of Lemma 2 it can be shown that

$$P(A_n) = O\big(na_n(\log n)^{d-1}\big),$$

which on account of (2.1b) yields (2.6). The details are omitted. [See (3.18)–(3.22) of Einmahl and Mason (1985).]

Now consider H_n^-. Notice that trivially we have for all $n \in \mathbb{N}$,

$$\frac{n^{1/2+\nu}H_n^-}{k_n^\nu(\log\log n)^{1/2}} \le \frac{n^{1/2+\nu}}{k_n^\nu(\log\log n)^{1/2}} \frac{(\log\log n)^{(1+2\nu)/4\nu}}{n^{1/2+\nu}} = \frac{(\log\log n)^{1/4\nu}}{k_n^\nu}.$$

This last inequality combined with (2.1b) yields

$$\limsup_{n\to\infty} n^{1/2+\nu}H_n^- / \big(k_n^\nu(\log\log n)^{1/2}\big) \le 0 \quad \text{a.s.} \tag{2.18}$$

Statements (2.3) and (2.18) imply (2.2). Hence Lemma 1 is proven. □

To continue the proof of Theorem 1, we require the following facts: Define ψ: $[0,\infty) \to [0,\infty)$ by

$$\psi(\lambda) = \begin{cases} 2\lambda^{-2}\int_0^\lambda \log(1+s)\,ds = 2\lambda^{-2}((1+\lambda)\log(1+\lambda)-\lambda), & \lambda > 0, \\ 1, & \text{when } \lambda = 0. \end{cases}$$

Observe that ψ is continuous nondecreasing and that it has the property

$$\psi(\lambda) \ge c\psi(c\lambda) \quad \text{for } 0 \le c \le 1 \text{ and } 0 \le \lambda < \infty. \tag{2.19}$$

FACT 2 [Einmahl (1987)]. Let $0 < \alpha \le \beta \le \frac{1}{4}$ and $0 < \varepsilon \le \frac{1}{2}$. Write $\gamma = 1 - \varepsilon$. Then for all $\lambda \ge 0$,

$$P\Big(\sup_{\alpha \le |t| \le \beta} |U_n(t)|/|t|^{1/2} \ge \lambda\Big)$$
$$\le C\big\{(\log(1/(\gamma\alpha)))^d - (\log(\gamma/\beta))^d\big\}\exp\big(-\tfrac{1}{2}\gamma\lambda^2\psi\big(\lambda/(n\alpha)^{1/2}\big)\big),$$

where $0 < C = C(d,\varepsilon) < \infty$.

FACT 3 [Einmahl (1986), page 20; see also Einmahl (1987)]. Let $0 < \alpha \le \beta \le 1$, $\varepsilon \in (0,1)$ and write $n_k = [(1+\varepsilon/12)^k]$, $k \in \mathbb{N}$. Then we have for all $k \in \mathbb{N}$ and for $\lambda > (2/\varepsilon)^{1/2}$,

$$P\Big(\max_{n_k < n \le n_{k+1}} \sup_{\alpha \le |t| \le \beta} |U_n(t)|/|t|^{1/2} \ge \lambda\Big)$$
$$\le 2P\Big(\sup_{\alpha \le |t| \le \beta} |U_{n_{k+1}}(t)|/|t|^{1/2} \ge (1-\varepsilon)\lambda\Big).$$

FACT 4 [Alexander (1987); for $d = 1$ see also Kiefer (1972)]. Let $k_n/n \to 0$. If $d = 1$, for $k_n/(\log\log n) \to \infty$, or if $d > 1$, for $k_n/(\log\log n) \to \infty$ and $\lim_{n\to\infty} \log\log(n/k_n)/(\log\log n) = \alpha$, we have

$$\limsup_{n\to\infty} \sup_{|t|=k_n/n} n^{1/2}|U_n(t)|/(k_n \log\log n)^{1/2} \geq (2(1 + \alpha(d-1)))^{1/2} \quad \text{a.s.}$$

