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Theory and Methodology

Production, inventory, and pricing under cost and demand learning effects

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Abstract

The paper considers a monopolist firm that plans its production, inventory, and pricing policy over a fixed and finite horizon. The problem is represented by an optimal control model which combines elements from three streams of literature. The first of these is a classical OR area and deals with optimal production and inventory under exogenously given demand conditions. The second is an area of marketing science which studies dynamic pricing under demand learning effects. Demand learning refers to the situation where current demand for a product is influenced by past demand. The third area belongs to microeconomics and industrial organization and is concerned with the effects of learning-by-doing in a firm's production process. Learning is reflected in a unit production cost that decreases with cumulative production. Using a path-synthezising procedure we obtain closed-form characterizations of optimal production, pricing, and inventory policies.

Keywords: Production; Inventory; Pricing; Optimal control theory

1. Introduction

This paper studies an optimal control problem positioned in the intersection between three streams of research. The first of these streams originates in the inventory and production model of Holt et al. (1960) where the problem was to determine an optimal production and inventory policy facing a time-varying but exogenously given demand. Examples of this type of modeling are found in Thompson et al. (1984) and Teng et al. (1984). Pekelman (1974) introduced product price as an explicit decision variable in the production-inventory problem but confined his interest to a simple specification of the demand function in which the demand rate at time t depends linearly on the price at that specific instant. Even if the coefficients

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in the demand function are allowed to vary over time, such a demand function cannot accommodate demand learning effects by which we mean that the current demand rate depends on past demand experience. Another contribution in the area of production and inventory planning is Feichtinger and Hartl (1985) who generalized the Pekelman (1974) model to include a nonlinear demand specification. With respect to inventory, Pekelman (1974), Thompson et al. (1984) and Teng et al. (1984) assumed a nonnegative inventory. Feichtinger and Hartl (1985) allowed for backlogging of demand. Other contributions to the production-inventory-pricing area are Jørgensen (1986) and Eliashberg and Steinberg (1987). However, neither of these papers employed the assumption of demand learning effects

As to the cost structure, all the above models typically assume a linear or convex production cost, being a function of the current production rate, and do not include cost experience. Cost experience means that a firm's unit production cost decreases with cumulative output and this decrease reflects the productivity increasing effects of a learning-by-doing process in production. Gaimon (1988), Clarke et al. (1982) and Dotan et al. (1985) include cost learning. In particular, the latter two papers assume a "microeconomics" cost function being dependent on current as well as cumulative production volume. These three papers, however, disregard the pricing policy.

Demand learning effects play an important role in marketing models of new product adoption. The idea is to let the demand rate at time t depend not only on price (or any other marketing variable) at that time but also on cumulative sales by time t. This allows one to model the impacts of various forms of demand or consumption experience, e.g., word-of-mouth and carry-over effects, saturation, habit formation, and bandwagon effects. There is a considerable literature on this topic in the marketing science literature (see, for example, Kalish, 1983; Mahajan et al., 1990). Some of these analyses also incorporate cost experience. However, a common feature of the marketing science literature on dynamic pricing with both demand and cost learning effects is the absence of inventories.

The aim of this paper is to analyze a model which combines the essential elements from the three streams of research discussed above. Our paper is perhaps most closely related to that of Clarke et al. (1982) who introduced demand learning and a cost function which depends on both current and cumulative output. Inventories, however, were not incorporated.

The paper is organized as follows. Section 2 presents the optimal control model and our assumptions. Formally, the model is a finite horizon optimal control problem with two controls and two states and a pure state constraint. In Section 3 we state a sufficient optimality condition for the firm's dynamic pricing, production, and inventory policy under hypotheses that are as general as possible. Then we employ a synthesizing procedure to generate an optimal sequence of the "regimes" that were identified by the optimality condition. Section 4 concludes and gives some avenues for future research.

2. An optimal control model

Consider a monopolistic firm that must plan its production, inventory and sales of a durable product during a period of time starting at time $t=0$. Let $X(t)$ denote the cumulative demand by time t, $0 \leqslant t \leqslant T$. The terminal instant T is fixed, finite and relatively short since we are dealing with a problem of operational/tactical planning. Let $p(t)$ denote the retail price at time t. The demand dynamics are given by

$$\dot{X}(t) = k(X(t))g(p(t)) \tag{1}$$

in which the left-hand side represents the demand rate at time t. Functions k and g take positive values and are twice continuously differentiable. We have assumed that the demand function is multiplicatively separable in X and p which is fairly standard in the literature (see, e.g., Kalish, 1983; Clarke et al., 1982;

Dockner and Jørgensen, 1988). According to Eq. (1), the current demand rate is influenced by the current price as well as the cumulative demand. As mentioned in Section 1, including cumulative demand in the demand specification allows one to take into consideration the often observed fact that for durable products, past demand influences current demand. Among these demand diffusion effects are word-of-mouth (positive or negative) and market saturation. The latter has a negative impact on current demand and reflects the fact that when cumulative demand increases toward the total market potential, the remaining untapped potential diminishes. In Eq. (1), demand learning is reflected in the derivative of function $k(X)$ such that learning effects are positive [negative] if $k'(X)$ is positive [negative]. In this paper we confine our interest to cases where

