Learning in Consumer Choice

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Universiteit van Tilburg, op gezag van de rector magnificus, prof. dr. F.A. van der Duyn Schouten, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de aula van de Universiteit op woensdag 16 januari 2008 om 14.15 uur door

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"The way that can be followed
is not the true way"

Lao Tzu

"Life can only be understood going backwards,
but it must be lived going forwards."

Søren Kierkegaard

"Der Weg entsteht im Gehen"

German Proverb

Preface

The aphorisms on the preceding page do not only apply to the subject of this dissertation, but certainly also to the writing of it. This thesis was not written with a big master-plan in mind. It was not until the final stages of writing this thesis that I could really see where I was going with this work (or where it was going with me), and what the purpose was of going there. As a consequence, writing this thesis has certainly not always been easy; it has been a long journey on roads that were from time to time quite bumpy. There were times when the end seemed far away and when the going got really tough, but I am happy that I kept going.

Still, throughout most of my time in Tilburg I felt fortunate and grateful to be a Ph. D. student. I found it something quite extraordinary and enriching. I liked teaching and I really enjoyed diving deep into my own thinking, and organizing and writing down my thoughts. And I feel that I have learned a lot while dedicating a large portion of my life to this work, also about myself, even (or maybe especially) from the setbacks and the difficult times. For instance, I have had to learn important lessons about patience, about trusting my intuition and my own inner process, about my doubting mind and about not giving up.

Therefore I want to express my gratitude to CentER as a whole for the wonderful opportunity to be a Ph.D. student. In particular I want to express my appreciation and gratitude to my supervisors Thijs ten Raa and Jan Potters. Thanks for giving me the space that I needed to follow my own path.

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Finally, I want to take this opportunity to thank my close friend Matthijs Spek and my family.

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Tilburg, October 2007.

Contents

1 Introduction

This dissertation is a mainly theoretical work. The questions that are raised in this work are fundamental questions about economic theorizing and modelling. Rather than focusing on the models that are erected from the building blocks that are used in economic theory (or on these models' predictions), this dissertation will focus on some of these building blocks themselves.

This dissertation will consider two fundamental parts of standard economic theory, i.e. two of the main building blocks that are used in economic modelling. This thesis will critically evaluate these standard building blocks and draw attention to some of their drawbacks. For this critical evaluation, we will again take a mainly conceptual point of view, rather than an empirical point of view. Then, this dissertation will attempt to provide an alternative building block that could again be used to build models from, and that does not (or not so much) suffer from these conceptual difficulties. Finally, this work will investigate the precise theoretical relations that exist between the standard and the alternative building blocks.

The two fundamental parts of standard economic theory, or the building blocks of economic modelling that were referred to, concern the models of consumer choice as used in neoclassical microeconomics, and the models of consumption/savings as used in macroeconomics.

The neoclassical model of consumer choice will be familiar to almost any economist, as it typically plays a very prominent role in microeconomics courses and in microeconomics textbooks (see e.g. Luenberger [28], Mas-Colell, Whinston and Green [30]). The second building block, the class of consumption/savings models, is actually a more specific case of the first. Still, in macroeconomics this more specific class of models has a special standing of its own, as it is one of the fundamental building blocks for macroeconomic modelling (see e.g. Romer [37]). This is why the two are mentioned separately here. Throughout this work we will present (sub)sections dedicated to consumption/savings models, that may be seen to function as more specific examples of the ideas that are developed in the main text that is set in the more general context of models of consumer choice.

Both models of consumer choice and models of consumption/savings decisions describe the choice problem for a consumer who has to make a decision as to how to spend his available resources (i.e. income) over all of the consumption opportunities that he will be confronted with presently and in the future. Both of these classes of models are based on viewing an economic decision-maker (or a consumer) as being rational, or as what is sometimes called a 'homo economicus'; standard economic theory assumes that a decision-maker, when confronted with a decision problem, will extensively consider all the options he can choose from (affordable consumption patterns or consumption plans), and all the possible consequences that may result from each of these options, and that he will finally choose that option that he thinks will give him his most preferred consequences. Thus, in this standard approach the objects of choice that a consumer is assumed to consider and choose from, are complete specifications of lifetime consumption.

The alternative building block that will be introduced in this dissertation attempts to provide a learning framework for consumer choice (and with it also for models of consumption/savings decisions). Now, why would we want to set up such a new learning framework? There are a number of reasons why this might be interesting.

Firstly, such a new framework might be interesting for its own sake, as this alternative approach may provide a new building block, that could be included in a modeller's toolbox, and from which (in principle) new types of models could again be constructed. While learning models have found their way into numerous areas of economics, such as game theory and macroeconomic dynamics, the same can hardly be said for the areas of consumer choice and consumption/savings models.[1] One reason for this may be that it is not at all straightforward how such a learning model of consumer choice should be set up, as some conceptual difficulties arise. Learning is usually modelled in settings where the same economic problem or situation is encountered repeatedly. For instance, in learning models in game theory it is typically assumed that the same game is repeatedly played. Then after every stage, the corresponding game is over, and the stage's payoffs are fully known. In principle, the same could be done in the context of consumer choice. We could assume that a consumer would go through a sequence of lifetimes, where in each of these lifetimes consumption decisions would have to be made, and where a consumer could learn from his experiences in previous lives. Of course, although such an approach may formally not be inconsistent, a model like that would probably not seem very convincing, as we only seem to live once.[2] Therefore, for learning consumption, what would really be needed is a framework in which learning takes place *within* a single decision problem or a lifetime. The difficulty then, is that choices in any period do affect the situation in all subsequent periods (how much is spent now will influence what will be affordable in all later periods), so that all the effects of a certain decision in a certain period will not become fully known until the very end of the lifetime. Our learning model should deal with this in one way or another. Thus, there are a few tough nuts to crack here, and from a conceptual point of view the problem of how to construct a learning model of consumer choice may by itself already prove to be quite interesting and rich. And although the idea of trying to model learning in consumer choice settings is not completely new[3], the approach towards learning consumption (and towards dealing with the above conceptual difficulties) that will be presented here is very different from other approaches and, to the best of our knowledge, it is new.

Secondly, in later chapters we will draw attention to some drawbacks associated

[1] A survey of related work is given in chapter 3.

[2] Even if reincarnation does exist, this type of learning would only work if consumers would remember the consumption decisions and their consequences from previous lives.

[3] We will review existing theoretical work on learning in consumption/savings models in section 3.6.

with the standard approach to consumer choice. There we will mainly focus on the very stringent conditions that these models pose on the cognitive abilities that a decision-maker should have in order to behave as these models suggest. Most notably, these cognitive abilities include being able to imagine everything that can happen in the (economic) future, and being able to deal with all this information about the future such as to arrive at an optimal choice. In any realistic description of a decision regarding an individual's (let alone a more complex household's) lifetime consumption pattern, the complexity of the choice problem faced is huge, both in terms of all the information that would be relevant for making such a decision, and in terms of the computational aspects of dealing with all this information. Thus there may be some tension between what economic models assume people to do, and what seems reasonable to expect them to be able to do.

Our new framework does not (as much) suffer from these drawbacks. In a learning model, a decision-maker does not consider and decide on complete lifetime consumption plans at once. In fact, learning seems to imply that there are multiple points in time in which choices will have to be made. Therefore, in each of these periods such a learning individual would only have to make consumption decisions regarding a limited time interval. This would require less ability to perform extensive surveys of what is yet to come, and correspondingly, a smaller computational burden.

A third reason, which is related to the second, is that we may wonder what links would exist between a learning framework and the standard framework. For instance, we could wonder whether making optimal consumption choices (as in the standard framework) could be learned. Milton Friedman [15] defended the assumptions of optimality in the context of consumption decisions (and thus here the standard framework) by arguing that agents could learn roughly optimal behaviour by a process of trial and error. Apparently Friedman felt that the standard framework needed some defending, quite possibly (partly) because of the aforementioned conceptual difficulties. More generally, many theorists feel that theoretical constructs such as economic equilibria (such as Nash equilibria), and similarly predicted behaviour resulting from optimization models, should be regarded as steady states of some dynamic system, rather than as being likely to come by in a one-shot situation. Thus, these constructs are judged to be more plausible when viewed as resulting outcomes in a dynamic setting (in which some form of learning may occur), rather than in a static one-shot setting (see e.g. Lucas [27]). In this dissertation we will formally provide such a dynamic setting where learning does occur, and we will extensively investigate whether, and if so when, we can establish that Friedman's claim, that roughly optimal consumption behaviour could be learned, will hold.

The alternative learning framework that will be presented here will also be called the 'ad hoc framework'. Obviously, as this new ad hoc framework tries to model learning about consumption, it starts from the idea that a decision-maker would cut up an all-encompassing decision of lifetime consumption into a series of subdecisions. That is, here we will model a consumer who would cut up his one lifetime into a number of periods, and who would try to learn from what he did in previous periods. Whereas in the standard model complete consumption horizons are decided upon at once, here it is assumed that in any such subproblem only a decision on present consumption and present savings is required. Then, in any such subproblem, decisions are supposed to be made much in the same vein as the standard microeconomic framework supposes the whole problem is dealt with: by means of (ad hoc) preference relations or (ad hoc) utility functions. However, now these ad hoc preference relations and ad hoc utility functions will not be defined on complete consumption horizons, but rather on combinations of present consumption and present savings. In any such subproblem decision-making is modelled as allocating the budget that is available in that particular subproblem over present consumption and present savings, such as to maximize ad hoc preferences.

However, these ad hoc preferences would have a somewhat different status than preferences in the standard framework. In the standard framework preferences are absolute, they are given and can never be wrong. In the ad hoc framework, preferences, and especially the implicit valuations for savings relative to present consumption, are not absolute and they can be viewed as guesses or as estimates. Then, some estimates may be better than others, and here the idea is that learning may improve these estimates over time. Thus, in the ad hoc framework we assume that essentially the valuations for money are learned over time.

The idea behind how this learning would take place is that if at a certain point in time a consumer would regret having spent too much in the recent past, he would adjust his valuations for money such as to value money more. And conversely, if at a certain point in time the consumer thinks that he could have spent more in the recent past, then adjusting would lead to lower valuations for money. A difficulty that arises here is how such retrospective evaluations of past expenditures should be established.

In a sense the present work is still somewhat sketchy. We do not want to present the new alternative framework as having a very definitive quality. Rather, this work should be seen as a theoretical exploration into largely uncharted territory. In setting up the new framework, many modelling choices have to be made, each of which may have considerable impact on the resulting framework and its implications. And although in instances where such modelling choices are needed we will try to justify or motivate the choices that we do end up making, there might of course be other ways to proceed that could also be fruitful. In the final chapter we will review some instances where certain modelling choices were made, and where certain alternative modelling choices could also have been made, and we will comment on how the framework and its' implications may change as a result of these alternative choices. Consequently, at this point we should probably still be careful to draw very firm conclusions about economic reality from this research.

Here we will present some notational conventions that will be used throughout this work. The set of strictly positive integers $\{1, 2, 3, ...\}$ is denoted by $\mathbb{N}$, the set of non-negative integers $\{0, 1, 2, 3, ...\}$ is denoted by $\mathbb{N}_0$, and $\mathbb{R}$ denotes the set of rational numbers. The set $\mathbb{R} \cup \{\infty\}$ is here called the **extended real numbers**, and denoted by $\overline{\mathbb{R}}$, and similarly $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, and $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$. For any $n \in \overline{\mathbb{N}}$, we denote the n-dimensional Euclidian space by $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$ and each $1 \leq i \leq n$ (or $1 \leq i < n$ if $n = \infty$), we will write $x^i \in \mathbb{R}$ to denote the i'th component of bundle x. The relation '$\geq$' is defined on the set $\mathbb{R}^n$ to mean that for all $x, y \in \mathbb{R}^n$ it holds that $x \geq y$ if in all dimensions i it holds that the corresponding components satisfy $x^i \geq y^i$. The relation $>$ is defined similarly on $\mathbb{R}^n$. By $\mathbb{R}^n_+$ we denote the non-negative Euclidian n-space $\{x \in \mathbb{R}^n : x \geq 0\}$, and by $\mathbb{R}_{++}$ we denote the strictly positive Euclidian n-space $\{x \in \mathbb{R}^n : x > 0\}$, where 0 simply denotes an n-dimensional vector of zeros. Consequently, $\mathbb{R}_+$ denotes the non-negative real numbers, $\mathbb{R}_{++}$ denotes the strictly positive real numbers, and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$.

Throughout this work we will refer to a consumer as a 'he', and similarly we will refer to 'his' preferences or 'his' choices. Of course the use of 'he' and 'his' could equally well be replaced by 'she' or 'her'.

This thesis consists of 10 chapters. Chapter 2 will present the preliminaries. This chapter will present the standard building blocks of the frameworks for consumer choice and for consumption/savings models. Also, this chapter will explicitly model time within the standard framework for consumer choice, and it will specify Expected Utility Theory, both in a general form and more specifically in the settings of consumer choice and consumption/savings. Also, this chapter will lay the formal groundwork that will be drawn from in later chapters. In chapter 3 the standard framework will be discussed and criticized, and the new, alternative approach towards modelling consumption behaviour that is chosen here, will be motivated. In chapter 4 a first component of the new ad hoc framework is introduced. This chapter will consider a single period in isolation, where it is modelled how in any such period choices would be made, namely by means of ad hoc preference relations and ad hoc utility functions. Chapter 5 investigates the links that exist between the concepts of ad hoc preference relations (and utility functions) as defined in the ad hoc framework, and total preference relations (and utility functions) as defined on complete consumption horizons in the standard framework. This chapter shows that the standard framework is in fact a special case of the ad hoc framework, by defining a specific way in which ad hoc preferences could be derived from total preferences, namely as summarizing total preferences consistently. Chapter 6 completes the ad hoc framework. First it provides a second component of the new ad hoc framework. It considers two subsequent periods, and it models how the ad hoc preferences in two such subsequent periods would be related, namely by means of a learning procedure by which valuations for savings from the previous period are adjusted into new valuations. These adjustments would depend on retrospective evaluations of past expenditures. Finally, this chapter completes the ad hoc framework, putting together all the modelling components, by considering lifetimes and formally modelling a learning algorithm. Chapters 7, 8 and 9 investigate Friedman's assertion that optimal behaviour could be learned. Chapter 7 considers convergence of sequences of ad hoc preferences, and it considers convergence towards optimality. This chapter shows that, given the specific adjustment procedure, ad hoc preferences will always converge, and it identifies conditions under which convergence towards optimality will occur. Chapters 8 and 9 are completely set in a consumption/savings setting, and these chapters show that under some (rather specific) conditions convergence towards optimality will occur in these settings. Chapter 8 deals with consumption/savings models under certainty, chapter 9 deals with expected utility models of consumption/savings decisions. The concluding chapter 10 looks back at previous chapters, and forward towards possible extensions and new research.

2 The standard framework for consumer choice

This chapter lays the formal groundwork for this thesis. The standard framework for consumer choice from microeconomics, and the standard framework for modelling consumption/savings decisions from macroeconomics are presented. There are a number of reasons why these standard frameworks are presented here. Firstly, the framework of learning consumer behaviour that is introduced in this dissertation will be contrasted with these standard static frameworks. Secondly, the standard frameworks will serve as a useful benchmark. And thirdly, many of the concepts and methods used in the standard framework will also be employed in the new, learning framework (although often in somewhat different ways).

In the first three sections the standard framework for consumer choice from microeconomics is presented. We consecutively present the objects from which this framework is constructed: preference relations, utility functions and basic consumer problems.

In the fifth section we will consider the standard framework for modelling consumption/savings decisions from macroeconomics, and the methods of dynamic programming that are very convenient for solving models in this specific class of models. The fourth section, then, tries to bridge the gap between the two frameworks, and establishes that, and in exactly what way, the second is actually a more specific case of the first.

Finally, in the sixth section choice under uncertainty is introduced. First the objects with which uncertainty is defined are presented in a very general and abstract way, after which they will be specifically interpreted in both standard frameworks.

2.1 Preference relations

The starting point in the formal models that accompany these analyses of consumer choice, are the elements of choice for the consumer. These objects of choice are called commodity bundles.

Definition 2.1.1 *A* ***commodity bundle*** *is an element of* $\mathbb{R}^n_+$*, for* $n \in \overline{\mathbb{N}}$*.*

Such a commodity bundle is an n-dimensional vector of non-negative amounts, with $n \in \overline{\mathbb{N}}$, and will typically be denoted by $x \in \mathbb{R}^n_+$. Here n is the total number of commodities that are (or will become) available, a commodity bundle can be thought of as a specification of the amounts of all available goods that a consumer owns or consumes. The set of all possible commodity bundles is called the commodity space.

Definition 2.1.2 *A* ***commodity space*** *is a set that consists of all commodity bundles.*

Typically, a commodity space will be denoted by $X = \mathbb{R}^n_+$.

The consumer can choose between elements from the set X. The set of all commodities may also be interpreted to include commodities that will become available at different points in time, and an identical good, but available at different points in time, may be seen as different commodities.

In order to model how our consumer will choose, he is assumed to have clear, stable preferences over elements in X. That is, he can compare pairs of elements from the commodity space and judge which element (commodity bundle) he prefers to own or consume. Formally, these assumptions are represented by the existence of a preference relation.

Definition 2.1.3 *A* ***preference relation*** *on a set X is a binary relation on X, that satisfies completeness and transitivity.*

Such a preference relation will typically be denoted by $\succsim$. Before the properties of completeness and transitivity are stated, we first take a look at the meaning of the concept of a binary relation.

A binary relation $\succsim$ is a relation between ordered pairs of elements (x, x'), with $x, x' \in X$ (ordered means that (x, x') is not the same as (x', x)). This means that if the elements in such a pair satisfy the binary relation $\succsim$, then they are related in some way, and we write $x \succsim x'$. Thus, each pair of elements may or may not satisfy this relation. The interpretation of such a relation $\succsim$ is as follows: if indeed the ordered pair (x, x'), with $x, x' \in X$, satisfies the relation $x \succsim x'$, then we will say that x is preferred to x', hence the name *preference* relation. Note, however, that such preferences should be interpreted in the weak sense of "at least as good as" ($x \succsim x'$ and $x' \succsim x$ is not impossible).

It is usual to extend the notation somewhat. We also write $x' \precsim x$ to mean $x \succsim x'$, and both notations may be interchanged. Furthermore, we write $x \succ x'$ and say that x is strictly preferred to x' if $x \succsim x'$ holds, and if $x' \succsim x$ does not hold. Similarly, we write $x \sim x'$, and say that x and x' are considered equivalent, or that the consumer is indifferent between x and x', if both $x \succsim x'$ and $x' \succsim x$ hold.

As we see in the definition, in order to qualify as a preference relation in the usual, microeconomic sense, the binary relation $\succsim$ must be assumed to satisfy the following properties.

Definition 2.1.4 *A binary relation $\succsim$ on a set X is called* ***complete*** *if for all pairs of elements $x, x' \in X$ we have $x \succsim x'$ or $x' \succsim x$ (or both).*

Definition 2.1.5 *A binary relation $\succsim$ on a set X is called* ***transitive*** *if for all $x, x', x'' \in X$ we have that $x \succsim x'$ and $x' \succsim x''$ also implies that $x \succsim x''$.*

The consumer was already assumed to be able to state a preference between some pairs of elements, and the completeness axiom now ensures that he can in fact do this for all pairs of elements. Thus it is assumed that for each pair of elements he can express whether he prefers the one or the other, or both (then he is indifferent, he finds them equally agreeable), and it cannot happen that the consumer is unable to compare the two.

The property of transitivity is related to the consistency and stability of preferences, it says that a combination of different preference statements always yields an ordering on the underlying set that is consistent. So the transitivity property makes sure that preference relations do not yield 'cycles' of preferences, where we would have elements $x, x', x'' \in X$, with $x \succ x'$, $x' \succ x''$, and $x'' \succ x$.

Another important assumption that a preference relation may satisfy is listed below.

Definition 2.1.6 *A preference relation $\succsim$ on a set X*[4] *is called* ***continuous*** *if for all $x' \in X$ the sets $\{x \in X : x \succsim x'\}$ and $\{x \in X : x' \succsim x\}$ are closed in X.*

In what follows, preference relations will usually be defined on some space $X = \mathbb{R}^n_+$, and we will simply assume that the Euclidian topology on $\mathbb{R}^n_+$ is used. In the instances where we have preference relations that are not defined on some multi-dimensional Euclidian space, we will come back to what topology is used.

The property of continuity essentially says that if a first element is strictly preferred to a second one, and we have a third element that is sufficiently close to the first, then the third will also be strictly preferred to the second. This axiom is a more technical assumption that enables us to use a much more convenient type of mathematics.

Other properties that a preference relation may or may not have are defined next.

Definition 2.1.7 *For a preference relation $\succsim$ on the space $X = \mathbb{R}^n_+$ (with $n \in \overline{\mathbb{N}}$), the commodity $j \in \mathbb{N}$, $j \leq n$ is called* ***weakly good*** *if for every $x \in X$, and for every $\lambda \in \mathbb{R}_+$, it holds that $x + \lambda 1^j \succsim x$. Here $1^j \in X$ denotes the vector with the j'th component equal to one, and all other components equal to zero.*

Definition 2.1.8 *A preference relation $\succsim$ on the space $X = \mathbb{R}^n_+$ (with $n \in \overline{\mathbb{N}}$), is such that the commodity $j \in \mathbb{N}$, $j \leq n$ is called* ***strongly good*** *if for every $x \in X$, and for every $\lambda \in \mathbb{R}_+$, it holds that $x + \lambda 1^j \succ x$.*

This property says that if, for $\succsim$ on the set $X = \mathbb{R}^n_+$, j is a good commodity if more of this commodity is always preferred to less.

2.2 Utility functions

Instead of using preference relations in economic analyses, for mathematical convenience these preferences are usually represented by means of utility functions.

Definition 2.2.1 *A* ***utility function*** *$u(.)$ is a function mapping a set X into the real numbers $\mathbb{R}$. Moreover, given a preference relation $\succsim$ on X, we say that the utility function $u(.)$* ***represents*** *the preference relation $\succsim$, if for all $x, x' \in X$ it holds that $u(x) \geq u(x')$ if and only if $x \succsim x'$.*

Such a utility function is typically denoted by $u : X \to \mathbb{R}$, and by $u(X) := \{\bar{u} \in \mathbb{R} | \exists x \in X : u(x) = \bar{u}\}$ we will denote the range of $u(x)$ as x varies over X.

A utility function expresses preferences in the sense that it gives for each element $x \in X$ a real number $u(x)$ that represents the utility that the consumer derives (or

[4] That is endowed with some topology.

thinks he will derive) from the element. So the consumer prefers one element to another if the one element will give a higher utility than the other.

The following theorem specifies when it is formally justified to use utility functions instead of preference relations.

Theorem 2.2.1 *If a preference relation $\succsim$ on the set X[5] is continuous, then it can be represented by a continuous utility function $u : X \to \mathbb{R}$.*

For a proof, see Rader [36].[6]

One feature of utility functions that follows immediately is that if the utility function $u(.)$ represents a consumer's preferences, and if $f : u(X) \to \mathbb{R}$ is a strictly increasing function, then we can see that $f(u(.))$ is also a utility function that represents the consumer's preferences. This implies that there is no cardinal meaning to the numbers $u(x)$ and $u(x')$, only an ordinal meaning. That is, only statements of the form $u(x) > u(x')$, $u(x) = u(x')$, or $u(x) < u(x')$ are really meaningful, and essentially only the sign of $u(x) - u(x')$ is informative, and not its magnitude. Some properties that utility functions may or may not satisfy are now listed.

Definition 2.2.2 *A utility function $u(.)$ on some space $X = \mathbb{R}^n_+$ (with $n \in \overline{\mathbb{N}}$) is called* ***strongly monotonic*** *if for all $x, x' \in X$ with $x' \geq x$ and $x' \neq x$ implies that $u(x') > u(x)$.*

Definition 2.2.3 *A utility function $u(.)$ on some space $X = \mathbb{R}^n_+$ (with $n \in \overline{\mathbb{N}}$) is called* ***quasi-concave*** *if for all $\bar{u} \in \mathbb{R}$ and all $x, x' \in X$, with $u(x) \geq \bar{u}$ and $u(x') \geq \bar{u}$ it holds that $u(\alpha x + (1-\alpha)x') \geq \bar{u}$, for all $0 < \alpha < 1$.*

Definition 2.2.4 *A utility function $u(.)$ on some space $X = \mathbb{R}^n_+$ (with $n \in \overline{\mathbb{N}}$) is called* ***strongly quasi-concave*** *if for all $\bar{u} \in \mathbb{R}$ and all $x, x' \in X$, that satisfy $x \neq x'$, $u(x) \geq x$ and $u(x') \geq x$, it holds that $u(\alpha x + (1-\alpha)x') > x$, for all $0 < \alpha < 1$.*

The strong monotonicity assumption says that no matter what someone owns, getting an additional amount of some of the commodities is always strictly preferred, so more is better. The two quasi-concavity assumptions essentially say that mixing commodities is never bad, or even good. That is, quasi-concavity says that mixtures of different goods are never worse than extreme outcomes where you get much of some goods and little of other goods, and strong quasi-concavity means that mixtures are always strictly preferred to extreme outcomes.

[5] That is endowed with some topology that has a countable base.

[6] This paper also shows that for every commodity space $X = \mathbb{R}^n_+$ (for $n \in \overline{\mathbb{N}}$) there exists a topology that has a countable base.

2.3 The basic consumer problem

The model of consumer choice is completed by the assumption that our consumer behaves as if he is perfectly rational: if he is faced with a certain set of possible consumption bundles that he can choose from, he will always choose a bundle that maximizes his preferences. That is, the consumption bundle he will end up choosing will always be a maximal element (if such a maximal element exists), i.e. an element that is preferred to every other consumption bundle he could have alternatively chosen. In terms of utility functions this assumption says that our consumer always chooses an element that maximizes utility over his choice set. So in maximizing utility the consumer is assumed to incorporate differences in all goods simultaneously (including all future commodities). Because of these assumptions we call our consumer a rational utility maximizer.

Whereas preference statements were only needed for pairs of commodity bundles, now we also implicitly assume the consumer to be able to make his preference judgements over all commodity bundles at once.

In the context of consumer choice, such a set of alternatives that can be chosen from would be a budget set.

Definition 2.3.1 *Given a commodity space $X = \mathbb{R}^n_+$, a* ***price vector*** *p is an element of $\mathbb{R}^n_+ \setminus \{0\}$, and a* ***budget*** *is a non-negative real number $m \in \mathbb{R}_+$. A* ***budget set*** *is a set of all commodity bundles $x \in X$ that are affordable, given prices and income(s).*

If for every available commodity our consumer knows the price that one unit of it will cost, then the price vector is an n-dimensional vector of non-negative numbers, where, for each $1 \leq i \leq n$ (or $1 \leq i < n$ if $n = \infty$), p^i is the price for one unit of the i'th commodity (note that we allow some but not all prices to equal zero). Given this price vector the monetary value of a commodity bundle $x \in X$ is given by $p \cdot x = \sum_i p^i x^i$. Hence, if the consumer has a budget of $m \geq 0$ monetary units to spend, then the corresponding budget set B of all of the commodity bundles that are affordable given prices p and income m can be denoted $\{x \in X : p \cdot x \leq m\}$.

Now we are ready to describe the basic consumer problem as maximizing utility over the budget set.

Definition 2.3.2 *Given a utility function $u : X \to \mathbb{R}$, and given a budget set $B \subset X$, the* ***basic consumer problem*** *is given by:*

$$\max_{x \in B} u(x)$$

In addition to the aforementioned assumptions about consumers' knowledge of all different commodities and about continuous preference relations, in using the above formulation to model consumer choice, two more implicit assumptions are that consumers know prices for all products (including all future commodities) and that they know the (lifetime) income they will be able to spend.

Now, for a consumer with utility $u(.)$ defined on the commodity space X, the solution to this basic consumer problem gives rise to the following functions, for every combination of prices p and budget m.

Definition 2.3.3 *Given a utility function* $u : X \to \mathbb{R}$, *prices* $p \in \mathbb{R}^n_+ \backslash \{0\}$ *and a budget* $m \geq 0$, *the* ***demand function*** $x(p, m)$ *is defined as:*

$$x(p,m) := \arg \max_{x \in \{x \in X : p \cdot x \leq m\}} u(x)$$

if this is well-defined.

The demand function $x(.,.)$ gives the commodity bundle that will solve the basic consumer problem corresponding to prices p and income m.

Definition 2.3.4 *Given a utility function* $u : X \to \mathbb{R}$, *prices* $p \in \mathbb{R}^n_+ \backslash \{0\}$ *and a budget* $m \geq 0$, *the* ***indirect utility function*** $v(p, m)$ *is defined as:*

$$v(p,m) = \sup_{x \in \{x \in X : p \cdot x \leq m\}} u(x)$$

The function $v(.,.)$ gives the maximal level of utility attainable given prices p and income m.

Note that if indeed $x(.,.)$ is well-defined for p and m, that is if the function u does assume a maximum on the set $\{x \in X : p \cdot x \leq m\}$, then we have $v(p, m) = u(x(p, m))$.

Now, if $u(.)$ is a differentiable function, and if $x(p, m) \in X$ is an internal solution to this constrained maximization problem, then by the Lagrange method there will be a Lagrange multiplier $\lambda \geq 0$ such that $x(p, m)$ is also an internal solution to the following unconstrained maximization problem: $\max_{x \in X} u(x) - \lambda(p \cdot x - m)$. The first order conditions for this last maximization problem are: $\frac{\partial u}{\partial x^i} = \lambda p^i, \forall i$, and $p \cdot x - m = 0$, and consequently these conditions also define the solution to our original problem. So in this setting, we see that the quantities of the commodities that the consumer will choose will be such that all marginal utilities are proportional to the corresponding prices. This implies that for every commodity i the Lagrange multiplier will be equal to the quotient of the marginal utility for this commodity and its price: $\lambda = \frac{\partial u}{\partial x^i} \frac{1}{p^i}$ for all $i \in \{1, 2, ..., n\}$. Therefore in an internal solution to the basic consumer problem it holds that $MRS_{ij} = \frac{\partial u / \partial x^i}{\partial u / \partial x^j} = \frac{p^i}{p^j}$ for all i and $j \in \{1, 2, ..., n\}$. So for any pair of commodities, the marginal rate of substitution between these goods is equal to the ratio of their prices. In a two dimensional case ($n = 2$) this equality of marginal rate of substitution and price ratio reflects the fact that the indifference curve that the solution point lies on $\{x \in X : u(x) = v(p, m)\}$ touches the budget line $\{x \in X : p \cdot x = m\}$ exactly in this solution point.

The above expressions also reflect simultaneity in the maximization process. In this situation, in determining how much of a good to buy, its benefits and costs are compared to the benefits and costs of all other goods simultaneously.

From the above equations we also see that the Lagrange multiplier λ plays a central role in determining the optimal values for all the x^i's. Now using the envelope theorem also gives us that $\frac{\partial v}{\partial m}(p, m) = \lambda$, that is, the Lagrange multiplier λ equals the derivative of the indirect utility function with respect to m at the optimum. So apparently the Lagrange multiplier equals the marginal utility of budget or money, and $\lambda = \frac{\partial u}{\partial x^i}\frac{1}{p^i}$ thus shows us that the quotient of the marginal utility with respect to a commodity and its price, must in an optimum be equal to the marginal utility of the budget m.

2.4 Modelling time

Thus far we have considered the standard framework for consumer choice from microeconomics. Later in this chapter we will consider the standard framework for consumption/savings decisions as used in macroeconomics. We will also see that indeed the second framework is a more specific example of the first, as stated in the introductory chapter. This section, then, tries to bridge the gap between the two frameworks, thereby establishing precisely in what way the second framework is a special case of the first.

While the generality of the above microeconomic framework is certainly one of its strengths, it also seems clear that this framework is so general that it does not permit many investigations apart from strictly theoretical ones. For empirical investigations into consumer spending behaviour the above framework with an appropriately chosen utility function could probably explain almost anything and therefore essentially not much. Thus, to be used in empirical applications the framework will necessarily have to be made much more specific. Therefore in order to bring consumer choice theory to the data a lot more structure will have to be imposed. Also, from a theoretical point of view, adding some structure to the model may greatly enhance tractability.

To motivate one of the directions in which the framework can be made more specific, first it is worth noting that in the microeconomic framework as presented above, time does not actually play any role. Although the commodities may not become available at the same points in time, in that framework this is irrelevant because at the first moment in time that a decision is required with respect to some of the commodities, all other commodities that will become available later are supposed to be known. Therefore, stating preferences regarding the first group of commodities should also be dependent on considerations regarding the other commodities that will become available later.

Thus, a first way to impose more structure on the framework could be to explicitly model time. In the context of consumer choice, adding a time structure may seem quite natural, since consumption behaviour is stretched out over a lifetime, and time considerations may seem important.

Therefore, to become more formal, here we introduce a discrete way to model time: a discrete time variable t is introduced, that progresses through one of the two sets $\{0, 1, 2, 3, ..., T\}$, if we want to model a situation with a finite number of periods, or $\{0, 1, 2, 3, ...\}$, if we want to model an infinite number of periods. Thus there is a finite or countably infinite number of periods.

Thus far preferences were defined on the total commodity space $X = \mathbb{R}^n_+$, where $n \in \overline{\mathbb{N}}$ denoted the total number of commodities that were assumed to become available for purchasing at some point in time. Here we may also assume that the consumption opportunities do not present themselves simultaneously, but rather according to the above discretization of time t. That is, at every stage t, by $X_t = \mathbb{R}_+^{n_t}$ we denote the space of all combinations of amounts of the commodities available at time t, where $n_t \in \mathbb{N}$ denotes this total number of commodities available at stage t. So corresponding to the subdivision of time, the total commodity space X is subdivided into $T+1$, or an infinite amount of distinct period-t commodity spaces X_t. And thus we would get that the total commodity space X is now a Cartesian product of all period-t commodity spaces

$$X = \times_t X_t = \mathbb{R}_+^n,$$

with $\sum_t n_t = n$.

Similarly, commodity bundles are decomposed into period-t commodity bundles: any $x \in X$ can be written as $x = (x_0, x_{1,} x_{2,} ..., x_T)$, or $x = (x_0, x_{1,} x_{2,} ...)$, where for any t the period-t consumption bundle is denoted $x_t \in X_t$.

2.4.1 Independence of preference relations

Of course, preference relations are still only defined on the total commodity space X. And although we will keep assuming this, we *can* now, along with the above decompositions over time, add more structure to such preference relations. In order to formally do this, some properties that a preference relation may satisfy, which are related to the subdivision of the total commodity space into smaller sets, are introduced in this subsection.

An important feature that a preference relation may or may not satisfy is that of independence. To be able to define this, first suppose we would for some reason like to partition the set of all available commodities into two disjoint subsets of commodities. Then, along with this partition of the set of available commodities, we can write the overall commodity space X as the Cartesian product of the respective commodity spaces X_1 and X_2 that correspond to the two subsets of commodities. Note that this situation would simply correspond to a model as in the preceding subsection, with only two periods.

And suppose that the preference relation $\succsim$ is defined on $X = X_1 \times X_2$. Then from the preference relation on X, a new preference relation on X_1 can be derived, by conditioning on a certain choice of an element $x_2 \in X_2$ (or vice versa a preference relation on X_2 can be derived, by conditioning on an element $x_1 \in X_1$).

Definition 2.4.1 *Given a preference relation $\succsim$ on $X = X_1 \times X_2$, and given an element $x_2 \in X_2$, the* ***conditional preference relation*** *$\succsim_{x_2}$ on X_1 is determined by: $x_1 \succsim_{x_2} x_1'$ if and only if $(x_1, x_2) \succsim (x_1', x_2)$, for all $x_1, x_1' \in X_1$.*

The new preference relation $\succsim_{x_2}$ is obtained from the original preference relation $\succsim$ by holding a certain element $x_2 \in X_2$ fixed. Given that this element x_2 is fixed, $\succsim_{x_2}$ can now be seen as a relation that gives preference statements for all elements in X_1.

It is easy to see that $\succsim_{x_2}$ is indeed a preference relation on X_1 (that is, $\succsim_{x_2}$ satisfies completeness and transitivity) because $\succsim$ is also a preference relation (and therefore satisfies completeness and transitivity). Hence for every $x_2 \in X_2$ we see that $\succsim$ on X

also defines a preference relation $\succsim_{x_2}$on X_1. In general, the precise specification of $\succsim_{x_2}$ will depend on the particular element $x_2 \in X_2$. In some cases, however, $\succsim_{x_2}$ will be the same for any element $x_2 \in X_2$.

Definition 2.4.2 *For a preference relation $\succsim$ on $X = X_1 \times X_2$, we say that x_1 is* ***independent*** *of x_2 in $\succsim$, if the conditional preference relations $\succsim_{x_2}$ are identical for all conditioning choices of $x_2 \in X_2$.*

Hence independence means that the particular choice of x_2 that would be made from X_2, would not affect relative preferences for the goods in X_1.

Similarly, as in the previous subsection, the set of all available commodities can be partitioned into $T \in \mathbb{N}$ or a countably infinite number of subsets, or categories, of commodities. And along with such a partition of the set of available commodities, we can write the overall commodity space X as the Cartesian product of the respective commodity spaces X_t that correspond to the subsets of commodities. Again, we suppose that the preference relation $\succsim$ is defined on $X = \times_t X_t$.

Now suppose that S is some proper subset of the set of categories of commodities $\{1, 2, ..., T\}$, or $\{1, 2, ...\}$ and let S^c denote its complement. Also, let X_S and X_{S^c} denote the Cartesian products of all the distinct commodity spaces corresponding to all the elements in S, and to S^c, respectively. Then for every $x_S \in X_S$ we can define the relation $\succsim_{x_S}$on X_{S^c}, by $x_{S^c} \succsim_{x_S} x'_{S^c}$ if and only if $(x_S, x_{S^c}) \succsim (x_S, x'_{S^c})$, for all $x_{S^c}, x'_{S^c} \in X_{S^c}$. The new preference relation $\succsim_{x_S}$ is again obtained from the original preference relation $\succsim$ by holding a certain element $x_S \in X_S$ fixed. And again, $\succsim_{x_S}$ is indeed a preference relation on X_{S^c}. This allows us to impose more structure on preference relations, which will prove very useful later on.

Definition 2.4.3 *A preference relation $\succsim$ on $X = \times_t X_t$ is called* ***strongly independent*** *with respect to the corresponding partition of commodities if, for any proper subset S of the set of partition indices t the variable x_S is independent of x_{S^c} in $\succsim$.*

2.4.2 Separability of utility functions

Utility functions, like preference relations, are defined on the total commodity space X. And, similar to preference relations, we can now use the extra structure we imposed on X, to add more structure to utility functions.

