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Semi-infinite assignment problems and related games

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Abstract. In this paper we look at semi-infinite assignment problems. These are situations where a finite set of agents of one type has to be assigned to an infinite set of agents of another type. This has to be done in such a way that the total profit arising from these assignments is as large as possible. An infinite programming problem and its dual arise here, which we tackle with the aid of finite approximations. We prove that there is no duality gap and we show that the core of the corresponding game is nonempty. Finally, the existence of optimal assignments is discussed.

Key words: Infinite programs, assignment, cooperative games, balancedness

1 Introduction

Since finite assignment games were introduced in Shapley and Shubik [12], much work related to these games has been developed. We point out the book of Roth and Sotomayor [9] as an important monograph on two-sided matching. Curiel [2] provides a thorough analysis of assignment games. In their work, Shapley and Shubik proved that the core of an assignment game is the non-empty set of solutions of the dual problem corresponding to the assignment problem. In [11], Sasaki gives axiomatic characterizations of the core of assignment games.

In this paper, we look at semi-infinite assignment problems where the number of one of the two types of agents involved is finite and the other is countable infinite and we prove that semi-infinite bounded assignment games are balanced. Fragnelli et al. [3], Tijs et al. [14] and Timmer et al. [15] have studied some kinds of semi-infinite balanced games arising from different lin-

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ear programming situations, where one of the factors involved in the problem is countable infinite but the number of players is finite. Here we tackle semi-infinite assignment games with the aid of some tools that are related to Tijs [13].

A more general problem is the transportation problem where demand for a single good at several locations has to be met from several supply points. An assignment problem is a transportation problem where all demands and supplies equal one unit. In Kortanek and Yamasaki [6, 7] semi-infinite transportation problems are studied with a finite number of supply points and an infinite number of demand locations. They assume that the total supply and the total demand for the good are equal and finite. This implies that semi-infinite assignment problems, as studied here with an infinite 'total demand', are not covered by their analysis. Further, their focus is on programs while we include a game-theoretic analysis.

This paper consists of four sections. In the next section we present the most relevant definitions and results for the assignment problem with two finite sets of agents. We extend these problems in section 3 to semi-infinite bounded assignment problems where one of the sets of agents is countable infinite and the set of values of matched pairs of agents is upper bounded. We show that the corresponding primal and dual program have no duality gap and that there exist optimal solutions to the dual program, which is equivalent to the non-emptiness of the core of the corresponding game. In section 4 we introduce the critical number and the existence of optimal assignments is discussed. Section 5 concludes.

2 Finite assignment problems

An assignment problem describes a situation in which there are two types of agents, for example, sellers and buyers or firms and workers. Denote by M and W respectively these two finite and disjoint sets of agents. Let m be the number of agents in M, i.e., $m = |M|$, and $n = |W|$. Assume without loss of generality that $m \leq n$. When agent $i \in M$ is matched to agent $j \in W$ then this gives the couple a value of $a_{ij} \geq 0$. An assignment problem is thus described by the triple (M, W, A) with $A = [a_{ij}]_{i \in M, j \in W}$. For ease of notation we denote this assignment problem by $\mathscr{A}$.

The maximal total value of paired agents, where each agent $i \in M$ is coupled to at most one agent $j \in W$ and vice versa, can be determined by the following integer program

$$P: \quad \begin{aligned} \max \quad & \sum_{i \in M} \sum_{j \in W} a_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i \in M} x_{ij} \leq 1, \quad \text{for all } j \in W \\ & \sum_{j \in W} x_{ij} \leq 1, \quad \text{for all } i \in M \\ & x_{ij} \in \{0, 1\}, \quad \text{for all } i \in M, j \in W \end{aligned}$$

with value $v_p(\mathscr{A})$. The assignment matrix $X \in \{0,1\}^{M \times W}$, $X = [x_{ij}]_{i \in M, j \in W}$, corresponds to the situation in which the agents $i \in M$ and $j \in W$ are matched if and only if $x_{ij} = 1$. An assignment or matching is an injective function $\pi : M \to W$ and such an assignment is optimal if $\sum_{i \in M} a_{i\pi(i)} \geq \sum_{i \in M} a_{i\pi'(i)}$ for all assignments π'.

Given an assignment problem $\mathscr{A}$, the corresponding assignment game (N, w) is a game with player set $N = M \cup W$. Let $S \subset N$ be a coalition of players. Then the worth $w(S)$ is defined to be the maximal value this coalition can obtain by matching its members. Define $M_S = S \cap M$ and $W_S = S \cap W$. If $M_S = \varnothing$ or $W_S = \varnothing$ then $w(S) = 0$ since no matchings can be made. Otherwise, if $M_S \neq \varnothing$ and $W_S \neq \varnothing$ then $w(S) = v_p(\mathscr{A}_S)$ where $\mathscr{A}_S$ refers to the assignment problem $(M_S, W_S, [a_{ij}]_{i \in M_S, j \in W_S})$. It is obvious that $\mathscr{A}_N = \mathscr{A}$.