$$k'(X) \geqslant 0 \quad \forall X. \tag{2}$$

If $k'(X) = 0\,\forall X$ there are no demand learning effects at all and the demand function essentially is static. If $k'(X) > 0\,\forall X$ there are positive demand learning effects throughout the planning period. This situation occurs when the firm has a product which, during the planning period, is subject to an adoption process having the property that the higher the past demand, the higher the current demand. This can come about, for instance, by positive imitation effects. We consider a planning horizon which is relatively short and our assumption thus is that during this interval of time, saturation effects are only of minor importance. (Clarke et al. (1982), studied a case in which there are initially no demand learning effects, $k' = 0$, but later on negative diffusion effects occur due to saturation, $k' < 0$.) We suppose that function $k(X)$ is concave. Hence we have positive demand learning throughout but cumulative demand has a decreasing marginal impact on current demand (see, for instance, Kalish, 1983; Dockner and Jørgensen, 1988). For function $g(p)$ we introduce the standard conditions

$$g'(p) < 0, g''(p) < \frac{2(g'(p))^2}{g(p)} \quad \forall p > 0$$

in which the second one assures that profits are maximized. See, for example, Feichtinger and Hartl (1986).

Let $Q(t)$ be the cumulative output by time t and denote by $q(t)$ the output rate at time t. By definition it holds that

$$\dot{Q}(t) = q(t). \tag{3}$$

Let $I(t)$ represent the inventory level at time t. The inventory evolves over time according to the standard dynamics

$$\dot{I}(t) = q(t) - k(X(t))g(p(t)), \quad I(0) = I_0 \geqslant 0. \tag{4}$$

If the initial inventory I_0 is positive we suppose that this has occurred because a terminal, positive inventory was transferred from the previous planning period. The inventory thus transferred acts as the initial inventory I_0 for the planning period under consideration, i.e., the one starting at time $t = 0$.

By Eqs. (1), (3) and (4) we have introduced a dynamical system described by the three state variables $X(t)$, $Q(t)$, and $I(t)$. The current inventory level satisfies the accounting identity

$$I(t) = I_0 + Q(t) - X(t), \tag{5}$$

which can be used to eliminate one of the state variables. In what follows we shall eliminate the inventory level $I(t)$. The firm has two control variables, price $p(t)$ and production rate $q(t)$ that both must remain nonnegative for all $t \in [0, T]$. Using Eqs. (1) and (3), and the control variable constraints it is easy to see that the state variables $X(t)$ and $Q(t)$ remain nonnegative for all t. There is, however, no guarantee that the inventory level will stay positive. Here we introduce the assumption that backlogging of demand is impossible. (In Section 4 we address briefly the possibility of backlogging unsatisfied demand.) Thus, we need to introduce the state variable constraint

$$I(t) \geqslant 0 \;\; \forall t \in [0,T] \quad \Longleftrightarrow \quad I_0 + Q(t) \geqslant X(t) \;\; \forall t \in [0,T]. \tag{6}$$

We only consider production and inventory costs. Assume that the inventory cost is linear: $\beta I(t) = \beta[I_0 + Q(t) - X(t)], \beta = \text{const.} > 0$ and define a total production cost function that depends both on cumulative and current output. Let a variable appearing as a subscript denote partial differentiation with respect to that variable. The firm's total production cost is given by a twice differentiable function $c(Q, q)$ the derivatives of which satisfy

$$c_Q < 0, c_q > 0; \quad c_{QQ} > 0, c_{qq} > 0, c_{Qq} \leqslant 0; \quad c_{QQ}c_{qq} - (c_{Qq})^2 > 0. \tag{7}$$

Hence, given the current production rate, the unit production cost is smaller the higher the cumulative production. This is the cost learning effect which is supposed to exhibit decreasing returns. The total cost is convex increasing in the current production rate, given the cumulative output. Function c is strictly convex. The assumption $c_{Qq} \leqslant 0$ means that the production process has the property that the marginal production cost c_q is nonincreasing in cumulative production. A similar assumption was employed in Dotan et al. (1985); see also case (ii) in Eq. (8) below.

To illustrate, the following two specifications of the general cost function were considered by Clarke et al. (1982) and Dotan et al. (1985):

$$\begin{aligned}
&\text{(i)}\;\; c(Q,q) = c_0 + c_{1a}(Q) + c_{2a}(q),\\
&\text{(ii)}\;\; c(Q,q) = c_0 + c_{1m}(Q)c_{2m}(q),\\
&\frac{\mathrm{d}c_{1j}}{\mathrm{d}Q} < 0, \;\; \frac{\mathrm{d}^2c_{1j}}{\mathrm{d}Q^2} > 0, \;\; \frac{\mathrm{d}c_{2j}}{\mathrm{d}q} > 0, \;\; \frac{\mathrm{d}^2c_{2j}}{\mathrm{d}q^2} > 0, \quad j \in \{a,m\},\\
&\frac{\partial^2 c}{\partial Q \partial q} \leqslant 0.
\end{aligned} \tag{8}$$

In the additively separable specification, the term $c_{1a}(Q)$ reduces the fixed cost c_0 as cumulative output Q increases. The last component is a standard variable cost c_{2a} which is a function of q alone. The second specification is multiplicatively separable which means that the variable cost is reduced as Q increases. The work by Clarke et al. (1982) and Dotan et al. (1985) shows that the optimal price and output trajectories are significantly affected by the choice of the cost function, i.e., it matters whether learning comes about by fixed or by variable cost reduction.

In this paper we will demonstrate that the conclusions of the earlier studies reviewed in Section 1 must be modified in a context of positive demand learning, cost experience, and nonnegative inventory. It is necessary to introduce one additional assumption which concerns the production and inventory cost functions. Thus, for all feasible q and Q we require that

$$c_Q(Q,q) + \beta > 0. \tag{9}$$

The inequality means that the cost learning effect is not dominant, in the sense that the reduction of production cost caused by an extra unit of cumulative output Q is always less than the inventory cost caused by that extra unit. The assumption restrains the firm in such a way that it would not find it optimal to expand its production beyond any limit in order to proceed very rapidly downward the cost learning curve, while building up a huge inventory. In the case of an insignificant holding cost, such a production policy might be optimal.