As in the previous subsection, we first suppose here that the set of all available commodities is partitioned into two disjoint subsets of commodities, and that, along with this partition, we can write the overall commodity space X as the Cartesian product of the respective commodity spaces X_1 and X_2 that correspond to the two subsets of commodities. We can again use this extra structure such as to impose more structure on utility functions that are defined on the overall set X.

Definition 2.4.4 *The utility function $u(.,.)$ defined on $X = X_1 \times X_2$, is called* ***separable*** *in x_1 if $u(x_1, x_2)$ can be written as $u(x_1, x_2) = U(v(x_1), x_2)$, for certain functions $v : X_1 \to \mathbb{R}$ and $U : v(X_1) \times X_2 \to \mathbb{R}$,[7] such that $U(v, x_2)$ is strictly increasing in v.*

[7] $v(X_1) = \{\bar{v} \in \mathbb{R} | \exists x_1 \in X_1 : v(x_1) = \bar{v}\}$ is the range of v.

Separability says that there is a separate function measuring the (sub)utility from the separable commodities, such that overall utility is then only determined from the amounts of the other commodities, and the level of subutility obtained from the separable commodities. Notice that this subutility function for the separable commodities is independent of all of the other commodities.

The set of all available commodities can also be partitioned into $T \in \mathbb{N}$ or a countably infinite number of subsets of commodities, so that the overall commodity space X can be written as the Cartesian product of the respective commodity spaces X_t. This allows for adding more structure to utility functions $u(.)$ that are defined on $X = \times_t X_t$.

Definition 2.4.5 *The utility function $u(.)$ defined on some set $X = \times_t X_t$, is called **additively separable** with respect to the corresponding partition of commodities, if $u(x)$ can be written as $u(x) = \sum_t u_t(x_t)$, for certain functions $u_t : X_t \to \mathbb{R}$.*

Additive separability is strong property that says that utility is separable in every x_t (in the usual sense), so for every t there exists a period-t utility function $u_t(x_t)$, that only depends on period-t consumption, and not on consumption in other periods. Moreover, the overall utility function (that weights all of these subutilities u_t) is additive in all the u_t's.

The following proposition links the properties of separability of utility functions with the properties of independence of preference relations.

Theorem 2.4.1 *(A)Let $\succsim$ be a preference relation on $X = X_1 \times X_2$, such that x_1 is independent of x_2 in $\succsim$, and let $u : X \to \mathbb{R}$ be a utility function representing $\succsim$. Then u will be separable in x_1.*

(B) Suppose that $\succsim$ is a preference relation on $X = \mathbb{R}^n_+$ (for $n < \infty$) that is continuous and strongly monotonic. Then $\succsim$ is strongly independent with respect to the partition of commodities corresponding to $X = X_1 \times X_2 \times ... \times X_T$ if and only if every utility function $u : X \to \mathbb{R}$ that represents $\succsim$ is additively separable (with respect to the same partition).

For proofs see Luenberger [28].

The first part of the theorem indicates that the separate (sub)utility function $v(.)$ represents the separate conditional preferences $\succsim_{x_2}$ on X_1.

2.4.3 The basic consumer problem

If the total commodity space $X = \mathbb{R}^n_+$ (with $n \in \overline{\mathbb{N}}$) can be written as the Cartesian product of all the per period commodity spaces $X = \times_t X_t$, and commodity bundles can be decomposed into period-t commodity bundles $x = (x_0, x_1, x_2, ..., x_T)$, or $x = (x_0, x_1, x_2, ...)$, then accordingly price vectors $p \in \mathbb{R}^n_+ \backslash \{0\}$ for all goods in X can be decomposed into period-t prices $p_t \in \mathbb{R}^{n_t}_+ \backslash \{0\}$, that denote the corresponding prices for the goods in X_t, so $p = (p_0, p_1, ..., p_T)$, or $p = (p_0, p_1, ...)$.[8] Then, the budget constraint $p \cdot x \leq m$ would in this more specific case read $\sum_t p_t \cdot x_t \leq m$.

However, now that we have modelled time explicitly, this also allows for a situation where income is not yet completely owned in period 0, but to be received as an income *stream*. We could model this as a situation where $m_0 \geq 0$ is the initial endowment owned in period 0, but where in every subsequent period an additional income $I_t \geq 0$ will be earned. Because we are still assuming certainty here, this model with an income stream, can also be captured by the former model without income streams, as long as total income is finite and if it is assumed that there is a perfect capital market. That is, if our consumer is able to borrow and save at a (constant) interest rate $r \geq 0$, then by certainty, this model can be captured by a model without income streams $I_t = 0$, but with initial endowment $m_0' = m_0 + \sum_t R^{-t} I_t$, where $R = 1 + r$. If there is no perfect capital market in place, then this equivalence will not hold. For instance, if we consider a situation with an income stream, where it is possible to save at the constant interest rate r, but where it is impossible to borrow, then the situation becomes more complicated because the budget set is no longer defined by just one budget constraint, but now by $T+1$ or an infinite number of budget constraints. These budget constraints would read $p_0 \cdot x_0 \leq m_0$, and $p_t \cdot x_t \leq m_t$, for all t, where m_t would be an implicit period-t budget, given the actual choices made in previous periods x_i for all $i < t$, as defined by:

$$m_t := R^t m_0 + \sum_{i=1}^{t} R^{t-i} I_i - \sum_{i=0}^{t-1} R^{t-i} (p_i \cdot x_i) \qquad (✠)$$

with $R = 1 + r$. In period t, the effective period-t budget constraint would read $p_t \cdot x_t \leq m_t$, which could also be rewritten as

$$\sum_{i=0}^{t} R^{t-i} (p_i \cdot x_i) \leq R^t m_0 + \sum_{i=1}^{t} R^{t-i} I_i.$$

Then, with additively separable utility and an income stream, the basic consumer problem would amount to

$$\max_x \sum_t u_t(x_t) \text{ sub to } p_t \cdot x_t \leq m_t, \text{ for all } t,$$

where m_t is the implicit budget as defined by formula (✠).

Note that this specification also still incorporates the possibility of only one budget constraint, by setting $I_t = 0$, for all $t > 0$, so that all different budget constraint are captured by: $\sum_t R^t (p_t \cdot x_t) \leq m_0$.

[8] Note that the condition that the overall price vector satisfies $p \in \mathbb{R}^n_+ \setminus \{0\}$ is not completely equivalent to $p_t \in \mathbb{R}^{n_t}_+ \setminus \{0\}$, for all t, therefore we have to slightly restrict the overall price space here.

2.4.4 Consumption levels

The above formulation immediately brings to light that under additively separable utility, the basic ad hoc consumer problem can be subdivided into distinct subproblems: firstly, the subdecision of choosing the levels of period-t consumption $c_t \in \mathbb{R}_+$ that determines how much can be spent in all periods t, and secondly, the subsequent subdecisions of allocating each of these c_t's over all of the commodities available in the corresponding period. If our consumer is indeed a rational utility maximizer, then in the second stage of the problem he will choose to allocate the period-t budgets c_t amongst the period-t commodities in an optimal manner. That is, given prices p_t and the period-t budget c_t, the period-t bundle $x_t(p_t, c_t)$ that he will end up choosing will be a solution to the subproblem of maximizing period-t utility over the period-t budget set:

$$x_t(p_t, c_t) \in \arg\max_{x_t} u_t(x_t) \text{ sub to } x_t \in X_t, p_t \cdot x_t \leq c_t.$$

And thus, under the assumption that the decision-maker would behave optimally in the second type of subproblems, we see that this optimal second stage utility would be given by: $u_t(x_t(p_t, c_t)) = v_t(p_t, c_t)$. Here $v_t(p_t, c_t)$ denotes the indirect utility function as derived from u_t, given prices p_t and the period-t budget c_t. Now, given that saving is possible at interest rate r and borrowing is not possible, if we write the profile of consumption levels as $c = (c_0, c_1, ..., c_T)$ or $c = (c_0, c_1, ...)$, then using this notation we can represent the first subdecision as

$$\max_c \sum_t v_t(p_t, c_t) \text{ sub to } c_t \leq m_t, \forall t.$$

Here m_t again denotes the implicit period-t budget from formula (✠). And under certainty we may regard all prices p_t as exogenous, therefore if, for every t, we define the function $\tilde{u}_t(c_t) := v_t(p_t, c_t)$, then we can let the first subdecision be represented by

$$\max_c \sum_t \tilde{u}_t(c_t) \text{ sub to } c_t \leq m_t, \forall t,$$

with m_t as in formula (✠).

It is exactly this first type of subdecisions that has been studied extensively in macroeconomics, both theoretically and empirically, where it is quite common to forget about the second type of subproblems and just focus on the first type of subdecisions. This is one of the reasons why in macroeconomics problems like these are typically just presented in their own right, without referring back to the 'original' problem that starts from commodity spaces. That is, in typical consumption/savings models from macroeconomics, preferences are typically expressed as $u(c) = \sum_t u_t(c_t)$, where $u_t : \mathbb{R}_+ \to \mathbb{R}$ is simply a function that directly gives utility from a level of period-t consumption c_t, rather than an indirect utility function that is derived from a more primitive utility function defined on some underlying commodity space. Here we will also follow this example in the subsections dealing with consumption/savings models.

Note that this type of situations, where utility is derived directly from consumption levels, can also be accounted for within the general microeconomic framework by simply assuming that all the period-t commodity spaces $X_t = \mathbb{R}_+$ are one-dimensional to begin with, so $X = \mathbb{R}_+^T$ (or $X = \mathbb{R}_+^\infty$, in the infinite case). Then, if all the $u_t(x_t)$'s are strongly monotonic functions, and all prices are set to unity $p_t = 1$, for all t, then we get that the distinction between direct utility and indirect utility is no longer necessary, so v may be replaced by u. Indeed, then we get

$$\tilde{u}_t(c_t) = v_t(p_t, c_t) = \max_{x_t : p_t x_t \leq c_t} u_t(x_t) = u_t(c_t/p_t) = u_t(c_t).$$

So in this case there would be no need for making reference to or worrying about underlying commodity spaces.

2.4.5 Exponential discounting

With this specification of utility we are getting closer to the type of models used in macroeconomics to study consumption/savings decisions. However, a setting like this is still too general to be of much use in empirical investigations. If we do not have a way in which the different functions $u_t : \mathbb{R}_+ \to \mathbb{R}$ can precisely be related, then just about anything can be explained from the above model by choosing the appropriate (or actually inappropriate) u_t functions.

So assumptions are needed on how the different period-t utility functions are related. Recall that each of the period-t utility functions $u_t(c_t)$ are defined on the same set $\mathbb{R}_+$. Moreover, these variables c_t have the same interpretation of levels of consumption in the concerning periods, which will simply be expressed in monetary terms. That is, each of the variables c_t can be measured identically.

A quite straightforward and popular way to relate these different period-t utility functions, is by introducing a discounting function $d(.)$, and basically assuming that the utility functions are the same for all periods, except for some discount factors $d(t)$, so that $u_t(c) = d(t) \cdot u_0(c)$. These discount factors can be said to represent time preference, which measures the difference in valuations of consumption in different periods. Typically we would expect that individuals would prefer consumption now over (the same amount of) consumption in the future. The most standard choice of these discount factors is that of exponential discounting.

Definition 2.4.6 *An additively separable utility function $u(c) = \sum_t u_t(c_t)$, satisfies **exponential discounting** if $u_t(c) = \delta^t \cdot u_0(c)$, for all t, and all $c \in \mathbb{R}_+$, and some discount factor $0 < \delta < 1$. The period-0 utility function $u_0(.)$ is called an **instantaneous utility** function.*

Hence, if such an additively separable utility function $u(c) = \sum_t u_t(c_t)$ satisfies exponential discounting, then we see that

$$u_{t+1}(c) = \delta^{t+1} \cdot u_0(c) = \delta \cdot \delta^t \cdot u_0(c) = \delta \cdot u_t(c).$$

Thus, the discount function $d(t)$ is a function that decreases exponentially in t: $d(t) = \delta^t$, where δ is a constant per-period discount factor. Essentially, exponential discounting assumes that money can buy the same levels of utility in every period (except for the decreasing desirability of consumption over time, which is only due to time discounting). And a consequence of exponential discounting, is that discounting between any two dates that are equally far apart is the same.

Under exponential discounting, a typical utility function will take the form: $u(c) = \sum_t \delta^t u_0(c_t)$.

2.5 Consumption/savings models and Dynamic programming

Thus far in this chapter we have considered the standard framework for consumer choice from microeconomic theory, as mentioned in the introductory chapter, and some ways to make this general framework more specific. The most important of these specifications are explicitly modelling time, assuming additive separability of utility functions with respect to time, and exponential discounting. And with these specifications, we have arrived at a basic consumer problem that can mathematically be expressed as

$$\max_c \sum_t \delta^t u_0(c_t) \text{ sub to } c_t \leq m_t, \forall t.$$

Here m_t denotes the implicit period-t budget as in formula (✠).

And with this class of models, we have arrived at the second fundamental part of economic theory we wanted to consider: the macroeconomic models of consumption/savings decisions. These consumption/savings models typically take the above form, or something similar. Thus, the preceding section has shown that (and in what way) the macroeconomic framework of consumption/savings is indeed a more specific case of the microeconomic framework of consumer choice, as also stated in the introduction.

But before we will take a look at the methods of dynamic programming that are used to solve consumption/savings models, we will still make a number of additional assumptions.

Firstly, from now on we will assume that the number of periods is (countably) infinite: $T = \infty$. This implies that preferences are represented by an infinite sum of discounted utilities: $u(c_0, c_1, ...) = \sum_{t=0}^{\infty} \delta^t u_0(c_t)$. We make this assumption for two reasons, the first of which being mathematical convenience. The second reason is that in this work we are trying to set up a learning model of consumer choice, and investigate the convergence properties of such a model. And in order to be able to study convergence we will generally need an infinite number of periods (we will come back to these issues later). And although this assumption may not seem completely realistic, note that if the discount rate $0 < \delta \leq 1$ that reflects pure time preference is given, we could additionally also assume that after every period it is only with probability $0 < \theta < 1$ that there will be a next period (so after any period with probability $(1 - \theta)$

this period may turn out to have been the last). We will see later that if our decision-maker is an expected utility maximizer, then his preferences may be represented by

$$E[u(c_0, c_1, ...)] = \sum_{t=0}^{\infty} (\delta\theta)^t u_0(c_t),$$

so then the combined factor $\delta\theta$ may also be used as a new discount factor to model this situation.

Secondly, we will assume that models are stationary. We want to model an infinite number of periods, in which case stationary problems are much more easy deal with (analytically). In the framework we are considering here, stationarity would mean that when income is not yet completely owned in period 0, but to be received as an income stream, then this income stream should be constant. That is, we need to have a situation with an initial endowment $m_0 \geq 0$, and where in every subsequent period the additional income I_t that will be earned, will have to be constant: $I_t = I \geq 0$, for all $t > 0$.

Thirdly, for notational simplicity, we will from now on assume that no interest is obtained from saving: $r = 0$, so $R = 1$. The analysis in this work can also be extended to allow for (constant) non-zero interest rates. However, conceptually the analysis would remain the same, and as it would make notation more complicated, we opt for a zero interest rate. Under this assumption, the mathematical expression of the consumer problem simplifies to

$$\max_c \sum_t \delta^t u_0(c_t) \text{ sub to} \sum_{i=0}^{t} c_i \leq m_0 + tI, \forall t.$$

2.5.1 Dynamic Programming

We saw that in macroeconomics consumption/savings models are usually represented by the maximization of a utility function that is additively separable over time and that satisfies exponential discounting. In this subsection we deal with how these problems can be *solved*. So in a sense, this subsection is more about mathematics than it is about economics. Still, the ideas and methods found here are also relevant for subsequent chapters.

Sequence problems In mathematics, the problem of maximizing an additively separable utility function that satisfies exponential discounting has a name of its own, and will be contrasted to another method later.

Definition 2.5.1 *Given instantaneous utility u_0, a discount factor $0 < \delta < 1$, and an income stream $(m_0, I, I, ...)$, the problem of solving*

$$\max_{(c_0, c_1, ...)} \sum_{t=0}^{\infty} \delta^t u_0(c_t) \ \textit{sub to} \ \sum_{i=0}^{t} c_i \leq m_0 + tI, \textit{ for all } t \geq 0,$$

is called a ***sequence problem****.*

Functional equations In principle we could attempt to solve a sequence problem directly by means of the Lagrange method as in section 2.3. However, in section 2.3 a single budget constraint was faced, so that only one Lagrange multiplier needed to be considered. Here we see that with $I > 0$ an infinite number of budget constraints are faced, so that an infinite number of Lagrange multipliers are needed. Therefore in general the Lagrange method is not very practical, and something else will have to be attempted.

The way to proceed would typically involve taking a recursive approach and using the fact that by stationarity, the problem faced in period 1 can be seen to be a copy of the one faced in period 0. In the above sequence problem our consumer is supposed to choose an infinite sequence of consumption levels $c = (c_0, c_1, c_2, ...)$ in (or before) period 0. In contrast to this approach, we can also think of the problem that a decision-maker faces at period 0, as one of simply dividing his resources m_0 between consumption c_0 and savings $s_0 = m_0 - c_0$ which, since here there is no budgetary uncertainty, would completely determine next period's budget $m_1 = s_0 + I = m_0 - c_0 + I$. Then, the decision of how to allocate m_1 could wait until period 1.

Given m_1, the maximal lifetime utility that could be obtained in all periods after time 0, would be given by $\max_{(c_1,c_2,c_3,....)} \sum_{t=1}^{\infty} \delta^{t-1} u_0(c_t)$ s.t. $\sum_{i=0}^{t} c_i \leq m_1 + (t-1)I$, for all $t \geq 1$. Now suppose that the value function $V^* : \mathbb{R}_+ \to \mathbb{R}$, would return for every budget level m_1 exactly the above maximum utility value that could be attained from budget m_1 from period 1 onwards, discounted to period-1 terms:

$$V^*(m_1) := \max_{(c_1,c_2,c_3,...)} \sum_{t=1}^{\infty} \delta^{t-1} u_0(c_t) \text{ s.t. } \sum_{i=1}^{t} c_i \leq m_1 + (t-1)I, \text{ for all } t \geq 1.$$

Then the problem faced at stage 0 could alternatively be represented as

$$\max_{(c_0,s_0)} [u_0(c_0) + \delta V^*(s_0 + I)] \text{ s.t. } c_0 + s_0 \leq m_0.$$

Since this is a maximization problem in only two variables, this looks a lot easier. Of course the question remains how to find such a value function V^*.

Now, by stationarity ($I_t = I$) the consumer's problems at dates 0 and 1 can be seen to be copies. That is, a change of variables shows that the value function V^* evaluated at m_0 can also be expected to give the maximum level of total lifetime utility attainable from this initial budget, that is: $V^*(m_0) = \max_{(c_0,c_1,c_2,...)} \sum_{t=0}^{\infty} \delta^t u_0(c_t)$ sub to $\sum_{i=0}^{t} c_i \leq m_0 + tI$, for all $t \geq 0$. Thus we would expect that the maximal utility levels would be the same in the two maximization problems, i.e. that: $V^*(m_0) = \max_{(c_0,s_0)}[u_0(c_0) + \delta V^*(s_0 + I)]$ sub to $c_0 + s_0 \leq m_0$. Indeed, mathematically this would be due to the fact that maximizing over all variables simultaneously would have to give the same result as first, for any given choice of c_0, determining the maximum over $(c_1, c_2, ...)$ while holding c_0 fixed, and then maximizing this conditional maximum over c_0. Again using the fact that the problem faced is identical in all different periods we search for a value function V^* that will solve $V^*(m) = \max[u_0(c) + \delta V^*(s + I)]$ sub to

$c + s \leq m$. This equation is also called a Bellman equation, and it is said to reflect the principle of optimality

Definition 2.5.2 *Given instantaneous utility* u_0*, a discount factor* $0 < \delta < 1$*, and an income stream* $(m_0, I, I, ...)$*, an equation such as*

$$V^*(m) = \max_{(c,s)}[u_0(c) + \delta V^*(s + I)] \text{ sub to } c + s \leq m,$$

that is to be solved in the unknown function V^* *is called a* ***functional equation****.*

The method of using and solving a functional equation can be contrasted with that of the sequence problem, as mentioned earlier in this section. The relations between sequence problems and functional equations, and how such these functional equations can be solved, is exactly what *dynamic programming* is concerned with.

For any instantaneous utility function $u_0(.)$, we define a value function by the optimal value of the sequence problem

$$V^*(m_0) := \max_{(c_0,c_1,c_2,...)} \sum_{t=0}^{\infty} \delta^t u_0(c_t) \text{ sub to } \sum_{i=0}^{t} c_i \leq m_0 + tI.$$

Now, it can be shown that if u_0 is an increasing function, then for any $\delta < 1$, the value function V^* indeed solves the functional equation

$$V^*(m') = \max_{(c,s)}[u_0(c) + \delta V^*(s + I)] \text{ sub to } c + s \leq m',$$

for any $m' \geq 0$. Moreover, any function that solves the functional equation must equal the value function that maximizes the sequence problem.

Similarly, any infinite sequence of consumption levels $(c_0^*, c_1^*, c_2^*, ...)$ that solves the first period-0 sequence problem, will also be such that for each $t \in \mathbb{N}_0$ the pair (c_t^*, s_t^*), with $s_t^* = m_t - c_t^*$, will maximize $\max_{(c_t,s_t)}[u_0(c_t) + \delta V^*(s_t + I)]$ s.t. $c_t + s_t \leq m_t^* = m_0 + tI - \sum_{i=0}^{t-1} c_i^*$. And conversely, for any infinite sequence of pairs $\{(\tilde{c}_t, \tilde{s}_t)\}_{t=0}^{\infty}$ each of which solve the corresponding maximization problem $\max_{(c_t,s_t)}[u_0(c_t) + \delta V^*(s_t + I)]$ s.t. $c_t + s_t \leq \tilde{m}_t = m_0 + tI - \sum_{i=0}^{t-1} \tilde{c}_i$, the sequence $(\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, ...)$ will also solve the first period-0 sequence problem $\max_{(c_0,c_1,c_2,...)} \sum_{t=0}^{\infty} \delta^t u_0(c_t)$ s.t. $\sum_{i=0}^{t} c_i \leq m_0 + tI$. (See e.g. Stokey and Lucas [43].)

Thus if the value function V^* is known, then the whole optimal profile of consumption levels $(c_0^*, c_1^*, c_2^*, ...)$ can be determined from V^* and m_0. Of course, such a value function V^* would still have to be found. Often it is easier to solve a functional equation than the corresponding sequence problem.

Also note that although we have included some intertemporal considerations in our framework, the final maximization problem that our consumer is facing is still basically a static problem. It still boils down to a one shot decision, where the different parts of this single decision will be implemented at different times. Therefore the actual decision is made at time $t = 0$, after which all that remains to be done for the consumer is to implement the chosen consumption plan.

2.6 Uncertainty

Thus far we have seen models that describe consumer choice as choosing a bundle of commodities from a budget set of affordable commodity bundles, such as to maximize the utility that will be obtained from consuming such a commodity bundle. Underlying all this are assumptions of perfect knowledge or of complete certainty about the elements in the model, or the different components of the (future) economic environment. It is assumed that the consumer knows the number of commodities that can possibly be purchased, he knows each of these commodities (including all future ones), he knows all prices and he knows his income (stream). These assumptions do not seem to agree with everyday life very well, because obviously in real life always some uncertainty is faced.

Whereas we can assume that this consumer does in fact know all the (relevant) goods that are available to him now, it does not seem very realistic to assume that a consumer knows all the goods that will be available at all future dates. Thus in reality, people seem to face uncertainty about the composition of the choice set, and also about the number of commodities in this choice set. In the more specific consumption/savings models, there is no need to know the composition of the commodity space, as utility is supposed to be derived from consumption levels instead of commodity bundles. But of course, the exact composition of the future commodity space is by no means the only source of uncertainty faced by someone who wants to plan his consumption. For instance, the assumption that a decision-maker would know his entire (lifetime) budget with certainty, seems to be very strong.

Similarly, we may wonder whether, when a consumer makes a choice that involves all levels of future consumption, he really has exact knowledge of all his own future utilities, as the standard model assumes. In some cases we can imagine that future preferences might change, or be influenced by the mood you will be in at that particular moment of consumption.

These obvious problems of the models of choice dealt with so far have made it necessary to extend these models and to develop ways to incorporate uncertainty into them. In order to properly describe what choice under uncertainty is, we need a few definitions.

Definition 2.6.1 *An* ***act*** *is a complete description of a particular course of action that a decision-maker can take in a certain choice problem.*

Definition 2.6.2 *An* ***outcome*** *is a complete description of the result that a decision-maker may obtain in a certain choice problem.*

A typical act is denoted a and the set of all possible acts for a certain choice problem is denoted A. Similarly, a typical outcome will be denoted o, and the set of all possible acts is denoted O. Note that the notion of an outcome refers to what a decision-maker will end up with, the precise result of a decision, and therefore describes the situation after the resolution of uncertainty. The notion of an act refers to everything that a decision-maker can do to influence what outcome will come about before the resolution of uncertainty.

Definition 2.6.3 ***Choice under certainty*** *refers to all models of decision-making where every act can lead to only one outcome.*

Definition 2.6.4 ***Choice under uncertainty*** *refers to all models of decision-making where there are acts that can lead to several outcomes.*

In models of choice under certainty, any act will lead to only one outcome, so it is usually not even necessary to distinguish acts from outcomes, they can simply be thought of as the same objects. However, in decision-theoretic terms choice under uncertainty means that there will be acts with which more than one possible outcome can be associated, and the distinction between acts and outcomes becomes crucial. Of course, only one outcome will finally occur after any particular act is chosen but at the moment the choice is made it remains unclear which one. Hence, with choice under uncertainty, outcomes are not completely determined by acts, but also influenced by forces beyond the decision-maker's control.

Thus, the setting is that of a decision maker who is faced with a choice problem for which there is missing information between acts and outcomes that prevents him from being able to view the problem in the prospective utility maximizing manner from the standard static framework.

Indeed, mathematically the correspondence that associates with any act the set of outcomes that may result from it, will under certainty only return sets containing a single element. Therefore mathematically under certainty this correspondence satisfies the criteria of being a function $f : A \to O$, with $f(a) \in O$. Under uncertainty, the correspondence that associates with any act the set of outcomes it may yield, will now also return sets with more than one element, and can therefore no longer be called a function, but should mathematically indeed be called a correspondence $g : A \rightrightarrows O$, with $g(a) \subset O$.

So such an acts-outcomes-correspondence, attaches to every act the set of outcomes that may result from this act. More structure can be added to the extent of uncertainty, and the properties of the acts-outcomes-correspondence, by introducing states of the world, that will help refine the above correspondences into functions.

Definition 2.6.5 *A* ***state of the world*** *is a complete description of all factors that are beyond the control of the decision-maker and that, together with the chosen act, uniquely determine the outcome that will occur.*

A typical state of the world will be denoted ω, and the set of all states of the world, that will also be called the **state space**, will be denoted Ω. States of the world are designed in such a way that any combination of an act and a state of nature will lead to one and only one outcome. Mathematically this could be represented by a function

$f : A \times \Omega \rightarrow O$, so $f(a, \omega) \in O$. Then, the above acts-outcomes-correspondence g can be linked to the acts-states-outcomes-function f by: $g(a) = \{f(a, \omega) : \omega \in \Omega\}$.

Since the set of outcomes O is known, everything that can happen is known. But of course, we generally do not know which element of O will result from any act, and this is where the states of the world or simply 'states' come in. We are now supposed to have a set of states Ω, one of which is going to materialize, so that the state $\omega \in \Omega$ that does materialize, together with the act that was taken, completely determines what outcome will result. Thus all remaining uncertainty is now embodied in uncertainty about which of the states in Ω will occur.

It is important to note here that within this general definition of choice under uncertainty, it has not yet been specified what properties the acts-outcomes-correspondence, or the acts-states-outcomes-function may have. It has also not been specified yet how much about this acts-outcomes-correspondence, or this acts-states-outcomes-function is known by the decision-maker, apart from the fact that acts may not necessarily lead to single outcomes. The modeller can choose models with different amounts of uncertainty, depending on how much the modellee within the model, and the modeller outside the model, know. Models can be devised in which a decision-maker has full knowledge of this acts-states-outcomes-function, or in which the modeller does have this information but the modellee does not. It is even possible to devise models in which even the modeller does not know the full acts-states-outcomes-function.

2.6.1 Expected utility

Of course, the above decision-theoretic view on how to specify uncertainty, does by itself not say anything about how decision-makers would deal with it, and make choices under uncertainty. As usual, there is not really one undisputed way to resolve these matters. Indeed, there is a long history of work on decision under uncertainty. However, the approach towards uncertainty that is by far the most used in economics nowadays is Expected Utility Theory.

From the definition of the acts-states-outcomes-function we see that any pair of an act and a state of the world will always lead to a single, unique outcome. Therefore, we can now also specify acts as functions from states of the world to outcomes. That is, an act can be viewed as a rule that specifies for each possible state of the world the outcome that will result from this act. Thus, mathematically we can now express an act here as a function $a(.)$, with $a : \Omega \rightarrow O$. Thus, the set of acts A is a set containing a number of functions. For any act $a : \Omega \rightarrow O$, the acts-outcomes-correspondence g would then be given by $g(a) = \{o \in O : \exists \omega \in \Omega \text{ s.t. } a(\omega) = o\}$, and the acts-states-outcomes-function f would be given by $f(a, \omega) = a(\omega)$, for all $\omega \in \Omega$.

Now, at the moment that a decision is required, our decision-maker does not know which state of the world will occur. However, in Expected Utility Theory it is assumed that our decision-maker knows the probabilities that each of the states of the world will occur. That is, a probability distribution $\pi(.)$ on the state space Ω is known, with $\pi(\omega) \geq 0$, for all $\omega \in \Omega$, and $\sum_{\omega \in \Omega} \pi(\omega) = 1$ (if Ω is a countable set), or $\int_{\omega \in \Omega} 1 \cdot d\pi(\omega) = 1$ (if Ω is not countable). The set of all probability distributions on a given state space is denoted by Π.

Then, in Expected Utility Theory pairs (a, π), that consist of an act $a : \Omega \to O$, and a probability distribution $\pi : \Omega \to [0, 1]$, are seen as the objects of choice, and called lotteries (or simple lotteries).

Definition 2.6.6 *A **lottery** l is a pair (a, π) that consists of an act $a(.)$ that maps the state space Ω into the set of outcomes O, and a probability distribution $\pi : \Omega \to [0, 1]$.*

Such a lottery (a, π) thus basically consists of a list of all outcomes $o \in O$ that may happen, and their associated probabilities of occurring $\tilde{\pi}_a(o) = \sum_{\omega \in \Omega : a(\omega) = o} \pi(\omega)$ for a countable state space, or $\tilde{\pi}_a(o) = \int_{\{\omega \in \Omega : a(\omega) = o\}} 1 \cdot d\pi(\omega)$, in the uncountable case. For notational brevity, we will also let such a lottery (a, π) be denoted by l, and the set of all such lotteries is denoted by $L = A \times \Pi$.

If two lotteries $l = (a, \pi) \in L$ and $l' = (a', \pi') \in L$ and a number $\mu \in [0, 1]$ are given, then we can define the **compound lottery** $\mu l \oplus (1 - \mu) l'$ to be the lottery that will result in lottery l with probability μ, and that will result in lottery l' with probability $(1 - \mu)$. Now it is assumed that the compound lottery $\mu l \oplus (1 - \mu) l'$ is identical to the simple lottery that will result in each of the outcomes $o \in O$ with the probabilities $\mu \tilde{\pi}_a(o) + (1 - \mu) \tilde{\pi}'_{a'}(o)$.

Effectively when choosing between acts, a decision-maker chooses between lists of outcomes that will occur with certain probabilities. So these lotteries are now the objects of choice, and similarly to the deterministic case, we assume that underlying the choices from any set of lotteries is a preference relation $\succsim$ on L.

Then if we would define a topology on the set of lotteries L, and assume a preference relation $\succsim$ on L to be continuous with respect to this topology, then by 2.2.1 preferences $\succsim$ could be represented by some utility function $U : L \to \mathbb{R}$. However, in Expected Utility Theory we want such utility functions to have an expected utility structure: there should be a function $u : O \to \mathbb{R}$ such that $U(l) = E_{\pi(\omega)}[u(a(\omega))]$, for all $l = (a, \pi) \in L$. In order to arrive at utility functions representing such preference relations that have an expected utility form two more requirements are needed.

Definition 2.6.7 *A preference relation $\succsim$ on the set of lotteries L is called **continuous** if for any l, $l', l'' \in L$, the sets $\{\mu \in [0, 1] : \mu l \oplus (1 - \mu) l' \succsim l''\}$, and $\{\mu \in [0, 1] : l'' \succsim \mu l \oplus (1 - \mu) l'\}$ are closed (in $\mathbb{R}$).*

Definition 2.6.8 *A preference relation $\succsim$ on the set of lotteries L satisfies **independence** if for any l, $l', l'' \in L$, and any $\mu \in [0, 1]$, we have $l \succsim l'$ if and only if $\mu l \oplus (1 - \mu) l'' \succsim \mu l' \oplus (1 - \mu) l''$.*

Now, for any given act, the probability distribution on the state space can be said to yield a probability distribution on the outcome set, where this probability distribution would specify the probabilities each of the outcomes will occur with. And, if $u : O \to \mathbb{R}$ is a function that gives the final utility as it would be derived from getting certain outcomes, then given an act, the probability distribution on the state space can also be said to yield a probability distribution of final utility. And in Expected Utility Theory it is the mathematical expectation of this probability distribution of final utility that is ultimately maximized. Presented next is the traditional view of Expected Utility Theory, also sometimes termed objective Expected Utility Theory.

Theorem 2.6.1 *Suppose that the preference relation $\succsim$ on the set of lotteries L satisfies continuity and independence. Then there exists a utility function $U : L \rightarrow \mathbb{R}$ that represents $\succsim$, and has an expected utility form: there is a (Bernoulli) utility function $u : O \rightarrow \mathbb{R}$ such that $U(l) = E_{\pi(\omega)}[u(a(\omega))]$, for all $l = (a, \pi) \in L$.*

For a proof, see von Neuman and Morgenstern [33].

So (objective) Expected Utility Theory says that if a preference relation on the set of lotteries satisfies continuity and independence, then this preference relation can be represented by a utility function that has the expected utility form (or von Neumann-Morgenstern form): i.e. there is a Bernoulli utility function $u(.)$ defined on the set of outcomes O so that, if the state space Ω is finite or countably infinite,

$$U(l) = E_\omega[u(a(\omega))] = \textstyle\sum_{\omega \in \Omega} \pi(\omega) u(a(\omega)),$$

or if the state space Ω is not countable,

$$U(l) = E_\omega[u(a(\omega))] = \int_{\{\omega \in \Omega\}} u(a(\omega)) d\pi(\omega).$$

Underlying this specification of objective Expected Utility Theory are assumptions that whereas a decision-maker doesn't know what *will* happen, he does somehow know or can imagine everything that *can* happen, and that he knows all probabilities with which everything that can happen will happen. That is, it is assumed that when our decision maker has to decide, this probability distribution that gives a complete description of the uncertainty that is faced, is objectively given to him. This really seems a strong assumption, since when we are dealing with uncertainty in real life we are usually not equipped with such knowledge. Here if we don't want to assume that our consumer actually knows the probability distribution of the states of the world, then we could still assume that somehow our consumer acts *as if* he does have such a probability distribution at his disposal. That is, the assumption of objective knowledge of each state's probability of occurring can be dispensed with. Then instead of starting from a preference relation defined on the space of lists of outcomes that will occur with known probabilities, a preference relation can alternatively be defined on the space of lists of outcomes directly, without the additional interpretation of known probabilities of occurrence. Again, it can be shown (see, e.g. Savage [38]) that if this new preference relation satisfies a number of axioms, then this preference relation can still be represented by a utility function that has the above expected utility form: $U(l) = E_\omega[u(a(\omega))]$. Not surprisingly though, since the objects of choice that would underlie this new type of preference relations are more general than lotteries, the axioms that such a subjective type of preference relation would have to satisfy should then be strengthened. This kind of expected utility that does not assume objectively known probabilities of occurrence is called subjective expected utility. In contrast with the above objective expected utility where the objective probabilities $\pi(\omega)$ of certain states occurring are exogenously given to the decision-maker, with subjective expected utility the subjective probabilities $\pi(\omega)$ of states occurring can endogenously be derived from the preference relation.

In the next subsections we will first apply uncertainty in the standard (static) microeconomic framework for consumer choice, then we will extend our explicit way to model time to include uncertainty, after which we will apply uncertainty in the standard macroeconomic framework for consumption/savings decisions.

2.6.2 Expected utility in consumer choice

The above framework for choice under uncertainty is very abstract and general indeed. Therefore more specification is required in order to apply it in some more concrete setting. How exactly the components of this framework (the acts, states and outcomes) should be interpreted in a specific economic problem is not always straightforward, and sometimes subject to the modeller's choice. Thus the question arises of how the above elements of the framework of choice under uncertainty should be interpreted within the case of consumer choice that we were considering.

The interpretation of an outcome is probably most straightforward here. An outcome should simply be a complete description of all consumption that a consumer would be able to attain, thus an outcome should be a commodity bundle $x \in \tilde{X}$, in the set of all commodity bundles that may result from all combinations of acts and states that can possibly occur.

Next, the states of the world are to be thought of as everything that is beyond the control of the decision-maker, but that does influence what outcome x will finally occur. One specification of these states of the world, and one that is particularly convenient, is a specification where the uncertainty about the future economic environment can simply be reduced to uncertainty about each of the components that this economic environment consists of. Thus we assume that the economic environment consists of a number of distinct components (such as available consumption opportunities, prices and income) that are uncertain, and we may define the states of the world to specify realizations of all the uncertain components of the consumer choice model. Thus a state of the world would consist of a realization of the commodity space, a realization of the price vector, and a realization of the consumer's budget, and we write $\omega = (X, p, m)$.

Such a realization of the commodity space is denoted by X, and such a space would be some Euclidian space: $X = \mathbb{R}^n_+$, where its dimension $n \in \overline{\mathbb{N}}$ does not have to be the same for all different realizations of the commodity space. Also note that for any two realizations X and X', even if their dimensions are equal, these spaces may very well represent different sets of commodities, so the interpretations of each of their dimensions may differ. In any case, each of these realizations is a full-blown commodity space. To be consistent with the interpretation of $\tilde{X}$, we should have that the set $\tilde{X}$ equals the union of all the possible realizations for X.