The vector (u, v), $u \in \mathbb{R}_+^M$ and $v \in \mathbb{R}_+^W$, is called a *feasible payoff* for the assignment problem $\mathscr{A}$ if there is an assignment π such that $\sum_{i \in M} u_i + \sum_{j \in W} v_j = \sum_{i \in M} a_{i\pi(i)}$. In this case, we say $((u, v), \pi)$ is a *feasible outcome* and it is *stable* if (u, v) is an element of the core $C(w)$ of the corresponding assignment game, where

$$C(w) = \left\{ (u,v) \in \mathbb{R}_+^M \times \mathbb{R}_+^W \;\middle|\; \begin{array}{l} \sum_{i \in M_S} u_i + \sum_{j \in W_S} v_j \geq w(S), S \subset N, \\ \sum_{i \in M} u_i + \sum_{j \in W} v_j = w(N) \end{array} \right\}.$$

If $(u, v) \in C(w)$ is proposed as payoff to the players, then each coalition $S \subset N$ gets at least as much as it can obtain on its own since $\sum_{i \in M_S} u_i + \sum_{j \in W_S} v_j \geq w(S)$. Thus no coalition has an incentive to break up with the grand coalition N. The following lemma by Roth and Sotomayor [9] tells something more about stable outcomes.

Lemma 2.1 (Roth and Sotomayor). *Let $((u, v), \pi)$ be a stable outcome for $\mathscr{A}$. Then*

(a) $u_i + v_j = a_{ij}$ *if* $\pi(i) = j$
(b) $u_i = 0$ *and* $v_j = 0$ *for all unassigned* i *and* j.

This result implies that at a stable outcome, the only utility transfers occur between agents in M and W who are matched to each other. It also shows that those players who remain unmatched in some optimal solution receive a zero payoff.

In e.g. [1] it is shown that if the integer condition $x_{ij} \in \{0, 1\}$ in the primal problem P is replaced by $x_{ij} \geq 0$ for all $i \in M$, $j \in W$, then all the optimal solutions will still have $x_{ij} \in \{0, 1\}$. Related to this problem is the following dual problem with value $v_d(\mathscr{A})$.

$$D: \quad \begin{array}{ll} \min & \sum_{i \in M} u_i + \sum_{j \in W} v_j \\ \text{s.t.} & u_i + v_j \geq a_{ij}, \quad \text{for all } i \in M, j \in W \\ & u_i, v_j \geq 0, \quad \text{for all } i \in M, j \in W \end{array}$$

Because the primal problem P has a solution, we know that also D must have a solution and the fundamental duality theorem asserts that these programs attain the same value.

By definition of $w(S)$ it holds that if (u,v) is an optimal solution of the dual program then $\sum_{i\in M_S} u_i + \sum_{j\in W_S} v_j \geq w(S)$ for any coalition S, which ensures that this coalition cannot improve by splitting off from N when (u,v) is proposed as payoff. The following theorem says that these conditions are exactly the conditions that determine the core of an assignment game.

Theorem 2.2 (Shapley and Shubik). *Let $\mathscr{A}$ be an assignment problem. Then the core $C(w)$ of the corresponding assignment game (N,w) is the nonempty set of optimal solutions of D.*

Moreover, if π is an optimal assignment then $((u,v),\pi)$ is a stable outcome for all core-elements (u,v). Vice versa, if $((u,v),\pi)$ is a stable outcome then π is an optimal assignment (see [9] for the proofs). So, we can concentrate on the payoffs to the agents rather than on the underlying assignment.

Let $\mathscr{A}=(M,W,A)$ be an assignment problem and let $j\in W$. By $B_i(j,A)$ we denote the set of agents in $W\backslash\{j\}$ who are at least as good as j for agent $i\in M$,

$$B_i(j,A)=\{k\in W\,|\,k\neq j, a_{ik}\geq a_{ij}\}.$$

The following proposition tells us that an agent $j\in W$ gets payoff zero in each core-element if for each $i\in M$ there are at least m agents in W whom he finds better than j.

Proposition 2.3. *For each assignment problem $\mathscr{A}=(M,W,A)$ and for each $j\in W$ such that $|B_i(j,A)|\geq m$ for all $i\in M$ it holds that $v_j=0$ for all $(u,v)\in C(w)$.*

Proof. Let $\mathscr{A}=(M,W,A)$ be an assignment problem and let $j\in W$ be such that $|B_i(j,A)|\geq m$ for all $i\in M$. Let π be an optimal assignment for P. If $j\notin\pi(M)=\{\pi(i)\,|\,i\in M\}$ then $v_j=0$ by item (b) of lemma 2.1.