In the case of an additively separable cost function (cf. case (i) in Eq. (8)) we have

$$\frac{dc_{1a}}{dQ}(0) > -\beta \Rightarrow \frac{dc_{1a}}{dQ}(Q) > -\beta \quad \forall Q$$

and hence, if Eq. (9) initially is satisfied, it is satisfied for all t. This observation follows from the fact that Q cannot decrease over time. However, in the case of a general or a multiplicatively separable cost function, the satisfaction of the inequality in Eq. (9) also depends on the level of q.

The payoff functional of the firm is the undiscounted profit stream over $[0, T]$:

$$J = \int_0^T \{p(t)k(X(t))g(p(t)) - c(Q(t), q(t)) - \beta[I_0 + Q(t) - X(t)]\}\,dt.$$

The reader will notice that we have disregarded discounting, a salvage value at the horizon date, and possible constraints on $Q(T)$. The first assumption is reasonable since we have assumed that the planning horizon is relatively short. The two other omissions can be handled but do not seem to add much in terms of interesting results.

3. Analysis of the optimal control model

We confine our interest to optimal solutions in which the price always is strictly positive, although one cannot exclude the possibility of having an optimal price of zero during an initial interval of time, provided that the initial inventory is sufficiently large. In that case the firm will give away the product for free, since it pays to reduce as fast as possible the initial inventory and thereby the inventory costs. Feichtinger and Hartl (1985, 1986) deal explicitly with the constraint $p(t) \geqslant 0$ and state a condition for having a zero price. (This is done, however, in a model that does not include the inventory nonnegativity constraint.)

Let $\mu(t)$ be a multiplier associated with the control constraint $q \geqslant 0$ and let $v(t)$ be a multiplier associated with the state constraint in Eq. (6). Denote by $\lambda(t)$ and $\omega(t)$ the costates associated with state variables X and Q, respectively, and let λ_0 be a nonnegative number. (If λ_0 is positive it can without loss of generality, be set equal to one.) Define a Lagrangian function by

$$L(X, Q, p, q, \lambda, \omega, \mu, v) = H(X, Q, p, q, \lambda, \omega) + \mu q + v(I_0 + Q - X),$$

where

$$H(X, Q, p, q, \lambda, \omega) = \lambda_0[pk(X)g(p) - c(Q, q) - \beta(I_0 + Q - X)] + \lambda k(X)g(p) + \omega q.$$

3.1. The optimality conditions

Sufficient optimality conditions for optimal control problems with pure state constraints can be found in Feichtinger and Hartl (1986), Seierstad and Sydsæter (1987) and Hartl et al. (1995). In general, an optimality condition should allow for the possibility of jump discontinuities in the costate variables at the terminal instant T as well as for $t < T$. The latter discontinuities typically occur at points at which a state constraint becomes (or ceases to be) active.

Here we state the sufficient condition only with the jump condition at $t = T$. The reason for this omission is that we can verify conditions that will guarantee continuity of the costates for all $t \in [0, T)$. To do so, first note that in a solution with $\lambda_0 = 1$ the following inequalities hold:

$$H_{pp} = \frac{k(X^*)}{g'(p^*)}[2(g'(p^*))^2 - g(p^*g''(p^*))] < 0,$$
$$H_{qq} = -c_{qq}(Q^*, q^*) < 0,$$
$$H_{pp}H_{qq} - (H_{qp})^2 = H_{pp}H_{qq} > 0,$$

where in the first line we have used the first-order condition for an optimal price, cf. Eq. (10) below. These inequalities guarantee that the maximum of H with respect to the controls is unique which in turn has the implication that the controls are continuous at all points of time.

Disregarding the extremely rare case in which a costate variable has a jump in the interior of a time interval on which the state constraint is active, it is known (Feichtinger and Hartl, 1986; Hartl et al., 1995) that the costate variables λ and ω are continuous if the following three conditions hold: the controls are continuous, the state constraint is of the first order, and a constraint qualification is satisfied, at least at points where the state constraint becomes (ceases to be) active.

A state constraint is said to be of the first order if the time-derivative of the constraint explicitly contains the control variables. This is satisfied in our model since by Eqs. (1) and (3) we have $\dot{Q} - \dot{X} = q - k(X)g(p)$. The constraint qualification requires that the following matrix must have full rank at all points where the state constraint becomes (ceases to be) active:

$$\begin{vmatrix} 1 & 0 & q & 0 \\ 1 & -k(X)g'(p) & 0 & I_0 + Q - X \end{vmatrix}.$$

The second element in the second row never vanishes and thus the matrix has full rank everywhere. With these remarks we can use the following sufficiency condition.