A realization of the price vector p of a state of nature $\omega = (X, p, m)$ should give the prices for all the commodities in the accompanying realization of the commodity space X, and should therefore be given by a vector $p \in \mathbb{R}^n_+ \backslash \{0\}$, where its dimension $n \in \overline{\mathbb{N}}$ equals the dimension of the accompanying realization of the commodity space.

And a realization of the consumer's budget from a state of nature $\omega = (X, p, m)$ should simply be a non-negative real number $m \in \mathbb{R}_+$. Thus indeed, a state ω could be denoted by a triple (X, p, m).

Then, for these specifications of outcomes and states of the world, we can now specify acts. Since a pair of an act and a state of the world will always lead to a single, unique outcome, we can also define acts as functions from states of the world to outcomes. That is, an act would have to be some rule that specifies a commodity bundle for each possible state of the world. Of course, given the interpretations of the components of states of the world, we should additionally require that such a chosen commodity bundle should be feasible. Here feasibility would mean that given the state of the world $\omega = (X, p, m)$, the commodity bundle x that would be chosen has to satisfy $x \in X$ and $p \cdot x \leq m$. Thus, a feasible act is some rule that specifies for each possible state of the world some course of action that will lead to a feasible outcome. And we can formally define an act here as a function $a(.)$, with $a : \Omega \to \tilde{X}$, and such that for any $\omega = (X, p, m)$, we have $a(\omega) \in X$, with $p \cdot a(\omega) \leq m$.

Now, when a Bernoulli utility function $u : X \to \mathbb{R}$ is given on the set of outcomes (commodity bundles), and if indeed a probability distribution π is given on Ω, then the expected utility of any act a can be expressed as $U(a) = E_\omega[u(a(\omega))]$.

Note that in the above framework for dealing with uncertainty, it seems that the consumer does not face any uncertainty about his future preferences or tastes, so that utility is deterministic. However, this framework for dealing with uncertainty does also allow for uncertainty of future preferences by choosing a different interpretation of what outcomes should be. The description of an outcome does not have to be confined to the physical properties of a given situation, but could also be enriched to include the psychological reactions to this situation. Thus the consumer could deal with this extra bit of uncertainty in the same way as he did with the other uncertain factors.

2.6.3 Modelling time

However, the above way to specify uncertainty may also not seem completely natural or helpful. In the above account, the decision problem under uncertainty is essentially only a series of decision problems under certainty. That is, an act would be a function from states of the world to outcomes, thus essentially here decision-making under uncertainty would boil down to, for every state that may occur, making a single choice as under certainty.

And here we encounter another reason for explicitly modelling time: it will prove helpful in extending the present framework to include uncertainty in a more natural way. Of course time plays a crucial role in considerations about uncertainty. Specifying the unveiling of uncertain information requires being able to specify what happens when. Therefore, it is probably more appropriate to further specify the extent of uncertainty, so that the uncertain elements are not unveiled at once, but instead little by little as time progresses.

To model this, we use the same discrete way to model time, as presented before. Thus, we have a time variable t, that progresses through one of the sets $\{0, 1, 2, ..., T\}$, in case of a finite number of periods, or $\{0, 1, 2, ...\}$, in case of a (countably) infinite number of periods. Now, above we saw that this time variable was used to break up the commodity space into period-t commodity spaces X_t, such that the overall commodity space X would equal the Cartesian product of all the X_t's: $X = \times_t X_t$. Similarly, overall price vectors p were broken up into sequences $(p_0, p_1, ..., p_T)$ or $(p_0, p_1, p_2, ...)$.

Accordingly, a state of the world ω is a triple (X, p, m), and can now also be written as

$$\omega = ((X_0, X_1, ..., X_T), (p_0, p_1, ..., p_T), m),$$

or

$$\omega = ((X_0, X_1, X_2, ...), (p_0, p_1, p_2, ...), m).$$

However, recall that explicitly modelling a time structure also allows for the possibility of an income stream $(m_0, I_1, ..., I_T)$, or $(m_0, I_1, I_2, ...)$. In this case a state of the world could thus be written as

$$\omega = ((X_0, X_1, ..., X_T), (p_0, p_1, ..., p_T), (m_0, I_1, ..., I_T)),$$

or

$$\omega = ((X_0, X_1, X_2, ...), (p_0, p_1, p_2, ...), (m_0, I_1, I_2, ...)).$$

Now we could also break up states of the world ω into sequences of time-t states ω_t, so that $\omega = (\omega_0, \omega_1, ..., \omega_T)$, or $\omega = (\omega_0, \omega_1, \omega_2, ...)$. Then by $\omega_t = (X_t, p_t, I_t)$ we would denote the prevailing period-t part of the state of the world, consisting of a period-t commodity space, a period-t price vector, and a period-t income (we define $I_0 := m_0$).

Recall that for such period-t commodity spaces $X_t = \mathbb{R}_+^{n_t}$, the dimension $n_t \in \mathbb{N}$ may vary across realizations. A realization of the period-t price vector $p_t \in \mathbb{R}_+^{n_t} \backslash \{0\}$ should be of the same dimension as the accompanying realized commodity space. A realization of the additional income in period t should again be some $I_t \in \mathbb{R}_+$.

As noted above, we also wanted to explicitly model time in order to allow for the possibility of a gradual unveiling of uncertainty. That is, since we can now write a state of the world ω as a sequence $\omega = (\omega_0, \omega_1, ..., \omega_T)$, or $\omega = (\omega_0, \omega_1, ...)$, different parts ω_t may be learned at different points in time. Indeed, here we will rather straightforwardly assume that the period-t part of the state of the world is in fact learned at (and never before) time t. Therefore, at time t, the subsequence $(\omega_0, \omega_1, ..., \omega_t)$, which we also denote here by ω_0^t, is already known. The future part of the state of the world ω_{t+1}^T, or ω_{t+1}^∞, is not known at time t. Thus for $t < T$, such a sequence $\omega_0^t = (\omega_0, \omega_1, ..., \omega_t)$

corresponds to several different states of the world ω that may finally occur. Here we denote the set of all final states of the world ω that can occur, given ω_0^t, by $\{\omega_0^t\} \times \Omega_{t+1}^T$, or $\{\omega_0^t\} \times \Omega_{t+1}^\infty$.

Given this new decomposition of states of nature, recall that an act is still a function $a(.)$ from states of the world into outcomes such that for any $\omega = (X, p, m)$, we have $a(\omega) \in X$, with $p \cdot a(\omega) \leq m$. But since $X = \times_t X_t$, all elements of the overall commodity space X can be broken down into sequences $(x_0, x_1, ..., x_T)$ or $(x_0, x_1, x_2, ...)$, and the same can be done with acts. We can now write acts $a(\omega)$ as sequences of functions: $a(\omega) = (a_0(\omega), a_1(\omega), ..., a_T(\omega))$, or $a(\omega) = (a_0(\omega), a_1(\omega), a_2(\omega), ...)$.

Now, we assume that an action taken at a certain point in time can only depend on the information available at that point in time. Mathematically, for any $t < T$, and all $\omega, \omega' \in \{\omega_0^t\} \times \Omega_{t+1}^T$, or $\omega, \omega' \in \omega \in \{\omega_0^t\} \times \Omega_{t+1}^\infty$, it should always hold that $a_i(\omega) = a_i(\omega')$, for all $i \leq t$. Therefore, the period-t action $a_t(.)$ can only depend on ω_0^t, so we should write $a_t(\omega_0^t)$.

Thus we would get that

$$a(\omega) = (a_0(\omega_0), a_1(\omega_0^1), ..., a_T(\omega_0^T)),$$

or

$$a(\omega) = (a_0(\omega_0), a_1(\omega_0^1), a_2(\omega_0^2), ...).$$

In this case the expected utility of any act a is still given by $U(a) = E_\omega[u(a(\omega))]$, and this can now be written as

$$U(a) = E_\omega[u(a(\omega))] = E_{\omega_0} E_{\omega_1} ... E_{\omega_T}[u(a(\omega))] =$$

$$E_{\omega_0} E_{\omega_1} ... E_{\omega_T}[u(a_0(\omega_0), a_1(\omega_0^1), ..., a_T(\omega_0^T))].$$

Or, in the infinite case,

$$U(a) = E_{\omega_0} E_{\omega_1} E_{\omega_2} ...[u(a_0(\omega_0), a_1(\omega_0^1), a_2(\omega_0^2), ...)].$$

2.6.4 Expected utility in consumption/savings models

Remember that in consumption/savings models, we set $T = \infty$ and $R = 1$. And rather than commodity bundles, we considered the objects of choice to be consumption patterns $c = (c_0, c_1, ...)$, that specify sequences of levels of period-t consumption c_t. And in these models we made the assumptions of additive separability over time and of exponential discounting so the utility function reads $u(c) = \sum_{t=0}^\infty \delta^t u_0(c_t)$. We have a situation with an initial budget m_0 and a stream of additional incomes $(I_1, I_2, ...)$, with $I_t \geq 0$, for all t, where no borrowing is possible, but saving is possible (at a zero interest rate). Therefore this utility function is then supposed to be maximized over all consumption patterns that satisfy the budget constraints:

$$c_t \leq m_t = m_0 + \sum_{i=1}^{t} I_i - \sum_{i=0}^{t-1} c_i, \text{ for all } t.$$

Next we consider consumption/savings models under uncertainty. These models then naturally fit the mould as introduced in the previous subsection. States of the world are decomposed into sequences over time $\omega = (\omega_0, \omega_1, ...)$, with $\omega_t = (X_t, p_t, I_t)$, for all t (here $I_0 = m_0$). Every ω_t is learned at time t, and acts $a : \Omega \to X$ can be decomposed in to sequences $a(\omega) = (a_0(\omega_0), a_1(\omega_0^1), a_2(\omega_0^2), ...)$.

But now, within this class of consumption/savings models we see that the role of uncertainty is limited by the fact that, for every $\omega_t = (X_t, p_t, I_t)$, the commodity spaces X_t are all one-dimensional and basically representing the same variable 'level of consumption', which will be valued equally across periods, except for the discount factor δ. Thus, all X_t's are basically known.

Also, since all c_t's denote levels of consumption, and since for simplicity we assumed no interest $R = 1$, in this setting the price for each period's level of consumption is equal to $p_t = 1$, so the p_t's are also certain.

Therefore, there may now only be uncertainty regarding the available budgets. In this setting, period-t states of the world ω_t can thus be equated with additional incomes I_t. These states of the world, and thus the I_t's would not be known in advance, but they would be gradually learned. It is this specification of budgetary uncertainty that we will consider extensively here in consumption/savings models.

In this case the expected utility of any act a is still given by $U(a) = E_\omega[u(a(\omega))]$, but can now be further specified to

$$U(a) = E_\omega[u(a(\omega))] = E_{\omega_0} E_{\omega_1} E_{\omega_2} ....[u(a_0(\omega_0), a_1(\omega_0, \omega_1), a_2(\omega_0, \omega_1, \omega_2), ...)] =$$

$$E_{m_0} E_{I_1} E_{I_2} ....[\sum_{t=0}^{\infty} \delta^t u_0(c_t(m_0, I_1, ..., I_t))] = E_{I_1} E_{I_2} E_{I_2} ....[\sum_{t=0}^{\infty} \delta^t u_0(c_t(m_0, I_1, ..., I_t))].$$

Here the acts $a(\omega) = (a_0(\omega_0), a_1(\omega_0, \omega_1), a_2(\omega_0, \omega_1, \omega_2), ...)$ are given by the sequences of functions $(c_0(m_0), c_1(m_0, I_1), c_2(m_0, I_1, I_2), ...)$. Each of the decision functions $c_t(m_0, I_1, ..., I_t)$ within such a sequence should satisfy the budget constraint $c_t(m_0, I_1, ..., I_t) \leq m_t$. And like in formula (✠), the implicit budgets m_t are specified according to $m_t = m_0 + \sum_{i=1}^{t} I_i - \sum_{i=0}^{t-1} c_i$, which can now be rewritten as

$$m_t = m_0 + \sum_{i=1}^{t} I_i - \sum_{i=0}^{t-1} c_i(m_0, I_1, ..., I_i).$$

Thus the resulting basic consumer problem reads:

$$\max E_{I_1} E_{I_2} E_{I_3} ...[\sum_{t=0}^{\infty} \delta^t u_0(c_t(m_0, I_1, ..., I_t))]$$

sub to

$$c_t(m_0, I_1, ..., I_t) \leq m + \sum_{i=1}^{t} I_i - \sum_{i=0}^{t-1} c_i(m_0, I_1, ..., I_i), \text{ for all } t$$

2.6.5 Dynamic programming in consumption/savings models with uncertainty.

A maximization problem as obtained in the previous subsection may look quite complicated, but fortunately we can extend the procedures from dynamic programming to include uncertainty. In fact, the procedures of dynamic programming become even more appropriate and convenient under uncertainty.

Then, the maximization problem that we ended up with in the previous subsection, is one of the following type.

Definition 2.6.9 *Given instantaneous utility u_0, a discount factor $0 < \delta < 1$, an initial income m_0, and probability distributions for additional incomes I_t, for $t \geq 1$, the problem of solving*

$$\max_{(c_0,c_1,c_2,...)} E_{I_1} E_{I_2}...[\sum_{t=0}^{\infty} \delta^t u_0(c_t(m_0, I_1, ..., I_t))]$$

over sequences of functions $c_t(m_0, I_1, ..., I_t)$ that satisfy the budget constraints

$$\sum_{i=0}^{t} c_i(m_0, I_1, ..., I_i) \leq m_0 + \sum_{i=1}^{t} I_i,$$

for all $t \geq 0$, is called a ***sequence problem****.*

If we would take a degenerate probability with only one state that can possibly occur ($\Omega = \{\omega\}$), we would have the old definition of a sequence problem under certainty back. Thus this definition is really a more general one than the definition of a sequence problem under certainty, which would justify using the same name twice.

However, instead of trying to solve such a sequence problem directly (by solving for all variables simultaneously), dynamic programming again considers functional equations.

Definition 2.6.10 *Given instantaneous utility u_0, a discount factor $0 < \delta < 1$, an initial income m_0, and probability distributions for additional incomes I_t, for $t \geq 1$, an equation such as*

$$V^*(m) = \max_{(c,s):c+s\leq m} \{u_0(c) + \delta E_I[V^*(s + I)]\},$$

that is to be solved in the unknown function V^* is called a ***functional equation****.*

Under certainty we saw that functional equations were of the form $V^*(m) = \max_{c+s\leq m}\{u_0(c) + \delta V^*(s + I)\}$. Under uncertainty, next period's additional income is uncertain, which is why the term $\delta E_I[V^*(s + I)]$ appears.

In the case we are considering here, the relations between sequence problems and the corresponding functional equations that existed under certainty, also hold under uncertainty (see e.g. Stokey and Lucas [43]). In the current setting, it can be shown that the value function V^* that is defined to maximize a sequence problem indeed solves the corresponding functional equation. Moreover, any function that solves the functional equation must equal the value function that maximizes the corresponding sequence problem (Stokey and Lucas [43]).

In this chapter we have introduced two fundamental economic building blocks: the standard microeconomic framework for consumer choice, and the models of consumption/savings decisions from macroeconomics. We have established that, and in exactly what way, the second is a special case of the first. And in presenting these building blocks we have laid a formal groundwork that will be drawn from in the rest of this work.

3 Problems and motivations

In the present chapter we will discuss and evaluate the standard frameworks of consumer choice and consumption/savings, and we will draw attention to some drawbacks of these standard frameworks. We will briefly go over some empirical problems, and then present some conceptual problems. These conceptual problems of the standard approach to modelling consumption behaviour will also motivate the new approach that is taken in the remainder of this work.

The choice problem that is faced in actual real-life consumption problems is extremely complex, as it seems to be influenced by thousands of different variables. As a result, modelling the process of deciding on actual real-life consumption behaviour could be influenced by thousands of different considerations. If we want to try to capture this complex consumption behaviour in an economic model (and especially if we prefer such a model to be relatively simple), then somewhere down the line assumptions will have to be made that may seem quite strong and not entirely realistic, or even very strong and plainly unrealistic. However, the fact that some assumptions that are made do not always agree with every day life does not necessarily have to be insuperable, because our models (hopefully) do not claim to fully capture all of the economic reality. Instead, the best we seem to be able to do is to very much simplify or stylize the relevant problem, while somehow capturing some features of the economic reality that seem particularly important or even essential to us. Of course, which features of the economic reality we judge as being important or essential may depend on, for instance, the kind of questions about the economic reality that we are trying to answer from the models we use. So the fact alone that an assumption underlying a theoretical model seems not (completely) realistic does not have to be a reason for dismissing the model altogether, as long as we feel that there are essential features of the economic reality that remain more or less intact, and that are somehow illuminated by the model.

Still, in this chapter we will argue that the standard approach to modelling consumption behaviour has some serious drawbacks, and that an alternative approach might not suffer as much from all of these drawbacks.

This chapter consists of six sections. The first section recaptures the assumptions underlying the standard way to model consumption behaviour, and some possible justifications for these assumptions. The second section briefly goes over some empirical problems of the standard framework, from both econometric and experimental work. The third section will present some conceptual problems of the standard approach. The fourth section will provide categorizations of types of economic models and of types of rationality, that will help place the alternative framework, and that will help to distinguish it from the standard framework. The fifth section will briefly sketch the alternative approach towards consumer choice that will be taken here, and it will look back at preceding sections to motivate this new approach. The final section will review some related literature.

3.1 The standard framework

In most microeconomics textbooks we find more or less the same (neoclassical) analysis of consumer choice, which was reproduced in the first three sections of the previous chapter. This analysis models a consumer who has a certain number of commodities he can choose from, that may also include commodities that will become available in the future. Given this total number of commodities, commodity bundles are specified as possible combinations of amounts of each of the available commodities that a consumer might conceivably own or be able to consume in a certain situation.

In order to make choices our consumer is assumed to be able to compare all of these bundles of commodities, and to state preferences over these bundles. Thus, our consumer is assumed to be able to make comparisons between bundles for all goods simultaneously (including all future commodities).

The model of consumer choice is completed by assuming that a consumer, when faced with a certain set of possible consumption bundles that he can choose from, will always choose a consumption bundle that maximizes his preferences. That is, he always chooses a consumption bundle that he prefers to every other consumption bundle he could otherwise have chosen.

This neoclassical theory of consumer choice is typically set under certainty (or perfect information), in its standard specification there is no uncertainty about any of the aspects relevant to the problem at hand: all commodities that will be available, all prices and budgets are supposed to be known. However, it seems apparent that in real-life consumption choice problems uncertainty is present. Of all the information about the future (economic) environment that is relevant to deciding an expenditure level now, only very little is in fact certain. And especially for information about the more distant future we might argue that hardly anything is really certain. Hence for more descriptive realism the setting of consumer choice should be extended to include uncertainty.

Then, in the face of uncertainty the assumption that a consumer knows everything that will happen is replaced by new assumptions. These new assumptions are that the consumer does know everything that can happen, and the probabilities that all these possibilities will happen with. This probabilistic information would enable a consumer to still view his problem in a fully prospective way.

Given the information that will allow for this prospective view, Expected Utility Theory seems to be the standard choice for incorporating uncertainty into economic models. Given the probability distributions that link actions to outcomes, and given a utility function on these outcomes, in Expected Utility Theory a consumer is supposed to choose that action that will maximize the expectation of his final utility. Expected Utility Theory was presented in section 2.6, both in its general form and in a more specific form applied to consumer choice.

However, it seems questionable that the thought processes that people actually go through in making consumption decisions, would resemble the procedures as described in the standard model of consumer choice. A consumer would have to be assumed to have (or have constructed) probability distributions for all variables for which the values are not entirely certain, and moreover, the consumer would have to be assumed to be able to deal with all this information in a way that corresponds to the maximization of expected utility.

From our own lives we know that we are not just given credible descriptions of our future environments from a source outside us. And from the simple thought experiment of trying to mentally construct such a complete description of our future economic environments, and consequently trying to make sense of all this information such as to choose an optimal consumption plan, it seems that such a task does not altogether come natural to us, and that our minds do not seem used to doing this. Thus it seems that the prospective description of the choice process is not very close to the way in which we mentally do consider these problems.

Even if there were no uncertainty whatsoever, then it would still not seem completely plausible that people would actually have such utility functions in mind when making choices. Therefore the question to what extent these utility functions actually exist seems a legitimate one. The answer that standard economic theory would give would be that these utility functions do not really have to exist in people's minds, because it is enough to assume that people make choices *as if* they use these functions. It doesn't matter if the concepts we employ are cognitively real, as long as people's behaviour is in line with our models' predictions. A justification for this view can be found by linking the unobservable concepts we use (utility functions) to certain regularities in people's observable behaviour (completeness, transitivity, continuity of preferences). One example of an axiomatization that expresses the possibility for a preference relation to be represented by a utility function in terms of the (in principle) observable characteristics of the underlying preference relation, is theorem 2.2.1 which applies under certainty. Theorem 2.6.1 shows a similar result for objective Expected Utility Theory, where the probabilities for all possible realizations are assumed to be given exogenously. As for subjective Expected Utility Theory, where probability distributions are not objectively given, axiomatizations like that of Savage [38] show that if the preference relations on lotteries that we assume to underlie the choices, satisfy certain axioms then these preference relations can be represented by utility functions that have an expected utility structure, and by subjective probability distributions.

Thus, when we model a consumer's consumption behaviour in the standard way, we don't have to claim that our consumer actually mentally constructs the complete model, including all probability distributions, and solves the problem by actually calculating the optimal solution. Instead, we may assume that our consumer makes choices *as if* he has such a complete structure of the problem at his disposal and *as if* he subsequently chooses such as to maximize (expected) utility.

However, there still remain problems with the standard way to model consumption behaviour, first we will present some empirical problems, and then we will focus on some conceptual problems.

3.2 Empirical Drawbacks

In section 2.4 we mentioned that the standard microeconomic framework for consumer choice is too general to be tested empirically. This standard framework does not yield enough clear-cut behavioural predictions that can be tested, as this framework could predict almost anything, from appropriately chosen utility functions.

3.2.1 Expected Utility Theory

However, some of the assumptions that are made in modelling consumer choice as in the previous chapter *can* be tested separately. For instance, Expected Utility Theory can be tested as a theory of choice under uncertainty. It would typically not be tested in choice situations as complicated as those that are modelled in the previous chapter, but in much more simple, highly stylized settings in laboratory experiments. Since Allais [1] a lot of experimental studies have been done to test the predictions made by Expected Utility Theory, and these studies have shown a number of biases in which most people systematically deviate from these predictions, see for instance Kahneman and Tversky [21].

Thus people do not always behave according to Expected Utility Theory and, to be more in line with these experimental findings, a whole line of different alternative theories of choice under uncertainty have been developed, that are appropriately grouped under the header non-expected utility theory. Probably the most well known of these is Kahneman and Tversky's Prospect Theory [21].

However, the experimental studies that proved Expected Utility Theory wrong (or not entirely right) usually stay within an Expected Utility Theory setting: before making a decision a subject is typically given an extensive list of all outcomes that can occur, and the probabilities each of them will occur with. As a result, most alternative theories of choice under uncertainty are still specified in this same setting where decisions are made based on probability distributions specifying the uncertainty that is faced. Therefore like with Expected Utility Theory, there are also some conceptual problems associated with using these alternative theories (in the context of consumer choice). Some of these conceptual problems will be discussed in the next section.

3.2.2 Consumption/savings models

As mentioned in section 2.4, and as stated above, the standard microeconomic framework for consumer choice cannot really be tested empirically in its completely general form. Therefore in section 2.4 a lot more structure was imposed. Time was explicitly modelled, and in all periods only spending levels were considered, so that in any period the commodity space would effectively be one-dimensional. Utility was assumed to be additively separable (with respect to time) and to satisfy exponential discounting. Thus the framework was very much narrowed down, and we ended up with the consumption/savings setting as used in macroeconomics.

Like with Expected Utility Theory, some of the assumptions that are made in modelling consumption behaviour as in consumption/savings models (such as exponential discounting) can be tested separately, see for instance Loewenstein and Thaler [26] or

Laibson [23]. However, the implications of consumption/savings models *can* also be tested empirically as a whole. In fact, these implications have been tested extensively, mainly in econometric work, and some empirical deviations from the predictions of these models have been documented.

Econometric work Before we can consider any deviations from theoretical predictions, we should first say something about the predictions that arise from consumption/savings models. And although the specific predictions will depend on the exact forms of the specific consumption/savings models, here we can identify two features at a more general level that would be predicted by these models in some form or another. These two features that are implied by consumption/savings models, are that consumption smoothing will occur, and that the growth or decline in consumption should be determined by preferences, and not by the particular pattern that the income stream takes.

The prediction that consumption growth or decline is determined by preferences (and maybe the interest rate), and not by the particular pattern of the income stream, will always hold in consumption/savings models if there is a perfect capital market, i.e. if saving and borrowing is possible at the same interest rate. If this last assumption holds, then for any income stream the corresponding budget set is determined by only a single constraint, namely that total (discounted) lifetime spending should not exceed total (discounted) lifetime income. Then, it is a simple mathematical fact that any two income patterns that (given the interest rate) yield the same total lifetime income, will also yield the same budget set and thus the same maximizing choice from this budget set. Thus, consumption growth or decline should not be related to the growth or decline in income. Of course, in the presence of liquidity constraints, where saving and borrowing is not possible at the same interest rate (for instance because of income uncertainty) this principle need not fully apply anymore.

As the term seems to suggest, the idea of consumption smoothing is that it is best to follow a smooth consumption pattern, rather than a pattern with large spending differences between periods (that are not too far apart). This property will typically hold in consumption/savings models in some form, the permanent income hypothesis (Friedman [16]) and the life-cycle hypothesis (Modigliani and Brumberg [31]) are two specific forms that consumption smoothing can take. Mathematically consumption smoothing is a result of the very common assumption of concavity of utility functions. Recall from section 2.2 that in the general consumer choice setting, (quasi-)concavity meant that mixing was good. This already suggests that situations in which nothing can be consumed in some period(s) will never be preferred and will thus, if possible, be avoided. A fortiori, in a setting of consumption/savings models, where utility is of the form $\sum_t \delta^t u_0(c_t)$, the per-period utilities are all very comparable. Therefore under certainty and if instantaneous utility is strictly concave, we get that any chosen consumption pattern (in an internal solution) will be very smooth over time. Under expected utility the same principle of consumption smoothing applies, although to a lesser extent, since in that case it is expected consumption that will typically follow a smooth pattern. Of course, under expected utility, consumption choices also depend on the realizations of the uncertain additional incomes. Still, a single favourable additional income realization in some period would simply raise the total expected lifetime income a bit, which would typically be spread out more or less evenly across all future periods, so this will not lead to dramatic changes in consumption.

These theoretical predictions are not always observed (to the extent that they are predicted) in empirical studies. The feature of excess sensitivity of consumption to transitory income entails that consumption patterns are not as smooth as theory would suggest. Excess sensitivity of consumption to transitory income is a well-known empirical phenomenon, see for instance Flavin [14], Zeldes [48] or Browning and Lusardi [5].

Also, the theoretical prediction that consumption growth or decline should not be related to income growth or decline does not always agree with empirical findings. Carroll and Summers [7] find that high income growth is typically associated with high consumption growth, both across countries and across occupational groups.

And more generally, most households hold rather small amounts of savings (see e.g. Wolff [46]), so that consumption approximately tracks income. In the terminology of Deaton (1991)), most households exhibit buffer-stock saving.

Zeldes [49] and a number of more recent, similar publications have shown that the above empirical observations are not necessarily incompatible with using optimal policies in consumption/savings models. Thus, there are at least some specifications of consumption/savings models under which buffer-stock saving behaviour *is* an optimal policy.

The idea of learning consumption could point in the same direction, and perhaps provide an alternative explanation. We will return to these questions in chapter 10.

Experimental work There have also been some, although relatively few, experimental studies investigating consumption/savings models. These studies show that in laboratory settings subjects' behaviour diverges from the theoretical predictions, in some way or another (see e.g. Hey and Dardanoni [19], Noussair and Matheny [34], Fehr and Zych [13].

Ballinger, Palumbo and Wilcox [3] experimentally study social learning in consumption/savings models. They group subjects into "families" of three, that choose consumption patterns sequentially, where later generations can observe choices and outcomes of earlier generations. The authors observe a strong tendency to save too little early on, so that not enough consumption smoothing can occur. However, later generations perform significantly better than earlier generations.

Chua and Camerer [8] study learning (both individual and social) in an intertemporal consumption setting. They consider a consumption/savings model of 30-period lifetimes with induced constant relative risk aversion utility, where subjects face income uncertainty and habit formation. Subjects are asked to choose consumption patterns for a sequence of lifetimes, in treatments with and without social learning. Subjects have the possibility to learn from previous lifetimes, so this setting gives rise to 'reincarnation learning', and is not chosen to closely mimic reality. The authors argue that if convergence towards optimality is slow (even in a relatively simple setting), this would justify skepticism about people being able to learn *within* a single lifetime. Chua and Camerer find that in first lifetimes, choices are far from optimal but that subjects do learn to approach optimality rather quickly (within about four lifetimes), even more so in the presence of social learning.

Johnson, Kotlikoff and Samuelson [20] present the results of an experimental study of a life-cycle model under certainty. Subjects are first asked to determine a consumption pattern for a 40-year lifetime, where an income stream and a 4% interest rate are given with certainty. Subsequently they are asked how much they would like to consume in some selected periods, given some particular combinations of current assets and future incomes. They find that "errors in consumption decision-making appear to be very substantial and, in many cases, systematic". For instance, this paper reports that subjects displayed significant inconsistencies in their consumption decisions. Many subjects chose consumption values that differed by at least twenty percent in pairs of economically equivalent, or even identical situations. This result contradicts standard economic theory, and it could point in the direction of learning. If we would not want to dismiss these errors altogether as signs of irrationality, then they would indicate that some valuations or framings would have changed (or have been learned) over the process of making these decisions.

Moreover, Johnson et al. find that most subjects seem to oversave, possibly because they underestimated the power of compound interest. As subjects approach the end of their lifetimes, they appear to realize that they have saved too much, and start spending much more. The authors call this "adaptive" consumption behaviour.

3.3 Conceptual Drawbacks

Besides the empirical drawbacks, there are also some conceptual problems associated with modelling consumption behaviour in the standard way as presented in the previous chapter. Here we will argue that the complexity of real-life consumption choice problems is simply overwhelming, that these problems seem to involve a more profound, more fundamental kind of uncertainty than is modelled in Expected Utility Theory, and that introspection may show us that the standard setting that is used to model consumer choice problems in, does not seem to resemble how people think about such consumption problems.

A choice problem that is faced in actual real-life consumption problems is extremely complex; we can argue that it is influenced by thousands of different variables. As a result, trying to find an optimal solution (as in the standard framework) in such choice problems would be extremely demanding. This difficulty is not only experienced by consumers, but also by economists who try to model consumer behaviour. Even in the rather stylized setting of consumption/savings models (that already assumes very stringent conditions on utility functions), under somewhat plausible assumptions about instantaneous utility and about income uncertainty, these models typically do not permit analytical solutions (see [7]). Therefore, finding solutions requires numerical approximation by computers. But, as Allen and Carroll [7][9] note: "One fact that is known by any economist who has attempted numerical solution of consumption models is that finding optimal behavior in these models is an extraordinarily computation-intensive task." These consumption/savings models are often too complicated for even our most powerful computers to solve; as computers become more powerful, this enables economists to make their models a bit more realistic, and still get a solution.

If finding (approximate) solutions in these models seems to require using a supercomputer, assuming that consumers simply behave in accordance with these solutions may seem troublesome. Therefore, we can wonder whether the way in which consumption choice problems are specified in standard economics is very representative of the way in which people perceive of and think about these problems.

3.3.1 Risk, uncertainty and structural ignorance

Also, real-life consumption problems seem to involve a more profound, more fundamental kind of uncertainty than is modelled in Expected Utility Theory. Recall that in section 2.6, choice under uncertainty was defined alongside the notions of acts, outcomes and states. Also recall that the Expected Utility Theory approach towards dealing with uncertainty required that a decision-maker would know all these acts, outcomes and states. However, it seems that in many real-life choice situations we don't have clear descriptions of the states of the world that might occur, or of the resulting outcomes, and we could wonder how appropriate Expected Utility Theory is in these situations. As Gilboa and Schmeidler [18] argue:

> "Yet it seems that in many situations of choice under uncertainty, the very language of expected utility models is inappropriate. For instance, states of the world are neither naturally given, nor can they be simply formulated. Furthermore, sometimes even a comprehensive list of all possible outcomes is not readily available or easily imagined."

The kind of situations that Gilboa and Schmeidler refer to here does not correspond to either of the categories of risk and uncertainty, as distinguished by Knight [22]. Under **risk**, a decision maker does not know what will happen, but he does know everything that possibly could happen, and he knows the probabilities with which each of those possibilities will occur. In models of **uncertainty**, a decision maker also does not know what will happen, he also knows anything that could possibly happen, but he doesn't know the probabilities with which each of those possibilities will occur. In the situations that Gilboa and Schmeidler have in mind, the decision maker does not even know everything that could possibly happen. Thus Gilboa and Schmeidler add a third category, which they call **structural ignorance**:

[9] Also see section 3.6.

> " "risk" refers to situations where probabilities are given; "uncertainty" - to situations in which states are naturally defined, or can be simply constructed, but probabilities are not. Finally, decision under "structural ignorance" refers to decision problems for which states are neither (i) naturally given in the problem; nor (ii) can they be easily constructed by the decision maker."[10]

It seems that the context of consumer choice that we are considering in this work, should be categorized as one of structural ignorance, as there is much fundamental uncertainty about future consumption opportunities. A comprehensive list of all possible outcomes does not seem readily available or easily imagined. These outcomes would in the present case be commodity bundles that can be consumed. Thus knowing all outcomes would mean knowing all different commodities available at each future date. But people don't seem to have this knowledge, technological progress and fashion are obvious sources for commodities that will be available in the future, but which we cannot even imagine now.

Moreover, in the context of consumer choice, states don't seem to be naturally given, nor can they be easily constructed. For instance, financial markets, political constellations (both domestic and internationally), climatological conditions and our medical situations (or that of close relatives) may change in ways we cannot even conceive of now. And what is more, these changes could be quite dramatic and have great implications for our personal consumption.

3.3.2 Case-Based Decision Theory

Gilboa and Schmeidler argue that in situations of risk Expected Utility Theory is appropriate, and also in the face of uncertainty Expected Utility Theory or one of its generalizations may still be used. However, Expected Utility Theory is not very appropriate in cases of structural ignorance:

> " Expected utility theory does not describe the way people actually think about such problems. Correspondingly, it is doubtful that expected utility theory is the most useful tool for predicting behavior in decision problems of this nature. A theory that will provide a more faithful description of how people think would have a better chance of predicting what they will do."

But then what else is there that we can do? How else can we perform a microeconomic analysis then by assuming thorough, prospective thinking? Gilboa and

[10]This new category of structural ignorance is not the same as the category of "unawareness" (see e.g. Dekel et al [11]). Unawareness refers to situations where there are things that one does not know, that one does not know that one does not know, and so on. Under unawareness, one thinks that one has a good understanding of the situation, where one really doesn't, so this is clearly different from the concept of structural ignorance. Structural ignorance does seem to be identical, or at least very similar, to the notion of "radical uncertainty", a term that can (for instance) be found in the literature on Austrian Economics (e.g. Vaughn [45]).

Schmeidler [18] develop an entirely new paradigm for modelling decision-making under uncertainty, and call it Case-Based Decision Theory. This new paradigm suggests that people may use the past in making choices concerning the future, and thus assumes that decision makers come to their choices in a predominantly retrospective way.

> "Case-based decision theory suggests that people make decisions by analogies to past cases: they tend to choose acts that performed well in the past in similar situations, and to avoid acts that performed poorly."

Unlike the usual prospective way of trying to solve decision problems, where whatever happened in the past may basically only be used for determining probabilities, Case-Based Decision Theory is mainly based on retrospective viewing. Case-Based Decision Theory asserts that decisions are made by drawing on similarities that exist between the problem at hand and previous choice problems, and this approach has the advantage that the information that decisions are based on are known and certain.

As argued above, it seems that the context of consumer choice that we are considering here, should be categorized as one of structural ignorance, as there seems to be a more fundamental kind of uncertainty about future consumption opportunities than is modelled in Expected Utility Theory. But although the new paradigm that Gilboa and Schmeidler introduce is set in a relatively general type of choice situation, it does not seem very appropriate for the application we have in mind here. This is because Case-Based Decision Theory models a series of basically unrelated choice problems. That is, choices and outcomes from previous cases seem to be independent of the outcomes in present and future choice problems, these past cases are only used as indications of how favourable or unfavourable the outcomes of certain actions might turn out to be in the problem at hand. In the context of consumer choice, however, all different (sub)choices are strongly linked, consumer choice is actually one big problem that is divided into different subproblems. Thus Case-Based Decision Theory as such does not seem to be very appropriate to be applied in the consumption choice problems we have in mind here.[11] Still, the approach we will apply here to consumer choice does have something of a similar flavour.

3.4 Rationality and models

This section will provide a categorization of types of economic models and a categorization of types of rationality that will help place the new alternative framework, and that will help to distinguish it from the standard framework.

[11] Case-Based Decision Theory is also extended to explicitly model dynamic settings. Gilboa and Schmeidler [17] introduce a dynamic theory of consumer choices, but this theory is restricted to the case of repeated small decisions. And Gilboa and Schmeidler [18] model a theory of case-based planning, but this theory does not seem very appropriate here either, because of its rather specific nature.

3.4.1 Types of rationality

Corresponding to a difference in the notions of rationality as they are employed in economics and in psychology, Simon [42] distinguishes two types of rationality, called '**substantive rationality**' and '**procedural rationality**'. Substantive rationality corresponds to what is in economics usually called rationality proper. By this definition, an act or choice is called (substantively) rational if it is the result of selecting a course of action that will be most appropriate to the achievement of given goals within the limits imposed by given conditions and constraints. In contrast, a choice is said to be procedurally rational simply if it is based on reasoning, rather than on emotional or affective responses. Substantive rationality focuses on the outcome of the choice procedure or on *what* decisions are made, whereas procedural rationality focuses on the choice procedure itself or on *how* decisions are made.

We may recognize the definition of substantive rationality in the descriptions of the standard framework as presented in the previous chapter. Case-Based Decision Theory is a theory of procedurally rational behaviour. The behaviour that will be modelled in the learning framework that is presented here will also fall in the category of procedural rationality.

When in later chapters we refer to rationality, we use this term in the way in which it is usually used in economics, to mean substantive rationality.