If $j=\pi(i^*)$ for some $i^*\in M$ then since $|B_{i^*}(j,A)|\geq m$ and $|\pi(M\backslash\{i^*\})|=m-1$ there exists a $k\in B_{i^*}(j,A)\backslash\pi(M\backslash\{i^*\})$. Since k is unassigned, $k\notin\pi(M)$, $v_k=0$ by lemma 2.1. Together with $k\in B_{i^*}(j,A)$ this gives

$$u_{i^*}=u_{i^*}+v_k\geq a_{i^*k}\geq a_{i^*j}=u_{i^*}+v_j$$

where the last equality follows from $\pi(i^*)=j$ and lemma 2.1. Thus $v_j\leq 0$ and because $v_j\geq 0$ according to the dual problem D we conclude that $v_j=0$. □

3 Semi-infinite bounded assignment problems

In this section we introduce semi-infinite bounded assignment problems (M,W,A), where $M=\{1,2,\ldots,m\}$, a finite set, $W=\mathbb{N}=\{1,2,\ldots\}$, the countable infinite set of natural numbers, and $0\leq a_{ij}\leq b$ for some $b\in\mathbb{R}$, for all $i\in M$, $j\in W$. The boundedness of the values a_{ij} is not a real restriction. It is clear that if the values a_{ij} would have no upper bound then the primal problem

$$P: \quad \begin{aligned} \sup \quad & \sum_{i \in M} \sum_{j \in W} a_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i \in M} x_{ij} \leq 1, \quad \text{for all } j \in W \\ & \sum_{j \in W} x_{ij} \leq 1, \quad \text{for all } i \in M \\ & x_{ij} \in \{0, 1\}, \quad \text{for all } i \in M, j \in W \end{aligned}$$

would have an infinite value, $v_p(\mathscr{A}) = \infty$, and no optimal solutions. We analyze the corresponding semi-infinite bounded assignment games by means of *finite approximation problems* $\mathscr{A}_n = (M, \{1, \ldots, n\}, A_n)$ where $A_n = [a_{ij}]_{i \in M, j=1,2,\ldots,n}$, and by means of the so-called *hard-choice number* of $\mathscr{A}$, to be introduced later.

We start by defining two types of agents in M. An agent $i \in M$ is of *type 1* if this agent can choose one-by-one m best elements $j \in W$ with respect to the largest reward a_{ij}. We denote by M_1 the set of agents of type 1. The remaining agents in $M_2 = M \backslash M_1$ are of *type 2*.

The *choice set* C_i of an agent i of type 2 is the set of all his chosen best elements in W. Since this agent cannot choose m best elements (otherwise he is of type 1), we have $0 \leq |C_i| < m$. The choice set C_i of an agent $i \in M_1$ consists of those m agents in W obtained in m steps by taking in each step that agent $j \in W$ not yet chosen by him and which gives him the maximal value a_{ij} over all non-chosen $j \in W$. In case there are more agents $j \in W$ that give the same maximal value a_{ij} then we choose that agent j with the smallest ranking number. The following example illustrates these concepts.

Example 3.1. Let $M = \{1, 2, 3\}$, $W = \mathbb{N}$ and

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 1 & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \end{bmatrix}.$$

Agent $1 \in M$ attains his maximal value of 3 if he is assigned to agent $1 \in W$. The second largest value he can obtain is $a_{12} = 2$ and $a_{13} = 1$ is the third largest value he can get. This agent has no problems with choosing his three best agents from W and therefore he is of type 1. His choice set thus equals $C_1 = \{1, 2, 3\}$.

The largest value that agent $2 \in M$ can attain is $a_{22} = 1$. However, there is no second largest value because a_{2n} reaches the value 1 from below when n goes to infinity. This agent can only choose one best agent from W and therefore he is of type 2. His choice set equals $C_2 = \{2\}$.

Finally, agent $3 \in M$ has an easy job, since for all $j \in W$ he gets the value $a_{3j} = 1$. All agents in W are best elements for him. We will choose those three agents with the smallest ranking number, thus $C_3 = \{1, 2, 3\}$. This agent is of type 1. We conclude that $M_1 = \{1, 3\}$ and $M_2 = \{2\}$.

We will now introduce the hard-choice number.

Definition 3.2. *The* hard-choice number $n^*(\mathscr{A})$ *is the smallest number in* $\mathbb{N} \cup \{0\}$ *such that* $\bigcup_{i=1}^{m} C_i \subset \{1, 2, \ldots, n^*(\mathscr{A})\}$.

Lemma 3.3. *For each semi-infinite bounded assignment problem* $\mathscr{A} = (M, W, A)$ *and for each* $j > n^*(\mathscr{A})$, $j \in W$, *there is an agent* $n(j) \geq j$, $n(j) \in W$, *such that* $|B_i(j, A_{n(j)})| \geq m$ *for all* $i \in M$.