Arrow-type sufficient condition: For all $t \in [0, T]$ let (p^*, q^*, X^*, Q^*) satisfy $p^* > 0$, $q^* \geqslant 0$, $X^* \geqslant 0$, $Q^* \geqslant 0$ and let the state constraint be satisfied. Assume that there exist costate functions $\lambda(t)$, $\omega(t)$ that are continuous and piecewise continuously differentiable, piecewise continuous multiplier functions $\mu(t)$ and $\nu(t)$, and a number γ such that the following conditions are satisfied with $\lambda_0 = 1$:

$$\begin{aligned} &L_p = g(p^*(t)) + [p^*(t) + \lambda(t)]g'(p^*(t)) = 0, \\ &L_q = -c_q(Q^*(t), q^*(t)) + \omega(t) + \mu(t) = 0, \\ &\mu(t) \geqslant 0, \ \ \mu(t)q^*(t) = 0, \quad \text{for v.e. } t, \end{aligned} \tag{10}$$

$$\begin{aligned} &\nu(t)(Q^*(t) - X^*(t) + I_0) = 0, \ \ \nu(t) \geqslant 0 \quad \text{for all } t, \\ &\lambda(T) = -\gamma, \ \ \omega(T) = \gamma, \ \ \gamma[I_0 + Q^*(T) - X^*(T)] = 0, \ \ \gamma \geqslant 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &\dot{\lambda}(t) = -k'(X^*(t))g(p^*(t))[p^*(t) + \lambda(t)] - \beta + \nu(t), \\ &\dot{\omega}(t) = c_Q(Q^*(t), q^*(t)) + \beta - \nu(t) \quad \text{for v.e. } t. \end{aligned} \tag{12}$$

In Eqs. (10) and (12), "for v.e. t" means "for virtually every t", that is, for all t except a countable number of t's. Then, if for all $t \in [0, T]$ the maximized Hamiltonian is concave in (Q, X) and the right-hand side of the state constraint is quasi-concave in (Q, X), then (p^*, q^*, X^*, Q^*) solves the problem.

The right-hand side of the state constraint is linear in the state and hence quasi-concave. To verify that the maximized Hamiltonian is concave in the state vector for all t we calculate the partial derivatives of the maximized Hamiltonian:

$$H^{\max} = k(X)g(p^*(\lambda))[p^*(\lambda) + \lambda] + \omega q^*(Q, \omega) - c(Q, q^*(Q, q^*(Q, \omega))) - \beta(Q - X + I_0),$$

where

$p^*(\lambda), q^*(Q,\omega)$

are given by (10).

$$H^{\max}_{QQ} = 0 \text{ for } q^* = 0, \quad H^{\max}_{QQ} = -\left[c_{QQ}(Q,q^*) + c_{Qq}(Q,q^*)\frac{\partial q^*}{\partial Q}\right] < 0 \text{ for } q^* > 0.$$

In the last line, $\partial q^*/\partial Q \geqslant 0$. To see this, note that the optimality condition for q^* in Eq. (10) implicitly defines q^* as a function of Q. Then we have $\partial q^*/\partial Q = -c_{Qq}/c_{qq} \geqslant 0$ which can be inserted into $H^{\max}_{QQ}$ to yield $H^{\max}_{QQ} = -c_{QQ} + (c_{Qq})^2(c_{qq})^{-1} < 0$. The latter inequality is satisfied by convexity of the production cost function. Next consider a Hessian matrix consisting of the four second order partial derivatives of the maximized Hamiltonian with respect to X and Q . In this matrix the off-diagonal elements are zero since $H^{\max}_{XQ} = 0$. Using the rule of principal minors we conclude that the maximized Hamiltonian is concave in (X, Q).

3.2. The optimal policies

In the sequel we often omit the star on the optimal controls and states. We also omit the time argument, when no confusion can arise. Consider any instant of time for which p is differentiable with respect to t and differentiate totally with respect to time in the first expresssion in Eq. (10). Using this result and Eq. (12) yields

$$\dot{p}\left[2g'(p) - \frac{g(p)g''(p)}{g'(p)}\right] = g'(p)[\beta - v] - k'(X)(g(p))^2. \tag{13}$$

. the left-hand side of Eq. (13), the bracketed term is always negative by our assumption on demand ıction $g(p)$. Thus, the sign of the time derivative of p is the opposite of the sign of the right-hand side. The ıgn of the first term on the right-hand side is ambiguous since it depends on the difference $\beta - v$. The second term on the right-hand side is always nonpositive. The following lemma is a direct consequence of Eq. (11) (the first line) and Eq. (13):

Lemma 1. *The optimal price is increasing on any interval of time during which inventory is positive.*

This result is expected for the case of positive demand learning (where function $k(X)$ is increasing) and is in accordance with previous findings in pricing models without inventory (Kalish, 1983; Dockner and Jørgensen, 1988). Such models recommend, when there are positive diffusion effects, that price be increased over time. A marketing manager would say that the firm employs a penetration pricing policy, the purpose of which is to increase demand in order to benefit from the positive learning effects. In the case where inventory is positive, our model recommends to increase price also in situations when there are no demand diffusion effects (i.e., function $k(X)$ is constant). This differs from the predictions of no-demand-learning models without inventory in which price is constant (e.g., Kalish, 1983). The reason for having a constant optimal price in those models is that the problem essentially is static and hence the price can be set at an optimal, constant level for the whole planning period. In our model, however, the presence of a positive inventory provides an incentive to start out with a low price in order to decrease inventory by the price-induced larger demand rate. As the inventory level is going to decrease over time, the incentive for pursuing a low-price policy gradually diminishes and consequently price can be increased over time.

Differentiating totally with respect to time in the second expression of Eq. (10) and using the second costate equation in Eq. (12) yields, at any instant of time at which the control q and the multiplier μ are differentiable,

$$\dot{q} = \frac{1}{c_{qq}}[-c_{Qq}q + c_Q + \beta - \nu + \dot{\mu}]. \tag{14}$$

Using Eqs. (9), (11) and (14) establishes the following lemma.