3.4.2 Types of models

Simon [41] also distinguishes two types of models: 'models of optimization' and 'models of adaptive behaviour'. The first type, the **models of optimization**

> "are those that employ as their central concepts the notions of: (1) a set of alternative courses of action presented to the individual's choice; (2) knowledge and information that permit the individual to predict the consequences of choosing any alternative; and (3) a criterion for determining which set of consequences he prefers."

Thus, it can be seen that models of optimization will yield substantively rational behaviour. Obviously, the standard framework for modelling consumer choice fits neatly into this category of models. As argued above, and as seen under (2) in the definition, models of optimization require detailed and extensive information about the alternatives and about their consequences, and possibly a considerable amount of analytic ability enabling the consumer to actually determine a preferred alternative.

The **models of adaptive behaviour** are those that are based

> "on the ability of the individual to distinguish "better" (or "preferred") from "worse" directions of change in his behavior and to adjust continually in the direction of the "better".

Learning behaviour as in models of adaptive behaviour is also sometimes called 'hill climbing' or 'gradient descent' learning, by analogy to the problem of trying to find the peak of a mountain (or the lowest point of a valley) while being blindfolded, by simply continuing to move in the direction with the steepest slope. Models of adaptive behaviour require much less information about the environment, and analytic ability of the decision maker. Whereas in models of optimization the considerations and evaluations are of a global type, in adaptive models only local considerations and evaluations are needed.

The behaviour in models of adaptive behaviour seems to fall outside the scope of substantive rationality and should be categorized as procedurally rational. The above categorization of models is not exhaustive, not all learning models fall in one of the above two categories, in section 3.6 we will encounter two learning models that are not really models of adaptive behaviour. The ad hoc framework that will be presented here, *does* more or less fall into this last category.

Also note that these models of adaptive behaviour do allow for learning *within* an episode, rather than between episodes. The term 'directions of change in behavior' can also be interpreted to mean changes during an ongoing effort. Moreover, the hill climbing analogy gives an example where learning would take place within one trip to a mountaintop.

In many circumstances it seems quite natural to link the two types of models. For a given system, a model of optimization could specify some optimal solution or equilibrium, and an adaptive model could specify behaviour out of equilibrium, where this behaviour may or may not lead towards equilibrium. For instance, consider a system that at any point in time will find itself in a certain situation or state θ, where there is a criterion θ_c, or an optimal state that the system is directed towards. In a model of optimization that we might construct corresponding to this situation, the solution or outcome would simply be that θ_c will be the prevailing state. We could also construct an adaptive model for this situation. For instance, a very simple adaptation rule would be that if $(\theta_c - \theta)$ is the system's **error** (or departure from its goal), the system would adapt its state in the direction of the system's error, according to the **error-correction** term $\frac{\partial \theta}{\partial t} = k(\theta_c - \theta)$. Models where adaptations are made in this way are also called error-correction models. If the adjustment coefficient $k > 0$ is chosen appropriately, we would expect the system to move towards its goal or equilibrium, if no further shocks occur.

Linking the two types of models in such a way, does presuppose the global kind of information needed in the static model, which we may not assume to be available in the dynamic model, so as modellers we would then place ourselves on a higher level of information than the adapting decision maker. The above learning procedure does depend on θ_c, which may be problematic, as this optimal state may not be known by the adaptive decision-maker.

Artificial intelligence is one scientific area where these models of adaptive behaviour are very prominent. Still, early in the development of Artificial intelligence as a scientific discipline, models of optimization were often used (see e.g. Crevier [10]). Early artificially intelligent systems (robots) were designed to perform certain tasks, by being equipped with complete inner representations of the outside world (exact information about, or a 'map' of the relevant environment) and enough calculation power to be able to solve the problem of finding an optimal way to complete a task. Many first generation Artificial Intelligence models worked according to this 'brute force' method. One important criticism directed towards these models was that they were not judged to be biologically very realistic, for instance because of the assumptions of complete inner representations of the outside world. In reality, many tasks are performed by actual intelligent beings through interacting with the outside world, so that information gathering and action are interrelated, rather than isolated elements of the process of completing a task. When faced with the task of walking to the refrigerator, we do not plan ahead the number of steps in the exact directions that are needed to reach it, before executing the plan. We simply start moving in what seems to be the right direction, when there is furniture or other obstacles on the way to the fridge we determine the exact behavioural changes that are needed to avoid them only when we come close to them, and when we approach the refrigerator we see that a behavioural change is called for and we stop walking. Thus the task performance of a real-life intelligent being does not seem to be based on complete inner representations (an exact inner map of your house specifying all distances). Another problem for these first generation models is that in a lot of situations (such as chess) things become so complex that a computer's calculating power is insufficient for running down all different possibilities in a reasonable amount of time, and the system ends up being paralyzed. Later developments in Artificial Intelligence showed that in a lot of situations it is not necessary to endow the system with complete inner representations of the outside world and with gigantic calculating power to be able to achieve a (near-)optimal performance. Most second and third generation Artificial Intelligence models are in fact adaptive models that use some learning algorithm similar to gradient descent learning (see Crevier [10]).

The similarities between the criticisms to the first generation Artificial Intelligence models and some criticisms raised to the standard models of optimization in economics are worth noting (for instance about the assumptions of the very detailed information about the outside world). Maybe some flavour of the alternatives to these criticized models employed in Artificial Intelligence may also prove useful in economics.

3.5 The ad hoc framework

To get an idea for the new approach towards consumer choice that will be presented here, imagine you need new shoes and in a shoe store you see a pair of shoes (that you haven't seen before) that you like but that is a bit expensive so you hesitate whether you should buy the pair. As we saw standard economic theory says that people's choices are based on preferences over all possible plans of present and future consumption that are affordable. You had not seen this particular pair of shoes before, so obviously it wasn't included in previous specifications (or possible realizations) of the commodity space, and you cannot decide whether or not to buy the shoes by simply implementing a previously chosen consumption plan. Therefore to at least stay in line with standard economic theory, after the information update of seeing these new shoes a new consumption plan would have to be chosen. Once again, you should consider your preferences over all possible plans of present and future consumption that will be affordable, and from this determine your choice. Now the problem is updated with respect to the last choice problem by enlarging the commodity space to include the new shoes (and possibly by adjusting for other changed circumstances). So when faced with the new shoes you will have to choose a whole new consumption horizon, including the subdecision to buy the shoes or not, but also including specifications of your total consumption future. This new consumption plan is totally based on what you know (or think) will be available, and is therefore also based on the knowledge you have about all other pairs of shoes you could alternatively buy. However, in reality when deciding whether or not to buy the shoes it is likely (or at least plausible) that you will not only use the information that you do have about other shoes, but that you would also take into account the fact that if you would continue your search you would probably also encounter shoes that you had never seen before, and didn't know existed. That is, you do need new shoes, so if you would decide not to buy the shoes, then this decision is probably not only based on what you know or imagine to be available, but it will also take into account the fact that there is an unknown and unknowable stream of information that you will receive when continuing your search. And especially when dealing with product categories where information regarding what will be available at what price at certain points in time is limited (for example in markets where supply changes quickly because of fashion) actual choices are probably not determined solely by what you know to be (or become) available.

Of course, a situation where a consumer realizes that when he does not buy, he will probably obtain new information about his future economic environment that he cannot even conceive of now, cannot be accounted for within the standard model. Hence maybe the sort of decision process that someone might go through in choice problems like these is much closer to a sort of satisficing procedure (see e.g. Simon (1955)) than to the standard prospective way to look at it.

And, like Gilboa and Schmeidler argued, if the way in which we view these choice problems does not resemble the prospective view that would underlie the standard account of consumer choice, then maybe we could come up with an alternative theory of consumer choice that is closer to the sort of thought processes and trade-offs that people do actually make in their minds.

In what follows, we choose a different approach to consumer choice that seems closer to the type of reasoning from the above shoe example, and that starts from a different type of choice sets.

In fact, every time we make a choice whether or not to purchase a certain item, we are (at least implicitly) making a trade-off between the utility gain that is associated with the consumption of the item and the fall in utility due to being able to spend a bit less in the future. And while we may imagine that people translate this spending potential immediately into the consumption plans that can be bought from it, we might say that this is somewhat hypothetical. What we do know is that people just do make choices between goods and money on a daily basis.

Therefore, here we assume that total consumption horizons are not really our choice variables, but that instead bundles of present consumption (until the next moment where some relevant uncertainty is faced) and present monetary resources are really what we choose from. Then we proceed in a similar way as in the standard framework. We suppose that in making these choices a consumer behaves sufficiently rational, so that these choices can be derived from a preference relation (or a utility function). Of course, now these preferences would not be defined on complete consumption horizons, but on bundles of present consumption and savings.

The idea of treating money as part of choice variables, and letting money enter into utility functions, is quite unusual in economics. As in the previous chapter, in the standard view money is not valued in itself, it is only valued as a means of reaching consumption. In the indirect utility function, the value for money is not a direct given, but rather indirectly derived from direct valuations of consumption.

There is some support from neuroeconomics for the alternative approach of letting money enter into utility functions. Camerer, Loewenstein and Prelec [6] review some findings from neuroeconomics, and they find that this research suggests that money provides direct reinforcement. That is, they find that money becomes what psychologists call a "primary reinforcer", which would indicate that people value money without carefully computing what they plan to buy with it. They state that

> "brain-scans conducted while people won or lost money, suggest that money activates similar reward areas as other "primary reinforcers" like food and drugs do, which implies that money confers *direct* utility, rather than simply being valued only for what it can buy."

Thus, our alternative approach starts from preferences or utility on bundles of present consumption and present monetary resources, rather than on complete consumption horizons. However, these two types of preferences do not have the same status. The preferences on complete consumption horizons from the standard framework are absolute, in the sense that they are given and can never be wrong. By definition, maximizing these preferences will always yield optimality and (substantive) rationality.

Instead, the new type of preferences that are defined on bundles of present consumption and money, are not absolute. In a sense these new preferences are guesses or estimates. More specifically, especially the implicit valuations for money, relative to present consumption, are guesses or estimates. As the money variable still does represent future consumption, and as this future is unknown, valuations for money do not at all have to agree with optimality or rationality. In fact, these notions of optimality and rationality may not even be well-defined. And since these valuations for money are in fact estimates, some estimates may be better than others, and here the idea is that over time these estimates may be improved by a learning procedure. Thus, in the new framework that we will present here, we assume that the valuations for money are learned over time.

The idea behind how such valuations for money could be learned, would be that if at a certain point in time a consumer would regret having spent too much in the recent past, then he would adjust his valuation for money such as to value money more. And conversely, if at a certain point in time the consumer thinks that he could have spent more in the recent past, then the adjustment should lead to a lower valuation for money.

The resulting learning model does more or less fit the structure of an error-correction model (see the previous section). Here the valuations for money would be the states, and regret about past spending would indicate an error. Adjusting the valuations for money in the above way such as to deal with regret can be seen as error-correction.

Some casual observations about how people speak about their expenditure patterns may be seen as supporting the view that consumer choice is not determined by isolated decisions that are unrelated to past decisions. When confronted with disappointing news about their economic futures, people tend to use terms as 'cutting down on expenses' or 'making cutbacks' and 'tightening your belt'. Using these terms already seems to imply the existence of some previous expenditure pattern relative to which adjustments have to be made. Maybe this use of language can be seen as an indication that in fact consumers do not look at consumption decisions as a series of isolated single decisions that try to achieve an optimal use of endowments to make the future and the present as good as possible independently from the past. Instead, maybe people approach consumer choice much more like a continuous effort that reacts to changes by making adjustments to previously chosen strategies.

Learning in consumption models could alternatively also be modelled by means of more basic rules of thumb, each of which would simply specify a course of action that would be taken in any situation that might arise, where more efficient rules could be learned over time. Instead of courses of action that are learned over time, the ad hoc framework models a setting where preferences are (to some extent) learned over time. This ad hoc framework could be called a 'hybrid' framework, where in the short term a decision-maker would behave exactly as standard economic theory says he would, but where in the long term he doesn't. Although the behaviour that is modelled in this ad hoc framework is boundedly rational, this type of behaviour may require some more rationality than rule of thumb learning models. In the ad hoc framework consumers would in a sense be locally rational, but not globally.

3.6 Related literature

As mentioned in the previous section, the new framework that is presented here will start from preferences or utility functions that are defined on bundles of present consumption and present monetary resources. It is not an entirely new approach to consider money as part of the objects of choice that a decision-maker's preferences are defined on, see for instance Morishima [32] or Patinkin [35]. There, however, the motivations for doing so, and the models that follow from these motivations, are very different from what we have in mind here. In that literature, the motivation for including money in the arguments of utility functions has nothing to do with bounded rationality. There the motivation is to generalize the standard neoclassical, non-monetary models of equilibrium theory into models with monetary equilibria, such as to generalize non-monetary economies into monetary economies.

In the equilibrium theory literature it is standard practice to model an economy with a number of consumers who can choose from several commodities such as to maximize their preferences over their budget sets, when prices are given. In equilibrium theory the focus then is on economic equilibrium, where at certain equilibrium prices the total demand for any of the goods (as aggregated over all individuals in the economy) should equal the total (aggregated) supply for this good. A well-known property of these equilibrium theory models is that if all equilibrium prices are multiplied by the same constant, this will not change the prevailing equilibrium. In that sense, the economy is non-monetary.

The models of Morishima and Patinkin, try to generalize these equilibrium theory models by considering money more explicitly. They do so by introducing different periods, so that consumers may have to save or borrow. In any period a consumer's demand for money simply equals his budget minus the monetary value of his demand for consumption goods. In this literature the equilibrium concept is expanded so that not only all (physical) commodities should be in equilibrium, but so that also the money variable should be in equilibrium (that is, the total demand for money should equal the total supply). In these models we no longer get that equilibrium is unaffected by a multiplication of all equilibrium prices by the same constant. In that sense, the economy has become a monetary economy.

Obviously, in the present work the motivation for letting money enter into utility functions is very different, and focussed more on an individual or micro level, rather than an aggregate or macro level. In this work no equilibrium concept is defined, although it would not be impossible to do so.

The standard way to model consumption behaviour in which consumers act rationally, as in the previous chapter, is by no means the only approach that can be and has been taken. For instance, models have been devised in which consumers use simple rules of thumb in choosing their consumption (see e.g. Shefrin and Thaler [39] or Cochrane [9]). These rules of thumb would usually consist of simply spending a fixed proportion of current income, or of cash-on-hand, in all periods. It is possible that such rules would do a reasonable job, but only for certain parameter values or proportions. In these models the rules of thumb would typically be somehow exogenously given, so the question of how such rules of thumb would arise remains unanswered.[12]

Of course, the use of learning models, or models of adaptive behaviour, is not new

[12] Lettau and Uhlig [25] do provide a model where rules of thumb are learned.

in economics. In the last decades a lot of research has incorporated learning into economic models, for instance in the areas of game theory (see e.g. Young [47]), finance (e.g. Lettau [24] or Timmermann [44]), and also in macroeconomics (e.g. Evans and Honkapohja [12]).

However, there have only been a few theoretical papers trying to incorporate learning into consumption/savings models. Hence Friedman's assertion, that (roughly) optimal behaviour could be learned, has hardly been substantiated by theoretical research investigating whether it is likely, or even possible, to hold. Here we list the papers that do model learning in consumption/savings settings.

Marcet and Sargent [29] model learning in an investment problem, but they assume that consumers do behave according to dynamic programming theory, and are learning only about the distribution of shocks.

Lettau and Uhlig [25] construct a model where decision-makers learn their behaviour in a context of dynamic decision problems, which they also apply in a consumption/savings setting. In their setting decision-makers face income uncertainty: in each period an additional income realization occurs. These realizations are also called "states": if in a certain period a particular additional income realization occurs, the consumer is said to find himself in the corresponding state in that period. The paper considers a boundedly rational decision-maker who in every period can choose to use one out of a set of exogenously given rules of thumb, or courses of action. Such rules of thumb directly determine how much to consume in the relevant period as a function of the last period's savings and of the current period's additional income (state). These rules of thumb may also include a rule that prescribes to consume according to the optimal consumption function (and thus to behave in accordance with standard theory), and not all of these rules of thumb need to be applicable in all states (i.e. in periods in which some particular additional income realization occurs).

In any state in which more than one rules are applicable, the consumer can choose which rule to use, and he is assumed to choose that rule that gave him the best average past experience. In any period in which a certain rule is used, the consumer's experience of that rule in that period is reflected by the instantaneous utility experienced in that period plus a term that reflects the discounted value of the resulting situation that is faced in the following period. Thus, choices are made according to average past experiences, which are updated in every period, and this will thus give rise to a learning model.

Lettau and Uhlig find that in situations where the optimal decision rule is included in the set of rules, and where this optimal rule is applicable in all states, it may still happen that in some favourable state(s) (in which a high realization occurs) the decision-maker likes another, suboptimal rule (or more than one) better than the optimal rule, so that in such a state the optimal rule is consistently not chosen. This result is due to the fact that the consumer fails to recognize that the favourable outcomes that are experienced in periods in which the suboptimal rule(s) is used should be attributed to being in a good state, rather than to the rule that is used. The consumer thus fails to distinguish between luck and smart behaviour, and Lettau and Uhlig call this the "good state bias", and present this as a candidate explanation of the empirical finding of excess sensitivity of consumption to transitory income.

Obviously, this model is very different from the idea we have in mind here. Lettau and Uhlig consider learning of rules of thumb, and they start from an exogenously given set of rules of thumb, without the possibility of generating new rules. Learning would mean learning which of these given rules of thumb is best. Moreover, as the optimal consumption rule would somehow be exogenously given to the consumer, this model essentially does not allow for investigating Friedman's claim that optimal behaviour could be learned.

In the paper "Individual learning about consumption", Allen and Carroll [2] set up a learning model of consumption, and they do theoretically try to investigate whether (near-) optimal behaviour could be learned. However, their approach is still quite different from the approach that we have in mind here. We will list four important differences.

Firstly, the setting in which Allen and Carroll specify their learning model is much more specific. They consider one particular consumption/savings model, with one particular instantaneous utility function, where all parameters are calibrated to what they argue to be realistic values. They consider an instantaneous utility function u_0 that is of a constant relative risk aversion type: $u_0(c) = c^{1-\rho}/(1-\rho)$, where the coefficient of relative risk aversion ρ is calibrated to $\rho = 3$. Furthermore, they assume that in all periods the uncertain additional income can take three values $(0.7, 1, 1.3)$ with the probabilities $(0.2, 0.6, 0.2)$. The discount rate δ is set to equal 0.95.

Secondly, their learning model involves learning of rules of thumb, or consumption functions, that directly specify how much to consume as a function of cash-on-hand. All the rules of thumb that they consider have the following two-part linear structure

$$c_{\gamma,\mu}(m) = \begin{cases} 1 + \gamma(m-\mu) & \text{if } c \geq (1-\gamma\mu)/(1-\gamma) \\ m & \text{if } c < (1-\gamma\mu)/(1-\gamma) \end{cases}$$

for two positive constants γ and μ. There are different rules of thumb that differ only in the constants γ and μ. For the above calibrations Allen and Carroll specify (a numerical approximation of) the optimal consumption function $c^*(m)$, and they show that this optimal consumption function can be closely approximated (in utility terms) by a two-part linear function $c_{\gamma^*,\mu^*}(m)$ of the above type, for certain specific constants γ^* and μ^*. Then they argue that since this approximation to the optimal consumption function has a simple structure, these parameter values γ^* and μ^* could perhaps be learned.

Thirdly, their learning procedure employs a grid search. For each of these two constants γ and μ, intervals which would reasonably contain the optimal values γ^* and μ^* are identified, and subdivided into 20 points. Thus they end up with a grid consisting of 400 points, each of which represents a specific combination of values of the two constants.

Fourthly, each of these grid points is evaluated by means of the utility that is obtained while living with the corresponding rule for $n \in \mathbb{N}$ periods. That is, any point (γ, μ) is evaluated according to the summed discounted utilities $\sum_{i=t+1}^{t+n} \delta^{i-t} u_0(c_{\gamma,\mu}(m_i))$ that are obtained in the n periods in which this rule is used, thereby not taking account of the situation that the process is left in after the n periods. This evaluation does not take into account what happens after the n periods, so it does not value savings in period $t+n$. Finally, after the 400 times n learning periods, the grid point that yielded the highest utility in the periods that it was used in, will be chosen.

Due to time discounting, the distortion of not valuing savings in period $t+n$ will be relatively small for n relatively large. However, Allen and Carroll find that the best (or a good) grid point will generally not be reached, not even for large n, as each of the obtained utilities from a certain rule is heavily influenced by the additional income realizations that will occur early on in the n periods. If instead each of the rules is lived with during m separate time intervals of n periods, then as m and n get large, the process *can* be expected to find the best (or a good) rule in the grid with a high probability.

However, in this paper Allen and Carroll also focus on *how long* learning roughly optimal behaviour would take. They interpret periods as years (which they argue is in line with setting $\delta = 0.95$), and they conclude that it would take this learning process more than a million periods before it could identify a good grid point with a high probability.

This chapter has provided a discussion of the standard approach to modelling consumption behaviour. We have seen some empirical and conceptual problems associated with the standard approach. Also, this chapter has provided a first sketch of the new alternative framework, and some motivations for the choices made in this particular approach. Finally, this chapter has reviewed some related literature.

4 Ad hoc preferences

In the present chapter the first part of the alternative framework for consumer choice is introduced. Here an all-encompassing decision of lifetime consumption is cut up into a series of subdecisions, and it is modelled how a consumer would solve any such subdecision within this framework. In the first section subproblems are considered as an alternative to the all-encompassing decisions of lifetime consumption patterns from the standard microeconomic framework. In the subsequent sections we model how a consumer would deal with any of these subdecisions within this framework, much in the same vein as the standard microeconomic framework supposes the whole problem is dealt with. Paragraph 2 defines ad hoc preference relations for any such subdecision, section 3 defines ad hoc utility functions, which may represent these ad hoc preferences, and section 4 puts the elements together to form a basic ad hoc consumer problem that models which choices are made in any such subproblem. Thus in this chapter any such subdecision is treated in isolation. In chapter 6 we will continue the set-up of the new, alternative framework by specifying how the different subproblems, and the decision-making that is used to solve them, would be related.

4.1 Considering subproblems

In this work we want to construct a learning model in the context of consumer choice. A first thing to note is that in order to do this, it seems inevitable to distinguish different subproblems and different subdecisions taken at them. We cannot set up a learning model without modelling time explicitly, and viewing consumption as a series of subdecisions, rather than as one big all-encompassing decision. This is in contrast with the standard framework, where all choices are made simultaneously, so where essentially only one all-encompassing consumption choice is made.

As we also want to be able to study the convergence properties of such a learning model, we will need an infinite (though countably infinite) amount of such subproblems, as convergence in finite time is quite unlikely to occur in any model.

Therefore, we use the same discrete way to model time as in chapter 2: a time variable t progresses through the set $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$. So, where in chapter 2 we assumed this discrete time set to be either finite or countably infinite, here we only model situations of the last type, where $T = \infty$. In chapter 2 the time variable was used to be able to distinguish between periods and arrive at models of consumption/savings, and to be able to specify a gradual way in which uncertainty unfolds. Here we will use the same time structure, and the same decompositions of commodity spaces, commodity bundles, prices, and budgets to set up a model of consumer learning.

In this chapter we will focus on only a single, isolated subdecision, say at time t, in which there is a subset of commodities that have to be decided upon at that time, and where there remains a set of other commodities that will have to be decided upon later. These other commodities would probably in turn be subdivided and decided upon at different points in time, as the above way to model time seems to suggest, but for now we can proceed by considering what happens at time t without making specific what will happen afterwards.

To become more formal, recall that in the standard model under certainty, preferences were defined on the total commodity space $X = \mathbb{R}^n_+$, where $n \in \overline{\mathbb{N}}$ denoted the total number of commodities that were assumed to be available for purchasing at some point in time. The elements that such a commodity space consists of are commodity bundles that specify amounts of all commodities, including all future commodities.

However, here we assume that the consumption opportunities do not present themselves simultaneously, but rather according to the above discrete model of time t. That is at every stage $t \in \mathbb{N}_0$, by $X_t = \mathbb{R}^{n_t}_+$ we denote the space of all combinations of amounts of commodities available at time t, where $n_t \in \mathbb{N}$ denotes this total number of commodities available at stage t. And thus we get that $X = \times_{t=0}^{\infty} X_t$. Since here we assumed the overall problem to consist of a countably infinite number of periods, here we should have that indeed $n = \infty$, so $X = \mathbb{R}^{\infty}_+$.

Axiom 4.1.1 *The total number of available commodities n is (countably) infinite. Moreover, there is a countably infinite number of periods, and for any period t the total number of commodities available in that period is denoted $n_t \in \mathbb{N}$. Accordingly, $X_t = \mathbb{R}^{n_t}_+$ denotes the commodity space that corresponds to all commodities available at time t, and the (total) commodity space can be written as $X = \times_{t=0}^{\infty} X_t$.*

The above decomposition of the total commodity space also allows for the following, alternative approach of cutting up the overall problem into distinct, smaller subproblems. Instead of making one big decision involving all stages at once, here we suppose that a consumer would make an infinite number of smaller decisions, one for each period t. Therefore when at stage t of the process, t periods have already passed, and t subdecisions have already been made. Then at stage t, another one of these smaller decisions will have to be made. Now, at stage t, the consumer is confronted with the period-t commodity space X_t of which one element x_t will have to be picked. Thus such a smaller problem involves determining what to choose from only the set $X_t = \mathbb{R}^{n_t}_+$. To extend the model, a few definitions are presented.

Definition 4.1.1 *At time t, a* ***present commodity space*** *is a space $\mathbb{R}^{n_t}_+$, where n_t is the number of commodities available at stage t. A* ***present commodity bundle*** *in some period is an element of the corresponding present commodity space.*

Definition 4.1.2 *At time t, a* ***past commodity space*** *is the Cartesian product of all the present commodity spaces of periods preceding time t. A* ***past commodity bundle*** *in some period is an element of the corresponding past commodity space.*

Definition 4.1.3 *At time t, a* ***future commodity space*** *is the Cartesian product of all the present commodity spaces of periods after time t. A* ***future commodity bundle*** *in some period is an element of the corresponding future commodity space.*

A typical present commodity space at time t is denoted by $X_t = \mathbb{R}_+^{n_t}$, a typical present commodity bundle is denoted by x_t. A past commodity space at time t will typically be denoted by $W_{t-1} := \times_{i=0}^{t-1} X_i = \mathbb{R}_+^{k_{t-1}}$, where $k_{t-1} = \sum_{i=0}^{t-1} n_i < \infty$ is the number of commodities that have been available to the consumer before stage t, a typical past commodity bundle is denoted by w_{t-1}. A future commodity space at time t is typically denoted by $Y_{t+1} := \times_{i=t+1}^{\infty} X_i = \mathbb{R}_+^{\infty}$ (the number of commodities that will become available to the consumer after stage t is infinite), and a typical future commodity bundle is denoted by y_{t+1}.

Our notation and interpretation of the formal objects introduced here will remain in line with that of chapter 2. For each $x_t \in \mathbb{R}_+^{n_t}$ and each $1 \leq i \leq n_t$, we will interpret the i'th component $x_t^i \in \mathbb{R}_+$ to represent the amount of commodity i in bundle x_t, and similarly for past and future commodity bundles.

Now, in accordance with the sequential unveiling of the commodity space as in chapter 2, we assume that there is never any uncertainty about what is presently (and was previously) available. Thus a consumer has full knowledge of what the sets W_{t-1} and X_t look like, and we do not yet make any assumptions here about what knowledge our consumer has with respect to Y_{t+1} (we will come back to this later). Since the sets $W_{t-1} = \mathbb{R}_+^{k_{t-1}}$ and $X_t = \mathbb{R}_+^{n_t}$ are known, their respective dimensions k_{t-1} and n_t are also known at time t. However, the sets $X_\tau = \mathbb{R}_+^{n_\tau}$ and also the dimensions n_τ for $\tau > t$ may not be known at time t. Still, since there is an infinite number of periods, and in each of these periods the number of commodities available will be a natural number, we see that the dimensions of the sets X and Y_{t+1} must be countably infinite, and therefore are basically known.

We are now considering a situation at time t where a previously chosen past commodity bundle w_{t-1} is given, and where a consumer has a certain period-t budget. We can now view the resulting choice problem as one of choosing an element in X_t and determining how much money to save for later consumption from the remaining commodities in Y_{t+1}. We propose that our consumer simply chooses how much to consume, and how much to save for the future. Thus here we do not take the ad hoc commodity bundles in X_t alone to be the elements of choice, but rather we attach to every such element x_t an amount of money that is saved for consumption from Y_{t+1}, so we add a dimension to X_t that represents a new money good.

Definition 4.1.4 *At time t, an* ***extended present commodity space*** *or alternatively, an* ***ad hoc choice set****, is a Cartesian product $X_t \times \mathbb{R}_+$ of a present commodity space X_t and the non-negative real numbers. Any element (x_t, s_t) of such an ad hoc choice set is called an* ***ad hoc choice pair****.*

Definition 4.1.5 *At time t, an* ***ad hoc commodity space*** *is a Cartesian product $W_{t-1} \times X_t \times \mathbb{R}_+$ of a past commodity space W_{t-1}, a present commodity space X_t and the non-negative real numbers. Any element (w_{t-1}, x_t, s_t) of such an ad hoc commodity space is called an* ***ad hoc commodity bundle****.*

As can be seen in the definitions, a typical ad hoc choice set is denoted $X_t \times \mathbb{R}_+$, a typical element of such an ad hoc choice set is denoted (x_t, s_t). A typical ad hoc commodity space is denoted $W_{t-1} \times X_t \times \mathbb{R}_+$, and a typical element of such an ad hoc commodity space is denoted (w_{t-1}, x_t, s_t). Here w_{t-1} is a past commodity bundle, x_t is simply a present commodity bundle from X_t, and s_t is a non-negative real number, that can be seen as a new auxiliary commodity, and that we will throughout interpret as an amount of money that is saved for consumption in the remaining periods. The lower bound of zero on the amounts of savings that are allowed reflects the fact that saving is possible but borrowing is not.

It is important to bear in mind that the amount s_t that is saved, does not have to be equal to the budget m_{t+1} with which the next period $t+1$ will be entered. After all, this depends on the specifications of how the m_t's are related, and thus ultimately on additional incomes (and interest rates). We may recall that there were several ways to do this. However, in this chapter we will treat only one period in isolation; the links between subsequent are dealt with in a later chapter. Therefore here we do not need to specify the links between budgets across periods. Still, it is important to distinguish s_t from m_{t+1}, and to remember that the extra dimension in ad hoc choice sets or ad hoc commodity spaces represents amounts of *savings*, and not next period budgets.

Hence in the elements of choice that are primitive to the model, the complete descriptions of all (future) consumption from Y_{t+1} from the standard framework are here replaced by amounts of money, that represent savings for consumption from Y_{t+1}.

4.1.1 Subproblems in dynamic programming

At this point it is worth noting a similarity between the ideas presented here of considering subproblems, and the dynamic programming approach towards solving consumption/savings models. Remember that the sequence problem approach to solving such a model involves solving for an optimal infinite sequence of consumption levels at once. In stead, in the dynamic programming approach to solving such a model, the maximization problem inside a functional equation considers a single subproblem, in which a decision-maker only has to decide how to divide his resources m_t between consumption c_t and savings $s_t = m_t - c_t$. To make this view consistent, a recursive approach was taken, thereby determining a value function that actually solves such a functional equation. In what follows, we will see that these similarities can be further extended.

4.2 Ad hoc preferences

Here we assume that when deciding upon which ad hoc choice pair to choose, given a past commodity bundle and a budget, our decision-maker would just make trade-offs between the benefits obtained from consuming the corresponding present commodity bundles and the costs of paying for them. Here we proceed in a way similar to that of the standard microeconomic framework, by assuming that a consumer can state preferences between any pair of elements from the ad hoc commodity space. In order to be sure to end up with stable and consistent preferences, we will assume that underlying the choices from such a set is a binary relation on this set that satisfies completeness and transitivity. Given a past commodity bundle that was chosen previously, and given the new ad hoc choice set $X_t \times \mathbb{R}_+$, we assume that any consumer's preferences can be represented by a preference relation on this choice set.

Here we also want to allow for the possibility that such preferences on the ad hoc choice set depend on the previously chosen past commodity bundle. Therefore here we will not just define preferences on the ad hoc choice set, but rather on the ad hoc commodity space. Note, however, that at period t where these new preferences become relevant, one specific past commodity bundle was already chosen, while a present commodity bundle is not. Therefore we could also simply view the particular past commodity bundle that is given, as a parameter that may influence preferences on the ad hoc choice set. And whereas in principle it would suffice to have preferences for only one given past commodity bundle, here we want to be able to track how these past choices may influence present preferences on the ad hoc choice set. Therefore we do not define ad hoc preferences on the Cartesian product of the present commodity space, the non-negative real numbers, and a set containing only a single past commodity bundle. Rather, here we define ad hoc preferences on the Cartesian product of the present commodity space, the non-negative real numbers, and the past commodity space. Note that this is not an extra requirement of the model. This assumption actually makes the model more general, not more specific. It also makes it possible to allow for things like habit formation, or more generally complementarities and substitutabilities over time.

Definition 4.2.1 *An* ***ad hoc preference relation*** *is a preference relation defined on an ad hoc commodity space.*

At time t, given the ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$, a typical ad hoc preference relation is denoted by $\succsim^{(t)}$.

Note that if in the standard microeconomic framework for consumer choice a situation would arise in which a decision is needed regarding what to choose from X_t, and how much money to keep for later consumption from Y_{t+1}, then a decision maker can only make such a decision by somehow imagining what Y_{t+1} will look like, because by assumption his decisions are derived from preferences over complete consumption bundles in X. So even if he does not in fact know yet which commodities will be available later in Y_{t+1}, he can only arrive at a solution to the problem at hand by imagining what Y_{t+1} might look like, since preferences on X are all that he has to reach a decision. Then he would arrive at a decision of which element of $X_t \times \mathbb{R}_+$ to choose, by translating money back into complete specifications of affordable future consumption.

In this chapter we just start from the fact that if a consumer is faced with the problem of what to choose from the set $X_t \times \mathbb{R}_+$, he will somehow have to make a choice, and from the assumptions that he can state preferences over the elements, and that these preferences can be represented by a preference ordering. Here we do not yet worry about whether these ad hoc preferences are in fact (or can be) derived from preferences on the whole commodity space X. Later we will come back to this question of where these new ad hoc preferences do come from.

Note that in the standard framework, if the commodity space is given then preferences (and utility) can be regarded as exogenous, there we do not have to worry where such preferences come from. In the context that we are considering here it does not suffice to treat ad hoc preferences as being completely exogenous, at least to some extent these preferences will have to be "explained". After all, it seems inevitable that preferences for money should somehow be related to future purchasing power, but in what way? Also note that while in the standard framework there is only one preference relation defined over all commodities at once, in the ad hoc framework a different preference relation is needed for every single period. Then we also need to answer the question of how these ad hoc preferences would (or should) be related to ad hoc preferences in different periods. We will come back to these questions later, but in the present chapter, we will only consider what happens in an isolated period, and we *will* treat ad hoc preferences as being exogenous.

4.3 Ad hoc utility

Again, for mathematical convenience ad hoc preferences will usually be represented by means of (ad hoc) utility functions.

Definition 4.3.1 *An **ad hoc utility function** is a utility function defined on an ad hoc commodity space.*

Such an ad hoc utility function (that is defined on an ad hoc commodity space corresponding to period t) is typically denoted by $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$.

Then, as in section 2.2, the question of whether such an ad hoc preference relation can always be represented by an ad hoc utility function can be posed. Now, note that although the remaining budget $s_t \in \mathbb{R}_+$ in this model has a new meaning, mathematically we still have a choice set in $\mathbb{R}^{k_t+1}$ and the whole mathematical analysis from the standard framework is still valid. So if we also assume an (ad hoc) preference relation $\succsim^{(t)}$ to be continuous[13] on $W_{t-1} \times X_t \times \mathbb{R}_+$, then theorem 2.2.1 tells us that it can be represented by a continuous ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$.

4.4 The basic ad hoc consumer problem

Similar to consumption bundles, price vectors p are broken up into sequences $p = (p_0, p_1, p_2, ...)$, where for every t the vector $p_t \in \mathbb{R}_+^{n_t} \setminus \{0\}$ denotes the vector of prices for the commodities in the corresponding present commodity space X_t. And similarly to the definitions of w_{t-1} and y_{t+1}, a **past price vector** will be denoted by $o_{t-1} := (p_0, p_1, ..., p_{t-1})$, and a **future price vector** is denoted $q_{t+1} := (p_{t+1}, p_{t+2}, p_{t+3}, ...)$.

At time t, our consumer is faced with prices $p_t \in \mathbb{R}_+^{n_t} \setminus \{0\}$ for the goods in X_t, and if he has an available budget of $m_t \geq 0$ monetary units, this determines what is feasible in that period. Here it seems natural to take $(p_t, 1)$ as a price vector for bundles in $X_t \times \mathbb{R}_+$, so that the monetary value of the bundle (x_t, s_t) in $X_t \times \mathbb{R}_+$ is given by

[13] With respect to the usual Euclidian topology.

$p \cdot x_t + s_t$. Thus, the consumer's choice of elements from the ad hoc choice set will have to satisfy the budget constraint $p \cdot x_t + s_t \leq m_t$.

Definition 4.4.1 *At time t, given an ad hoc choice set $X_t \times \mathbb{R}_+$, prices p_t for present consumption and a budget m_t, an* ***ad hoc budget set*** *is a set of all ad hoc choice pairs (x_t, s_t) that satisfy $p_t \cdot x_t + s_t \leq m_t$.*

In the framework as developed in the previous sections we arrived at ad hoc utility functions from ad hoc preference relations. These ad hoc preference relations were defined on ad hoc commodity spaces that consist of ad hoc commodity bundles. Now at time t, a past commodity bundle is given and fixed, therefore the decision problem that a consumer would face should not consist of maximizing ad hoc utility over all ad hoc commodity bundles that are feasible. Rather, the most straightforward way to put the elements of the framework together in order to describe the basic (ad hoc) consumer problem would be as the problem of maximizing ad hoc utility over the elements of the ad hoc choice set that satisfy a feasibility condition, given the fixed past commodity bundle.