Proof. Let $\mathscr{A} = (M, W, A)$ be a semi-infinite bounded assignment problem and let $j > n^*(\mathscr{A})$, $j \in W$. Notice that $j > n^*(\mathscr{A})$ implies $j \notin C_i$ for all $i \in M$. If $i \in M_1$ then $B_i(j, A) \cap \{1, 2, \ldots, n^*(\mathscr{A})\} \supset C_i$ thus $|B_i(j, A) \cap \{1, 2, \ldots, n^*(\mathscr{A})\}| \geq |C_i| = m$ and we define $n_i(j) = j$. If $i \in M_2$ then $|C_i| < m$ and there are an infinite number of agents in $W \setminus \{1, 2, \ldots, n^*(\mathscr{A})\}$ strictly better than j. So, for n sufficiently large, say $n_i(j) \geq j$, there are (at least) m agents in $\{1, 2, \ldots, n_i(j)\}$ better than j. Take $n(j) = \max\{n_i(j) \mid i \in M\}$. Then $|B_i(j, A_{n(j)})| \geq m$ for all $i \in M$. □

Remark 3.4. *From lemma 3.3 and from proposition 2.3 it follows that for all* $j > n^*(\mathscr{A})$ *and for all optimal dual solutions* (u, v) *for* $\mathscr{A}_n$, $n \geq n(j)$, *we have* $v_j = 0$.

The games corresponding to these semi-infinite bounded assignment problems are defined as follows. The player set $N = M \cup W$ consists of an infinite number of players. The value of coalition S, $w(S)$, equals 0 if $S \subset M$ or $S \subset W$ and $w(S) = v_p(\mathscr{A}_S)$, the value of the finite or infinite assignment problem when restricted to coalition S, otherwise. Just as in the previous section, the value $w(N) = v_p(\mathscr{A})$ of the grand coalition N can be determined by the program P. The following problem is the dual if he integer condition in the primal problem P is replaced by nonnegativity (see [10]).

$$D: \quad \begin{aligned} \inf \quad & \sum_{i \in M} u_i + \sum_{j \in W} v_j \\ \text{s.t.} \quad & u_i + v_j \geq a_{ij}, \quad \text{for all } i \in M, j \in W \\ & u_i, v_j \geq 0, \quad \text{for all } i \in M, j \in W. \end{aligned}$$

Notice that both the primal and the dual program have an infinite number of variables and an infinite number of restrictions. In general, $\infty \times \infty$-programs show a gap between the optimal primal and dual value. There is a large literature on the existence or absence of so-called duality gaps in (semi-)infinite programs. See e.g. the books by Glashoff and Gustafson [4] and Goberna and López [5]. Our goal is to prove that here the primal and the dual problem have the same value and that there exist optimal solutions of the dual problem. We achieve this result in some steps starting with a limit process in the finite space $\mathbb{R}^m \times \mathbb{R}^{n^*}$, where for the sake of brevity we will write n^* instead of $n^*(\mathscr{A})$ in a subscript or a superscript.

We take for each $n \in \mathbb{N}$ with $n > n^*(\mathscr{A})$, a pair (u^n, v^n) that is optimal for $D(\mathscr{A}_n)$, the dual problem of $\mathscr{A}_n$. Then we remove all coordinates of v^n with index larger than $n^*(\mathscr{A})$ and obtain $(u^n, s^{n^*}(v^n)) \in \mathbb{R}^M \times \mathbb{R}^{n^*}$, where $s^{n^*} : \mathbb{R}^n \to \mathbb{R}^{n^*}$ is the map defined by $s^{n^*}(v_1^n, \ldots, v_{n^*}^n, \ldots, v_n^n) = (v_1^n, \ldots, v_{n^*}^n)$, for all $n > n^*(A)$. Note that $\{(u^n, s^{n^*}(v^n)) \mid n \in \{n^*(A)+1, n^*(A)+2, \ldots\}\}$ is a bounded set in the finite dimensional space $\mathbb{R}^M \times \mathbb{R}^{n^*}$ since A is a bounded matrix and (u^n, v^n) is optimal for $D(\mathscr{A}_n)$.

$$u_i^n \leq \max\{a_{ij} \mid i \in M, j \in \{1, 2, \ldots, n\}\} \leq \sup\{a_{ij} \mid i \in M, j \in \mathbb{N}\}$$

and similarly $v_j^n \leq \sup\{a_{ij} \mid i \in M, j \in \mathbb{N}\}$.

Without loss of generality, assume that $\lim_{n\to\infty}(u^n, s^{n^*}(v^n))$ exists (otherwise take a subsequence) and denote this limit by $(\bar{u}, \bar{v}) \in \mathbb{R}^M \times \mathbb{R}^{n^*}$. With the aid of $(\bar{u}, \bar{v})$ we construct the vector $(\hat{u}, \hat{v}) \in \mathbb{R}^M \times \mathbb{R}^W$ by taking $\hat{u} = \bar{u}$ and $\hat{v} = \alpha_{n^*}(\bar{v})$, where $\alpha_k : \mathbb{R}^k \to \mathbb{R}^W$ is the map defined by $\alpha_k(x) = (x_1, \ldots, x_k, 0, 0 \ldots)$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^k$. So, $\hat{v}$ is obtained from $\bar{v}$ by adding an infinite number of zeros. Later we will see that $(\hat{u}, \hat{v})$ is a core-element of the corresponding semi-infinite bounded assignment game but we start with showing that $(\hat{u}, \hat{v})$ is feasible for the dual problem.