We now proceed with the proof of Theorem 1(ii). First observe that for large n

$$\begin{aligned} D_{n,\nu}(k_n)/(\log\log n)^{1/2} &\\ \leq E_{n,\nu}(k_n)/(\log\log n)^{1/2} &\\ + (\log k_n)^{-\nu} \sup_{(\log\log n)^{1/(2\nu)}/n \leq |t| \leq l_n/n} |U_n(t)|/(|t|\log\log n)^{1/2} &\qquad (2.20)\\ + \sup_{l_n/n \leq |t| \leq k_n/n} |U_n(t)|/(|t|\log\log n)^{1/2} &\\ := E_{n,\nu}(k_n)/(\log\log n)^{1/2} + \Delta^{(1)}_{n,\nu}(k_n)/(\log k_n)^{\nu} + \Delta^{(2)}_{n,\nu}(k_n), & \end{aligned}$$

where $l_n = k_n/\log k_n$. Notice that $l_n/n \downarrow 0$.

Fact 2 in combination with Fact 3 [the routine details are omitted; see, e.g., Einmahl (1987)] yields

$$\limsup_{n\to\infty} \Delta^{(1)}_{n,\nu}(k_n) \leq (2(d+1))^{1/2} \quad \text{a.s.} \tag{2.21}$$

and

$$\limsup_{n\to\infty} \Delta^{(2)}_{n,\nu}(k_n) \leq (2d)^{1/2} \quad \text{a.s.} \tag{2.22}$$

From Lemma 1, (2.21) and (2.22), we have (1.2). For the case $d = 1$, Fact 4 along with (1.2) completes the proof of Theorem 1(ii). □

PROOF OF THEOREM 1(iii). Under the additional assumption of (iii), Fact 2 along with Fact 3 shows that, instead of (2.22), we have

$$\limsup_{n\to\infty} \Delta^{(2)}_{n,\nu}(k_n) \leq (2(1 + \alpha(d-1)))^{1/2} \quad \text{a.s.} \tag{2.23}$$

Lemma 1, (2.21) and (2.23) now yield

$$\limsup_{n\to\infty} D_{n,\nu}(k_n)/(\log\log n)^{1/2} \leq (2(1 + \alpha(d-1)))^{1/2} \quad \text{a.s.} \tag{2.24}$$

From (2.24) and Fact 4 we infer (1.3). This completes the proof of Theorem 1. □

PROOF OF THEOREM 3(i). We require the following fact.

FACT 5 [Kiefer (1972)]. Let $1 \leq k < \infty$ be fixed and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive constants with $n\alpha_n \downarrow$. Then

$$\sum_{n=1}^{\infty} n^{k-1}\alpha_n^k = \infty \quad \text{implies} \quad P(X_{k,n} \leq \alpha_n \text{ i.o.}) = 1.$$

Observe that for any $0 < \varepsilon < 1$, $\varepsilon a_n \wedge k/2n \wedge k_n/n = \varepsilon a_n$ for all large enough n. Combining this with Fact 5, we have

$$P\left(X_{k,n} \le \varepsilon a_n \wedge \frac{k}{2n} \wedge \frac{k_n}{n} \text{ i.o.}\right) = 1.$$

This immediately yields as in the proof of Theorem 1(i),

$$\limsup_{n\to\infty} D_{n,\nu}^{(k)}(k_n)/(\log\log n)^{1/2} \ge k/(2\varepsilon^{1/2-\nu}) \quad \text{a.s.}$$