Lemma 2. *The optimal production rate is increasing on any interval of time during which there is a positive inventory.*

To characterize the optimal production and inventory policies consider the following "regimes". By a regime we mean a specific combination of q, Q, and X which prevails during a nonzero interval of time:

$$\begin{aligned}
&R0: \quad I_0 + Q = X, \ q = 0 \Rightarrow \mu \geqslant 0, \ \nu \geqslant 0,\\
&R1: \quad I_0 + Q > X, \ q = 0 \Rightarrow \nu = 0, \ \mu \geqslant 0,\\
&R2: \quad I_0 + Q > X, \ q > 0 \Rightarrow \mu = \nu = 0,\\
&R3: \quad I_0 + Q = X, \ q > 0 \Rightarrow \mu = 0, \ \nu \geqslant 0.
\end{aligned}$$

Regime *R0* can only occur if price is set so high that it drives demand to zero. Since we consider the inclusion of positive demand learning in a production-inventory model as a major contribution of the paper, we shall disregard *R0* as being a less interesting regime. Note that positive demand learning throughout the planning period means that past demand stimulates current demand at no cost whatsoever. Then it seems less obvious why it could be optimal to price so as to drive demand to zero, thus refraining completely from the benefits of the positive demand learning effects.

With respect to the three remaining regimes we say that an initial regime is a regime which starts at tim zero and satisfies the initial condition. A final regime is a regime ending at time T and which satisfies th transversality conditions. It is useful to start with a brief characterization of the optimality conditions that must prevail along each of the three regimes.

Regime 1. This regime is a feasible initial regime only if the initial inventory is positive. If *R1* is a final regime we must have $c_q(Q(T), 0) = \mu(T)$, that is, $\mu(T) > 0$. *R1* occurs if $\omega \leqslant c_q(Q, 0)$; in economic terms, if the marginal cost of producing the first unit exceeds the imputed value (the shadow price) of the stock Q. Along *R1*, the inventory level is strictly decreasing. Cumulative output Q is constant with respect to time which implies that the cost $c_Q(Q, 0)$ is constant. It holds that $\dot{\omega} = c_Q(Q, 0) + \beta, \dot{\lambda} = -k'(X)g(p)(p + \lambda) - \beta$. If *R1* is a final regime, the transversality conditions $\lambda(T) = \omega(T) = 0$ must be satisfied.

Regime 2. This regime is a feasible initial regime only if the initial inventory is positive. *R2* cannot be a final regime. To see this, note that we would have $-c_q\,(Q(T), q(T)) = 0$ in the optimality condition (10). This would, however, contradict the assumption $c_q > 0$. The regime occurs if $\omega > c_q(Q, 0)$ and the optimal and positive q is given by $\omega = c_q(Q,q)$. Along *R2* it holds that $\dot{\omega} = c_Q(Q, q) + \beta, \dot{\lambda} = -k'(X)g(p)(p + \lambda) - \beta$. If *R2* is a final regime, the constant multiplier γ is zero and we obtain the usual transversality conditions for the costates.

Regime 3. *R3* cannot be an initial regime unless the initial inventory equals zero. Since inventory is zero, and since backlogging is not permitted, the production rate equals the demand rate: $q = k(X)g(p) > 0$. The optimal production rate q is given by the equation $\omega = c_q$ which determines (implicitly) q as a function of costate ω and state Q. Along *R3* it holds that $\dot{\omega} = c_Q(Q, q) + \beta - \nu, \ \dot{\lambda} = -k'(X)g(p)(p + \lambda) - \beta + \nu$. Since the costate ω is positive along *R3*, the constant multiplier γ must be positive if *R3* is to qualify as a final regime. The following condition is satisfied along *R3*:

$$\beta + c_Q \geqslant c_{Qq}(Q, q)q + c_{qq}(Q, q)\dot{q}$$

in which the left-hand side is the marginal inventory cost adjusted downward by the marginal cost learning effect c_Q. The right-hand side of this inequality could be seen as a growth condition on the demand rate (which along *R3* equals the production rate). Thus, if demand grows sufficiently slow and/or the holding cost is sufficiently large, no inventory should be kept. (If there is no cost experience, the two terms c_Q and c_{Qq} vanish and the above inequality is similar to one in Pekelman (1974).) See also Feichtinger and Hartl (1986).

To obtain the optimal sequence of regimes we apply a formal synthesizing procedure. The procedure determines which regime(s) can precede a given regime, exploiting the continuity of state variables and costates as well as the optimality conditions. Van Hilten et al. (1993) provide the details. The procedure starts out by identifying the set of feasible final regimes which in our case consists of *R1* and *R3*. For each element in this set the procedure works backward and determines which regime(s) can precede each final regime, and so forth. The procedure ends when it turns out that no regime can precede a given sequence of regimes and such that the initial regime of the sequence satisfies the initial conditions. In general there may be more than one optimal sequence. Our next result is somewhat extreme but occurs only for "large" initial inventory levels.

Proposition 1. *If* $I_0 \geqslant X(T)$ *then R1 is the optimal solution for all* $t \in [0, T]$.