Lemma 3.5. *Let $\mathscr{A} = (M, W, A)$ be a semi-infinite bounded assignment problem and let $(\hat{u}, \hat{v})$ be as defined above. Then $(\hat{u}, \hat{v})$ is a feasible solution for D.*

Proof. By definition of $(\hat{u}, \hat{v})$ it holds that all its coordinates are nonnegative. Furthermore, $\hat{u}_i + \hat{v}_j \geq a_{ij}$ for all $i \in M$, $j \in \{1, 2, \ldots, n^*(\mathscr{A})\}$ since $u_i^n + v_j^n \geq a_{ij}$ for all $i \in M$, $j \in \{1, 2, \ldots, n^*(\mathscr{A})\}$. For $i \in M$, $j > n^*(A)$, we know from remark 3.4 that $\lim_{n\to\infty} v_j^n = 0$. Together with $u_i^n + v_j^n \geq a_{ij}$ for all $j \in \{1, 2, \ldots, n\}$ it follows by taking the limit for $n \to \infty$ that $\hat{u}_i + \hat{v}_j \geq a_{ij}$. So $(\hat{u}, \hat{v})$ is a feasible solution of the dual problem. □

The next lemmas deal with the relations between the values of the finite subproblems and the infinite problems.

Lemma 3.6. $v_d(\mathscr{A}) \leq \lim_{n\to\infty} v_d(\mathscr{A}_n)$

Proof. For $n > n^*(\mathscr{A})$ and (u^n, v^n) optimal for $D(\mathscr{A}_n)$ we have $\sum_{i=1}^m u_i^n + \sum_{j=1}^n v_j^n = v_d(\mathscr{A}_n)$. We construct $(\hat{u}, \hat{v})$ as we did before and so, $\sum_{i=1}^m \hat{u}_i + \sum_{j=1}^n \hat{v}_j = \lim_{n\to\infty} v_d(\mathscr{A}_n)$. Then, from lemma 3.5 $v_d(\mathscr{A}) \leq \sum_{i=1}^m \hat{u}_i + \sum_{j=1}^\infty \hat{v}_j = \lim_{n\to\infty} v_d(\mathscr{A}_n)$. □

Lemma 3.7. $v_p(\mathscr{A}) = \lim_{n\to\infty} v_p(\mathscr{A}_n)$

Proof. Clearly for $n \geq m$ we have $v_p(\mathscr{A}_n) \leq v_p(\mathscr{A})$ because each matching $\pi : M \to \{1, 2, \ldots, n\}$ in the finite problem is also feasible in the infinite problem. Furthermore, $\{v_p(\mathscr{A}_n) \mid n \geq m\}$ is an increasing sequence. So, $\lim_{n\to\infty} v_p(\mathscr{A}_n)$ exists and $\lim_{n\to\infty} v_p(\mathscr{A}_n) \leq v_p(\mathscr{A})$.

For the converse inequality, take $\varepsilon > 0$ and a matching $\pi^\varepsilon : M \to \mathbb{N}$ such that $\sum_{i=1}^m a_{i\pi^\varepsilon(i)} \geq v_p(\mathscr{A}) - \varepsilon$. Let $k \in \mathbb{N}$ be such that $\{\pi^\varepsilon(i) \mid i \in M\} \subset$

$\{1, 2, \ldots, k\}$. Then for all $n \geq k : v_p(\mathscr{A}_n) \geq \sum_{i=1}^{m} a_{i\pi^{\varepsilon}(i)} \geq v_p(\mathscr{A}) - \varepsilon$. This implies that $\lim_{n\to\infty} v_p(\mathscr{A}_n) \geq v_p(\mathscr{A})$. □

Now we formulate the main result in this section, which tells us that there is no duality gap and that there exists optimal solutions for D.

Theorem 3.8. *Let $\mathscr{A} = (M, W, A)$ be a semi-infinite bounded assignment problem. Then $v_p(\mathscr{A}) = v_d(\mathscr{A})$ and there exist optimal solutions for D.*

Proof. First, we prove that there is no duality gap using the fact that finite problems have no duality gap. From lemmas 3.6 and 3.7 follows,

$$v_d(\mathscr{A}) \leq \lim_{n\to\infty} v_d(\mathscr{A}_n) = \lim_{n\to\infty} v_p(\mathscr{A}_n) = v_p(\mathscr{A}).$$

Conversely, weak duality, $v_p(\mathscr{A}) \leq v_d(A)$, holds. So $v_p(\mathscr{A}) = v_d(\mathscr{A}) = \lim_{n\to\infty} v_d(\mathscr{A}_n)$.