Again by letting $\varepsilon \downarrow 0$, we have (1.8). □

PROOF OF THEOREM 3(ii). The proof is a mixture of the proof of Theorem 1(ii) and Theorem 1 in Einmahl, Haeusler and Mason (1985). Observe that the series in (ii) being finite along with $k_n \uparrow$ implies that

$$na_n(\log n)^{1/k} \to 0 \quad \text{as } n \to \infty. \tag{2.25}$$

Thus for large enough n,

$$D_{n,\nu}^{(k)}(k_n) \le E_{n,\nu}^{(k)}(k_n) + \sup_{(\log\log n)^{1/(2\nu)}/n \le t \le k_n/n} (n/k_n)^{\nu}|U_n(t)|/t^{1/2-\nu},$$

where

$$E_{n,\nu}^{(k)}(k_n) = \begin{cases} \displaystyle\sup_{X_{k,n} \le t \le (\log\log n)^{1/(2\nu)}/n} \left(\frac{n}{k_n}\right)^{\nu} \frac{|U_n(t)|}{t^{1/2-\nu}}, & \text{if } X_{k,n} \le \dfrac{(\log\log n)^{1/2\nu}}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

We now prove

LEMMA 3. *$\sum_{n=1}^{\infty} n^{k-1}a_n^k < \infty$ implies*

$$\lim_{n\to\infty} E_{n,\nu}^{(k)}(k_n)/(\log\log n)^{1/2} = 0 \quad a.s. \tag{2.26}$$

PROOF. Write for either choice of sign

$$\tilde{H}_n^{\pm} = \begin{cases} \displaystyle\sup_{X_{k,n} \le t \le (\log\log n)^{1/(2\nu)}/n} \pm \frac{(F_n(t) - t)}{t^{1/2-\nu}}, & \text{if } X_{k,n} \le \dfrac{(\log\log n)^{1/(2\nu)}}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

We first show that

$$\limsup_{n\to\infty} n^{1/2+\nu}\tilde{H}_n^{+}/\left(k_n^{\nu}(\log\log n)^{1/2}\right) \le 0 \quad \text{a.s.} \tag{2.27}$$

To verify (2.27), it suffices to prove that

$$\limsup_{n\to\infty} n^{1/2+\nu}\tilde{H}_n^{+}/\left(k_n^{\nu}(\log\log n)^{1/2}\right) \le 1 \quad \text{a.s.} \tag{2.28}$$

Define A_n, C_n, $x_{i,n}$, $y_{i,n}$ and i_n as in the proof of Lemma 1 and write

$$B_{n,i}' = \{(n-1)F_{n-1}(y_{i,n}) \ge i - 1;\ X_n \le y_{i,n}\}.$$

We can then show by the methods used in the proofs of Lemmas 2.1, 2.2 and 2.3 of Einmahl, Haeusler and Mason (1985) that

$$P(C_n) \le \sum_{i=k}^{i_n} P(B'_{n,i}). \tag{2.29}$$

To prove (2.28), we need to show that (2.5) and (2.6) hold.

By (2.29) and Stirling's formula, we have

$$P(C_n) \le \sum_{i=k}^{i_n} \binom{n-1}{i-1} y_{i,n}^i \le \sum_{i=k}^{i_n} (n/i)^{i-1} e^i y_{i,n}^i. \tag{2.30}$$

Using (2.11) and $i \le i_n$ it is easily seen that the right side of (2.30) is less than or equal to

$$a_n^k n^{k-1} \sum_{i=k}^{i_n} e^k i^{k(1+2\nu)/(1-2\nu)+1} \left(k_n^{-\nu} e (\log\log n)^{1/[2\nu(1-2\nu)]} \right)^{i-k}. \tag{2.31}$$

We now have by (2.25) that for large n, the expression in (2.31) is less than or equal to

$$a_n^k n^{k-1} \sum_{i=k}^{i_n} \left(\tfrac{1}{2}\right)^{i-k} e^k i^{k(1+2\nu)/(1-2\nu)+1} \le a_n^k n^{k-1} C \tag{2.32}$$

with $0 < C = C(k, \nu) < \infty$. This verifies (2.5).

Similarly we can show $P(A_n) = O((na_n)^k)$, which proves (2.6). Thus we have (2.27).