Proof. We know that *R1* and *R3* are candidates for being a final regime. On *R3* there is a zero inventory and hence *R3* can be excluded since the premise $I_0 \geqslant X(T)$ implies that inventory will always stay positive. This leaves *R1* as the only feasible final regime. Along *R1* we know that the costate ω is increasing over time and that $\omega(T) = 0$ by the transversality conditions. This implies that the costate ω is negative along *R1* for $t < T$. We also know that ω is strictly positive along the other regimes (*R2* and *R3*) and that ω is continuous for all $t \in [0, T)$. Hence, neither *R2* nor *R3* can precede *R1* and the result of the proposition is established. □

The result of the proposition may be expected. When the firm foresees that its initial inventory is sufficient to meet the accumulated demand throughout the planning period, there is no need to produce. Production is not economically profitable since $\omega(t) \leqslant c_q(Q, 0)$: whatever the level of Q, the marginal cost of producing the first unit exceeds the shadow price of the stock Q. Producing at any positive level would only make the production cost even higher. Similar results as the one in Proposition 1 have been reported in, for example, Teng et al. (1984), Thompson et al. (1984) and Feichtinger and Hartl (1985) for the case of no demand learning. Thus, the result of the proposition seems to be quite robust against the choice of the demand law.

On the other hand, if the condition $I_0 \geqslant X(T)$ is not satisfied, *R3* is the only feasible final regime and the firm will always end up with zero inventory at the horizon date. This case, i.e., $I_0 < X(T)$, is considered in the next proposition. To formulate the proposition we call a sequence of regimes a "master trajectory" if its final regime is feasible and if no regime can precede the initial regime of the sequence. It may happen (mainly depending on the initial level of inventory) that subsequences of the master trajectory become optimal trajectories.

Proposition 2. *Let* $I_0 < X(T)$ *and suppose that either* (A): $k'(X) = 0\ \forall X$ *or* (B): $k'(X) > 0\ \forall X$, $g''(p) \geqslant (g'(p))^2/g(p)$. *Then the sequence R1 → R2 → R3 is the master trajectory.*

Proof. See Appendix A. □

Case A is the situation in which there are no demand learning effects, that is, the demand function essentially is static. This is the case treated in most of the literature dealing with problems of pricing, production, and inventories (e.g., Pekelman, 1974; Feichtinger and Hartl, 1985; Jørgensen, 1986; Eliashberg and Steinberg, 1987). Our result for Case A is an extension of this literature since we include learning effects in the cost function. In the references cited, the cost function only depends on current output rate (i.e., cost learning effects are absent). The result in Case B can be considered as a main contribution of our paper. It yields a characterization of the optimal pricing, production, and inventory policies in a situation where both demand learning and cost learning are present. Admittedly, the kind of demand learning assumed is not the most general one since we have positive learning effects throughout the planning period. However, our assumption may be plausible in a short term planning problem for a product that has not yet reached saturation. Notice that to obtain the result in Case B we needed to strengthen our assumptions on the pricing component of the demand function so that our assumptions on this component now are

$$\frac{(g'(p))^2}{g(p)} \leqslant g''(p) < \frac{2(g'(p))^2}{g(p)}.$$

As mentioned in Section 2, the right-hand inequality is a standard assumption in pricing literature. The left-hand inequality puts a positive lower bound on the second order derivative of function $g(p)$. This excludes linear or concave demand functions. For the case of convex functions, the inequality may be interpreted as being a requirement of "sufficient" convexity. Since, $g'(p) < 0\ \forall p > 0$, the inequality implies that even for a very high price, some positive amount of demand remains. It is readily checked that the left-hand inequality is satisfied, for example, in the case of exponential and isoelastic functions. We now provide a qualitative characterization of the master trajectory.

Inventory is strictly decreasing along *R1* and *R2*. The reason is that due to the presence of an initial inventory, the current inventory remains sufficiently large. However, the firm wishes to reduce its inventory costs by selling from the inventory. On the supply side, the reduction of the initial inventory is accomplished by not producing at all along *R1*, and by having a low production level during an initial interval of time on *R2*. Moreover, along *R2* the production rate is less than the demand rate which also causes the inventory to decrease. In terms of production costs, such a production policy does not initially increase cumulative output very rapidly and hence does not exploit fully the cost learning effects. Note, however, that the production rate is increasing over time along *R2*. Recalling the assumption $c_Q + \beta > 0$ (which states that the marginal cost learning effect does not dominate the unit inventory cost) we see that our scenario is one in which holding an inventory is rather costly. It is for this reason that the firm employs various instruments in order to reduce the initial inventory. On the demand side, the desired reduction in inventory is accomplished by quoting a low price during an initial interval of time. Using a low price policy in the beginning of the planning period also serves the purpose of stimulating current sales with a view to increase cumulative demand, to obtain the benefits of the positive demand learning effects.

At the end of *R2*, the inventory is exhausted and during a final interval of time before the horizon date T, the firm applies *R3* on which the firm produces just to meet demand. In this way, the relatively high inventory costs can be escaped completely. Along *R3* it holds that $\dot{X} = k(X)g(p) = q = \dot{Q}$ and differentiating this expression with respect to time yields

$$\dot{q} = k(X)[k'(X)g(p)^2 + g'(p)\dot{p}]. \tag{15}$$

In the case of positive demand learning effects, Eq. (15) does not provide an unambiguous determination of the time derivatives of output and price along *R3*. (Note that along *R3* the optimal price and output rate are interdependent.) What we can say is the following. If price is decreasing over time, output is increasing. If output is decreasing over time, then price is increasing. However, if price [output] is increasing, nothing can be inferred about output [price].