Definition 4.4.2 *Given a past commodity bundle $\overline{w}_{t-1} \in W_{t-1}$, an ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$, a present price vector $p_t \in \mathbb{R}_+^{n_t} \backslash \{0\}$ and a budget $m_t \geq 0$, the* ***basic ad hoc consumer problem*** *is given by:*

$$\max_{(x_t, s_t)} u^{(t)}(\overline{w}_{t-1}, x_t, s_t) \text{ sub to } (x_t, s_t) \in X_t \times \mathbb{R}_+, p_t \cdot x_t + s_t \leq m_t.$$

If the function $u^{(t)}(\overline{w}_{t-1}, ., .)$ is differentiable in the second and third argument, then again such an ad hoc consumer problem can be solved by the Lagrange method. If $(x_t^*, s_t^*) \in X_t \times \mathbb{R}_+$ is an internal solution to this maximization problem, then it will hold that there is a Lagrange multiplier $\lambda \geq 0$ such that $\frac{\partial u^{(t)}}{\partial x_t^i} = \lambda p^i$, $\forall\, i \in \{1, 2, ..., n_t\}$, and such that $\frac{\partial u^{(t)}}{\partial s_t} = \lambda$. Therefore we see that $\frac{\partial u^{(t)}}{\partial x_t^i} \frac{1}{p^i} = \frac{\partial u^{(t)}}{\partial x_t^j} \frac{1}{p^j} = \lambda = \frac{\partial u^{(t)}}{\partial s_t}$, for all i and $j \in \{1, 2, ..., n_t\}$. So in an internal solution, the quantities of each of the commodities in X_t will be such that the marginal utility with respect to such a commodity, divided by its price, is the same for each of these commodities, and is equal to the marginal utility of money. Within this framework it also still holds that $\lambda = \frac{\partial v^{(t)}}{\partial m_t}(p, m_t)$, where $v^{(t)}(p, m_t)$ is the indirect utility function associated with $u^{(t)}$. Hence we see that because we set the price of s_t to be 1, it holds that at the optimum the marginal utility of money $\frac{\partial u^{(t)}}{\partial s_t}$ is equal to the marginal value of budget $\frac{\partial v^{(t)}}{\partial m_t}$. So here the marginal utility of money $\frac{\partial u^{(t)}}{\partial s_t}$ plays the same role as the marginal utility of budget did in the standard framework.

We also see that

$$MRS_{ij} = \frac{\partial u^{(t)} / \partial x_t^i}{\partial u^{(t)} / \partial x_t^j} = \frac{p^i}{p^j}$$

for all i and $j \in \{1, 2, ..., n_t\}$, and that

$$MRS_{ir} = \frac{\partial u^{(t)} / \partial x_t^i}{\partial u^{(t)} / \partial s_t} = p^i$$

for all $i \in \{1, 2, ..., n_t\}$. So for any pair of commodities in X_t, the marginal rate of substitution between these goods is equal to the ratio of their prices. The marginal rate of substitution between any of the X_t-goods and money s_t is equal to the price of this good.

Remember that while in the standard framework this equality of marginal rates of substitution and price ratios holds for all pairs of commodities (in X), here this doesn't have to be true. Here it holds only for all pairs of goods in X_t, and doesn't have to hold for other goods outside of X_t. And while this quotient of marginal utility and price will be constant again within other sets of commodities that are decided upon simultaneously, within this new framework it is possible that this quotient may differ between these different sets.

So from this analysis we also see that money can be seen as sort of an auxiliary good that helps to distribute consumption or utility efficiently over the different periods. We also see that here utility might not be distributed in such a completely efficient way as in the standard framework.

All the assumptions that are made thus far in this chapter are summarized into the following axiom.

Axiom 4.4.1 *In every period, a consumer is assumed to be endowed with a continuous ad hoc preference relation that can be represented by a continuous utility function, and to choose an element from the corresponding ad hoc choice set, that solves the relevant basic ad hoc consumer problem.*

4.4.1 Ad hoc utility and basic ad hoc consumer problems in dynamic programming

In the above subsection dealing with dynamic programming, it was already noted that the ad hoc framework presented here, and the approach towards solving consumption/savings models using dynamic programming, are somewhat similar in the sense that they both start from considering subproblems, rather than taking all variables into account at once. Now, the similarity between the ideas presented here and the ideas behind dynamic programming can be further extended to the notions of ad hoc utility (and ad hoc preferences) and of the basic ad hoc consumer problem.

Dynamic programming starts from problems such as

$$\max_{(c_t, s_t)}[u_0(c_t) + \delta V(s_t + I)] \text{ sub to } c_t + s_t \leq m_t. \tag{1}$$

(Remember that to make these models stationary we set $I_t = I$, for all t.) This maximization problem looks like a basic ad hoc consumer problem, as the space $\mathbb{R}_+ \times \mathbb{R}_+$ over which the maximization is done, can be seen as an ad hoc choice set. The set of all elements of this space that satisfy the constraint $c_t + s_t \leq m_t$ can be seen as an ad hoc budget set. And the function $u_0(c_t) + \delta V(s_t + I)$ that is being maximized in this problem looks like an ad hoc utility function. Of course, in our account at period t an ad hoc utility function $u^{(t)}$ would have to be defined on the ad hoc commodity space $W_{t-1} \times \mathbb{R}_+ \times \mathbb{R}_+$ and should therefore also include past consumption. The above function $u_0(c_t) + \delta V(s_t + I)$ is only defined on the set $\mathbb{R}_+ \times \mathbb{R}_+$.

Still, recall that the methods of dynamic programming are only used in models with additively separable total utility $\sum_{t=0}^{\infty} \delta^t u_0(c_t)$. Therefore the utility of future consumption after period t also enters this total utility function additively. And since the function V is devised exactly to collapse the desirability of the whole consumption future after period t into the single variable of period-t savings, it seems that the function $u^{(t)} : W_{t-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, as defined by

$$u^{(t)}(w_{t-1}, c_t, s_t) = u^{(t)}(c_0, c_1, ..., c_t, s_t) = \sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1} V(s_t + I), \tag{2}$$

would be a good candidate to specify an ad hoc utility function.

Then, for a past commodity bundle $w_{t-1} = (c_0, c_1, ..., c_{t-1}) \in W_{t-1}$ and a period-t budget $m_t \in \mathbb{R}_+$ given, the corresponding basic ad hoc consumer problem could be represented as a maximization of the ad hoc utility function in (2) over the set $\{(c_t, s_t) \in \mathbb{R}_+ \times \mathbb{R}_+ : c_t + s_t \leq m_t\}$. And as the past commodity bundle $w_{t-1} \in W_{t-1}$ is fixed, this basic ad hoc consumer problem is indeed very similar to the problem in (1). In fact, for a fixed w_{t-1} the function in (2) equals the function in (1) plus a constant, so it is clear that the pair (c_t^*, s_t^*) will solve the problem of maximizing $u^{(t)}$ in (2) (for $w_{t-1} \in W_{t-1}$ given) over the ad hoc choice set $c_t + s_t \leq m_t$, if and only if it will solve the maximization problem from the functional equation in (1). Thus, the maximization problems inside functional equations are indeed very similar to basic ad hoc consumer problems.

However, the methods of dynamic programming are made consistent by finding not just any, but one particularly appropriate choice for the value function V, namely the value function V^* that solves the functional equation

$$V^*(m_t) = \max_{(c_t, s_t): c_t + s_t \leq m_t} [u_0(c_t) + \delta V^*(s_t + I)],$$

and that consequently also gives the maximally attainable additional future utility.

Therefore as we will see more clearly in the next chapter, a decision-maker who chooses in basic ad hoc consumer problems by means of ad hoc preferences as in (2), using an optimal value function that solves the corresponding functional equation, is no less rational than a decision-maker solving the problem at once in a sequence problem. This solution to a functional equation is equally well based on perfect foresight and perfect rationality; it is simply a more convenient way to find a solution to the same problem. Still, as the problem that is being solved inside a functional equation can be seen as a basic ad hoc consumer problem, the ideas behind the ad hoc framework are somewhat similar to the ideas behind dynamic programming.

The ad hoc framework in consumption/savings models Aside from the similarities that exist between the ad hoc framework and the dynamic programming approach to solving consumption/savings models, of course the ad hoc framework can also just be modelled in consumption/savings models. Solving a dynamic program can really be quite a complicated task, which is the reason why finding ways to solve them is a science by itself. Therefore in such settings we may also want to consider the possibility of boundedly rational behaviour, as postulated in the above axiom.

We would again start from a decision-maker who subdivides the problem of what to decide in a consumption/savings problem into different subproblems, or ad hoc problems. Any of these subproblems that our boundedly rational consumer would face, say at time t, would in fact consist of dividing resources into period-t consumption c_t and period-t savings s_t. And each of these trade-offs are supposed to be made by using ad hoc preferences $u^{(t)}(w_{t-1}, c_t, s_t)$.

Total utility, which our consumer would apparently find too complicated to maximize at once, would still be of the additive form $\sum_{t=0}^{\infty} \delta^t u_0(c_t)$, and would therefore also be additive in future utility. And, because of this additive separability of the function he should optimally be trying to solve, it seems rather straightforward to keep using this additive separability assumption for his ad hoc preferences. That is, we may assume here that the ad hoc preferences will take the form:

$$u^{(t)}(w_{t-1}, c_t, s_t) = u^{(t)}(c_0, c_1, ..., c_t, s_t) = \sum_{i=0}^{t} \delta^i u_0(c_i) + \widetilde{V}(s_t)$$

for some value function $\widetilde{V} : \mathbb{R}_+ \to \mathbb{R}$. In order to stay closer to the notation and interpretation of the value function as in dynamic programming, here we choose the more specific form $\delta^{t+1} V(s_t + I)$ for $\widetilde{V}(s_t)$:

$$u^{(t)}(w_{t-1}, c_t, s_t) = \sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1} V(s_t + I).$$

Trying to maximize this function over some budget constraint $c_t + s_t \leq m_t$, given past choices w_{t-1}, indeed does look simpler. However, the problem of what value function V to use, still remains. As we saw, the value function V^* that solves the corresponding functional equation would be the best choice, as it simply collapses the whole consumption future into one single monetary variable in an optimal way.

Here we do not yet try to answer the question of which value function is used and simply treat such a value function as given. In later chapters we will come back to the questions of which value function will (or could) be used. ■

In this chapter we started the formal set-up for a new, alternative framework. We postulated that series of subproblems and subdecisions are considered, and that each of these subproblems consists of choosing how much to consume and how much to save in that period. These choices are supposed to be arrived at, from maximizing (ad hoc) preferences over all affordable combinations of present consumption and savings.

5 Total and ad hoc preferences

In the previous chapter a first component of the alternative ad hoc framework was introduced. In that chapter we distinguished sequences of consumption subdecisions, and modelled how choices would be made in any period (subdecision) in isolation. The previous chapter did not yet specify how the individual periods would be linked; these specifications will follow in chapter 6.

But before we close the model, this chapter investigates the links and relations that exist between the alternative ad hoc framework as presented so far, and the standard framework. That is, this chapter focuses on the links between the alternative view of ad hoc preference relations and ad hoc maximization problems on the one hand, and the standard view of total preference relations and corresponding maximization problems on the other hand. We will see that the alternative view is not incompatible with the standard view. More specifically, we will see that the standard framework can be seen as a special case of the alternative ad hoc framework, or equivalently, that the ad hoc framework introduced here can be seen as an extension of the standard framework.

In order to demonstrate this, we will show that it is always possible to consistently model ad hoc preferences within the standard framework, and by showing that is not always possible to consistently model a standard framework 'around' a system of ad hoc preferences. That is, we will see that total preferences within the standard framework can always be summarized consistently into ad hoc preferences. And we will also see that for a single given ad hoc preference relation there usually exists total preferences such that the ad hoc preferences summarize these total preferences consistently, but that for multiple given ad hoc preference relations there generally does not exist total preferences that can consistently be summarized into each of the ad hoc preference relations. Only if these different ad hoc preferences are related in some specific way, then there will be total preferences that can be consistently summarized into each of the ad hoc preference relations.

The exact way in which the different ad hoc preferences should be related in order to be consistent with the standard framework also gives a first answer as to how the different periods could be linked within the alternative ad hoc framework. That is, the extra assumptions needed to make the ad hoc framework agree exactly with the standard framework, provide a way in which the links across periods can be specified. In fact, this specification of the links between periods will serve as a benchmark in later chapters.

This chapter consists of four sections. The first section will specify what is meant by the aforementioned property that total preferences would be consistently summarized into ad hoc preferences, by providing formal definitions for consistency between ad hoc preferences (or utility) and total preferences (or utility).

The second section shows that from total preferences as specified in the standard microeconomic framework, we can always derive ad hoc preferences in a consistent way. Essentially this result says that the global problem from the standard framework can be correctly summarized into local problems in the ad hoc framework.

The third section investigates the converse direction: are the specifications of the ad hoc framework non-contradictory with the specifications of the standard framework? In case only a single period's ad hoc preferences would be given, the answer will turn out to be affirmative: we can generally find total preferences within the standard framework with which the given ad hoc preferences are consistent. In case multiple periods' ad hoc preferences would be given, the answer is less clear-cut: there will exist total preferences within the standard framework with which all of the given ad hoc preferences are consistent, only if these different ad hoc preference relations are related in a certain way.

The fourth section investigates whether properties of functional separability would be carried over from total utility functions to consistent ad hoc utility functions.

5.1 Consistency

In this chapter we investigate how the ad hoc framework, or the notion of ad hoc preferences, is related to the standard framework, or the notion of total preferences. We investigate whether from a given total preference ordering on a total commodity space, we can always derive an ad hoc preference ordering on some smaller ad hoc commodity space, such that these ad hoc preferences basically reflect the same preferences as the original total preference ordering. And conversely, we investigate whether for a given ad hoc preference ordering on some ad hoc commodity space, there exists a total preference ordering on some larger total commodity space, such that the ad hoc preferences reflect the same preferences as the total preference ordering. In order to be able to make these investigations precise, we will first need to specify what is exactly meant by this property that total preferences can be consistently summarized into ad hoc preferences, so that ad hoc preferences would reflect the same preferences as total preferences.

A precise definition of this property can only be given within a setting where total preferences are well-defined: in the standard static framework, either under certainty or under uncertainty. First we will define the property of consistency in models of certainty, and later in this section we will do the same in expected utility models.

Hence, the present section is set within the standard microeconomic framework under certainty, and we assume that a commodity space $X = \mathbb{R}_+^\infty$ is known and given. Also given is a discrete time variable t that progresses through the set $\{0, 1, 2, ...\}$. Accordingly, the commodity space X is written as a Cartesian product $X = \times_{t=0}^\infty X_t$, where $X_t = \mathbb{R}_+^{n_t}$ and $n_t \in \mathbb{N}$. Similarly, the prevailing price vector p is known, and broken down into a sequence $p = (p_0, p_1, p_2, ...)$ with $p_t \in \mathbb{R}_+^{n_t} \backslash \{0\}$, and an income stream $\{m_0, I_1, I_2, ...\}$ is given. Also given is a continuous preference relation $\succsim$ on X, that can be represented by a continuous utility function $u : X \rightarrow \mathbb{R}$.

Now, within this setting we consider the situation at time t. Not all commodities are to be purchased at the same time, and here we suppose that particular interest is in just the subset of all goods that can be purchased in period t. At time t, $\bar{w}_{t-1} := (\bar{x}_0, \bar{x}_1, ..., \bar{x}_{t-1})$ was already chosen from the set $W_{t-1} := X_0 \times X_1 \times ... \times X_{t-1}$, and a decision is required with respect to how much of each of the n_t goods to purchase, and how much of the budget m_t should be saved for later. Thus, $X_t \times \mathbb{R}_+$ denotes the ad hoc choice set at time t, where the last dimension of this set denotes savings for consumption from $Y_{t+1} = \times_{i=t+1}^{\infty} X_i$. In subsequent periods decisions will have to be made as to how to spend the remaining budget on the goods in Y_{t+1}. All of these decisions are, however, still based on $u : X \to \mathbb{R}$, so preferences are stated for all goods simultaneously.

The problem of translating money (savings) into optimally chosen future consumption bundles (and with it the idea of consistency) was already touched on in those (sub)sections of the previous chapter dealing with dynamic programming. Here we will further specify these ideas in a dynamic programming setting, to motivate the formal definition of consistency in more general settings.

5.1.1 Dynamic programming

Recall that in consumption/savings models total preferences are additively separable and satisfy exponential discounting, so that they can be represented by $\sum_{t=0}^{\infty} \delta^t u_0(c_t)$. Such a utility function would then have to be maximized over a budget set. If the model were stationary (i.e. $I_t = I$, $\forall t$), then a convenient way to solve such a maximization problem would be by using the methods of dynamic programming, more specifically by a functional equation

$$V^*(m_t) = \max_{c_t + s_t \leq m_t} \{u_0(c_t) + \delta V^*(s_t + I)\}.$$

As seen in the previous chapter, the function that is maximized inside the functional equation $u_0(c_t) + \delta V^*(s_t + I)$ is like an ad hoc utility function, and the maximization problem inside the functional equation is like a basic ad hoc consumer problem.

More precisely, in the previous chapter we saw that the only difference with a basic ad hoc consumer problem is the function $u_0(c_t) + \delta V^*(s_t + I)$ that is being maximized inside the functional equation. That is, given a value function $V^*(.)$ that gives the maximally attainable discounted future utility, at time t a full-blown ad hoc utility function $u^{(t)} : W_{t-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ would optimally be of the form

$$u^{(t)}(w_{t-1}, c_t, s_t) = u^{(t)}(c_0, c_1, ..., c_t, s_t) = \sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1} V^*(s_t + I).$$

Past consumption does enter this optimal ad hoc utility function, but it does not enter the function that is being maximized inside the functional equation. However, the past also enters the above optimal ad hoc utility function in an additive way. And at time t the past commodity bundle $w_{t-1} = (c_0, c_1, ..., c_{t-1})$ is fixed, so that this ad hoc utility function is to be maximized over c_t and s_t only. Therefore in maximization, past utility $\sum_{i=0}^{t-1} \delta^i u_0(c_i)$ will drop out. Hence the difference between the maximization problem from the above functional equation and the basic ad hoc consumer problem is inconsequential as both yield the same choices.

And in the second chapter we saw that in general, a decision-maker who would solve the appropriate functional equations with the optimal value function V^* to make choices, will end up making exactly the same decisions as a decision-maker solving the sequence problem in $(c_0, c_1, c_2, ...)$ directly. This equivalence is due to the fact that this V^* that solves the above functional equation, also gives the maximal value for the sequence problem in which the utility function $\sum_{t=0}^{\infty} \delta^t u(c_t)$ is directly maximized over the appropriate budget set, and the fact that by stationarity the period-t problem is an exact copy of the period-0 problem. Therefore implicit in using the ad hoc preferences $u^{(t)}(c_t, s_t) = \sum_{i=0}^{t} \delta^i u(c_i) + \delta^{t+1} V^*(s_t + I)$ is the property that any budget that will be saved into period t will also be spent optimally afterwards.

So in a sense, in the present case we see that the analysis on the total commodity space X can be correctly 'summarized' to an analysis on the ad hoc choice set $\mathbb{R}_+ \times \mathbb{R}_+$. From total preferences, ad hoc preferences can be found that basically represent the same preferences (and that yield the same choices), so that these ad hoc preferences can be called consistent with the original total preferences. ■

Thus, the notion of consistency is related to the property that money saved will always be spent optimally afterwards. Therefore in order to define consistency it needs to be specified what future consumption would be feasible in a certain situation with savings, a future commodity space, future income and future prices given. Recall that in period t a future price vector is given by a vector $q_t := (p_t, p_{t+1}, p_{t+2}, ...)$, with $p_\tau \in \mathbb{R}_+^{n_\tau} \setminus \{0\}$ for all $\tau \geq t$, that specifies all prices as of period t. Similarly, we can define a **future income stream** as a vector $J_t := (I_t, I_{t+1}, I_{t+2}, ...)$, with $I_\tau \in \mathbb{R}_+$ for all $\tau \geq t$ that specifies all additional incomes that will be obtained from period t onwards. For $t = 0$, a future income stream is simply equal to an income stream $J_0 := (m_0, I_1, I_2, ...)$.

Definition 5.1.1 *Given a future price vector q_t, an amount of savings $s_{t-1} \in \mathbb{R}_+$, and a future income stream J_t, a **future budget set** as of period t is a set*

$$B_t(q_t, s_{t-1}, J_t) = \{(x_t, x_{t+1}, ...) \in Y_t : \sum_{i=t}^{\tau} p_i \cdot x_i \leq s_{t-1} + \sum_{i=t}^{\tau} I_i, \forall \tau \geq t\},$$

*and a **strict future budget set** is a set*

$$\mathring{B}_t(q_t, s_{t-1}, J_t) = \{(x_t, x_t, ...) \in Y_t : \sum_{i=t}^{\tau} p_i \cdot x_i < s_{t-1} + \sum_{i=t}^{\tau} I_i, \forall \tau \geq t\}.$$

For period 0, there is no previous period so there will also be no savings brought into this period, and a future budget set for this first period can be denoted by $B_0(p, 0, J_0)$, or by $B_0(q_0, 0, J_0)$.

Thus, a future budget set specifies all the future commodity bundles that will be affordable after period $t-1$, given a future price vector q_t, an amount of savings $s_{t-1} \in \mathbb{R}_+$ that is brought over from the previous period, and a future income stream

J_t. And similarly for the strict future budget set, where the only difference is that all budget constraints will have to hold strictly.

The following definitions specify the property of consistency, both in terms of preference relations and in terms of utility functions, in the general setting under certainty. In the next subsection consistency will also be formally defined under uncertainty.

Definition 5.1.2 *An ad hoc preference relation $\succsim^{(t)}$ on the ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$ is called **consistent** with a (total) preference relation $\succsim$ on the commodity space $X = W_{t-1} \times X_t \times Y_{t+1}$, given some future price vector q_{t+1} and some future income stream J_{t+1}, if for all $(w_{t-1}, x_t, s_t), (w'_{t-1}, x'_t, s'_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$ it holds that $(w_{t-1}, x_t, s_t) \succsim^{(t)} (w'_{t-1}, x'_t, s'_t)$ if for every $y'_{t+1} \in \mathring{B}_{t+1}(q_{t+1}, s'_t, J_{t+1})$ there is a $y_{t+1} \in \mathring{B}_{t+1}(q_{t+1}, s_t, J_{t+1})$ such that $(w_{t-1}, x_t, y_{t+1}) \succsim (w'_{t-1}, x'_t, y'_{t+1})$.*

Definition 5.1.3 *An ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ is called **consistent** with a total utility function $u : X \to \mathbb{R}$, given a future price vector q_{t+1} and a future income stream J_{t+1}, if there is some strictly increasing function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that*

$$u^{(t)}(w_{t-1}, x_t, s_t) = f(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}, x_t, y_{t+1})),$$

for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$.

These definitions may need some explaining.

Firstly, notice that in the second definition we have $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ (recall that $\overline{\mathbb{R}}$ denotes the extended real numbers $\mathbb{R} \cup \{\infty\}$), rather than (for instance) $f : \mathbb{R} \to \mathbb{R}$. At this point it may not be clear why we would allow the $+\infty$ to be both in the domain and in the range of the strictly increasing functions f.

A first thing to note here is that the specific form of such a function f will depend on the given functions u and $u^{(t)}$. Therefore such a function f need not in all cases take the value $+\infty$. Still, we do want to allow for it.

As for why $\overline{\mathbb{R}}$ is the domain of f, since the future commodity space Y_{t+1} has infinite dimension, it would get quite complicated to ensure that the suprema $\sup_{y_{t+1}} u(w_{t-1}, x_t, y_{t+1})$ would always be finite-valued. This problem can be avoided by allowing for $f(\infty)$ to be well-defined.

The fact that we also allowed for the possibility to include the $+\infty$ into the range of f may seem even more puzzling. After all, since $u^{(t)}$ is defined into the real numbers $\mathbb{R}$, we see that $f(\sup_{y_{t+1}} u(w_{t-1}, x_t, y_{t+1})) = \infty$ can never occur. The reason for defining f into $\overline{\mathbb{R}}$ has to do with the fact that we also want to allow for such a function f to be unbounded on $\mathbb{R}$. By strict increasingness of f, it must hold that $f(c) < f(\infty) \leq \infty$, for all $c \in \mathbb{R}$. Therefore $f : \overline{\mathbb{R}} \to \mathbb{R}$ would give that $f(\infty) \in \mathbb{R}$, and thus that f would be bounded from above. And while this may not be problematic or undesirable in some cases, it *would* for instance exclude the possibility to have the identity function $f(c) = c$, even in cases where the suprema $\sup_{y_{t+1}} u(w_{t-1}, x_t, y_{t+1})$ would always be finite-valued. This limitation can be avoided by allowing for $f(\infty) = \infty$.

Thus in cases where all suprema $\sup_{y_{t+1}} u(w_{t-1}, x_t, y_{t+1})$ would be finite-valued, it would be inconsequential what value we would assign to $f(\infty)$, in the sense that this value will never be attained by $u^{(t)}$ anyway. If f would be bounded from above on $\mathbb{R}$, then $f(\infty)$ may either be finite (but larger that the upper bound) or infinite. If f would not be bounded from above on $\mathbb{R}$, then $f(\infty)$ must be infinite.

In cases where not all suprema $\sup_{y_{t+1}} u(w_{t-1}, x_t, y_{t+1})$ would be finite-valued, $f(\infty)$ must be finite, as $f(\sup_{y_{t+1}} u(w_{t-1}, x_t, y_{t+1}))$ must always be finite.

Secondly, also notice that whereas in the first definition sets such as $\mathring{B}_{t+1}(q_{t+1}, s_t, J_{t+1})$ (with strict budget constraints) are used, in the second definition suprema are taken over the budget sets $B_{t+1}(q_{t+1}, s_t, J_{t+1})$ that are closed. Also note that for continuous utility functions we could also have used the first type of sets in the second definition, as a supremum of a continuous function over any set will always be the same as the supremum of the same function over the closure of this first set. Still, it might be a bit counter-intuitive to see the first type of sets $\mathring{B}_{t+1}(q_{t+1}, s_t, J_{t+1})$ appear in any of the definitions. The reason why strict budget sets are used in the first definition is that, if we had used the closed budget sets in both definitions, then these definitions would not permit links as in the next proposition, because of problems with suprema being attained or not.

From the definition of $u^{(t)}$ being consistent with u, it also follows immediately that such a consistent $u^{(t)}(w_{t-1}, x_t, s_t)$ will be non-decreasing in s_t. After all, for $s_t > s'_t$ we know that $B_{t+1}(q_{t+1}, s'_t, J_{t+1})$ is a subset of $B_{t+1}(q_{t+1}, s_t, J_{t+1})$. And a supremum of a function over a set is never larger than the supremum of the same function over a larger set. Thus indeed the consistent $u^{(t)}(w_{t-1}, x_t, s_t)$ is non-decreasing in s_t.

Similarly, if $\succsim^{(t)}$ is consistent with $\succsim$, then s_t is weakly good in $\succsim^{(t)}$. To see this, for $s_t > s'_t$ we also get that $\mathring{B}_{t+1}(q_{t+1}, s'_t, J_{t+1}) \subseteq \mathring{B}_{t+1}(q_{t+1}, s_t, J_{t+1})$, so $y'_{t+1} \in \mathring{B}_{t+1}(q_{t+1}, s'_t, J_{t+1})$ implies that $y'_{t+1} \in \mathring{B}_{t+1}(q_{t+1}, s_t, J_{t+1})$. Therefore for all $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ we see that for every $y'_{t+1} \in \mathring{B}_{t+1}(q_{t+1}, s'_t, J_{t+1})$ there esists a $y'_{t+1} \in \mathring{B}_{t+1}(q_{t+1}, s_t, J_{t+1})$ such that $(w_{t-1}, x_t, y'_{t+1}) \succsim (w_{t-1}, x_t, y'_{t+1})$. This implies that $(w_{t-1}, x_t, s_t) \succsim (w_{t-1}, x_t, s'_t)$, and thus s_t is weakly good in $\succsim^{(t)}$.

The next proposition justifies why the term consistency is used for both preference relations and utility functions.

Proposition 5.1.1 *Assume given for some time $t \in \mathbb{N}_0$ an ad hoc preference relation $\succsim^{(t)}$ on the ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$ for which the last argument s_t is strongly good in $\succsim^{(t)}$, and a continuous ad hoc utility function $u^{(t)}$ that represents $\succsim^{(t)}$. Also assume given a preference relation $\succsim$ on the commodity space $X = W_{t-1} \times X_t \times Y_{t+1}$ for which at least one of the commodities in Y_{t+1} is strongly good in $\succsim$, a continuous utility function u that represents $\succsim$, a future price vector q_{t+1}, and a future income stream J_{t+1}. Then $\succsim^{(t)}$ is consistent with $\succsim$, given q_{t+1} and J_{t+1}, if and only if $u^{(t)}$ is consistent with u, given q_{t+1} and J_{t+1}.*

Proof. ▲ First we prove the 'if' part. Suppose that $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ is consistent with $u : X \to \mathbb{R}$, given $q_{t+1} = (p_{t+1}, p_{t+2}, ...)$ (with $p_\tau \in \mathbb{R}_+^{n_\tau} \setminus \{0\}$, $n_\tau \in \mathbb{N}$, for all $\tau \geq t+1$) and $J_{t+1} := (I_{t+1}, I_{t+2}, ...)$ (with $I_\tau \in \mathbb{R}_+$ for all $\tau \geq t+1$). Then there exists some strictly increasing function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$, it holds that

$$u^{(t)}(w_{t-1}, x_t, s_t) = f(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}, x_t, y_{t+1})).$$

Since both the future price vector q_{t+1} and the future income stream J_{t+1} are fixed, throughout this proof we will use the notation $B_{t+1}(s_t)$ rather than $B_{t+1}(q_{t+1}, s_t, J_{t+1})$. (Similarly we write $\mathring{B}_{t+1}(s_t)$ in stead of $\mathring{B}_{t+1}(q_{t+1}, s_t, J_{t+1})$.)

We now want to show that indeed $\succsim^{(t)}$ is consistent with $\succsim$, given q_{t+1} and J_{t+1}. Let $(w_{t-1}, x_t, s_t), (w'_{t-1}, x'_t, s'_t) \in X_t \times \mathbb{R}_+$ be such that $(w_{t-1}, x_t, s_t) \succsim^{(t)} (w'_{t-1}, x'_t, s'_t)$. We then have that

$$(w_{t-1}, x_t, s_t) \succsim^{(t)} (w'_{t-1}, x'_t, s'_t) \Leftrightarrow u^{(t)}(w_{t-1}, x_t, s_t) \geq u^{(t)}(w'_{t-1}, x'_t, s'_t) \Leftrightarrow$$

$$f(\sup_{y_{t+1} \in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1})) \geq f(\sup_{y_{t+1} \in B_{t+1}(s'_t)} u(w'_{t-1}, x'_t, y_{t+1})) \Leftrightarrow$$

$$\sup_{y_{t+1} \in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1}) \geq \sup_{y_{t+1} \in B_{t+1}(s'_t)} u(w'_{t-1}, x'_t, y_{t+1}) \Leftrightarrow$$

$$\forall y'_{t+1} \in \mathring{B}_{t+1}(s'_t), \exists y_{t+1} \in \mathring{B}_{t+1}(s_t) \text{ such that } u(w_{t-1}, x_t, y_{t+1}) \geq u(w'_{t-1}, x'_t, y'_{t+1})$$

$$\Leftrightarrow \forall y'_{t+1} \in \mathring{B}_{t+1}(s'_t), \exists y_{t+1} \in \mathring{B}_{t+1}(s_t) \text{ such that } (w_{t-1}, x_t, y_{t+1}) \succsim (w'_{t-1}, x'_t, y'_{t+1}).$$

Note that if the above equivalences are correct, the first statement holds if and only if the last statement holds, which exactly specifies the definition of $\succsim^{(t)}$ being consistent with $\succsim$, given q_{t+1} and J_{t+1}. The third of the above equivalences follows by strict increasingness of f. The next to last '$\Leftrightarrow$' may need a bit of explanation.

For the '$\Rightarrow$', suppose that

$$\sup_{y_{t+1} \in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1}) \geq \sup_{y_{t+1} \in B_{t+1}(s'_t)} u(w'_{t-1}, x'_t, y_{t+1})$$

is given. The preference relation $\succsim$ is such that at least one of the commodities in Y_{t+1} is strongly good in $\succsim$, and the function u must be strictly increasing in this commodity. Therefore since the set $\mathring{B}_{t+1}$ is defined by means of strict budget inequalities, it must hold that

$$\sup_{y_{t+1} \in B_{t+1}(s'_t)} u(w'_{t-1}, x'_t, y_{t+1}) > u(w'_{t-1}, x'_t, y'_{t+1}), \forall y'_{t+1} \in \mathring{B}_{t+1}(s'_t).$$

This also implies that

$$\sup_{y_{t+1} \in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1}) > u(w'_{t-1}, x'_t, y'_{t+1}), \forall y'_{t+1} \in \mathring{B}_{t+1}(s'_t).$$

Then by continuity of u we get that for all $y'_{t+1} \in \mathring{B}_{t+1}(s'_t)$, there must exist a $y_{t+1} \in \mathring{B}_{t+1}(s_t)$, such that $u(w_{t-1}, x_t, y_{t+1}) \geq u(w'_{t-1}, x'_t, y'_{t+1})$, which proves '$\Rightarrow$'.

Now, suppose that the other direction '$\Leftarrow$' does not hold. That is, suppose that the last statement holds: for all $y'_{t+1} \in \mathring{B}_{t+1}(s'_t)$, there exists a $y_{t+1} \in \mathring{B}_{t+1}(s_t)$, such that $u(w_{t-1}, x_t, y_{t+1}) \geq u(w'_{t-1}, x'_t, y'_{t+1})$. And, contradictory to '$\Leftarrow$', suppose that

$$\sup_{y_{t+1} \in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1}) < \sup_{y_{t+1} \in B_{t+1}(s'_t)} u(w'_{t-1}, x'_t, y_{t+1}).$$

Then by continuity there must be some $y'_{t+1} \in \mathring{B}_{t+1}(s'_t)$, such that

$$\sup_{y_{t+1} \in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1}) < u(w'_{t-1}, x'_t, y'_{t+1}).$$

But now, from the statement we started with, we know that for this $y'_{t+1} \in \mathring{B}_{t+1}(s'_t)$ there must also be a $y_{t+1} \in \mathring{B}_{t+1}(s_t)$ such that

$$u(w_{t-1}, x_t, y_{t+1}) \geq u(w'_{t-1}, x'_t, y'_{t+1}) > \sup_{y_{t+1} \in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1}).$$

This is obviously impossible.

Thus indeed, '$\Leftrightarrow$' holds, and we see that $\succsim^{(t)}$ is consistent with $\succsim$, given q_{t+1} and J_{t+1}.

▲ For the 'only if' part, suppose that $\succsim^{(t)}$ on $W_{t-1} \times X_t \times \mathbb{R}_+$ is consistent with $\succsim$ on X, given q_{t+1} and J_{t+1}. And suppose given some utility function u that represents $\succsim$, and some ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ that represents $\succsim^{(t)}$. We now want to prove that $u^{(t)}$ is also consistent with u, given q_{t+1} and J_{t+1}, i.e. that $u^{(t)}$ can be written as:

$$u^{(t)}(w_{t-1}, x_t, s_t) = f\left(\sup_{y_{t+1} \in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1})\right),$$

for some strictly increasing function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

It is given that for all (w_{t-1}, x_t, s_t) and $(w'_{t-1}, x'_t, s'_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$ it holds that

$$(w_{t-1}, x_t, s_t) \succsim^{(t)} (w'_{t-1}, x'_t, s'_t) \Leftrightarrow$$

$$\forall y'_{t+1} \in \mathring{B}_{t+1}(s'_t), \exists y_{t+1} \in \mathring{B}_{t+1}(s_t) \text{ such that } (w_{t-1}, x_t, y_{t+1}) \succsim (w'_{t-1}, x'_t, y'_{t+1}) \Leftrightarrow$$

$$\forall y'_{t+1} \in \mathring{B}_{t+1}(s'_t), \exists y_{t+1} \in \mathring{B}_{t+1}(s_t) \text{ such that } u(w_{t-1}, x_t, y_{t+1}) \geq u(w'_{t-1}, x'_t, y'_{t+1})$$

$$\Leftrightarrow \sup_{y_{t+1} \in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1}) \geq \sup_{y_{t+1} \in B_{t+1}(s'_t)} u(w'_{t-1}, x'_t, y_{t+1}).$$

Here the first equivalence holds by definition of consistency of $\succsim^{(t)}$ with $\succsim$. The last equivalence is the same as the next to last equivalence needed in the 'if' part of this proof, which was already established to hold above. Now we see that for all (w_{t-1}, x_t, s_t) and $(w'_{t-1}, x'_t, s'_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$ we have

$$u^{(t)}(w_{t-1}, x_t, s_t) \geq u^{(t)}(w'_{t-1}, x'_t, s'_t) \Leftrightarrow (w_{t-1}, x_t, s_t) \succsim^{(t)} (w'_{t-1}, x'_t, s'_t) \Leftrightarrow$$

$$\sup_{y_{t+1}\in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1}) \geq \sup_{y_{t+1}\in B_{t+1}(s'_t)} u(w'_{t-1}, x'_t, y_{t+1}).$$

Then it is possible to construct a function $f : \overline{\mathbb{R}} \to \mathbb{R}$ such that

$$u^{(t)}(w_{t-1}, x_t, s_t) = f(\sup_{y_{t+1}\in B_{t+1}(s_t)} u(w_{t-1}, x_t, y_{t+1})),$$

and this function will necessarily have to be strictly increasing. Hence indeed, $u^{(t)}$ is consistent with u, given q_{t+1} and J_{t+1}. ■

5.1.2 Expected utility

Next, we will define consistency of ad hoc utility functions with total (Bernouilli) utility functions in expected utility models.

Recall from chapter 2 that in models of expected utility a state space Ω, and an accompanying probability distribution $\pi : \Omega \to [0, 1]$ would be given. In a setting where time is explicitly modelled, the states $\omega \in \Omega$ would be broken up into sequences of ω_t's, so that $\omega = (\omega_0, \omega_1, \omega_2, ...)$. Then by $\omega_t = (X_t, p_t, I_t)$ we would denote the prevailing period-t part of the state of the world, consisting of a period-t commodity space, a period-t price vector, and a period-t additional income ($I_0 = m_0$). These period-t states of the world ω_t were supposed not to become known until time t.

For two periods $t < \tau$, by ω_t^τ we denote the vector of all period-i states of the world, where i ranges from period t up to period τ: $\omega_t^\tau = (\omega_t, \omega_{t+1}, ..., \omega_\tau)$. By ω_t^∞ we denote a complete future state of the world $(\omega_t, \omega_{t+1}, ...)$.