Again we have trivially that

$$\limsup_{n\to\infty} n^{1/2+\nu} H_n^- / \left(k_n^\nu (\log\log n)^{1/2} \right) \le 0 \quad \text{a.s.},$$

which completes the proof of Lemma 3. □

We now resume the proof of Theorem 3(ii). Observe that we have for large n,

$$\begin{aligned} D_{n,\nu}^{(k)}/(\log\log n)^{1/2} &\le E_{n,\nu}^{(k)}(k_n)/(\log\log n)^{1/2} \\ &\quad + (\log k_n)^{-\nu} \Delta_{n,\nu}^{(1)}(k_n) + \Delta_{n,\nu}^{(2)}(k_n), \end{aligned} \tag{2.33}$$

where the last two terms are defined exactly as in the proof of Theorem 1(ii). From (2.21), (2.22) and Lemma 3, we have immediately that

$$\limsup_{n\to\infty} D_{n,\nu}^{(k)}(k_n)/(\log\log n)^{1/2} \le 2^{1/2} \quad \text{a.s.} \tag{2.34}$$

Fact 4 along with $P(X_{k,n} > k_n/n \text{ i.o.}) = 0$ completes the proof of Theorem 3(ii). □

For the proof of Theorem 2 we require a number of additional facts.

FACT 6 [Alexander (1987); for $d = 1$ see also Kiefer (1972)]. If $k_n/\log\log n \to 0$ and $k_n/n \downarrow 0$, then

$$\limsup_{n\to\infty} \sup_{|t|=k_n/n} \frac{n^{1/2}|U_n(t)|}{\log\log n} \log(\log\log n/k_n) \geq 1 \quad \text{a.s.}$$

FACT 7 [Alexander (1987); Einmahl (1987)]. Let $0 < c < \infty$. Then

$$\limsup_{n\to\infty} \sup_{|t|=(c\log\log n)/n} \frac{n^{1/2}|U_n(t)|}{\log\log n} \geq c(\beta_{c/d} - 1) \quad \text{a.s.}$$

FACT 8 [e.g., Einmahl (1987)]. Let $x \in [0,1]^d$ such that $|s| \leq \frac{1}{2}$. For each $0 < \varepsilon < 1$, there exists $0 < C_1 = C_1(d, \varepsilon) < \infty$ such that for all $0 \leq \lambda < \infty$,

$$P\Big(\sup_{t\leq s}|U_n(t)| \geq \lambda\Big) \leq C_1 \exp\left(\frac{-(1-\varepsilon)\lambda^2}{2|s|}\psi\left(\frac{\lambda}{n^{1/2}|s|}\right)\right).$$

Also we need the following lemma.

LEMMA 4. *Let* $0 < \varepsilon < \frac{1}{2}$ *and* $0 < \alpha < \frac{1}{4}$. *Then for all* $0 \leq \lambda < \infty$,

$$P\Big(\sup_{|t|\leq\alpha}|U_n(t)| \geq \lambda\alpha^{1/2}\Big) \leq C\left(\log\left(\frac{1}{\alpha}\right)\right)^{d-1} \exp\left(\frac{-(1-\varepsilon)\lambda^2}{2}\psi\left(\frac{\lambda}{(n\alpha)^{1/2}}\right)\right),$$

where $0 < C = C(d, \varepsilon) < \infty$.

PROOF. Let θ be defined by $\theta^{d+1} = (1-\varepsilon)^{1/2}$ and let l be the integer such that $\theta^{l+1} < \alpha \leq \theta^l$. Notice that $l \leq \log(1/\alpha)/\log(1/\theta)$. For $|s| \leq \alpha/(1-\varepsilon)^{1/2}$, we have by Fact 8 and (2.19),

$$P\Big(\sup_{t\leq s}|U_n(t)| \geq \lambda\alpha^{1/2}\Big) \leq C_1 \exp\left(\frac{-(1-\varepsilon)\lambda^2}{2}\psi\left(\frac{\lambda}{(n\alpha)^{1/2}}\right)\right). \tag{2.35}$$

For integers $0 \leq k < \infty$, set

$$\mathscr{P}(\theta, k) = \big\{s \in [0,1]^d \colon |s| = \theta^k \text{ and for all } 1 \leq i \leq d,\ s_i = \theta^{k_i} \text{ for some } 0 \leq k_i < \infty\big\}.$$

It is easily seen that

$$\#\mathscr{P}(\theta, k) = \binom{k+d-1}{d-1}, \tag{2.36}$$

where # denotes number of elements.