Second, we prove that $(\hat{u}, \hat{v})$ is optimal for D. From the proof of lemma 3.6 and from the first part of this proof $\sum_{i=1}^{m} \hat{u}_i + \sum_{j=1}^{\infty} \hat{v}_j = \lim_{n\to\infty} v_d(\mathscr{A}_n) = v_d(\mathscr{A})$. Furthermore, by lemma 3.5, $(\hat{u}, \hat{v})$ is feasible for D. So, $(\hat{u}, \hat{v})$ is optimal for D. □

Since Llorca [8, page 34] shows that the core of the corresponding assignment game is equivalent to the set of optimal solutions for D, it follows from theorem 3.8 that all semi-infinite bounded assignment games have a nonempty core.

4 The critical number and related concepts

In this section, we present the *critical number* of a semi-infinite bounded assignment game. It turns out to be a key concept because, as we will show, it is related to the hard-choice number, introduced in section 3, and to the finite approximation problems.

Definition 4.1. *The* critical number $c(\mathscr{A})$ *equals* $\min\{n \in \mathbb{N} \mid v_p(\mathscr{A}_n) = v_p(\mathscr{A})\}$, *if there exists an $n \in \mathbb{N}$ with $v_p(\mathscr{A}_n) = v_p(\mathscr{A})$. Otherwise, $c(\mathscr{A}) = \infty$.*

First, we present some results for finite critical numbers. The next proposition shows a relation between the hard-choice number and the critical number.

Proposition 4.2. *Let $\mathscr{A} = (M, W, A)$ be a semi-infinite bounded assignment problem. Then $c(\mathscr{A}) < \infty$ if and only if P has optimal solutions, and $c(\mathscr{A}) \leq n^*(\mathscr{A})$ if $c(\mathscr{A}) < \infty$.*

Proof. Let $\mathscr{A} = (M, W, A)$ be a semi-infinite bounded assignment problem. The first statement follows immediately from the definition of the critical number.

To prove the second statement, let π be an optimal assignment for P. If $\pi(i) \notin C_i$ for some $i \in M_1$, then since the size of the set $\pi(M_1)$ is smaller than the size of C_i, there is a $j \in C_i$ such that $j \notin \pi(M_1)$. If we redefine $\pi(i) = j$ then the assignment π remains optimal for P and agent i restricts his choice to C_i.

For $i \in M_2$ there is no optimal matching π with $\pi(i) \notin C_i$ since any such a matching can be improved using a different value of $\pi(i)$. We conclude that there exists an optimal matching π for P that is also optimal for $P(\mathscr{A}_{n^*})$, the primal problem of $\mathscr{A}_{n^*}$. Thus $c(\mathscr{A}) \leq n^*(\mathscr{A})$. □

As the next example shows, an optimal assignment can use agents $j \in W$ for which $j > n^*(\mathscr{A})$.

Example 4.3. Let $M = \{1, 2, 3\}$, $W = \mathbb{N}$, and

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 1 & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \end{bmatrix}.$$

We have seen in example 3.1 that $C_1 = \{1, 2, 3\}$, $C_2 = \{2\}$, $C_3 = \{1, 2, 3\}$, $M_1 = \{1, 3\}$ and $M_2 = \{2\}$. Also, $n^*(\mathscr{A}) = 3$, $v_p(\mathscr{A}) = 5$ and each π_k, with $k \geq 3$, defined by $\pi_k(1) = 1$, $\pi_k(2) = 2$, $\pi_k(3) = k$, is optimal. For $k > 3$ we have optimal matchings with $\pi_k(3) \notin C_3$, but the assignment π_3 is optimal and uses only elements in A_{n^*}. So, $c(\mathscr{A}) = n^*(\mathscr{A}) = 3$.

In the theorem below we characterize the structure of the sets of optimal primal and dual solutions when the critical number is finite. Recall that $P(\mathscr{A}_n)$ and $D(\mathscr{A}_n)$ are the primal and dual problem of the finite assignment problem $\mathscr{A}_n = (M, \{1, \ldots, n\}, A_n)$, respectively.

Theorem 4.4. *Let $\mathscr{A} = (M, W, A)$ be a semi-infinite bounded assignment problem. If $c(\mathscr{A}) < \infty$ then*

(i) *an assignment π is optimal for P if and only if it is optimal for $P(\mathscr{A}_n)$ for some $n \geq n^*(\mathscr{A})$,*
(ii) *for each pair (u, v) that is optimal for D, $v_j = 0$ for $j > n^*(\mathscr{A})$,*
(iii) *a pair (u, v) is optimal for D if and only if $(u, s^n(v))$ is optimal for $D(\mathscr{A}_n)$ for all $n \geq n^*(\mathscr{A})$.*

Proof. Let $\mathscr{A} = (M, W, A)$ be a semi-infinite bounded assignment problem with $c(\mathscr{A}) < \infty$.