Then, between periods t and $t+1$ (for any t), the vector of past states $\omega_0^t = (\omega_0, \omega_1, ..., \omega_t)$ are known, and the vector of future states ω_{t+1}^∞ that is yet to occur, is not. We denote the set of all final states of the world ω that can occur, given ω_0^t, by $\{\omega_0^t\} \times \Omega_{t+1}^\infty$. Such a future state space Ω_{t+1}^∞ would also give rise to a set $\tilde{Y}_{t+1}$ that denotes the union of all the future commodity spaces $Y_{t+1} = \times_{i=t+1}^\infty X_i$ that may occur as part of some future state of the world ω_{t+1}^∞.

In subsection 2.6.3 we also wrote acts as sequences:

$$a(\omega) = (a_0(\omega_0), a_1(\omega_0^1), a_2(\omega_0^2), ...).$$

Between periods t and $t+1$, the vector of past states ω_0^t is given, and the actions $w_t = (a_0, a_1, ..., a_t)$ were already taken. Hence between periods t and $t+1$, ω_0^t and w_t (and an amount of savings s_t) are given, and a choice $b_{t+1} = (a_{t+1}, a_{t+2}, ...) \in \tilde{Y}_{t+1}$ of what to purchase from period $t+1$ onwards will still have to be made. What can possibly be chosen here, depends on the prevailing state of the world $\omega = (\omega_0^t, \omega_{t+1}^\infty)$. Thus such an act can now be denoted by a function $b_{t+1} : \{\omega_0^t\} \times \Omega_{t+1}^\infty \to \tilde{Y}_{t+1}$. However, the way in which this choice b_{t+1} may depend on ω, should satisfy a few conditions.

Firstly, for $\tau \geq t+1$ the period-τ state ω_τ does not become known until period τ, so as in chapter 2 the period-τ act a_τ may only depend on ω_0^τ. Thus

$$b_{t+1}(\omega) = (a_{t+1}(\omega_0^{t+1}), a_{t+2}(\omega_0^{t+2}), ...).$$

Secondly, if ω would be such that the period-τ commodity space X_τ would be part of ω_τ, then obviously $a_\tau(\omega_0^\tau) \in X_\tau$. That is, if the future commodity space Y_{t+1} is part of the future state ω_{t+1}^∞, then obviously $b_{t+1}(\omega) \in Y_{t+1}$ should hold.

Thirdly, the choice $b_{t+1}(\omega) = (a_{t+1}(\omega_0^{t+1}), a_{t+2}(\omega_0^{t+2}), ...)$ should also be feasible. That is, for any vector of realizations ω_{t+1}^τ of the states of the world in periods $t+1$ through τ, the prices $(p_{t+1}(\omega_{t+1}), ..., p_\tau(\omega_\tau))$ of consumption from the corresponding space $X_{t+1} \times ... \times X_\tau$, and the additional incomes $(I_{t+1}(\omega_{t+1}), ..., I_\tau(\omega_\tau))$ obtained in the periods between $t+1$ and τ, help determine what is affordable in period $\tau \geq t+1$. As before, for an amount of savings s_t, and the previous choices $(a_{t+1}(\omega_0^{t+1}), ..., a_{\tau-1}(\omega_0^{\tau-1}))$ given, any choice $a_\tau(\omega_0^\tau)$ should be such that what is spent $p_\tau(\omega_\tau) \cdot a_\tau(\omega_0^\tau)$ in period τ should never exceed the budget

$$m_\tau := s_t + \sum_{i=t+1}^{\tau} I_i(\omega_i) - \sum_{i=t+1}^{\tau-1} p_i(\omega_i) \cdot a_i(\omega_0^i) \tag{‡}$$

that is available in period τ.

All these requirements on which choices $b_{t+1}(\omega)$ can be made in later periods are summarized in the specification of the future budget set.

Definition 5.1.4 *Given an amount of savings $s_t \in \mathbb{R}_+$, a past state of the world ω_0^t, a future state space Ω_{t+1}^∞ and a set of future consumption possibilities $\tilde{Y}_{t+1}$, a **future budget set** as of period $t+1$ is a set*

$$B_{t+1}(s_t) = \left\{ b_{t+1} : \{\omega_0^t\} \times \Omega_{t+1}^\infty \to \tilde{Y}_{t+1} : \begin{array}{c} b_{t+1}(\omega) = (a_{t+1}(\omega_0^{t+1}), a_{t+2}(\omega_0^{t+2}), ...), \\ \forall \tau \geq t+1 : a_\tau(\omega_0^\tau) \in X_\tau \text{ if } \omega_\tau = (X_\tau, p_\tau, I_\tau), \\ \forall \tau \geq t+1 : p_\tau(\omega_\tau) \cdot a_\tau(\omega_0^\tau) \leq m_\tau. \end{array} \right\}$$

Here m_τ is as in (‡).

With this definition we can now also define consistency of ad hoc utility functions in expected utility models. Analogously to the corresponding definition in models under certainty, consistency would mean that money is optimally translated into future consumption.

Definition 5.1.5 *Given a past state of the world ω_0^t, a future state space Ω_{t+1}^∞ and a probability distribution $\pi : \Omega \to [0,1]$, an ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ is called **consistent** with a (total) Bernouilli utility function $u : \tilde{X} \to \mathbb{R}$, if there exists some strictly increasing function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that*

$$u^{(t)}(w_{t-1}, x_t, s_t) = f\Big(\sup_{b_{t+1} \in B_{t+1}(s_t)} E_{\pi(\omega|\omega_0^t)} u(w_{t-1}, x_t, b_{t+1}(\omega)\Big),$$

for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$.

The remainder of this chapter is completely set under certainty. This definition of consistency in expected utility models is presented here because we will need a formal definition in later chapters.

5.2 From total to ad hoc preferences

In this section we investigate whether in general within the standard microeconomic framework under certainty, for a given preference structure on a total commodity space X there exists an ad hoc preference structure on some ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$, that is consistent with the original total preference structure.

In the above dynamic programming example we argued that the answer was affirmative. This section shows that in a more general setting the answer is affirmative as well: for future prices and a future income stream as of next period given, total preferences on X can also be seen to define consistent ad hoc preferences on $W_{t-1} \times X_t \times \mathbb{R}_+$.

Proposition 5.2.1 *Suppose a (total) utility function $u : X \to \mathbb{R}$ is given. Then, for any time $t \in \mathbb{N}_0$, any future price vector q_{t+1} and any future income stream J_{t+1} given, there exists an ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ that is consistent with u, given q_{t+1} and J_{t+1}.*

Proof. The function $u(.)$ is a (total) utility function on the (total) commodity space X. For a given time $t \in \mathbb{N}_0$, some system of future prices $q_{t+1} = (p_{t+1}, p_{t+2}, ...)$ (with $p_\tau \in \mathbb{R}_+^{n_\tau} \backslash \{0\}$, $n_\tau \in \mathbb{N}$, for all $\tau \geq t+1$) and a future income stream $J_{t+1} = (I_{t+1}, I_{t+2}, ...)$ (with $I_\tau \in \mathbb{R}_+$ for all $\tau \geq t+1$) are given. Given t, we can also write $X = W_{t-1} \times X_t \times Y_{t+1}$.

Then the new function $g : W_{t-1} \times X_t \times \mathbb{R}_+ \to \overline{\mathbb{R}}$, given by

$$g(w_{t-1}, x_t, s_t) := \sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}, x_t, y_{t+1}),$$

for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$, can be derived from u. The set $B_{t+1}(q_{t+1}, s_t, J_{t+1})$ is never empty as it always contains the zero-vector in Y_{t+1}, so that g is a well-defined function mapping $W_{t-1} \times X_t \times \mathbb{R}_+$ into the extended real numbers $\overline{\mathbb{R}}$. Thus, for any strictly increasing function $f : \overline{\mathbb{R}} \to \mathbb{R}$, we see that the function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$, as defined by $u^{(t)}(w_{t-1}, x_t, s_t) := f(g(w_{t-1}, x_t, s_t))$ is a well-defined ad hoc utility function. Moreover, by definition $u^{(t)}$ satisfies the criteria for being consistent with u, given q_{t+1} and J_{t+1}, which concludes the proof.

Thus within the standard framework, from given total preferences on total commodity bundles we can derive consistent ad hoc preferences on ad hoc commodity bundles. Moreover, this property holds for all periods. However, as mentioned before, in this setting comparisons are still essentially made for all goods simultaneously; the actual introspective process of determining what you like is still done all goods at once.

The next proposition shows that a profile of choices $(x_0^*, x_1^*, ...) \in X$ solves the basic consumer problem corresponding to certain total preferences if and only if for every period t the corresponding pair (x_t^*, s_t^*), with $s_t^* = m_0 + \sum_{i=1}^t I_i - \sum_{i=0}^t p_i \cdot x_i^*$, solves the corresponding basic ad hoc consumer problem for any ad hoc preferences that are consistent with the total preferences.

Proposition 5.2.2 *Suppose given a total utility function $u : X \to \mathbb{R}$ on the commodity space $X = \times_t X_t$, a price vector $p = (p_0, p_1, ...)$ and an income stream $J_0 = (m_0, I_1, I_2, ...)$.*

(A) Suppose that the function u is such that for all t it is strictly increasing in at least one commodity from Y_{t+1}, and that the profile $(x_0^*, x_1^*, ...) \in X$ solves the basic consumer problem corresponding to u, p and J_0. Then, for any $t \in \mathbb{N}_0$, and any ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ that is consistent with u, given $q_{t+1} = (p_{t+1}, p_{t+2}, ...)$ and $J_{t+1} = (I_{t+1}, I_{t+2}, ...)$, the pair $(x_t^*, m_t^* - p_t \cdot x_t^*)$ will solve the basic ad hoc consumer problem corresponding to $u^{(t)}$, $(p_t, 1)$ and $m_t^* = m_0 + \sum_{i=1}^t I_i - \sum_{i=0}^{t-1} p_i \cdot x_i^*$, given $w_{t-1}^* = (x_0^*, x_1^*, ..., x_{t-1}^*)$.

(B) Suppose that the profile $(x_0^*, x_1^*, ...) \in X$ is such that for every $t \in \mathbb{N}_0$ and for any ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ that is consistent with u, given q_{t+1} and J_{t+1}, the pair $(x_t^*, m_t^* - p_t \cdot x_t^*)$ solves the basic ad hoc consumer problem corresponding to $u^{(t)}$, $(p_t, 1)$ and $m_t^* = m_0 + \sum_{i=1}^t I_i - \sum_{i=0}^{t-1} p_i \cdot x_i^*$, given w_{t-1}^*. Then $(x_0^*, x_1^*, ...)$ will also solve the basic consumer problem corresponding to u, p and J_0.

Proof. Given are a utility function $u : X \to \mathbb{R}$ on the commodity space X, a price vector $p = (p_0, p_1, ...)$ (with $p_t \in \mathbb{R}_+^{n_t} \setminus \{0\}$, $n_t \in \mathbb{N}$, for all t), and an income stream $J_0 = (m_0, I_1, I_2, ...)$ (with $m_0 \in \mathbb{R}_+$ and $I_t \in \mathbb{R}_+$ for all $t > 0$).

(A) For the first part of this proposition, suppose that the profile $x^* = (x_0^*, x_1^*, x_2^*, ...) \in X$ solves the basic consumer problem corresponding to u, p and J_0. That is, x^* is such that $u(x^*) = \max u(x)$ sub to $x \in B_0(p, 0, J_0)$.

Then, for some specific $t \in \mathbb{N}_0$, let $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ be some ad hoc utility function that is consistent with u, given $q_{t+1} = (p_{t+1}, p_{t+2}, ...)$ and $J_{t+1} = (I_{t+1}, I_{t+2}, ...)$. That is, $u^{(t)}$ is a function that can be written as

$$u^{(t)}(w_{t-1}, x_t, s_t) = f_t(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}, x_t, y_{t+1})),$$

for all $(w_{t-1}, x_t, s_t) \in X_t \times \mathbb{R}_+$, and for some strictly increasing function $f_t : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. We now want to show that the pair $(x_t^*, m_t^* - p_t \cdot x_t^*)$ solves the basic ad hoc consumer problem corresponding to $u^{(t)}$, $(p_t, 1)$ and $m_t^* = m_0 + \sum_{i=1}^t I_i - \sum_{i=0}^{t-1} p_i \cdot x_i^*$, given the previous choices $w_{t-1}^* = (x_0^*, x_1^*, ..., x_{t-1}^*)$.

In period t this basic ad hoc choice problem corresponding to $u^{(t)}$, $(p_t, 1)$ and m^*, for w_{t-1}^* given, reads

$$\max_{(x_t, s_t): p_t \cdot x_t + s_t \le m_t^*} u^{(t)}(w_{t-1}^*, x_t, s_t) \tag{1}$$

This can now be expanded to:

$$\max_{p_t \cdot x_t + s_t \le m_t^*} f_t(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}^*, x_t, y_{t+1})) =$$

$$\sup_{p_t \cdot x_t + s_t \le m_t^*} f_t(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}^*, x_t, y_{t+1})) =$$

$$f_t(\sup_{p_t \cdot x_t + s_t \le m_t^*} [\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}^*, x_t, y_{t+1})]) =$$

$$f_t(\sup_{y_t \in B_t(q_t, m_t^* - I_t, J_t)} u(w_{t-1}^*, y_t)). \tag{2}$$

These equalities may need a bit of explaining. The first equality of replacing a maximum by a supremum can be justified if indeed this supremum is attained, which we will shortly see is the case here. The second equality follows from strict increasingness of f_t.

The last equality should hold if we have specified the budget sets correctly. To see that this is so, note that with the identity $m_t^* = s_{t-1}^* + I_t$, the constraint $p_t \cdot x_t + s_t \leq m_t^*$ under which the outer supremum is taken, can be broken down into $p_t \cdot x_t \leq s_{t-1}^* + I_t$ and into $s_t \leq s_{t-1}^* + I_t - p_t \cdot x_t$. The inner supremum is taken over the future budget set

$$B_{t+1}(q_{t+1}, s_t, J_{t+1}) = \{(x_{t+1}, x_{t+2}, ...) \in Y_{t+1} : \sum_{i=t+1}^{\tau} p_i \cdot x_i \leq s_t + \sum_{i=t+1}^{\tau} I_i, \forall \tau \geq t+1\}.$$

Together, the constraint $s_t \leq s_{t-1}^* + I_t - p_t \cdot x_t$ and the budget constraints from $B_{t+1}(q_{t+1}, s_t, J_{t+1})$ imply that $\sum_{i=t}^{\tau} p_i \cdot x_i \leq s_{t-1}^* + \sum_{i=t}^{\tau} I_i$ must hold for all $\tau \geq t+1$. Therefore, any feasible bundle $y_t = (x_t, y_{t+1})$ must satisfy the latter system of budget constraints and the constraint $p_t \cdot x_t \leq s_{t-1}^* + I_t$. Thus, any feasible bundle y_t must belong to the following set:

$$\{(x_t, x_{t+1}, ...) \in Y_t : \sum_{i=t}^{\tau} p_i \cdot x_i \leq s_{t-1}^* + \sum_{i=t}^{\tau} I_i, \forall \tau \geq t\}.$$

In this set we recognize the future budget set as of period t, for the level of savings $s_{t-1}^* = m_t^* - I_t$ given: $B_t(q_t, m_t^* - I_t, J_t)$. It is exactly this last set over which the supremum in (2) is taken. Hence the last of the above equalities holds.

We now want to show that $y_t^* = (x_t^*, x_{t+1}^*, ...)$ attains the maximum in (2). In (2) it is given that the tuple $w_{t-1}^* = (x_0^*, x_1^*, ..., x_{t-1}^*)$ was chosen in the periods before t, and the supremum is taken over the budget set $B_t(q_t, m_t^* - I_t, J_t)$. By strict increasingness, no budget is ever wasted, and the choices w_{t-1}^* resulted in the period-$(t-1)$ level of savings $s_{t-1}^* = m_0 + \sum_{i=1}^{t-1} I_i - \sum_{i=0}^{t-1} p_i \cdot x_i^*$ (which equals $m_t^* - I_t$). Therefore the set of bundles $(x_t, x_{t+1}, ...)$ that are feasible from period t onwards, given w_{t-1}^*, is indeed exactly given by the budget set $B_t(q_t, m_t^* - I_t, J_t)$. Since we know that $u(x_0^*, x_1^*, ...) = \max u(x)$ sub to $x \in B_0(p, 0, J_0)$, the truncated profile $y_t^* = (x_t^*, x_{t+1}^*, ...)$ must now also attain the supremum in $\sup_{y_t \in B_t(q_t, m_t^* - I_t, J_t)} u(w_{t-1}^*, y_t)$.

And since $y_t^* = (x_t^*, x_{t+1}^*, ...)$ maximizes (2), by strict increasingness the pair $(x_t^*, m_t^* - p_t \cdot x_t^*)$ must maximize (1). And indeed $(x_t^*, m_t^* - p_t \cdot x_t^*)$ will solve the basic ad hoc choice problem corresponding to $u^{(t)}$, $(p_t, 1)$ and m_t^*, for w_{t-1}^* given.

(B) The profile $(x_0^*, x_1^*, ...) \in X$ is such that for every $t \in \mathbb{N}_0$ and for any ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ that is consistent with u, given $q_{t+1} = (p_{t+1}, p_{t+2}, ...)$ and $J_{t+1} = (I_{t+1}, I_{t+2}, ...)$, the pair $(x_t^*, m_t^* - p_t \cdot x_t^*)$ solves the basic ad hoc choice problem corresponding to $u^{(t)}$, $(p_t, 1)$ and $m_t^* = m_0 + \sum_{i=1}^{t} I_i - \sum_{i=0}^{t-1} p_i \cdot x_i^*$, given $w_{t-1}^* = (x_0^*, x_1^*, ..., x_{t-1}^*)$.

Such an ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ is consistent with u, given q_{t+1} and J_{t+1}, so it can be written as

$$u^{(t)}(w_{t-1}, x_t, s_t) = f_t(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}, x_t, y_{t+1})),$$

for all $(w_{t-1}, x_t, s_t) \in X_t \times \mathbb{R}_+$, and for some strictly increasing function $f_t : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

Now, we want to show that the profile $x^* = (x_0^*, x_1^*, ...)$ also solves the basic consumer problem corresponding to u, p and J_0, i.e. that $u(x^*) = \sup_{x \in B_0(p,0,J_0)} u(x)$. (Note that it also still remains to be shown that the last supremum is indeed attained.) For every t the truncated profile $(x_0^*, x_1^*, ..., x_t^*)$ will satisfy $\sum_{i=0}^{t} p_i \cdot x_i \leq m_0 + \sum_{i=1}^{t} I_i$, so that it must be the case that $(x_0^*, x_1^*, ...) \in B_0(p, 0, J_0)$. Then obviously we get that $u(x^*) \leq \sup_{x \in B_0(p,0,J_0)} u(x)$.

Suppose that $u(x^*) < \sup_{x \in B_0(p,0,J_0)} u(x)$. Then there must be some (earliest) period $\tau \in \mathbb{N}$ at which x_τ^* is suboptimal. That is, there must be some time τ in which

$$\sup_{y_{\tau+1} \in B_{\tau+1}(q_{\tau+1}, m_\tau^* - p_\tau \cdot x_\tau^*, J_{\tau+1})} u(w_{\tau-1}^*, x_\tau^*, y_{\tau+1}) < \sup_{x \in B_0(p,0,J_0)} u(x).$$

However, we know that for $t = 0$ it holds that $(x_0^*, m_0 - p_0 \cdot x_0^*)$ attains $\max_{p_0 \cdot x_0 + s_0 \leq m_0} u^{(0)}(x_0, s_0)$ for any consistent $u^{(0)}$. So for $s_0^* = m_0 - p_0 \cdot x_0^*$, we have that

$$u^{(0)}(x_0^*, s_0^*) = \sup_{p_0 \cdot x_0 + s_0 \leq m_0} u^{(0)}(x_0, s_0) = \sup_{p_0 \cdot x_0 + s_0 \leq m_0} f_t(\sup_{y_1 \in B_1(q_1, s_0, J_1)} u(x_0, y_1)) =$$

$$f_t(\sup_{p_0 \cdot x_0 + s_0 \leq m_0} [\sup_{y_1 \in B_1(q_1, s_0, J_1)} u(x_0, y_1)]) = f_t(\sup_{x \in B_0(p,0,J_0)} u(x)).$$

Again, the third equality follows from f_t being strictly increasing. The last equality was already justified in the first part of this proof, where we saw that $\sup_{p_t \cdot x_t + s_t \leq m_t^*} \sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})}$ can alternatively be written as $\sup_{y_t \in B_t(q_t, m_t^* - I_t, J_t)}$.

Moreover, by consistency it holds that

$$u^{(0)}(x_0^*, s_0^*) = f_t(\sup_{y_1 \in B_1(q_1, s_0^*, J_1)} u(x_0^*, y_1)).$$

Hence we see that $\sup_{y_1 \in B_1(q_1, s_0^*, J_1)} u(x_0^*, y_1) = \sup_{x \in B_0(p,0,J_0)} u(x)$, and thus that $\tau > 0$.

Similarly, for any $t \in \mathbb{N}$ it holds that (x_t^*, s_t^*), with $s_t^* = m_t^* - p_t \cdot x_t^*$, attains $\max_{p_t \cdot x_t + s_t \leq m_t^*} u^{(t)}(w_{t-1}^*, x_t, s_t)$ for any consistent $u^{(t)}$, so that

$$u^{(t)}(w_{t-1}^*, x_t^*, s_t^*) = \sup_{(x_t, s_t): p_t \cdot x_t + s_t \leq m_t^*} u^{(t)}(w_{t-1}^*, x_t, s_t) =$$

$$\sup_{p_t \cdot x_t + s_t \leq m_t^*} f_t(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}^*, x_t, y_{t+1})) =$$

$$f_t(\sup_{p_t \cdot x_t + s_t \leq m_t^*} [\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}^*, x_t, y_{t+1})]) = f_t(\sup_{y_t \in B_t(q_t, m_t^* - I_t, J_t)} u(w_{t-1}^*, y_t)).$$

And again by consistency it holds that

$$u^{(t)}(w_{t-1}^*, x_t^*, s_t^*) = f_t(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t^*, J_{t+1})} u(w_{t-1}^*, x_t^*, y_{t+1}))$$

and with $s_{t-1}^* = m_t^* - I_t$ we thus see that

$$\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t^*, J_{t+1})} u(w_{t-1}^*, x_t^*, y_{t+1}) = \sup_{y_t \in B_t(q_t, s_{t-1}^*, J_t)} u(w_{t-1}^*, y_t).$$

This same principle applies for every period before t, so by induction we find that for every t it holds that

$$\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t^*, J_{t+1})} u(w_{t-1}^*, x_t^*, y_{t+1}) = \sup_{x \in B_0(p, 0, J_0)} u(x).$$

This contradicts the existence of τ, and indeed we find that $u(x^*) = \max_{x \in B_0(p,0,J_0)} u(x)$. ■

The above proposition is based on the simple mathematical fact that

$$\max_{x_t, y_{t+1}} u(\overline{w}_{t-1}, x_t, y_{t+1}) = \max_{x_t}(\max_{y_{t+1}} u(\overline{w}_{t-1}, x_t, y_{t+1})).$$

Because of this Bellman-like equation we see that it doesn't matter when choices for the future commodities are made. A choice between goods in X_t and remaining budget is based on the fact that this remaining budget is indeed afterwards spent optimally, so the order in which purchases are made is irrelevant.

Next we provide a few cases where we can explicitly specify consistent ad hoc utility functions for some standard examples of utility functions that are typically found in microeconomic texts. Moreover, we show that in some cases consistent ad hoc utility inherits its functional structure from the functional structure of the underlying total utility.

Example 5.2.1 ***Additively separable utility***

Like in definition 2.4.5, a total utility function u on $X = \times_{t=0}^{\infty} X_t$ is additively separable (with respect to the partition of X corresponding to $\times_{t=0}^{\infty} X_t$, with $X_t = \mathbb{R}_+^{n_t}$ for some $n_t \in \mathbb{N}$ and all t) if there are functions $u_t : X_t \to \mathbb{R}$, such that $u(x_0, x_1, ...) = \sum_{t=0}^{\infty} u_t(x_t)$.

Suppose that for some specific time $t \in \mathbb{N}_0$ a future price vector $q_{t+1} = (p_{t+1}, p_{t+2}, ...)$ and a future income stream $J_{t+1} = (I_{t+1}, I_{t+2}, ...)$ are given. Then we can define the function $u^{(t)}$ on $W_{t-1} \times X_t \times \mathbb{R}_+$ by

$$u^{(t)}(w_{t-1}, x_t, s_t) = \sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}, x_t, y_{t+1})$$

Thus $u^{(t)}(w_{t-1}, x_t, s_t)$ *equals*

$$\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} \sum_{i=0}^{\infty} u_i(x_i) =$$

$$\sum_{i=0}^{t} u_i(x_i) + \sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} \sum_{i=t+1}^{\infty} u_i(x_i) = \sum_{i=0}^{t} u_i(x_i) + v_{t+1}(s_t).$$

Here $v_{t+1}(s_t)$ *denotes the function* $\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} \sum_{i=t+1}^{\infty} u_i(x_i)$. *If this function* $v_{t+1}(.)$ *is finite-valued, we see that* $u^{(t)}$ *is a well-defined ad hoc utility function, and by definition it is consistent with* u. *Moreover, the consistent ad hoc utility function* $u^{(t)}$ *is also additively separable.*

Example 5.2.2 ***Linear utility***

A more specific example of additively separable utility is that of linear utility. For all t *it holds that* $X_t = \mathbb{R}_+^{n_t}$ *for some* $n_t \in \mathbb{N}$. *Then, a (total) utility function* u *on* $X = \times_{t=0}^{\infty} X_t$, *is called linear if there are vectors* $\alpha_t \in \mathbb{R}_+^{n_t}$ *such that* $u(x) = \sum_{t=0}^{\infty} \alpha_t \cdot x_t$. *This would mean that all goods are perfect substitutes.*

Then for some specific time $t \in \mathbb{N}_0$, *suppose that a strictly positive future price vector* $q_{t+1} = (p_{t+1}, p_{t+2}, ...) \in \mathbb{R}_{++}^{\infty}$ *is given, and that no additional income will be obtained after period* t: $J_{t+1} = \vec{0} = (0, 0, ...)$. *In this case, the budget set can be written as*

$$B_{t+1}(q_{t+1}, s_t, \vec{0}) = \{(x_{t+1}, x_{t+2}, ...) \in Y_{t+1} : \sum_{i=t+1}^{\tau} p_i \cdot x_i \leq s_t, \forall \tau \geq t+1\} =$$

$$\{(x_{t+1}, x_{t+2}, ...) \in Y_{t+1} : \sum_{i=t+1}^{\infty} p_i \cdot x_i \leq s_t\} = \{y_{t+1} \in Y_{t+1} : q_{t+1} \cdot y_{t+1} \leq s_t\}.$$

Then we can define the function $u^{(t)}$ *on* $W_{t-1} \times X_t \times \mathbb{R}_+$ *by*

$$u^{(t)}(w_{t-1}, x_t, s_t) = \sup_{q_{t+1} \cdot y_{t+1} \leq s_t} u(w_{t-1}, x_t, y_{t+1}) = \sup_{q_{t+1} \cdot y_{t+1} \leq s_t} \sum_{i=0}^{\infty} \alpha_i \cdot x_i =$$

$$\sum_{i=0}^{t} \alpha_i \cdot x_i + \sup_{q_{t+1} \cdot y_{t+1} \leq s_t} \sum_{i=t+1}^{\infty} \alpha_i \cdot x_i = \sum_{i=0}^{t} \alpha_i \cdot x_i + \beta_{t+1} s_t.$$

Here $\beta_{t+1} := \sup_{i \geq t+1} \max_{0 \leq j \leq n_i} \{\frac{\alpha_i^j}{p_i^j}\}$ *denotes the maximal quotient of* α*-coefficients and prices, across all commodities from* Y_{t+1}. *The last equality is due to the fact that all commodities are perfect substitutes, so that savings can most efficiently be spent on that commodity with the largest quotient of* α*-coefficients and prices. Then if* $\beta_{t+1} < \infty$, *the function* $u^{(t)}$ *is a well-defined ad hoc utility function, that is by definition consistent with* u. *Moreover, this ad hoc utility function also has a linear form.*

In the more general case where $J_{t+1} \neq \vec{0}$, the shape of the budget set will be more complicated. However, s_t will still be spent on that commodity from Y_{t+1} with the largest quotient of α-coefficients and prices, across all commodities from Y_{t+1}. Similarly, for any period $\tau \geq t+1$, the additional income I_τ will be spent on that commodity from Y_τ with the largest quotient of α-coefficients and prices. Therefore, if $J_{t+1} \neq \vec{0}$ then

$$u^{(t)}(w_{t-1}, x_t, s_t) = \sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}, x_t, y_{t+1}) =$$

$$\sum_{i=0}^{t} \alpha_i \cdot x_i + \beta_{t+1} s_t + \sum_{\tau=t+1}^{\infty} \beta_\tau I_\tau,$$

so that $u^{(t)}$ is still (more or less) linear.

Example 5.2.3 ***Cobb-Douglas utility***

For all t it holds that $X_t = \mathbb{R}_+^{n_t}$ for some $n_t \in \mathbb{N}$, and for notational simplicity we first suppose that $n_t = 1$ for all t. Then a (total) utility function u on $X = \times_{t=0}^{\infty} X_t$ is of Cobb-Douglas form if there are constants $\gamma > 0$ and $\alpha_t > 0$, with $\alpha := \sum_{t=0}^{\infty} \alpha_t < \infty$, such that

$$u(x) = u(x_0, x_1, x_2, ...) = \gamma \cdot \prod_{t=0}^{\infty} {x_t}^{\alpha_t}.$$

For some specific time t, suppose that no additional income will be obtained after period t so that $J_{t+1} = \vec{0} = (0, 0, ...)$, and that all prices are strictly positive $q_{t+1} \in \mathbb{R}_{++}^{\infty}$. As seen in the linear utility example, the budget set $B_{t+1}(q_{t+1}, s_t, \vec{0})$ reduces to $\{y_{t+1} \in Y_{t+1} : q_{t+1} \cdot y_{t+1} \leq s_t\}$. Again, we define the function $u^{(t)}$ on $W_{t-1} \times X_t \times \mathbb{R}_+$ by

$$u^{(t)}(w_{t-1}, x_t, s_t) = \sup_{q_{t+1} \cdot y_{t+1} \leq s_t} u(w_{t-1}, x, y_{t+1}) =$$

$$\sup_{q_{t+1} \cdot y_{t+1} \leq s_t} \gamma \cdot \prod_{i=0}^{\infty} x_i^{\alpha_i} = \gamma \cdot \prod_{i=0}^{t} x_i^{\alpha_i} \cdot \sup_{q_{t+1} \cdot y_{t+1} \leq s_t} \prod_{i=t+1}^{\infty} x_i^{\alpha_i}.$$

It is a well-known fact that under Cobb-Douglas utility the last supremum is attained by spending a (α_i/β_{t+1})-proportion of the available budget s_t in any period $i \geq t+1$, for $\beta_{t+1} := \sum_{i=t+1}^{\infty} \alpha_i$. That is, the optimal time-i choices are given by $x_i^* = (\frac{\alpha_i s_t}{\beta_{t+1} p_i})$, for all $i \geq t+1$. Thus we get that

$$u^{(t)}(w_{t-1}, x_t, s_t) = \gamma \cdot \prod_{i=0}^{t} x_i^{\alpha_i} \cdot \prod_{i=t+1}^{\infty} (\frac{\alpha_i s_t}{\beta_{t+1} p_i})^{\alpha_i} =$$

$$\gamma \cdot \prod_{i=0}^{t} x_i^{\alpha_i} \cdot (\frac{s_t}{\beta_{t+1}})^{\beta_{t+1}} \prod_{i=t+1}^{\infty} (\frac{\alpha_i}{p_i})^{\alpha_i} = \kappa \cdot \prod_{i=0}^{t} x_i^{\alpha_i} \cdot s_t^{\beta_{t+1}},$$

where $\kappa := \gamma \cdot (\frac{1}{\beta_{t+1}})^{\beta_{t+1}} \cdot \prod_{i=t+1}^{\infty} (\frac{\alpha_i}{p_i})^{\alpha_i}$. Then if $\kappa < \infty$ the function $u^{(t)}$ is a well-defined ad hoc utility function is, and by definition it is consistent with u. Moreover, the resulting consistent ad hoc utility function is also of a Cobb-Douglas form.

The same principle applies in the case where $n_t \geq 1$ for all t. Then if we denote x_t^j to be the j'th component of x_t, and given coefficients $\alpha_t^j > 0$ for all components, *the (total) utility would be given by* $\gamma \cdot \prod_{t=0}^{\infty} \prod_{j=0}^{n_t} (x_t^j)^{\alpha_t^j}$. *Then the same reasoning will show that the similarly defined consistent ad hoc utility function* $u^{(t)}$ *will still be of a Cobb-Douglas form.*

However, the assumption that no additional income is obtained after period t, is *necessary to obtain this result. If* $J_{t+1} \neq \vec{0}$, *the shape of the budget set will be more complicated, and the rule that* $x_i^* = (\frac{\alpha_i s_t}{\beta_{t+1} p_i})$ *no longer applies.*

Example 5.2.4 ***Leontief utility***

For all t *it holds that* $X_t = \mathbb{R}_+^{n_t}$ *for some* $n_t \in \mathbb{N}$, *here we first assume that* $n_t = 1$ *for all* t. *A (total) utility function* u *on* $X = \times_{t=0}^{\infty} X_t$ *is of a Leontief (or fixed-proportions) form if there are scalars* $\alpha_t > 0$ *for all* t, *such that* $u(x) = \inf\{\frac{x_0}{\alpha_0}, \frac{x_1}{\alpha_1}, \frac{x_2}{\alpha_2}, ...\}$. *This means that all goods are perfect complements.*

We suppose that after time t *no additional income will be obtained so that* $J_{t+1} = \vec{0} = (0, 0, ...)$. *Then for any future price vector* $q_{t+1} = (p_{t+1}, p_{t+2}, ...)$, *again the budget set* $B_{t+1}(q_{t+1}, s_t, \vec{0})$ *reduces to* $\{y_{t+1} \in Y_{t+1} : q_{t+1} \cdot y_{t+1} \leq s_t\}$. *Now, a consistent ad hoc utility function is given by*

$$u^{(t)}(w_{t-1}, x_t, s_t) = \sup_{q_{t+1} \cdot y_{t+1} \leq s_t} u(w_{t-1}, x_t, y_{t+1})$$

$$= \sup_{q_{t+1} \cdot y_{t+1} \leq s_t} \inf\{\frac{x_1}{\alpha_1}, \frac{x_2}{\alpha_2}, ..., \frac{x_t}{\alpha_t}, \frac{x_{t+1}}{\alpha_{t+1}}, \frac{x_{t+2}}{\alpha_{t+2}}, ...\}.$$

Given w_{t-1} *and* x_t, *the objective would be to have* $\frac{x_i}{\alpha_i} \geq \mu = \min_{0 \leq j \leq t} \frac{x_j}{\alpha_j}$, *for all periods* $i \geq t+1$. *This is affordable if* $\sum_{i=t+1}^{\infty} p_i \alpha_i \mu \leq s_t$. *If this is not affordable, then the best that can be done is to set* $\frac{x_i}{\alpha_i} = \lambda$, *for all periods* $i \geq t+1$, *with* λ *such that* $\sum_{i=t+1}^{\infty} p_i \alpha_i \lambda = s_t$. *In that case,* $\inf_i \frac{x_i}{\alpha_i}$ *would equal* $\lambda = s_t (\sum_{i=t+1}^{\infty} p_i \alpha_i)^{-1} = \frac{s_t}{\zeta_{t+1}}$, *for* $\zeta_{t+1} = \sum_{i=t+1}^{\infty} p_i \alpha_i$. *Thus,*

$$u^{(t)}(w_{t-1}, x_t, s_t) = \begin{cases} \mu & \text{if } \mu\zeta_{t+1} \leq s_t \\ \frac{s_t}{\zeta_{t+1}} & \text{if } \mu\zeta_{t+1} > s_t \end{cases}$$

$$= \min\{\mu, \frac{s_t}{\zeta_{t+1}}\} = \min\{\frac{x_1}{\alpha_1}, \frac{x_2}{\alpha_2}, ..., \frac{x_t}{\alpha_t}, \frac{s_t}{\zeta_{t+1}}\},$$

and the consistent ad hoc utility function also has a Leontief form.

The same principle applies in the case where $n_t \geq 1$ *for all* t. *Then if* x_t^j *is the* j*'th component of* x_t, *and given coefficients* $\alpha_t^j > 0$ *for all components, total utility would be given by* $u(x) = \inf_t \min_{1 \leq j \leq n_t} \frac{x_t^j}{\alpha_t^j}$. *Then the same reasoning will show that the consistent ad hoc utility function will still be of a Leontief form.*

Again, the assumption that $J_{t+1} = \vec{0}$, is *necessary to obtain this result. If* $J_{t+1} \neq \vec{0}$, *the shape of the budget set and consequently the shape of the consistent ad hoc utility function, will be more complicated.*

5.3 From ad hoc to total preferences

In the previous section we found that, given total preferences on some given total commodity space, we could always find consistent ad hoc preferences defined on the ad hoc commodity space that corresponds to a certain period. In this section we try the opposite direction: we alternatively start from a consumer, who has for every (or for some) period t, ad hoc preferences defined on the corresponding ad hoc commodity spaces. Then for the total commodity space that would be constructed as the Cartesian product of all the present commodity spaces, we investigate if it is possible to find total preferences with which (all of) these ad hoc preferences would be consistent. In other words: we investigate whether such given ad hoc preferences could be obtained from the total preferences of a rational utility maximizer who would be able to consistently summarize total preferences into ad hoc preferences, and thus whether the notion of ad hoc preferences can always be made compatible with the notion of maximizing total preferences. We find that for a single period's ad hoc preferences we can generally find total preferences with which the ad hoc preferences are consistent. However, this same result does generally not hold for several periods' ad hoc preferences, only if these different ad hoc preferences are related in some specific way (that resembles consistency), then there will be total preferences, each of the ad hoc preferences will be consistent with.

Here we first start with one single period t for which ad hoc preferences are given, and investigate the converse of proposition 5.2.1. For some period $t \in \mathbb{N}_0$ an ad hoc utility function $u^{(t)}(w_{t-1}, x_t, s_t)$ is defined on the corresponding ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$ (with $W_{t-1} = \mathbb{R}_+^{k_{t-1}}$ ($k_{t-1} \in \mathbb{N}$) and $X_t = \mathbb{R}_+^{n_t}$ ($n_t \in \mathbb{N}$)). Then for any future commodity space $Y_{t+1} = \mathbb{R}_+^{\infty}$ for the remaining periods, the corresponding total commodity space would be $X = W_{t-1} \times X_t \times Y_{t+1}$. And for any future price vector q_{t+1} and any future income stream J_{t+1} given, the question is can we always find total preferences on X represented by some $u(x)$, such that the ad hoc utility function $u^{(t)}$ is consistent with u, given q_{t+1} and J_{t+1}?