In the case of no demand learning, the output rate and the price evolve in different directions over time. The reason is that an increasing price will decrease demand and therefore less output is needed. In the case of an additively separable production cost function we shall show in Appendix A that price is strictly increasing over time along *R3*. Hence, in the case of no demand learning and an additively separable cost function we obtain that the output rate is strictly decreasing on *R3*. The increase in price along *R3* continues the increase that was made along *R1* and *R2* but the output rate starts to decrease when the firm switches from *R2* to *R3*. The result obtained for the particular case of no demand learning and additively separable cost function confirms the pricing results of Clarke et al. (1982) and output results of Dotan et al. (1985). This is not unexpected since these papers did not include an inventory. (Recall that along *R3* there is no inventory.) To see the intuition behind the falling output and rising price, notice that we are considering a case in which cost learning only affects the fixed cost, not the instantaneous marginal cost of production. In such a case, larger current output may be desirable in order to obtain future reductions of fixed costs. Moreover, there are two incentives for choosing a decreasing production policy. The value of cost reductions decreases as we approach the end of the planning period, and increases in current production lead to higher marginal costs. Thus, increased production is not warranted and since the firm holds no inventory, demand must be restrained by a policy of increasing prices.

As already said, the optimal trajectory will be the master trajectory or a subsequence of this. It is easy to see that if the initial inventory is zero, the subsequence consisting of *R3* for all $t \in [0, T]$ is the optimal solution. For medium-sized initial inventory levels, the subsequence *R2* $\rightarrow$ *R3* is the optimal solution and for sufficiently large initial inventory levels, the master trajectory *R1* $\rightarrow$ *R2* $\rightarrow$ *R3* is the optimal solution.

Finally, one might ask what would happen if the assumption expressed by the inequality in Eq. (9) were dropped such that we could have $c_Q(Q, q) + \beta \leqslant 0$? Then the cost learning effect could be significant enough to outweigh the holding cost and now it could be optimal to have *R1* as a final regime even if $I_0 < X(T)$. In that case, to exploit such significant learning effects, the firm chooses larger production rates: indeed, production would be so large that the final inventory becomes positive. This is, however, not undesirable in view of the relatively small holding cost.

4. Conclusions

The main results obtained in this paper are stated and interpreted after Proposition 2 where we also pointed to the differences of our results compared to the literature. In Section 1 in particular we discussed how our model differs from what has been analyzed so far in the literature.

This paper has developed a dynamic model of pricing, production and inventory management which allows for learning effects both on the demand and the production side. Although the individual components of the model have been studied in separate streams of the literature, this paper is the first attempt to integrate demand learning, cost experience effects, and inventory management in one single model. We showed that within this more general framework, the conclusions of previous studies had to be modified. It is worth noticing that it was possible to give a characterization of the optimal policies without having to introduce very specific assumptions on the functional forms.

Some extensions of the model could be topics of future research. We assumed a short planning horizon, mainly in order to make plausible that saturation effects are not significant. The incorporation of such effects may be necessary, however, if one wishes to study the problem in a longer term perspective. (In this case, one should also introduce discounting.) It seems, however, that to study a model which also includes saturation effects one would have to introduce special functional forms for demand and cost functions and possibly rely on numerical solutions. We also assumed a dynamic cost function taking into account learning-by-doing effects, but the cost function does not allow for economies of scale that also may be important in a longer term perspective. The introduction of such a feature might have important impacts on optimal production policies, caused by the trade-off between increasing production to capture the benefits of economies of scale and incurring higher inventory costs.

Finally, we assumed that backlogging of demand is impossible. However, as in static inventory models, one can assume that backlogging is permitted at a cost. Introduction of backlogging means that one can discard the nonnegativity constraint on the state variable (the inventory level) in Eq. (6). One implication would be that the solution can be characterized in a more standard manner (in particular, avoiding the regime-synthesizing exercise). Backlogging was considered in Feichtinger and Hartl (1985) who used the cost function

$$h(0) = 0,\ h(I) > 0 \quad \text{for} \quad I \neq 0,\ h''(I) > 0,$$

which for $I > 0$ represents an inventory holding cost, for $I < 0$ a shortage (backlogging) cost. These authors dealt with both convex and linear costs. Incorporating backlogging in our setup would yield a model with two controls and two states, the latter evolving according to the dynamics in Eqs. (1) and (3). The analysis of such a model is outside the limits of this paper but presents an interesting avenue for future research.

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Appendix A

A.1. Proof of Proposition 2

The results on the development over time of price and production rate need no proof as they have already been established in Lemmas 1 and 2.

We start by showing that price is increasing on *R3* in the case of an additively separable cost function (cf. Case (i) in Eq. (8)). Along *R3* (in fact, along any regime) it holds that

$$\dot{\lambda} = \frac{g(p)g''(p) - 2(g'(p))^2}{(g'(p))^2}\dot{p} \Rightarrow \operatorname{sgn}(\dot{\lambda}) = -\operatorname{sgn}(\dot{p}). \tag{A.1}$$

From Eqs. (10) and (12) we derive

$$\dot{\lambda} = \frac{k'(X)(g(p))^2}{g'(p)} - \beta + v. \tag{A.2}$$

Since the production rate is strictly positive, Eq. (14) yields

$$\beta - v = \dot{q}c_{qq} + c_{Qq}q - c_Q. \tag{A.3}$$

Substitution of Eq. (A.3) into (A.2), and then into (A.1), yields

$$\left[\frac{g(p)g''(p) - 2(g'(p))^2}{(g'(p))^2}\right]\dot{p} = \frac{k'(X)(g(p))^2}{g'(p)} - \dot{q}c_{qq} - c_{Qq}q + c_Q. \tag{A.4}$$

Using Eq. (15) in Eq. (A.4) yields the following expression for the time derivative of p (where for simplicity of notation we have omitted the arguments of functions k, g, and c and their derivatives):

$$\dot{p} = \frac{k'g'(g)^2 - c_{qq}k'k(g)^2(g')^2 + c_Q(g')^2 - c_{Qq}(g')^2q}{gg'' - 2(g')^2 + c_{qq}k(g')^3}. \tag{A.5}$$

From Eq. (A.5) we conclude that price is strictly increasing on *R3* if $c_{Qq}=0$, which is the case when the cost function is additively separable.