(i) First, let $n \geq n^*(\mathscr{A}) \geq c(\mathscr{A})$ and let π be an optimal assignment for $P(\mathscr{A}_n)$. Then $\sum_{i=1}^{m} a_{i\pi(i)} = v_p(\mathscr{A}_n) = v_p(\mathscr{A})$ and π is also optimal for P. Second, let π be an optimal assignment for P and let $n \geq n^*(\mathscr{A})$ be such that $\pi(M) \subset \{1, \ldots, n\}$. Then π is feasible for $P(\mathscr{A}_n)$ and $\sum_{i=1}^{m} a_{i\pi(i)} = v_p(\mathscr{A}) = v_p(\mathscr{A}_n)$. So, π is optimal for $P(\mathscr{A}_n)$.

(ii) Let (u, v) be optimal for D. According to theorem 3.8

$$\begin{aligned} v_p(\mathscr{A}) &= v_d(\mathscr{A}) \\ &= \sum_{i \in M} u_i + \sum_{j \in W} v_j \end{aligned}$$

$$\geq \sum_{i\in M} u_i + \sum_{j=1}^{n^*} v_j$$

$$\geq \min\left\{ \sum_{i\in M} u'_i + \sum_{j=1}^{n^*} v'_j \,|\, u'_i + v'_j \geq a_{ij}, u'_i, v'_j \geq 0 \right\}$$

$$= v_p(\mathscr{A}_{n^*}) = v_d(\mathscr{A}_{n^*}) = v_p(\mathscr{A})$$

where the last equality follows from $n^*(\mathscr{A}) \geq c(\mathscr{A})$. Thus

$$\sum_{i\in M} u_i + \sum_{j\in W} v_j = \sum_{i\in M} u_i + \sum_{j=1}^{n^*} v_j$$

or, equivalently, $\sum_{j=n^*+1}^{\infty} v_j = 0$. Because (u, v) is optimal for D, $v_j \geq 0$ for all j. We conclude that $v_j = 0$ for all $j > n^*(\mathscr{A})$.

(iii) Let (u, v) be optimal for D. By part (ii) $v_j = 0$ for $j > n^*(\mathscr{A})$. This means that $(u, s^n(v))$ is optimal for $D(\mathscr{A}_n)$ for $n \geq n^*(\mathscr{A})$. Conversely, let $(u, s^n(v))$ be optimal for $D(\mathscr{A}_n)$ where $n \geq n^*(\mathscr{A})$ and $v_j = 0$ for $j > n$. If π is an optimal assignment for $P(\mathscr{A}_n)$ then π is also optimal for P since $v_p(\mathscr{A}) = v_p(\mathscr{A}_n)$. Hence, $v_p(\mathscr{A}) = \sum_{i=1}^{m} a_{i\pi(i)} = \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j = \sum_{i=1}^{m} u_i + \sum_{j=1}^{\infty} v_j$ and so (u, v) is optimal for D. □

In case $c(\mathscr{A}) = \infty$, there are no optimal solutions for P, we construct an auxiliary problem $\mathscr{H} = (M, \{1, \ldots, n^*(\mathscr{A}) + |M_2|\}, H)$ corresponding to $\mathscr{A}$. With the help of $\mathscr{H}$ we can find ε-optimal assignments for $\mathscr{A}$, that is, assignments π such that $\sum_{i\in M} a_{i\pi(i)} \geq v_p(\mathscr{A}) - \varepsilon$. The matrix H is defined by $H = [A_{n^*}\ T]$ where for each $i \in M_2$ we have a column $t_i e^i$ in T with $t_i = \sup\{a_{ij} \,|\, j \in \mathbb{N}\backslash C_i\}$, the largest value outside player i's choice set. The vector e^i is the ith unit vector in $\mathbb{R}^m$ defined by $e^i_k = 1$ if $k = i$ and $e^i_k = 0$ otherwise. We illustrate these concepts in the next example.

Example 4.5. Let $M = \{1, 2, 3\}$, $W = \mathbb{N}$ and

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \cdots \\ 0 & 2 & 1\frac{2}{3} & 1\frac{3}{4} & 1\frac{4}{5} & 1\frac{5}{6} & \cdots \end{bmatrix}.$$

Then $C_1 = \{1, 2, 3\}$, $C_2 = \{1\}$, $C_3 = \{2\}$, $M_1 = \{1\}$, $M_2 = \{2, 3\}$ and $n^*(\mathscr{A}) = 3$. The feasible matching π with $\pi(1) = 3$, $\pi(2) = 1$, $\pi(3) = 2$ has the property $\pi(i) \in C_i$ for each $i \in M$. But this assignment is not optimal since $\sum_{i=1}^{m} a_{i\pi(i)} = 4 < 6 = v_p(\mathscr{A})$. In this example $c(\mathscr{A}) = \infty$, no optimal assignment exists. Using the auxiliary problem $\mathscr{H}$ with

$$H = \begin{bmatrix} \mathbf{3} & 2 & 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{2}{3} & \mathbf{1} & 0 \\ 0 & \mathbf{2} & 1\frac{2}{3} & 0 & 2 \end{bmatrix},$$

results in $v_p(\mathscr{H}) = 6$ and the matching π', with $\pi'(1) = 1$, $\pi'(2) = n$ $(n \geq 3)$,

$\pi'(3) = 2$, is an $\frac{1}{n}$-optimal assignment for $\mathscr{A}$, that is, $\sum_{i \in M} a_{i\pi'(i)} = v_p(\mathscr{A}) - 1/n$.