Of course it is possible that there are several different total utility functions that give rise to the same consistent ad hoc utility function, so finding one total utility function that does this already proves that the ad hoc utility function is not incompatible with the assumptions from the standard framework.

Proposition 5.3.1 *Suppose given for some $t \in \mathbb{N}_0$ an ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$ and an ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$, which is such that $u^{(t)}(w_{t-1}, x_t, s_t)$ is non-decreasing in s_t. Then for any future commodity space $Y_{t+1} = \mathbb{R}_+^{\infty}$, any strictly positive future price vector $q_{t+1} \in \mathbb{R}_{++}^{\infty}$, and any future income stream $J_{t+1} = (I_{t+1}, I_{t+2}, ...)$ with $\sum_{i=t+1}^{\infty} I_i < \infty$, there exists some total utility function $u : W_{t-1} \times X_t \times Y_{t+1} \to \mathbb{R}$, such that $u^{(t)}$ is consistent with u, given q_{t+1} and J_{t+1}.*

Proof. Given are an ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$ (with $W_{t-1} = \mathbb{R}_+^{k_{t-1}}$, $k_{t-1} \in \mathbb{N}$, and $X_t = \mathbb{R}_+^{n_t}$, $n_t \in \mathbb{N}$), and an ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to$

$\mathbb{R}$. Also suppose given some future commodity space $Y_{t+1} = \mathbb{R}_+^\infty$, some strictly positive future price vector $q_{t+1} = (p_{t+1}, p_{t+2}, ...) \in \mathbb{R}_{++}^\infty$ and some future income stream $J_{t+1} = (I_{t+1}, I_{t+2}, ...) \in \mathbb{R}_+^\infty$ that satisfies $\iota := \sum_{i=t+1}^\infty I_i < \infty$. We now want to show that there exists some total utility function u on the commodity space $X = W_{t-1} \times X_t \times Y_{t+1}$, such that $u^{(t)}$ is consistent with u, given q_{t+1} and J_{t+1}. That is, we want to show that there exists some function $u : X \to \mathbb{R}$, such that

$$u^{(t)}(w_{t-1}, x_t, s_t) = f(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}, x_t, y_{t+1})),$$

for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$, and some strictly increasing function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

To prove this proposition, it suffices to explicitly provide one total utility function that satisfies the above relation. This is exactly what we will do here, and we start from a ((quasi-)linear) function $\upsilon : Y_{t+1} \to \mathbb{R}$, as defined by

$$\upsilon(y_{t+1}) = \sum_{i=t+1}^{\infty} \alpha_i \cdot x_i - \iota,$$

where for all $i \geq t+1$ the vector $\alpha_i = \mathbb{R}_+^{n_i}$ (with $n_i \in \mathbb{N}$) is such that the coefficients for all commodities in X_i are the same: $\alpha_i^j = \overline{\alpha}_i := \min_{1 \leq k \leq n_i} \{p_i^k\}$, for all $1 \leq j \leq n_i$. This function $\upsilon : Y_{t+1} \to \mathbb{R}$ can be interpreted as a separate (sub)utility function that represents preferences for the commodities in Y_{t+1}, independent of whatever choices (w_{t-1}, x_t) have or will be made from the past and present commodity spaces. The above specification of υ means that all commodities in Y_{t+1} are perfect substitutes.

Now we are ready to define the function $u : X \to \mathbb{R}$ by

$$u(w_{t-1}, x_t, y_{t+1}) := u^{(t)}(w_{t-1}, x_t, \upsilon(y_{t+1})).$$

This new function u has the property that, for the future price vector q_{t+1} and the future income stream J_{t+1} given, it holds that

$$\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}, x_t, y_{t+1}) = u^{(t)}(w_{t-1}, x_t, s_t), \tag{1}$$

for all $(x_t, s_t) \in X_t \times \mathbb{R}_+$.

To prove this claim, we first take a look at the left-hand side, which can be expanded to

$$\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u^{(t)}(w_{t-1}, x_t, \upsilon(y_{t+1})).$$

The function $u^{(t)}(w_{t-1}, x_t, s_t)$ is non-decreasing in s_t, so that the supremum can be brought inside to get

$$u^{(t)}(w_{t-1}, x_t, \sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} \upsilon(y_{t+1})).$$

Now, the innermost term reads

$$\sup_{y_{t+1}\in B_{t+1}(q_{t+1},s_t,J_{t+1})} \upsilon(y_{t+1}) = \sup_{y_{t+1}\in B_{t+1}(q_{t+1},s_t,J_{t+1})} \sum_{i=t+1}^{\infty} \alpha_i \cdot x_i - \iota.$$

For any period $i \geq t+1$ the coefficients for all goods $1 \leq j \leq n_i$ available in this period are equal: $\alpha_i^j = \overline{\alpha_i}$. Therefore by perfect substitutability, within each period there may only be consumption from that commodity that has the lowest price. And since for any period i the coefficients α_i^j for all period-i commodities equal the smallest price in this period ($\alpha_i^j = \min_{1\leq k\leq n_i}\{p_i^k\}$), again by perfect substitutability consumption will be equally desirable in all periods (when of course this means consumption from the best priced commodity in any period). Then, like in the linear utility example from the previous section, we get that

$$\sup_{y_{t+1}\in B_{t+1}(q_{t+1},s_t,J_{t+1})} \sum_{i=t+1}^{\infty} \alpha_i \cdot x_i - \iota = \sup_{y_{t+1}\in B_{t+1}(q_{t+1},s_t+\iota,\vec{0})} \sum_{i=t+1}^{\infty} \alpha_i \cdot x_i - \iota =$$

$$\sup_{y_{t+1}\in B_{t+1}(q_{t+1},s_t+\iota,\vec{0})} \alpha_{t+1} \cdot x_{t+1} - \iota = \overline{\alpha}_{t+1}((s_t+\iota)/\overline{\alpha}_{t+1}) - \iota = s_t + \iota - \iota = s_t.$$

Here the first two equalities follow by perfect substitutability. The third equality follows from the optimal policy of spending the total budget $s_t + \iota$ on that period-$(t+1)$ commodity $j' \in \{1, 2, ..., n_{t+1}\}$ that satisfies $p_{t+1}^{j'} = \overline{\alpha}_{t+1} = \min_{1\leq k\leq n_{t+1}}\{p_{t+1}^k\}$.

Hence, the above equalities prove our claim that (1) holds, and (with the function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ defined by $f(x) = x$, for all $x \in \overline{\mathbb{R}}$) indeed we have established that there exists a total utility function u such that $u^{(t)}$ is consistent with u, given q_{t+1} and J_{t+1}. ■

The above proof uses one specific (and especially convenient) example for the subutility function υ. And while this particular example suffices to prove the proposition, here we point out that the same procedure can be followed with other subutility functions. More generally, given some $q_{t+1} \in \mathbb{R}_{++}^{\infty}$ and some J_{t+1}, we could for any function $\upsilon : Y_{t+1} \to \mathbb{R}$ define the corresponding indirect utility function ν by

$$\nu(q_{t+1}, m, J_{t+1}) = \sup_{y_{t+1}\in B_{t+1}(q_{t+1},m,J_{t+1})} \upsilon(y_{t+1}).$$

And we can define, for q_{t+1} and J_{t+1} fixed, the function $g : \mathbb{R}_+ \to \mathbb{R}$, by $g(m) = \nu(q_{t+1}, m, J_{t+1})$ for all $m \in \mathbb{R}_+$. Now, if we can show that g is a strictly increasing, real valued function, then there exists an inverse function $g^{-1} : \mathbb{R} \to \mathbb{R}_+$, which will also be strictly increasing. In that case, as in the above proof, it can be shown that the function $u(w_{t-1}, x_t, y_{t+1}) := u^{(t)}(w_{t-1}, x_t, g^{-1}(\upsilon(y_{t+1})))$ would be consistent with $u^{(t)}$, given q_{t+1} and J_{t+1}.

However, in general here the hard part is to indeed demonstrate that the function g is real valued (and thus finite valued), and strictly increasing. Since the domain for g is an infinite-dimensional space, this can get tricky. The choice of $\upsilon(y_{t+1}) = \sum_{i=t+1}^{\infty} \alpha_i \cdot x_i - \iota$, used in the proof, is especially convenient as

$$g(s_t) = \sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} \sum_{i=t+1}^{\infty} \alpha_i \cdot x_i - \iota = s_t.$$

In this proposition we see that ad hoc utility being non-decreasing in savings is sufficient for the existence of a total utility function on any total commodity space, with which $u^{(t)}$ is consistent. Recall from the discussion after the definition of consistency that ad hoc utility being non-decreasing in savings was also a necessary condition for being consistent with some total utility function.

If we would be given ad hoc preferences for several periods we could ask the same question. Under what circumstances can we find a total preference relation such that each of the given ad hoc preference relations is consistent with the (same) total preference relation?

To answer this question, here we look at two different periods $t, t' \in \mathbb{N}_0$, with $t < t'$, for which we suppose given an ad hoc utility function $u^{(t)}$ on the ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$, and an ad hoc utility function $u^{(t')}$ on the ad hoc commodity space $W_{t'-1} \times X_{t'} \times \mathbb{R}_+$. Here $W_{t-1} = \mathbb{R}_+^{k_{t-1}}$ (with $k_{t-1} \in \mathbb{N}$) and $W_{t'-1} = \mathbb{R}_+^{k_{t'-1}}$ ($k_{t'-1} \in \mathbb{N}$) are the respective past commodity spaces, $X_t = \mathbb{R}_+^{n_t}$ (with $n_t \in \mathbb{N}$) and $X_{t'} = \mathbb{R}_+^{n_{t'}}$ (with $n_{t'} \in \mathbb{N}$) are the respective present commodity spaces.

Now, we denote the set Z to represent consumption opportunities in all periods after t, but not after t'. That is, $Z := \times_{i=t+1}^{t'} X_i = \mathbb{R}_+^l$, for $l := \sum_{i=t+1}^{t'} n_i$, so that we can write $W_{t-1} \times X_t \times Z = W_{t'-1} \times X_{t'}$, and we must have that $k_{t'-1} + n_{t'} = k_{t-1} + n_t + l$.

Then, for any given future commodity space $Y_{t'+1}$, the period-t future commodity space is given by $Y_{t+1} = Z \times Y_{t'+1}$. As usual we write the total commodity space X as $W_{t'-1} \times X_{t'} \times Y_{t'+1}$ or as $W_{t-1} \times X_t \times Y_{t+1}$. For the earlier period t, we suppose given a future price vector $q_{t+1} = (p_{t+1}, p_{t+2}, ...)$, with $p_i \in \mathbb{R}_+^{n_i} \backslash \{0\}$ for all $i \geq t+1$, and a future income stream $J_{t+1} = (I_{t+1}, I_{t+2}, ...)$, with $I_i \geq 0$ for all $i \geq t+1$. For the later period we denote the future price vector $q_{t'+1} = (p_{t'+1}, p_{t'+2}, ...)$, and the future income stream $J_{t'+1} = (I_{t'+1}, I_{t'+2}, ...)$, to be the truncated versions of q_{t+1} and J_{t+1}, respectively. Similarly, the price vector $p_z = (p_{t+1}, p_{t+2}, ..., p_{t'})$ and the income stream $I_z = (I_{t+1}, I_{t+2}, ..., I_{t'})$ are truncated versions of q_{t+1} and J_{t+1}, so that we can also write $q_{t+1} = (p_z, q_{t'+1})$, and $J_{t+1} = (I_z, J_{t'+1})$.

Then given the future price vectors q_{t+1} and $q_{t'+1}$, and the future income streams J_{t+1} and $J_{t'+1}$, we investigate whether (or when) we can find a total utility function $u : X \to \mathbb{R}$ such that $u^{(t)}$ is consistent with u, given q_{t+1} and J_{t+1}, and such that $u^{(t')}$ is consistent with u, given $q_{t'+1}$ and $J_{t'+1}$.

To answer this question we define a budget set for commodity bundles in Z and period-t' savings by

$$C_z(p_z, s_t, I_z) := \left\{ (z, s_{t'}) \in Z \times \mathbb{R}_+ : \begin{array}{l} \sum_{i=t+1}^{\tau} p_i \cdot x_i \leq s_t + \sum_{i=t+1}^{\tau} I_i, \forall t+1 \leq \tau \leq t', \\ s_{t'} = s_t + \sum_{i=t+1}^{t'} I_i - \sum_{i=t+1}^{t'} p_i \cdot x_i \end{array} \right\}$$

With this set $C_z(p_z, s_t, I_z)$ we can now specify that, if $u^{(t')}$ is consistent with a utility function u, $u^{(t)}$ will also be consistent with u if and only if $u^{(t)}$ and $u^{(t')}$ are related according to the equation

$$u^{(t)}(w_{t-1}, x_t, s_t) = g(\sup_{(z, s_{t'}) \in C_z(p_z, s_t, I_z)} u^{(t')}(w_{t-1}, x_t, z, s_{t'})),$$

for some strictly increasing function $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. Note the similarity with the definition of consistency of an ad hoc utility function.

Proposition 5.3.2 *Suppose that for some periods $t, t' \in \mathbb{N}_0$, with $t < t'$, we are given past commodity spaces $W_{t-1} = \mathbb{R}_+^{k_{t-1}}$, $W_{t'-1} = \mathbb{R}_+^{k_{t'-1}}$ (with $k_{t-1}, k_{t'-1} \in \mathbb{N}$, $l = k_{t'-1} - k_{t-1} \in \mathbb{N}$), and present commodity spaces $X_t = \mathbb{R}_+^{n_t}$, $X_{t'} = \mathbb{R}_+^{n_{t'}}$ (with $n_t, n_{t'} \in \mathbb{N}$). Also suppose we are given ad hoc utility functions $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$, and $u^{(t')} : W_{t'-1} \times X_{t'} \times \mathbb{R}_+ \to \mathbb{R}$, such that $u^{(t')}$ is consistent with a total utility function $u : X \to \mathbb{R}$, given a future price vector $q_{t'+1}$ and a future income stream $J_{t'+1}$. We define the set Z as $\times_{i=t+1}^{t'} X_i$ $(= \mathbb{R}_+^l)$. Then, for a price vector $p_z \in \mathbb{R}_+^l$ and an income vector $I_z \in \mathbb{R}_+^{t'-t}$ given, it holds that $u^{(t)}$ is also consistent with u, given $q_{t+1} = (p_z, q_{t'+1})$ and $J_{t+1} = (I_z, J_{t'+1})$, if and only if*

$$u^{(t)}(w_{t-1}, x_t, s_t) = g(\sup_{(z, s_{t'}) \in C_z(p_z, s_t, I_z)} u^{(t')}(w_{t-1}, x_t, z, s_{t'})), \tag{1}$$

for some strictly increasing function $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

Proof. Given is that the ad hoc utility function $u^{(t')}$ is consistent with a total utility function $u : X \to \mathbb{R}$ (on some commodity space $X = W_{t-1} \times X_t \times Z \times Y_{t'+1}$), given a future price vector $q_{t'+1} = (p_{t'+1}, p_{t'+2}, ...)$ (with $p_\tau \in \mathbb{R}_+^{n_\tau} \setminus \{0\}$, $n_\tau \in \mathbb{N}$, for all $\tau > t'$), and a future income stream $J_{t'+1} = (I_{t'+1}, I_{t'+2}, ...)$ (with $I_\tau \in \mathbb{R}_+$ for all $\tau > t'$).

▲ Then, for the 'if' part of the proposition, suppose that (1) holds for some strictly increasing function $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. We then want to show that $u^{(t)}$ is consistent with u, given $q_{t+1} = (p_z, q_{t'+1})$ and $J_{t+1} = (I_z, J_{t'+1})$.

It was given that $u^{(t')}$ is consistent with u, given $q_{t'+1}$ and $J_{t'+1}$, so that there exists a strictly increasing function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that

$$u^{(t')}(w_{t'-1}, x_{t'}, s_{t'}) = f(\sup_{y_{t'+1} \in B_{t'+1}(q_{t'+1}, s_{t'}, J_{t'+1})} u(w_{t'-1}, x_{t'}, y_{t'+1})). \tag{2}$$

Now, we can enter this into (1) to obtain

$$u^{(t)}(w_{t-1}, x_t, s_t) = g(\sup_{(z, s_{t'}) \in C_z(p_z, s_t, I_z)} u^{(t')}(w_{t-1}, x_t, z, s_{t'})) =$$

$$g(\sup_{(z, s_{t'}) \in C_z(p_z, s_t, I_z)} f(\sup_{y_{t'+1} \in B_{t'+1}(q_{t'+1}, s_{t'}, J_{t'+1})} u(w_{t-1}, x_t, z, y_{t'+1}))) =$$

$$g(f(\sup_{(z, s_{t'}) \in C_z(p_z, s_t, I_z)} [\sup_{y_{t'+1} \in B_{t'+1}(q_{t'+1}, s_{t'}, J_{t'+1})} u(w_{t-1}, x_t, z, y_{t'+1})])) =$$

$$g(f(\sup_{y_{t+1}\in B_{t+1}(q_{t+1},s_t,J_{t+1})} u(w_{t-1},x_t,y_{t+1}))). \tag{3}$$

The third equality follows by strict increasingness of f. The last equality results from joining the two suprema $\sup_{(z,s_{t'})\in C_z(p_z,s_t,I_z)}$ and $\sup_{y_{t'+1}\in B_{t'+1}(q_{t'+1},s_{t'},J_{t'+1})}$ into a single supremum $\sup_{y_{t+1}\in B_{t+1}(q_{t+1},s_t,J_{t+1})}$, like in the proof of proposition 5.2.2.

To see that this is allowed, we can use the equality $s_{t'} = s_t + \sum_{i=t+1}^{t'} I_i - \sum_{i=t+1}^{t'} p_i \cdot x_i$ from the set C_z, to rewrite

$$B_{t'+1}(q_{t'+1}, s_{t'}, J_{t'+1}) = \{y_{t'+1} \in Y_{t'+1} : \sum_{i=t'+1}^{\tau} p_i \cdot x_i \leq s_{t'} + \sum_{i=t'+1}^{\tau} I_i, \forall\tau \geq t'+1\}$$

as

$$\{y_{t'+1} \in Y_{t'+1} : \sum_{i=t'+1}^{\tau} p_i \cdot x_i \leq s_t + \sum_{i=t+1}^{t'} I_i - \sum_{i=t+1}^{t'} p_i \cdot x_i + \sum_{i=t'+1}^{\tau} I_i, \forall\tau \geq t'+1\}$$

$$= \{y_{t'+1} \in Y_{t'+1} : \sum_{i=t+1}^{\tau} p_i \cdot x_i \leq s_t + \sum_{i=t+1}^{\tau} I_i, \forall\tau \geq t'+1\}$$

The remaining constraints from the set C_z read $\sum_{i=t+1}^{\tau} p_i \cdot x_i \leq s_t + \sum_{i=t+1}^{\tau} I_i$, for all $t+1 \leq \tau \leq t'$.

Therefore, since the vectors $(z, y_{t'+1}) = (x_{t+1}, x_{t+2}, ...) \in Y_{t+1}$ over which total utility is maximized in (3) should satisfy all the constraints from the sets C_z and $B_{t'+1}$, we get that these vectors should satisfy

$$\sum_{i=t+1}^{\tau} p_i \cdot x_i \leq s_t + \sum_{i=t+1}^{\tau} I_i, \text{ for all } \tau \geq t+1.$$

In this description of the set of all feasible vectors $y_{t+1} \in Y_{t+1}$ we may recognize the budget set $B_{t+1}(q_{t+1}, s_t, J_{t+1})$, which indeed justifies joining the two suprema.

Thus, if we denote the function $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by $h(x) = g(f(x))$, then by the above equalities from (1) and (2) we obtain

$$u^{(t)}(w_{t-1}, x_t, s_t) = h(\sup_{y_{t+1}\in B_{t+1}(q_{t+1},s_t,J_{t+1})} u(w_{t-1}, x_t, y_{t+1})), \tag{4}$$

for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$. And since both f and g are strictly increasing functions, so is their composition h, and we see that indeed $u^{(t)}$ is consistent with u, given q_{t+1} and J_{t+1}.

▲ For the 'only if' part, suppose that $u^{(t)}$ is consistent with u, given future prices $q_{t+1} = (p_z, q_{t'+1})$ and the future income stream $J_{t+1} = (I_z, J_{t'+1})$. That is, (4) holds for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$, and for some strictly increasing function $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. And it was already given that $u^{(t')}$ is consistent with u, given $q_{t'+1}$ and $J_{t'+1}$, so that (2) holds for some strictly increasing function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. We now want to show that there is some strictly increasing function $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ for which (1) holds.

We can now relate the consistent ad hoc utility from the later stage $u^{(t')}(w_{t'-1}, x_{t'}, s_{t'})$, to the consistent ad hoc utility from the earlier stage $u^{(t)}(w_{t-1}, x_t, s_t)$. That is, for $q_{t+1} = (p_z, q_{t'+1})$ and $J_{t+1} = (I_z, J_{t'+1})$ given, we see that

$$u^{(t)}(w_{t-1}, x_t, s_t) = h(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1}\}} u(w_{t-1}, x_t, y_{t+1})) =$$

$$h(\sup_{(z,s_{t'}) \in C_z(p_z, s_t, I_z)} [\sup_{y_{t'+1} \in B_{t'+1}(q_{t'+1}, s_{t'}, J_{t'+1})} u(w_{t-1}, x_t, z, y_{t'+1})]) =$$

$$h(\sup_{(z,s_{t'}) \in C_z(p_z, s_t, I_z)} f^{-1}(u^{(t')}(w_{t-1}, x_t, z, s_{t'}))) =$$

$$h(f^{-1}(\sup_{(z,s_{t'}) \in C_z(p_z, s_t, I_z)} u^{(t')}(w_{t-1}, x_t, z, s_{t'}))).$$

Here the last three equalities may need to be explained. The second equality follows from splitting up the supremum $\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1}\}}$ into $\sup_{(z,s_{t'}) \in C_z(p_z, s_t, I_z)}$ and into $\sup_{y_{t'+1} \in B_{t'+1}(q_{t'+1}, s_{t'}, J_{t'+1})}$, which was already shown to be equivalent in the 'if' part. As for the third equality, f^{-1} denotes the inverse of the function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, which can indeed be inverted by strict increasingness. This third equality uses the equivalence

$$u^{(t')}(w_{t'-1}, x_{t'}, s_{t'}) = f(\sup_{y_{t'+1} \in B_{t'+1}(q_{t'+1}, s_{t'}, J_{t'+1})} u(w_{t'-1}, x_{t'}, y_{t'+1})) \iff$$

$$f^{-1}(u^{(t')}(w_{t'-1}, x_{t'}, s_{t'})) = \sup_{y_{t'+1} \in B_{t'+1}(q_{t'+1}, s_{t'}, J_{t'+1})} u(w_{t'-1}, x_{t'}, y_{t'+1}).$$

The last equality follows from strict increasingness of f^{-1}, which in turn follows from strict increasingness of f.

Thus if we define the function $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ as the composition $g(x) = h(f^{-1}(x))$, then from the above equalities we see that (1) holds, for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$. And since both h and f^{-1} are strictly increasing functions, so is g, which concludes the proof. ∎

The interpretation of the above proposition is quite intuitive, it says that both ad hoc utility functions $u^{(t)}$ and $u^{(t')}$ are consistent with the same total utility function if and only if the ad hoc utility at stage t for a given choice of (w_{t-1}, x_t, s_t), is equal to the maximum of stage t' ad hoc utility over all bundles (from C_z) that are feasible in the stages between the periods t and t', given this stage t choice of (w_{t-1}, x_t, s_t). This is a consequence of the fact that by definition consistent time-t (and time-t') ad hoc utility is derived from final utility by assuming optimal choices in stage t and beyond (in stage t' and beyond), or of the fact that

$$\max_{z, y_{t'+1}} u(w_{t-1}, x_t, z, y_{t'+1}) = \max_z(\max_{y_{t'+1}} u(w_{t-1}, x_t, z, y_{t'+1})).$$

So this says that the number $u^{(t)}(w_{t-1}, x_t, s_t)$ does not only correspond to an optimal final utility level that may ultimately be obtained after choosing to consume x_t from X_t

and keeping s_t for later consumption. But it also says that $u^{(t)}(w_{t-1}, x_t, s_t)$ corresponds to a maximal level of ad hoc utility that may be reached in a next stage after choosing x_t and keeping s_t.

Remember that we were trying to set up a learning model of consumer behaviour, and that we were interested in investigating the convergence properties of the learning model with respect to the standard model. The definition of consistency relates ad hoc utility to total utility in a specific way in the benchmark case of the standard framework. And although we argued that the usual prospective view on consumer choice might descriptively not be very appropriate, this ideal situation may still somehow serve as a reference point. In the next chapters we will deviate from the assumptions of the standard framework, so that ad hoc preferences can no longer be explicitly calculated from final utility of optimal commodity bundles (or so that there are no total preferences given from which anything can be derived in the first place). Still, throughout we will keep using the standard framework and its relations between total preferences and consistent ad hoc preferences as a normative benchmark. We may even suggest that while people are not able to perform the sort of analysis that the standard framework predicts they do, they still try to sort of mimic the corresponding behaviour.

5.4 Separability

The present section investigates whether separability properties are carried over from total utility functions and to consistent ad hoc utility functions. These separability properties will play an important role in the following chapters. The following proposition shows that some type of (quasi-)separability carries over from total utility to consistent ad hoc utility.

Proposition 5.4.1 *Suppose given a commodity space X, that can for some $t \in \mathbb{N}$ be written as $X = W_{t-1} \times X_t \times Y_{t+1}$, and a utility function $u : X \to \mathbb{R}$, that can be written as $u(x) = u(w_{t-1}, x_t, y_{t+1}) = U_t(v_t(w_{t-1}, x_t), y_{t+1})$, for some functions $v_t : W_{t-1} \times X_t \to \mathbb{R}$ and $U_t : \mathbb{R} \times Y_{t+1} \to \mathbb{R}$. Then for any ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ that is consistent with u, given some future price vector q_{t+1} and some future income stream J_{t+1}, there exists a function $U^{(t)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ such that $u^{(t)}$ can be written as $u^{(t)}(w_{t-1}, x_t, s_t) = U^{(t)}(v_t(w_{t-1}, x_t), s_t)$.*

Proof. Given is a utility function u on the set $X = W_{t-1} \times X_t \times Y_{t+1}$, which can be written as $u(w_{t-1}, x_t, y_{t+1}) = U_t(v_t(w_{t-1}, x_t), y_{t+1})$, for all $(w_{t-1}, x_t, y_{t+1}) \in W_{t-1} \times X_t \times Y_{t+1}$, and for some functions $v_t : W_{t-1} \times X_t \to \mathbb{R}$ and $U_t : \mathbb{R} \times Y_{t+1} \to \mathbb{R}$. Also suppose given some period-t ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ on the ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$ that is consistent with u, given some system of future price vectors $q_{t+1} = (p_{t+1}, p_{t+2}, ...)$ (with $p_\tau \in \mathbb{R}_+^{n_\tau} \setminus \{0\}$, $n_\tau \in \mathbb{N}$, for all $\tau > t$) and some future income stream $J_{t+1} = (I_{t+1}, I_{t+2}, ...)$ (with $I_\tau \in \mathbb{R}_+$ for all $\tau > t$). That is, $u^{(t)}$ can be written as

$$u^{(t)}(w_{t-1}, x_t, s_t) = f(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} u(w_{t-1}, x_t, y_{t+1})),$$

for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$, and some strictly increasing function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

What needs to be shown now, is that for the given ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$, and the given function $v_t : W_{t-1} \times X_t \to \mathbb{R}$, there exists some function $U^{(t)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, such that

$$u^{(t)}(w_{t-1}, x_t, s_t) = U^{(t)}(v_t(w_{t-1}, x_t), s_t),$$

for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$.

The function $u^{(t)}$ can also be written as

$$u^{(t)}(w_{t-1}, x_t, s_t) = f(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} U_t(v_t(w_{t-1}, x_t), y_{t+1})).$$

Now, given q_{t+1} and J_{t+1}, we can define the function $g : \mathbb{R} \times \mathbb{R}_+ \to \overline{\mathbb{R}}$ by

$$g(v_t, s_t) := f(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} U_t(v_t, y_{t+1})),$$

for all $(v_t, s_t) \in \mathbb{R} \times \mathbb{R}_+$. For any q_{t+1}, any s_t and any J_{t+1}, the set $B_{t+1}(q_{t+1}, s_t, J_{t+1})$ at least contains the zero vector in Y_{t+1}, so that $B_{t+1}(q_{t+1}, s_t, J_{t+1})$ is never empty, and the supremum $\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} U_t(v_t, y_{t+1})$ is a well-defined element of $\overline{\mathbb{R}}$. Thus we see that g is a well-defined function mapping $\mathbb{R} \times \mathbb{R}_+$ into $\overline{\mathbb{R}}$. The function g so defined has the property that

$$u^{(t)}(w_{t-1}, x_t, s_t) = g(v_t(w_{t-1}, x_t), s_t), \ \forall (w_{t-1}, x_t, s_t) \tag{1}$$

Thus, g would be a good candidate for the function $U^{(t)}$ that we are looking for, except that g is defined into $\overline{\mathbb{R}}$, rather than into $\mathbb{R}$. That is, it may happen that $g(v_t, s_t) = \infty$ for some $(v_t, s_t) \in \mathbb{R} \times \mathbb{R}_+$. Still, because of (1) and the fact that $u^{(t)}$ is a real-valued function on $W_{t-1} \times X_t \times \mathbb{R}_+$, we see that it must be the case that $g(v_t(w_{t-1}, x_t), s_t) < \infty$ for all $(w_{t-1}, x_t, s_t) \in W_{t-1} \times X_t \times \mathbb{R}_+$. Thus, we can define a new function $\widetilde{g} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, so that $\widetilde{g}(v, s_t) := g(v, s_t)$ for all v in the range of the function v_t (i.e. for all $v \in v_t(W_{t-1} \times X_t) \times \mathbb{R}_+$), and so that $\widetilde{g}(v, s_t)$ is somehow finite otherwise (for all $v \notin v_t(W_{t-1} \times X_t) \times \mathbb{R}_+$). This new function $\widetilde{g}$ would indeed be real-valued, and like g it would satisfy the property described in (1). Therefore the function $\widetilde{g}$ satisfies all properties that were needed for the function $U^{(t)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, which completes the proof. ■

The following corollary applies the above proposition in a setting with some recursive type of (quasi-)separability. It establishes that this property carries over from total utility functions to consistent ad hoc utility functions.

Corollary 5.4.1 *Suppose given a commodity space X, that can for some $t \in \mathbb{N}_0$ be written as $X = X_0 \times ... \times X_t \times Y_{t+1}$, and a utility function $u : X \to \mathbb{R}$, that can be written as $u(x) =$*

$$u(x_0, x_1, ..., x_t, y_{t+1}) = U_t(v_t(v_{t-1}(...v_1(v_0(x_0), x_1)..., x_{t-1}), x_t), y_{t+1}),$$

for some functions $v_0 : X_0 \to \mathbb{R}$, $v_\tau : \mathbb{R} \times X_\tau \to \mathbb{R}$, $\forall\ 0 < \tau < t$, *and* $U_t : \mathbb{R} \times Y_{t+1} \to \mathbb{R}$. *Then, for any* $0 \leq \tau \leq t$, *and any ad hoc preference relation* $u^{(\tau)}$ *on the ad hoc commodity space* $W_{\tau-1} \times X_\tau \times \mathbb{R}_+$, *that is consistent with* u, *given some* $J_{\tau+1} = (I_{\tau+1}, I_{\tau+2}, ...)$ *and some* $q_{\tau+1} = (p_{\tau+1}, p_{\tau+2}, ...)$, *there exists a function* $U^{(\tau)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, *such that*

$$u^{(\tau)}(x_0, ..., x_\tau, s_\tau) = U^{(\tau)}(v_\tau(v_{\tau-1}(...v_1(v_0(x_0), x_1)..., x_{\tau-1}), x_\tau), s_\tau).$$

The functional structures that the functions in the above proposition and corollary satisfy were referred to as quasi-separability. Note that for separability proper it was also required that the functions $U_t(v, y_{t+1})$ and $U^{(t)}(v, s_t)$ (and similarly v_1 through v_t) would be strictly increasing in the first argument v. We could of course simply assume that the function $U_t(v, y_{t+1})$ would indeed be strictly increasing in v. However, this would not imply that in the above proposition there would exist a strictly increasing function $U^{(t)}$ such that the consistent ad hoc utility function could be separated into $u^{(t)}(w_{t-1}, x_t, s_t) = U^{(t)}(v_t(w_{t-1}, x_t), s_t)$. Here the problem would be that the function $U^{(t)}$ should satisfy

$$U^{(t)}(v_t, s_t) = f(\sup_{y_{t+1} \in B_{t+1}(q_{t+1}, s_t, J_{t+1})} U_t(v_t, y_{t+1})),$$

and that the right-hand side need not be strictly increasing in v_t. To see this, if $(v_t', s_t) \in \mathbb{R} \times \mathbb{R}_+$ is such that $\sup_{y_{t+1} \in B_{t+1}} U_t(v_t, y_{t+1})$ is infinite (which is not impossible), then we will get that

$$f(\sup_{y_{t+1} \in B_{t+1}} U_t(v_t'', y_{t+1})) = f(\sup_{y_{t+1} \in B_{t+1}} U_t(v_t', y_{t+1}))$$

for all $v_t'' > v_t'$.

In this chapter we have modelled ad hoc preferences in the standard framework, in which total preferences are also defined. This chapter has shown that, and exactly in what way, the standard framework is a special case of the alternative ad hoc framework.

6 Preference adjustment and the learning algorithm

In the previous chapters we considered situations where consumption opportunities are not encountered simultaneously, but rather sequentially. Moreover, we assumed that consumption choices are made sequentially, rather than simultaneously. In such situations each of these consumption choices would consist of choosing present consumption, and deciding how much of the available budget to save for the possibly uncertain (or even unknowable) future consumption opportunities. In every period a preference relation is assumed to underlie these choices. Thus, in order to model decision-making in an isolated period, ad hoc preferences were introduced.

In the previous chapters these ad hoc preferences were just supposed to exist, they were introduced without worrying about where they would come from. In the previous chapters ad hoc preferences were essentially treated as being exogenous. But as noted before, ultimately it does not suffice to treat ad hoc preferences as being completely exogenous; at least to some extent these ad hoc preferences will have to be "explained", because it seems inevitable that preferences for money should somehow be related to future purchasing power.

Also, in the previous chapters we only considered such subdecisions (and the underlying preferences) in isolation. No general answer was yet given as to how these ad hoc preferences would (or should) be related across periods.

You could say that up to now we have dodged the bullet, in just assuming that these ad hoc preferences exist without worrying about where they come from or how they are determined or related. In this chapter we propose to answer these questions, of how ad hoc preferences come about and how they are related across periods, at the same time. We will do this in such a general way that it may be applicable in models of choice under certainty, under uncertainty, or even under structural ignorance.

Of course, as can be seen in the previous chapter, the standard microeconomic framework for consumer choice would give a rather straightforward answer to the above questions of how ad hoc preferences come about and how they are related across periods: since money that is not spent yet is saved for future consumption, savings will simply be translated into optimally chosen future consumption bundles, and (ad hoc) utility for savings will be derived from (total) utility of optimally chosen future consumption bundles. In this standard microeconomic approach, judgments regarding the value of money are still based on prospective viewing and on full rationality.

Here, however, we assume bounded rationality, so that a consumer would lack the foresight and/or the rationality to tackle the lifetime consumption problem as standard microeconomic theory suggests. Thus, in all periods decisions are based on ad hoc preferences, and we proceed by assuming that the ad hoc preferences in the very first period are exogenous, and that in all later periods the ad hoc preferences are endogenously determined by updating the ad hoc preferences from the period before that. Thus, ad hoc preferences in the first period are somehow obtained or invented by the consumer, they reflect some initial guess at what might be reasonable, and they are assumed to be exogenous. In later stages, ad hoc preferences are assumed to be endogenously determined from ad hoc preferences in the period before that, where adjustments are made to account for the changing perspective or the additional information that is obtained since this previous period. Ad hoc preferences from the previous period serve as a basis for new ad hoc preferences, and old ad hoc preferences are adjusted according to a retrospective evaluation of the actual choices that were arrived at from these old preferences. Thus, these assumptions specify a learning model of consumer choice, where ad hoc preferences are learned over time. Ad hoc preferences in the first period are given, and if the process of adjusting ad hoc preferences is in some way efficient, then this process may 'improve' subsequent ad hoc preferences, and their derived choices. We will return to these questions of efficiency and improvements in later chapters.

In chapter 4 a first component of the ad hoc framework that this dissertation presents, was introduced by considering single periods in isolation. The present chapter will provide a second component that will link (preferences in) any two subsequent periods, and it will close the model by putting together all components, such as to arrive at a learning algorithm. Thus, we consider whole lifetimes where within each separate period a basic ad hoc consumer problem as introduced in chapter 4 is solved, and where between any pair of subsequent periods preferences are updated in a way that will be presented in the present chapter.

This chapter consists of six sections. The setting of two subsequent periods is specified in the first section. In the second section ad hoc preferences in any period are broken down into two different types of (sub)preferences: (1) instantaneous preferences that specify preferences for consumption in the corresponding period, independently of savings, and (2) time preferences that specify preferences between instantaneous preferences (or instantaneous consumption as a whole) in the corresponding period, and savings in this period. We use this distinction to assume that instantaneous preferences are exogenously given, while time preferences have to be determined endogenously by the learning procedure. The third section is somewhat technical, it links instantaneous preferences across two subsequent periods, in order to mathematically justify the adjustment procedure introduced in section 4. The fourth section provides an exact way in which time preferences (and thus ad hoc preferences) can be adjusted from the time preferences (and the ad hoc preferences) that were used in the preceding period. To that end, an adjustment function is defined that is based on retrospective evaluations of the choices made in the preceding periods. The fifth section puts all elements of the ad hoc framework together to form a learning algorithm, and the sixth section discusses and motivates some assumptions that are made in this chapter, and underlie the learning algorithm.