Now we have by (2.35) and (2.36),

$$P\Big(\sup_{|t|\le\alpha}|U_n(t)| \ge \lambda\alpha^{1/2}\Big) \le P\Big(\sup_{|t|\le\theta^l}|U_n(t)| \ge \lambda\alpha^{1/2}\Big)$$

$$\le P\Big(\max_{s\in\mathscr{P}(\theta,l-d)}\sup_{t\le s}|U_n(t)| \ge \lambda\alpha^{1/2}\Big)$$

$$\le \sum_{s\in\mathscr{P}(\theta,l-d)} C_1\exp\left(\frac{-(1-\varepsilon)\lambda^2}{2}\psi\left(\frac{\lambda}{(n\alpha)^{1/2}}\right)\right)$$

$$= C_1\binom{l-1}{d-1}\exp\left(\frac{-(1-\varepsilon)\lambda^2}{2}\psi\left(\frac{\lambda}{(n\alpha)^{1/2}}\right)\right).$$

Using

$$\binom{l-1}{d-1} \le l^{d-1}/(d-1)! \le C_2(d,\varepsilon)(\log(1/\alpha))^{d-1}$$

completes the proof of the lemma. □

PROOF OF THEOREM 2(i). Assertion (1.4) is an immediate consequence of Fact 6. □

PROOF OF THEOREM 2(ii). First note that we can assume without loss of generality that $k_n = c\log\log n$. A maximal inequality, like the inequality in Fact 3, combined with the inequality in Lemma 4 shows that $c^{1/2}(\beta_{c/d}-1)$ is almost surely an upper bound for the lim sup in (1.5). The routine details are omitted. That it is almost surely a lower bound follows from Fact 7. □

PROOF OF THEOREM 2(iii) AND (iv). Again the upper bounds follow from a maximal inequality combined with Lemma 4; the lower bounds follow from Fact 4. □

We shall now provide a proof for Theorem 4.

PROOF OF THEOREM 4. Theorem 4 will be derived from Theorem 2 ($d = 1$) and its proof. It is well known [e.g., Kiefer (1972)] that for all $0 < \varepsilon < \infty$,

$$(2.37) \qquad P\big(X_{k,n} \ge ((1+\varepsilon)\log\log n)/n \text{ i.o.}\big) = 0.$$

Using (2.37), we obtain (iii) and (ii) for $c > 1$ from Theorem 2(iii) and (ii) and Facts 4 and 7.

We next consider (ii) for the case $0 < c \le 1$. For almost every $\omega \in \Omega$, we can take a subsequence $\{n_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} n_j|F_{n_j}(k_{n_j}/n_j,\omega) - k_{n_j}/n_j|/\big(k_{n_j}\log\log n_j\big)^{1/2} = c^{1/2}(\beta_c - 1).$$

Notice that $0 < c \le 1$ implies $\beta_c \ge e$. From this it follows that $X_{k,n_j} \le k_{n_j}/n_j$ for all large j. Hence

$$\liminf_{j\to\infty} D^{(k)}_{n_j,1/2}(k_{n_j})/(\log\log n_j)^{1/2} \ge c^{1/2}(\beta_c - 1).$$

Theorem 2(ii) completes the proof of (ii).