Theorem 4.6. *Let $\mathscr{A} = (M, W, A)$ be a semi-infinite bounded assignment problem with $c(\mathscr{A}) = \infty$ and let $\mathscr{H}$ be the corresponding auxiliary problem. Then*

(i) $v_p(\mathscr{A}) = v_p(\mathscr{H})$
(ii) *For each π that is optimal for $P(\mathscr{H})$ and each $\varepsilon > 0$ there is a matching π^ε optimal for P such that $\pi^\varepsilon(i) = \pi(i)$ for all $i \in M_1$ and $\pi^\varepsilon(i) \in \{n^*(\mathscr{A}) + 1, n^*(\mathscr{A}) + 2, \ldots\}$ such that $a_{i\pi^\varepsilon(i)} \geq t_i - \varepsilon/m$, for $i \in M_2$.*

Proof. To prove (i) and (ii) it is sufficient to show that $v_p(\mathscr{H}) \geq v_p(\mathscr{A})$ and $v_p(\mathscr{A}) \geq v_p(\mathscr{H}) - \varepsilon$ for all $\varepsilon > 0$.

First we show that $v_p(\mathscr{H}) \geq v_p(\mathscr{A})$. Let π be a feasible matching for P. Construct a feasible assignment π^* for $P(\mathscr{H})$ as follows. Let $i \in M$. If $\pi(i) \in C_i$ then $\pi^*(i) = \pi(i)$. If $\pi(i) \notin C_i$ and $i \in M_1$ then we can choose a partner $\pi^*(i) = j^* \in C_i$ because C_i is large enough. (See the proof of proposition 4.2.) If $\pi(i) \notin C_i$ and $i \in M_2$ then define $\pi^*(i) = j^*$, where j^* corresponds to column $t_i e^i$ in T. Thus for all $i \in M$ we have $h_{i\pi^*(i)} \geq a_{i\pi(i)}$, so, $v_p(\mathscr{H}) \geq v_p(\mathscr{A})$.

Second, we show that $v_p(\mathscr{A}) \geq v_p(\mathscr{H}) - \varepsilon$ for all $\varepsilon > 0$. Let $\varepsilon > 0$ and let π be feasible for $P(\mathscr{H})$. We will construct a matching π^ε that is feasible for $P(\mathscr{A})$ as follows. Take one-by-one elements $i \in M$. Note that $\pi(i) \notin \{1, 2, \ldots, n^*(\mathscr{A})\} \backslash C_i$ since otherwise player i can improve by choosing t_i. If $\pi(i) \notin T$ then define $\pi^\varepsilon(i) = \pi(i)$. If $\pi(i) \in T$ then take $j^* > n^*(\mathscr{A})$ such that $a_{ij^*} \geq t_i - \varepsilon/m$ and $j^* \neq \pi(i')$ for all $i' \neq i$ and define $\pi^\varepsilon(i) = j^*$. This can be done such that all $i \in M$ are matched to m different elements in W. Then

$$\begin{aligned}\sum_{i \in M} a_{i\pi^\varepsilon(i)} &= \sum_{i \in M : \pi^\varepsilon(i) \in C_i} a_{i\pi^\varepsilon(i)} + \sum_{i \in M : \pi^\varepsilon(i) \notin C_i} a_{i\pi^\varepsilon(i)} \\ &\geq \sum_{i \in M : \pi(i) \in C_i} h_{i\pi(i)} + \sum_{i \in M : \pi(i) \in T} (t_i - \varepsilon/m) \\ &\geq \sum_{i \in M} h_{i\pi(i)} - \varepsilon,\end{aligned}$$

where the last inequality holds because $|\{i \in M \mid \pi(i) \in T\}| \leq m$. Thus $v_p(\mathscr{A}) \geq v_p(\mathscr{H}) - \varepsilon$. □

5 Concluding remarks

In this paper we analysed semi-infinite assignment problems from a game-theoretic viewpoint. We started by showing that semi-infinite assignment problems have no duality gap and that there always exists an optimal solution for the dual problem. Consequently, the corresponding semi-infinite assignment games have a nonempty core, that is, they are balanced. Further, if there does not exist an optimal solution for the primal problem then an auxiliary assignment problem $\mathscr{H}$ can be used to derive ε-optimal assignments which are close to the optimum.

Future directions for research include extending these results to infinite assignment problems, where the two sets of agents are infinite, and to infinite transportation problems, which are generalizations of assignment problems.

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