Next we construct the master trajectory, working backward in time. Starting out on the final regime *R3* we first check which regime(s) can precede *R3*.

R1 $\rightarrow$ *R3*? We know that q is continuous which implies that $q=0, \dot{q} \geqslant 0$ must hold just after the time instant, say, t_{13}, at which *R1* passes into *R3*. Hence, at $t=t_{13}^{+}$ it must be true that

$$\dot{q}|_{q=0} = \frac{1}{c_{qq}}[c_Q + \beta - \nu] \geqslant 0 \Rightarrow c_Q + \beta \geqslant \nu \geqslant 0 \Rightarrow \beta > \nu. \tag{A.6}$$

From Eq. (15) we have $\dot{q}|_{q=0} = k(X)g'(p)\dot{p}$ which implies

$$\dot{p}(t_{13}^{+}) \leqslant 0. \tag{A.7}$$

Using Eqs. (A.1) and (A.7) yields $\dot{\lambda}(t_{13}^{+}) \geqslant 0$ and then (cf. (12))

$$\dot{\lambda}(t_{13}^{+}) = -k'(X)g(p)(p+\lambda) - \beta + \nu \geqslant 0.$$

This inequality cannot hold, however, since $p + \lambda > 0$ by Eq. (10) and $-\beta + \nu < 0$ due to Eq. (A.6). We conclude that *R1* cannot precede *R3*.

R2 $\rightarrow$ *R3*? This sequence is feasible since neither continuity requirements nor optimality conditions are violated.

Before proceeding to investigate which regime(s) can precede the sequence *R2* $\rightarrow$ *R3* it is convenient to characterize the evolution of the inventory along *R2*. In particular, we will show that during this regime, the inventory is decreasing in a convex way. From (1) and (3) we obtain

$$\frac{\mathrm{d}(\dot{Q}-\dot{X})}{\mathrm{d}t} = \dot{q} - k'(X)k(X)(g(p))^2 - k(X)g'(p)\dot{p}. \tag{A.8}$$

From Eq. (A.2) it follows (by strict positivity of inventory) that along *R2* it holds that

$$\dot{\lambda} = \frac{k'(X)(g(p))^2}{g'(p)} - \beta. \tag{A.9}$$

Using Eqs. (A.1) and (A.9) yields

$$\dot{p} = \frac{k'g'(g)^2 - \beta(g')^2}{gg'' - 2(g')^2}. \tag{A.10}$$

Substitution from Eq. (A.10) into Eq. (A.8) provides

$$\frac{\mathrm{d}(\dot{Q}-\dot{X})}{\mathrm{d}t} = \dot{q} + \frac{k'kg^2}{gg''-2(g')^2}[(g')^2 - gg''] + \frac{\beta k(g')^3}{gg''-2(g')^2}. \tag{A.11}$$

Eq. (A.11) shows that $\mathrm{d}(\dot{Q}-\dot{X})/\mathrm{d}t > 0$, that is, inventory $I(t)$ is convex in t. We already know that q is strictly increasing on *R2*.

We now return to the synthesizing procedure. Recall that we proved that the inventory is positive and convex decreasing in t on *R2*, and we know that inventory is zero on *R3*. Continuity of state and control variables implies that at the end of *R2*, the inventory decreases and approaches the level of zero in a tangential way. Now, since inventory is convex in t on *R2*, we can conclude that inventory decreases all along the interval of time on which *R2* is applied. On *R1*, inventory is positive and decreasing since production is zero. The next question to be answered is which regime(s) that can precede the string *R2* $\rightarrow$ *R3*.

R1 → *R2* → *R3*? No contradictions of continuity requirements and optimality conditions are found and hence this sequence is feasible.

R3 → *R2* → *R3*? Continuity of the state variables requires that the inventory is equal to zero at the beginning of *R2*. The proposed sequence is infeasible since we have proved that inventory decreases in a convex way along *R2* and hence it cannot be true that inventory is zero at the start of *R2*.

We proceed to verify which regime(s) can precede the string *R1* → *R2* → *R3*.

R2 → *R1* → *R2* → *R3*? Along *R2* we know that

$$\dot{q}|_{q=0} = \frac{1}{c_{qq}}[c_Q + \beta] > 0 \tag{A.12}$$

but if *R2* were to precede *R1*, continuity of the control q would require that $\dot{q}|_{q=0} \leqslant 0$ at the end of *R2*. This would, however, contradict the inequality in Eq. (A.12) and hence the proposed sequence is infeasible.

R3 → *R1* → *R2* → *R3*? Along *R1*, inventory is decreasing over time and it is readily concluded that the inventory is strictly positive for all t along *R1*. Thus, *R3* cannot precede *R1* and the proposed sequence is infeasible.

This completes the coupling procedure and we are left with a single feasible sequence, viz., the master trajectory *R1* → *R2* → *R3*. We know that the optimal trajectory is the master trajectory or a subsequence of this. Depending on the size of the initial inventory, two subsequences of the master trajectory can be optimal. If $I_0 = 0$, applying *R3* for all $t \in [0, T]$ is optimal. For "medium" sized initial inventory levels, the subsequence *R2* → *R3* is the optimal solution and for "large" initial inventory levels, the master trajectory itself is the optimal solution. □

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