Finally we consider (i). For almost every $\omega \in \Omega$, we can take a subsequence $\{n_j\}_{j=1}^{\infty}$ such that (see Fact 6)

$$\lim_{j\to\infty} n_j|F_{n_j}(k_{n_j}/n_j, \omega) - k_{n_j}/n_j|/(k_{n_j}\log\log n_j)^{1/2} = \infty.$$

Now $k_n/\log\log n \to 0$ implies

$$\lim_{j\to\infty} n_j F_{n_j}(k_{n_j}/n_j, \omega)/(k_{n_j}\log\log n_j)^{1/2} = \infty. \tag{2.38}$$

The limit in (2.38) implies that $U_{k,n_j} \le k_{n_j}/n_j$ for all large j. Thus we have

$$\lim_{j\to\infty} D^{(k)}_{n_j,1/2}(k_{n_j})/(\log\log n_j)^{1/2} = \infty.$$

This completes the proof of (i) and hence the proof of Theorem 4. □

Acknowledgments. We are grateful to the referee for his careful reading of the manuscript and to Arnoud van Rooij for a useful remark.

REFERENCES

ALEXANDER, K. S. (1987). Rates of growth and sample moduli of weighted empirical processes indexed by sets. *Probab. Theory Related Fields* **75** 379–424.

CSÁKI, E. (1975). Some notes on the law of the iterated logarithm for empirical distribution function. *Colloq. Math. Soc. János Bolyai. Limit Theorems of Probability Theory* **11** 47–58. North-Holland, Amsterdam.

CSÁKI, E. (1982). On the standardized empirical distribution function. *Colloq. Math. Soc. János Bolyai. Nonparametric Statistical Inference* **32** 123–138. North-Holland, Amsterdam

CSÖRGŐ, M. and MASON, D. M. (1985). On the asymptotic distribution of weighted uniform empirical and quantile processes in the middle and on the tails. *Stochastic Process. Appl.* **21** 119–132.

DEHEUVELS, P., HAEUSLER, E. and MASON, D. M. (1986). On the almost sure behavior of sums of extreme values from a distribution in the domain of attraction of a Gumbel law. Unpublished.

EINMAHL, J. H. J. (1986). Multivariate empirical processes. Ph.D. dissertation, Katholieke Universiteit, Nijmegen.

EINMAHL, J. H. J. (1987). A general form of the law of the iterated logarithm for the weighted multivariate empirical distribution function. *Statist. Probab. Lett.* **5** 181–185.

EINMAHL, J. H. J., HAEUSLER, E. and MASON, D. M. (1985). An extension of a theorem of Csáki with application to the study of the almost sure stability of sums of extreme values. Unpublished.

EINMAHL, J. H. J. and MASON, D. M. (1985). Bounds for weighted multivariate empirical distribution functions. *Z. Wahrsch. verw. Gebiete* **70** 563–571.

EINMAHL, J. H. J., RUYMGAART, F. H. and WELLNER, J. A. (1984). A characterization of weak convergence of weighted multivariate empirical processes. *Acta Sci. Math.* (*Szeged*). To appear.

GAENSSLER, P. and STUTE, W. (1979). Empirical processes: A survey of results for independent and identically distributed random variables. *Ann. Probab.* **7** 193–243.

GEFFROY, J. (1958 / 1959). Contributions à la théorie des valeurs extrêmes. *Publ. Inst. Statist. Univ. Paris* **7 / 8** 37–185.

HAEUSLER, E. and MASON, D. M. (1987). A law of the iterated logarithm for sums of extreme values from a distribution with a regularly varying upper tail. *Ann. Probab.* **16** 932–953.

KIEFER, J. (1972). Iterated logarithm analogues for sample quantiles when $p_n \downarrow 0$. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **1** 227–244. Univ. California Press.

SHORACK, G. R. and WELLNER, J. A. (1986). *Empirical Processes with Applications to Statistics.* Wiley, New York.

DEPARTMENT OF MEDICAL INFORMATICS
AND STATISTICS
UNIVERSITY OF LIMBURG
P.O. BOX 616
6200 MD MAASTRICHT
THE NETHERLANDS

DEPARTMENT OF MATHEMATICAL SCIENCES
501 EWING HALL
UNIVERSITY OF DELAWARE
NEWARK, DELEWARE 19716