6.1 The setting

The formal description of the dynamics of our framework can be said to continue both logically and chronologically from the descriptions in the previous chapters. We start from the same story as before by supposing that axiom 4.1.1 holds. There is a discrete time variable t that progresses through the set $\{0, 1, 2, ...\}$. In every period t, a corresponding present commodity space $X_t = \mathbb{R}_+^{n_t}$ (with $n_t \in \mathbb{N}$), a price vector $p_t \in \mathbb{R}_+^{n_t} \backslash \{0\}$, and an additional income $I_t \geq 0$ (here we also write $I_0 = m_0$) will be given.

However, there may be uncertainty. Here the assumptions made about the unveiling of uncertainty are that X_t, p_t and I_t are known at time t (at the latest). It is important to note that this may also mean that (some of) this information is known before time t. In fact, here we want to set up a learning model that can be applied in situations with certainty, uncertainty or structural ignorance. Under certainty, we would have that X_t, p_t and I_t are known at time 0 for every period t. Under expected utility, we would have that all the realizations of X_t, p_t and I_t that might possibly occur (and the corresponding probabilities) would be known before time t, but that the actual realizations that do occur are only learned at time t. Under structural ignorance, we would have that X_t, p_t and I_t are learned at time t, and that nothing about these realizations is known before time t.

Now, we suppose that the process has reached some period $t + 1 \in \mathbb{N}$, and that in every period $i < t$ before time $t+1$, our consumer was faced with a present commodity space X_i, a present price vector p_i, and an additional income I_i that gave rise to a budget m_i, and we suppose that axiom 4.4.1 holds. From these elements an (ad hoc) budget set was constructed. Our consumer made choices in every such previous period i based on an ad hoc preference relation $\succsim^{(i)}$, which is represented here by an ad hoc utility function $u^{(i)}$, defined on the corresponding ad hoc commodity space $W_{i-1} \times X_i \times \mathbb{R}_+$. Furthermore, by $(\bar{x}_i, \bar{s}_i) \in X_i \times \mathbb{R}_+$ we denote the ad hoc choice pair that the consumer ended up choosing in that previous period i.

Here we continue the story in the next stage, at time $t + 1$, where our consumer finds himself confronted with the next set of consumption opportunities X_{t+1}, with prices p_{t+1} for these goods, and with the additional income $I_{t+1} \geq 0$. At that time, $\bar{w}_t := (\bar{x}_0, \bar{x}_1, ..., \bar{x}_t)$ was already chosen from the set $W_t := X_0 \times X_1 \times ... \times X_t$, and since we modelled a situation where no borrowing is possible but where saving is possible (at a zero interest rate), we would have that the implicit budget in period $t+1$ is given by $m_{t+1} = m_0 + \sum_{i=1}^{t+1} I_i - \sum_{i=1}^{t} p_i \cdot \bar{x}_i$.

Then, at time $t + 1$ the ad hoc choice set is given by $X_{t+1} \times \mathbb{R}_+$, where the last dimension of this set denotes savings for future consumption. The relevant ad hoc budget set consists of all elements that satisfy the budget constraint $p_{t+1} \cdot x_{t+1} + s_{t+1} \leq m_{t+1}$, and a decision is required with respect to how much to purchase of each of the n_{t+1} goods, and how much of the budget m_{t+1} should be saved for later. By axiom 4.4.1 a basic ad hoc consumer problem, that consists of the maximization of ad hoc preferences over the ad hoc budget set, should be solved. So this stage of the model can only be completed with an ad hoc preference relation or an ad hoc utility function. And such a new (endogenous) ad hoc utility function $u^{(t+1)} : W_t \times X_{t+1} \times \mathbb{R}_+ \to \mathbb{R}$ is exactly what is needed here.

6.1.1 The setting in consumption/savings models

Recall that in consumption/savings models we considered profiles of consumption levels $c = (c_0, c_1, c_2, ...)$ rather than commodity bundles. Thus in every period t we have that

$n_t = 1$, and that $X_t = \mathbb{R}_+$. The variables c_t denote consumption levels, that are measured in monetary terms, and since saving will yield no interest, we can now set $p_t = 1$ for every period t. And at time t an additional income $I_t \geq 0$ (here we also write $I_0 = m_0$) is given. Thus, there is no uncertainty about commodity spaces and prices, but additional incomes may not be certain.

In any period t, given the past choices $\bar{w}_{t-1} = (\bar{c}_0, \bar{c}_1, ..., \bar{c}_{t-1})$, our consumer finds himself confronted with the implicit budget $m_t = m_0 + \sum_{i=1}^{t} I_i - \sum_{i=1}^{t-1} \bar{c}_i$. Then our decision-maker is supposed to make trade-offs between consumption and savings such as to arrive at a decision of how to distribute this relevant budget. These trade-offs are supposed to based on some ad hoc utility function $u^{(t)} : W_{t-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$. In consumption/savings models we supposed that (total) utility functions are additively separable and satisfy exponential discounting. Therefore, it may seem reasonable to assume that ad hoc utility at time t could similarly be written as

$$u^{(t)}(w_{t-1}, c_t, s_t) = u^{(t)}(c_0, c_1, ..., c_t, s_t) = \sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1} V^{(t)}(s_t),$$

where u_0 still denotes the (same) instantaneous utility function, and $V^{(t)}$ denotes some value function that collapses the whole future into one dimension.

Then, if the process has reached period $t+1$, a new ad hoc utility function on the ad hoc commodity space $W_t \times \mathbb{R}_+ \times \mathbb{R}_+$ is needed. This new ad hoc utility function should still be of the form in the above formula. And while $\sum_{i=0}^{t+1} \delta^i u_0(c_i)$ is exogenously given, the function $V^{(t+1)}$ should be endogenous, so that determining new ad hoc preferences would boil down to determining a new value function $V^{(t+1)}$.

6.2 Instantaneous preferences and time preference

In the previous section we saw that at time $t+1$, given a past commodity bundle $\bar{w}_t \in W_t$, a decision is required with respect to what element to choose from the period-$(t+1)$ budget set

$$\{(x_{t+1}, s_{t+1}) \in X_{t+1} \times \mathbb{R}_+ : p_{t+1} \cdot x_{t+1} + s_{t+1} \leq m_{t+1}\}.$$

In order to make such a decision of how much to purchase of each of the n_{t+1} goods in X_{t+1}, and how much of the budget m_{t+1} should be saved for later, a new (endogenous) ad hoc utility function $u^{(t+1)} : W_t \times X_{t+1} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is needed. The question that remains here is how to form new ad hoc preferences.

As mentioned before, here we assume that ad hoc preferences in the very first period are exogenously given, and that in later periods ad hoc preferences (at least to some extent) are endogenously determined (i.e. learned) from ad hoc preferences in the stage before that. Updating these ad hoc preferences would be done by making adjustments to account for newly gained additional information or a somehow changed perspective. Thus, ad hoc preferences from the previous period serve as a basis for new ad hoc preferences.

So how should preferences be adapted, or how can the new ad hoc preferences, that the current choice from $X_{t+1} \times \mathbb{R}_+$ will have to be based on, be determined from the information available at that point in time? Here in trying to answer this question we distinguish two aspects of these ad hoc preferences in order to add some more structure to the problem.

We consider relative preferences for ad hoc commodity bundles separately from the question of what the preferences for money should be. We divide the process of forming preferences into two distinct types of considerations, and we may also think of these as being two different stages of the process. The first type of considerations, which will here also be referred to as instantaneous preferences, would specify relative preferences on the ad hoc commodity space, without considering preferences for money or the trade off between instantaneous consumption from this ad hoc commodity space and savings for remaining periods. The second type of consideration would, given the first type, only make the remaining trade off between instantaneous consumption and money, or essentially between the present and the future, and can therefore also be termed time preference. So the instantaneous preferences define preferences over all ad hoc commodity bundles independent of the money variable. Time preference trades off instantaneous consumption and money, where now no distinction is made anymore between ad hoc commodity bundles that were judged to be indifferent according to the instantaneous preferences.

Note that the distinction made above has a ring of independence (or separability) to it. Indeed, if for some period t ad hoc preferences $\succsim^{(t)}$ are defined on the ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$, then making the above distinction can mathematically be justified if (w_{t-1}, x_t) is independent of s_t,[14] as defined in section 2.4. Remember that from the preference relation $\succsim^{(t)}$ that is defined on $W_{t-1} \times X_t \times \mathbb{R}_+$, we can derive the preference relation $\succsim^{(t)}_{s_t}$, that gives preferences for elements in $W_{t-1} \times X_t$, given the fixed element $s_t \in \mathbb{R}_+$. And (w_{t-1}, x_t) was defined to be independent of s_t in $\succsim^{(t)}$ if the preference relations $\succsim^{(t)}_{s_t}$ are identical for all $s_t \in \mathbb{R}_+$. Thus indeed, making this distinction is mathematically valid if the conditional preference relations $\succsim^{(t)}_{s_t}$ do not depend on the particular conditioning choice of $s_t \in \mathbb{R}_+$.[15]

Definition 6.2.1 *If the ad hoc preference relation $\succsim^{(t)}$ on $W_{t-1} \times X_t \times \mathbb{R}_+$ is such that (w_{t-1}, x_t) is independent of s_t in $\succsim^{(t)}$, then the resulting conditional preference relation $\succsim^{(t)}_{s_t}$ on $W_{t-1} \times X_t$ is called an* ***instantaneous preference relation****.*

[14] A typical element of $W_{t-1} \times X_t \times \mathbb{R}_+$ is denoted (w_{t-1}, x_t, s_t).

[15] Similarly, s_t will be independent of (w_{t-1}, x_t) in $\succsim^{(t)}$ if s_t is strongly good in $\succsim^{(t)}$ (or if the corresponding utility function $u^{(t)}$ is strictly increasing in s_t). Then we would have two full-fledged preference relations $\succsim^{(t)}_{s_t}$ and $\succsim^{(t)}_{(w_{t-1},x_t)}$ that are independent of each other. And the considerations of time preference (determining the relative importance of $\succsim^{(t)}_{s_t}$ and $\succsim^{(t)}_{(w_{t-1},x_t)}$) could then be seen as a third preference relation that trades off 'levels' of both types of independent preferences, such as to complete the overall preference relation $\succsim^{(t)}$.

Such an instantaneous preference relation $\succsim_{s_t}^{(t)}$ can be seen to partition the set $W_{t-1} \times X_t$ into a number of subsets or indifference classes, in such a way that all elements contained in any such a subset are judged as equally desirable. The relation $\succsim_{s_t}^{(t)}$ also orders all of these different subsets, or indifference classes, into more and less desirable ones. That is, it seems natural to say that one indifference class is preferred to a second indifference class if any element from the first indifference class is preferred to any element from the second indifference class. Note that if we define such an indifference class as containing *all* of the elements of $W_{t-1} \times X_t$ that are judged equally desirable as a certain element of $W_{t-1} \times X_t$, it cannot happen that there are two or more of these indifference classes that are judged as equally desirable. So $\succsim_{s_t}^{(t)}$ defines a partition on $W_{t-1} \times X_t$, and a strict ordering of the elements (indifference classes) of this partition. Consequently, time preference makes a trade off between the desire to obtain higher or more desirable indifference classes of instantaneous consumption and the desire to keep money for next periods.

6.2.1 Instantaneous utility

In the previous subsection we considered ad hoc preference relations that satisfy independence in order to mathematically justify distinguishing the two types of aspects or considerations underlying ad hoc preferences. In the present subsection we will use the equivalence of preference relations satisfying independence, and utility functions satisfying separability, to further elaborate on the aforementioned distinction. This equivalence allows us to describe the distinction between the two types of considerations (or stages) underlying ad hoc preferences more conveniently in a mathematically precise way.

According to theorem 2.4.1, if the ad hoc preference relation $\succsim^{(t)}$ on $W_{t-1} \times X_t \times \mathbb{R}_+$ is represented by the utility function $u^{(t)}(w_{t-1}, x_t, s_t)$, then independence of the preference relation holds if and only if the utility function satisfies separability. Hence, if some ad hoc preference relation $\succsim^{(t)}$ on some ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$ is such that (w_{t-1}, x_t) is independent of s_t in $\succsim^{(t)}$, then we know that any utility function $u^{(t)}(w_{t-1}, x_t, s_t)$ that represents these ad hoc preferences $\succsim^{(t)}$ can be decomposed into $u^{(t)}(w_{t-1}, x_t, s_t) = U^{(t)}(v^{(t)}(w_{t-1}, x_t), s_t)$, for certain functions $v^{(t)} : W_{t-1} \times X_t \to \mathbb{R}$ and $U^{(t)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, with $U^{(t)}(v^{(t)}, s_t)$ strictly increasing in $v^{(t)}$.

Definition 6.2.2 *For a separable ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$, as given by $u^{(t)}(w_t, x_t, s_t) = U^{(t)}(v^{(t)}(w_t, x_t), s_t)$, the function $v^{(t)} : W_{t-1} \times X_t \to \mathbb{R}$ is called an* ***instantaneous utility function****, and the function $U^{(t)} : v^{(t)}(W_{t-1} \times X_t) \times \mathbb{R}_+ \to \mathbb{R}$ is called a* ***time preference function****.*

Now, the fact that both the preference relation $\succsim_{s_t}^{(t)}$ and the utility function $v^{(t)}$ carry the prefix 'instantaneous' is not a coincidence. In fact, if $\succsim_{s_t}^{(t)}$ is an instantaneous preference relation, as derived from the ad hoc preference relation $\succsim^{(t)}$, and if $v^{(t)}$ is an instantaneous utility function, as derived from the ad hoc utility function $u^{(t)}$, then if $u^{(t)}$ represents $\succsim^{(t)}$, it will also hold that $v^{(t)}$ represents $\succsim_{s_t}^{(t)}$. To see this, for all

$(w_{t-1}, x_t), (w'_{t-1}, x'_t) \in W_{t-1} \times X_t$ we have that

$$v^{(t)}(w_{t-1}, x_t) \geq v^{(t)}(w'_{t-1}, x'_t) \Leftrightarrow$$

$$U^{(t)}(v^{(t)}(w_{t-1}, x_t), s_t) \geq U^{(t)}(v^{(t)}(w'_{t-1}, x'_t), s_t), \forall s_t \in \mathbb{R}_+ \Leftrightarrow$$

$$u^{(t)}(w_{t-1}, x_t, s_t) \geq u^{(t)}(w'_{t-1}, x'_t, s_t), \forall s_t \in \mathbb{R}_+ \Longleftrightarrow$$

$$(w_{t-1}, x_t, s_t) \succsim^{(t)} (w'_{t-1}, x'_t, s_t), \forall s_t \in \mathbb{R}_+ \Leftrightarrow (w_{t-1}, x_t) \succsim^{(t)}_{s_t} (w'_{t-1}, x'_t).$$

Here the first equality holds by strict increasingness of $U^{(t)}(v^{(t)}, s_t)$ in $v^{(t)}$. Thus we see that the function $v^{(t)} : W_{t-1} \times X_t \to \mathbb{R}$ is indeed a utility function that represents the preference relation $\succsim^{(t)}_{s_t}$.[16]

Axiom 6.2.1 *For any period $t \in \mathbb{N}_0$ and any ad hoc commodity space $W_{t-1} \times X_t \times \mathbb{R}_+$, every ad hoc preference relation $\succsim^{(t)}$ on $W_{t-1} \times X_t \times \mathbb{R}_+$ is assumed to be such that (w_{t-1}, x_t) is independent of s_t in $\succsim^{(t)}$, and every ad hoc utility function $u^{(t)}$ is assumed to be separable in (w_{t-1}, x_t).*

Given an instantaneous utility function $v^{(t)} : W_{t-1} \times X_t \to \mathbb{R}$, we could define a preference relation on the set $v^{(t)}(W_{t-1} \times X_t) \times \mathbb{R}_+$[17] to represent time preferences. Then, the composition of the instantaneous preference relation 'inside' this second (time) preference relation would specify an ad hoc preference relation. As the instantaneous preferences would represent the first type of considerations from the above distinction, this second preference relation would represent the second type of considerations, which we associated with time preference before. A function $U^{(t)} : v^{(t)}(W_{t-1} \times X_t) \times \mathbb{R}_+ \to \mathbb{R}$ that would weight the relative importance of instantaneous utility and savings, or of the present and the future, would then essentially be another utility function that represents the second preference relation over elements $(v^{(t)}, s_t)$.

The above distinction of the process of forming preferences, into two types of considerations or two stages, now allows us to consider the first stage to be basically exogenous, and the second stage to be endogenous. This may seem reasonable since the considerations of the first stage are only influenced by the forms and shapes of $W_{t-1} \times X_t$, that are known and given at time t, and not by what may happen in the (uncertain) future, so they don't have to be related to considerations about money or the future. The second type of considerations would then obviously have to be endogenous, and would somehow have to be invented or constructed by the consumer.

Also note here, that an explicit assumption that instantaneous preferences should be seen as exogenous also implies that instantaneous utility would be exogenous.

[16] Also note that if we assume $u^{(t)}$ to be strictly increasing in s_t, then the identity function $i : \mathbb{R}_+ \to \mathbb{R}_+$ ($i(s_t) = s_t$) can also be seen as a separate utility function that represents $\succsim^{(t)}_{(w_{t-1}, x_t)}$.

[17] The set $v^{(t)}(W_{t-1} \times X_t)$ denotes the range of $v^{(t)}$.

Axiom 6.2.2 *For any period $t \in \mathbb{N}_0$, any instantaneous preference relation $\succsim_{s_t}^{(t)}$ on $W_{t-1} \times X_t$ is supposed to be exogenous. Similarly, for any t instantaneous utility $v^{(t)} : W_{t-1} \times X_t \to \mathbb{R}$ is supposed to be exogenous. Except for in period 0, time preference functions are endogenous.*

At a first glance the assumption of separability, or independence, that is made here may seem quite strong, and maybe it is from a micro-economic point of view. However, this separability assumption may also be seen as a mathematical, or technical assumption that will improve tractability. The separability and exogeneity assumptions do add a lot of structure to the problem, and enable us to explain something from something else. If we would not make these assumptions it would be very hard to specify a way how the ad hoc preferences could be related across periods.

Next we will see that of the examples of utility functions of specific forms that were presented in the previous chapter, some do satisfy these conditions of independence and separability, and some don't.

Example 6.2.1 ***Additively separable utility***

Suppose an ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ is additively separable, so that there are functions $u_i : X_i \to \mathbb{R}$, for all $i \leq t$, and $u_s : \mathbb{R}_+ \to \mathbb{R}$ such that $u^{(t)}(w_{t-1}, x_t, s_t) = \sum_{i=0}^{t} u_i(x_i) + u_s(s_t)$. In this case it is obvious that the function $u^{(t)}(w_{t-1}, x_t, s_t)$ can be separated into the functions $v^{(t)}(w_{t-1}, x_t) := \sum_{i=0}^{t} u_i(x_i)$ and $U^{(t)}(v^{(t)}, s_t) := v^{(t)} + u_s(s_t)$, where indeed $U^{(t)}$ is strictly increasing in $v^{(t)}$. If $\succsim^{(t)}$ is an ad hoc preference relation that is represented by $u^{(t)}$, then by theorem 2.4.1 (w_{t-1}, x_t) is independent of s_t in $\succsim^{(t)}$. It may also be instructive to show directly that the relative preferences for (w_{t-1}, x_t) are independent of s_t. To see this, for all $(w_{t-1}, x_t), (w'_{t-1}, x'_t) \in W_{t-1} \times X_t$ we have

$$(w_{t-1}, x_t) \succsim_{s_t}^{(t)} (w'_{t-1}, x'_t) \Leftrightarrow (w_{t-1}, x_t, s_t) \succsim^{(t)} (w'_{t-1}, x'_t, s_t) \Leftrightarrow$$

$$\sum_{i=0}^{t} u_i(x_i) + u_s(s_t) \geq \sum_{i=0}^{t} u_i(x'_i) + u_s(s_t) \Leftrightarrow \sum_{i=0}^{t} u_i(x_i) \geq \sum_{i=0}^{t} u_i(x'_i) \Leftrightarrow$$

$$\sum_{i=0}^{t} u_i(x_i) + u_s(s'_t) \geq \sum_{i=0}^{t} u_i(x'_i) + u_s(s'_t) \Leftrightarrow$$

$$(w_{t-1}, x_t, s'_t) \succsim^{(t)} (w'_{t-1}, x'_t, s'_t) \Leftrightarrow (w_{t-1}, x_t) \succsim_{s'_t}^{(t)} (w'_{t-1}, x'_t).$$

This will hold for all $s'_t \in \mathbb{R}_+$, so indeed we get independence.

Example 6.2.2 ***Example: linear utility***

An ad hoc utility function $u^{(t)} : W_{t-1} \times X_t \times \mathbb{R}_+ \to \mathbb{R}$ is called linear if there are vectors $\alpha_i \in \mathbb{R}_+^{n_i}$ for all periods $i \leq t$, and a scalar $\beta \geq 0$, such that $u^{(t)}(w_{t-1}, x_t, s_t) = \sum_{i=1}^{t} (\alpha_i \cdot x_i) + \beta s_t$. This means that all goods are perfect substitutes. Obviously this ad hoc utility function satisfies additive separability, so that the reasoning of the preceding example applies.

Example 6.2.3 ***Example: Cobb-Douglas utility***

As before, an ad hoc utility function $u^{(t)}$ *is defined on the set* $W_{t-1} \times X_t \times \mathbb{R}_+$*, where* $W_{t-1} = \times_{i=0}^{t-1} X_i$*, with* $X_i = \mathbb{R}_+^{n_i}$ *for some* $n_i \in \mathbb{N}$ *and for all* $i < t$*. First suppose that* $n_i = 1$ *for all* t*. Then such an ad hoc utility function is of Cobb-Douglas form if there are coefficients* $\alpha_i \geq 0$ *for all* $i \leq t$*, and* $\beta > 0$*, such that*

$$u^{(t)}(w_{t-1}, x_t, s_t) = \gamma \cdot \prod_{i=0}^{t} x_i^{\alpha_i} \cdot s_t^{\beta}.$$

We see that the function $u^{(t)}(w_{t-1}, x_t, s_t)$ *can (for instance) be separated into the functions* $v^{(t)}(w_{t-1}, x_t) := \prod_{i=0}^{t} x_i^{\alpha_i}$ *and* $U^{(t)}(v^{(t)}, s_t) = \gamma \cdot v^{(t)} \cdot s_t^{\beta}$*. However, the function* $U^{(t)}$ *is only strictly increasing in* $v^{(t)}$ *for strictly positive* s_t*, as* $U^{(t)}(v^{(t)}, 0) = 0$ *for all* $v^{(t)}$*. This does not depend on the specific functional form of ad hoc utility, as any strictly monotone transformation of* $u^{(t)}$ *will also have these properties. Hence* $u^{(t)}$ *would only really satisfy the separability property if it could be restricted to the set* $W_{t-1} \times X_t \times \mathbb{R}_{++}$*. Similarly, if* $\succsim^{(t)}$ *is an ad hoc preference relation that is represented by* $u^{(t)}$*, we get that* (w_{t-1}, x_t) *is independent of* s_t *in* $\succsim^{(t)}$ *if it would be restricted to* $W_{t-1} \times X_t \times \mathbb{R}_{++}$*. Indeed, we see that for all* $(w_{t-1}, x_t), (\tilde{w}_{t-1}, \tilde{x}_t) \in W_{t-1} \times X_t$*, the equivalences*

$$(w_{t-1}, x_t) \succ_{s_t}^{(t)} (\tilde{w}_{t-1}, \tilde{x}_t) \Leftrightarrow (w_{t-1}, x_t, s_t) \succ^{(t)} (\widetilde{w}_{t-1}, \widetilde{x}_t, s_t) \Leftrightarrow$$

$$\gamma \cdot \prod_{i=0}^{t} x_i^{\alpha_i} \cdot s_t^{\beta} > \gamma \cdot \prod_{i=0}^{t} \tilde{x}_i^{\alpha_i} \cdot s_t^{\beta} \Leftrightarrow \gamma \cdot \prod_{i=0}^{t} x_i^{\alpha_i} \cdot \tilde{s}_t^{\beta} > \gamma \cdot \prod_{i=0}^{t} \tilde{x}_i^{\alpha_i} \cdot \tilde{s}_t^{\beta}$$

$$\Leftrightarrow (w_{t-1}, x_t, \widetilde{s}_t) \succ^{(t)} (\widetilde{w}_{t-1}, \widetilde{x}_t, \widetilde{s}_t) \Leftrightarrow (w_{t-1}, x_t) \succ_{\tilde{s}_t}^{(t)} (\tilde{w}_{t-1}, \tilde{x}_t),$$

will hold if and only if s_t *and* $\tilde{s}_t$ *are strictly positive.*

With a bit of extra notation, the same result would also hold in case the period-i commodity spaces are more-dimensional: $n_i \geq 1$ *for all* $i \leq t$*.*

Example 6.2.4 ***Leontief utility***

Suppose that an ad hoc utility function $u^{(t)}$ *is defined on the set* $W_{t-1} \times X_t \times \mathbb{R}_+$*, with* $W_{t-1} = \times_{i=0}^{t-1} X_i$*, where every* X_i *is one-dimensional:* $X_i = \mathbb{R}_+$ *(so that* $n_i = 1$*). Then ad hoc utility is of a Leontief form if there are coefficients* $\alpha_i > 0$ *for all* $i \leq t$*, and* $\beta > 0$ *such that* $u^{(t)}(w_{t-1}, x_t, s_t) = \min\{\frac{x_1}{\alpha_1}, \frac{x_2}{\alpha_2}, ..., \frac{x_t}{\alpha_t}, \frac{s_t}{\beta}\}$*. In this case it seems hard to find a functional separation. In fact, if* $\succsim^{(t)}$ *is an ad hoc preference relation that is represented by* $u^{(t)}$*, then* $\succsim_{s_t}^{(t)}$ *certainly does depend on* s_t*. To see this, consider an 'efficient' point* $(\hat{w}_{t-1}, \hat{x}_t, \hat{s}_t)$ *that satisfies* $\frac{\hat{x}_1}{\alpha_1} = ... = \frac{\hat{x}_t}{\alpha_t} = \frac{\hat{s}_t}{\beta}$*. Then for any* $(w'_{t-1}, x'_t) \in W_{t-1} \times X_t$ *with* $x'_i > \hat{x}_i$ *for all* $i \leq t$*, we see that* $(w'_{t-1}, x'_t) \sim_{\hat{s}_t}^{(t)} (\hat{w}_{t-1}, \hat{x}_t)$*, but that* $(w'_{t-1}, x'_t) \succ_{s'_t}^{(t)} (\hat{w}_{t-1}, \hat{x}_t)$ *for any* $s'_t > \hat{s}_t$*. Then indeed by theorem 2.4.1 we find that there will be no functionally separable ad hoc utility function that represents* $\succsim^{(t)}$*.*

Obviously then, in the Leontief case independence and separability properties will also not hold if present commodity spaces have higher dimensions $n_i \geq 1$ *for all* i*.*

Consumption/savings In models of consumption/savings, we arrived at ad hoc utility functions $u^{(t)} : W_{t-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ of the following form:

$$u^{(t)}(w_{t-1}, c_t, s_t) = u^{(t)}(c_0, c_1, ..., c_t, s_t) = \sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1} V^{(t)}(s_t),$$

where u_0 denotes an instantaneous utility function, and where $V^{(t)}$ denotes some value function measuring the utility for money.

Such an ad hoc utility function is an instance of an additively separable utility function, as in the above example, so that it is indeed separable. For instance, this ad hoc utility function can be separated into

$$v^{(t)}(w_{t-1}, c_t) = \sum_{i=0}^{t} \delta^{i-t} u_0(c_i)$$

and

$$U^{(t)}(v^{(t)}, s_t) = \delta^t(v^{(t)} + \delta V^{(t)}(s_t)).$$

Here $U^{(t)}$ is indeed strictly increasing in $v^{(t)}$.

Such an ad hoc utility function could alternatively have been separated into $\widetilde{v}^{(t)}(w_{t-1}, c_t) = \sum_{i=0}^{t} \delta^i u_0(c_i)$ and into $\widetilde{U}^{(t)}(\widetilde{v}^{(t)}, s_t) = \widetilde{v}^{(t)} + \delta^{t+1} V^{(t)}(s_t)$. Mathematically, both ways of separating ad hoc utility are fine. In what follows we will keep using the first way, as it will turn out more convenient for intertemporal comparisons. We will come back to these issues later. ■

The assumptions of separability of ad hoc utility, and of exogeneity of instantaneous utility are made to be able to further specify how ad hoc preferences from one stage can be used to determine ad hoc preferences in a next stage. Since instantaneous utility is supposed to be given exogenously, in order to arrive at a new ad hoc utility function it suffices to find a new time preference function. In our specification instantaneous utility will be used to help determine time preference functions.

As mentioned before, we assume that ad hoc preferences in the first period are given, and that in any later stage ad hoc preferences are determined from ad hoc preferences in the preceding stage. Thus, by the above separability and exogeneity assumptions, this must mean that old time preferences are used to determine new time preferences. In stead of coming up with entirely new time preferences, here the idea is that old time preferences may serve as a basis for new time preferences, so that new time preferences are obtained by improving, or adjusting, old time preferences. Thus time preferences would be adjusted to account for newly gained additional information, or a changed perspective.

In period $t+1$, an old time preference function $U^{(t)}(.,.)$ and a new instantaneous utility function $v^{(t+1)} : W_t \times X_{t+1} \to \mathbb{R}$ would be given. And as $U^{(t)}$ will serve as a basis for the new time preference function $U^{(t+1)}$, the composite function $U^{(t)}(v^{(t+1)}, s_{t+1})$ will implicitly serve as a basis for the new ad hoc utility function $u^{(t+1)}$. Here 'serving as a basis' will mean that adjustments may still be necessary. Thus, the new auxiliary function $U^{(t)}(v^{(t+1)}, s_{t+1})$ may still need updating, and since instantaneous utility $v^{(t+1)}$ is given exogenously, this means that it is the time preference function $U^{(t)}$ that may have to be updated.

Therefore adjusting ad hoc preferences over time would basically only boil down to making adjustments in the valuations for money and instantaneous utility.

6.3 Linking subsequent stages

However, if we want to use the composite function $U^{(t)}(v^{(t+1)}, s_{t+1})$ as a basis for the new ad hoc utility function $u^{(t+1)}$, as stated in the previous section, then a first thing we need to convince ourselves of, is that the composite function $U^{(t)}(v^{(t+1)}, s_{t+1})$ is actually well-defined, and that it really makes sense. Are we simply allowed to insert the new period's instantaneous utility into the old period's time preference function? And how do we know that the instantaneous utility functions from the old and from the new period are scaled in such ways that comparisons are meaningful, and thus that interchanging them is allowed? Therefore, the question that arises here, is how the new instantaneous utility function $v^{(t+1)}$ should be scaled, in order to allow for inserting it into the old time preferences function.

Moreover, this scaling of $v^{(t+1)}$ should also take into account time discounting. In comparing instantaneous utility across periods it should not be forgotten that consumption at later dates might be valued less than consumption at earlier dates. Between any two subsequent periods this time discounting effect may be present. Therefore it will have to be dealt with in the particular way in which comparisons of instantaneous utility across periods are established, and thus in the scaling of $v^{(t+1)}$.

First we will clarify why it is needed that the scalings of instantaneous utility in different periods are in line with each other, in order to justify using last period's trade-offs between instantaneous utility and money as a basis for the current period's trade-offs between instantaneous utility and money. Suppose that instantaneous preferences $\succsim_{s_t}^{(t)}$ in period t are represented by the given instantaneous utility function $v^{(t)} : W_{t-1} \times X_t \to \mathbb{R}$, and suppose that $\hat{v}^{(t+1)} : W_t \times X_{t+1} \to \mathbb{R}$ would be some function that represents $\succsim_{s_{t+1}}^{(t+1)}$. Then the question would be how to compare the utility $v^{(t)}(w_{t-1}, x_t)$ of a certain bundle (w_{t-1}, x_t) from $W_{t-1} \times X_t$, with the utility $\hat{v}^{(t+1)}(w_t, x_{t+1})$ of another bundle (w_t, x_{t+1}) in the different set $W_{t+1} \times X_{t+1}$. Of course, the instantaneous utilities $v^{(t)}(w_{t-1}, x_t)$ and $\hat{v}^{(t+1)}(w_t, x_{t+1})$ are simply real numbers, and from any pair of real numbers the greater one can be determined. However, if $\hat{v}^{(t+1)}$ represents $\succsim_{s_{t+1}}^{(t+1)}$, then so does $f(\hat{v}^{(t+1)})$, for any strictly increasing function $f : \mathbb{R} \to \mathbb{R}$. Therefore, as there is great freedom in choosing the scaling for the new instantaneous utility function, in general we are not allowed to compare its levels to the levels of the old instantaneous utility function. Thus we generally can't extrapolate any meaning of these utility numbers to outside the domains of each of these instantaneous utility functions.

Recall that, at the more basic level of preference relations, time preference trades off amounts of savings with 'levels' of instantaneous preferences. That is, time preference from the previous period trades off instantaneous preferences with savings in this previous period. And we now want to use the time preferences from this previous period as a basis for making trade-offs between instantaneous preferences and savings in the current period. The savings variables could rather straightforwardly be modelled as the non-negative real numbers, that simply represent amounts of money that are saved. Thus it seems that savings can be quite easily compared across periods. Therefore, in order to be able to use and adapt a previous period's time preference for the new period, it has to be the case that a decision-maker can also somehow compare the 'levels' of instantaneous preferences across periods.

The set $W_t \times X_{t+1} = W_{t-1} \times X_t \times X_{t+1}$ that $v^{(t+1)}$ (and $\succsim_{s_{t+1}}^{(t+1)}$) would have to be defined on, is a bigger set than the set $W_{t-1} \times X_t$ that $v^{(t)}$ (and $\succsim_{s_t}^{(t)}$) is defined on. However, note that the set $W_{t-1} \times X_t$ is *not* a subset of $W_{t-1} \times X_t \times X_{t+1}$, because elements of $W_{t-1} \times X_t$ are not elements of the set $W_{t-1} \times X_t \times X_{t+1}$. Of course, with the element $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ we could (for instance) associate the element $(w_{t-1}, x_t, 0) \in W_t \times X_{t+1}$. However, it is important to distinguish between these two bundles $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ and $(w_{t-1}, x_t, 0) \in W_t \times X_{t+1}$ as they are not the same object.

In sections after the next, we will simply assume that a consumer *is* able to establish links between instantaneous preferences across periods, and that new instantaneous utility would always be scaled in correspondence with the scaling of old instantaneous utility, whereby taking into account the effects of time discounting. In the following subsections we will provide a mathematical justification for these assumptions, by supposing that the new instantaneous preferences are actually derived from a preference relation that is defined on the union of both sets $W_{t-1} \times X_t$ and $W_t \times X_{t+1}$. Still, in sections after these, we will continue without explicitly making use of this last new preference relation. We will simply proceed by supposing that levels of instantaneous preferences and utility *can* be directly compared.[18]

[18] This could for instance be justified in case we suppose that previous instantaneous preferences are used to determine new ones. That is, while we do assume new instantaneous preferences to be exogenous in the sense that these are unaffected by considerations of time preference, we may still assume that new instantaneous preferences are being determined by the consumer by reference to old instantaneous preferences. It might be the case that whenever a consumer is faced with the problem of determining new instantaneous preferences on $W_{t-1} \times X_t \times X_{t+1}$, he would do this by comparing the commodities in X_{t+1} to the commodities that he already knew. The new opportunity set X_{t+1} may contain commodities that he already encountered before, so these will probably not be too hard to categorize or classify in terms of the commodities that were known before the realization of X_{t+1}. But X_{t+1} may also contain commodities that our consumer did not previously know existed, and similarly there may be commodities that were included in $W_{t-1} \times X_t$ but that are not included in X_{t+1}. Hence we could assume that he approaches the problem of assessing the new set X_{t+1} precisely by trying to compare the commodities contained in it to the old commodities. For instance, this could be established by imagining all the complementarities and substitutabilities that may exist between the commodities that are encountered for the first time and the commodities that were encountered before. Thus, determining new instantaneous preferences by reference to old ones may also make the ability to compare levels of instantaneous preferences across stages more plausible.

6.3.1 Consumption/savings

In our special case encountered in consumption/savings models, these comparisons between commodity bundles obtainable in different periods seem less problematic. By assumption, in every period a present commodity bundle only consists of a one-dimensional variable that denotes the *amount* of consumption in that period, and these amounts of consumption are simply represented by their monetary values. Moreover, underlying this class of models is the assumption of exponential discounting, which says that money can buy the same levels of instantaneous utility in different periods, except for the differences due to time discounting. However, this last exception of time discounting is an important one, as the way in which intertemporal comparisons of instantaneous utilities are made should also take this into account.

Remember that in the consumption/savings examples we modelled a consumer who uses ad hoc utility of an additive form

$$u^{(t)}(w_{t-1}, c_t, s_t) = \sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1} V^{(t)}(s_t).$$

In the previous subsection on consumption/savings models we separated this ad hoc utility into exogenous instantaneous utility $v^{(t)}(w_{t-1}, c_t) = \sum_{i=0}^{t} \delta^{i-t} u_0(c_i)$ and an endogenous time preference function $U^{(t)}(v^{(t)}, s_t) = \delta^t(v^{(t)} + \delta V^{(t)}(s_t))$.

This way to separate ad hoc utility deals with time discounting in such a way that it does allow for letting new instantaneous utility $v^{(t+1)}(w_t, c_{t+1}) = \sum_{i=0}^{t+1} \delta^{i-t-1} u_0(c_i)$ enter into the old time preference function $U^{(t)}$. To see this, $U^{(t)}(v^{(t+1)}, s_{t+1}) =$

$$\delta^t(\sum_{i=0}^{t+1} \delta^{i-t-1} u_0(c_i) + \delta V^{(t)}(s_{t+1})) = \sum_{i=0}^{t+1} \delta^{i-1} u_0(c_i) + \delta^{t+1} V^{(t)}(s_{t+1}).$$

Here the exponent $t+1$ of the discount factor for valuing savings is one higher than the exponent of the discount factor for instantaneous utility in the relevant period $t+1$. This seems appropriate, as it is the same for ad hoc utility functions.

In the previous subsection on consumption/savings models, we also noted that the ad hoc utility function could alternatively have been separated into $\widetilde{v}^{(t)}(w_{t-1}, c_t) = \sum_{i=0}^{t} \delta^i u_0(c_i)$ and into $\widetilde{U}^{(t)}(\widetilde{v}^{(t)}, s_t) = \widetilde{v}^{(t)} + \delta^{t+1} V^{(t)}(s_t)$. This way to separate ad hoc utility does not deal with time discounting in a convenient way, and letting new instantaneous utility enter into the old time preference function becomes problematic. To see this, $\widetilde{U}^{(t)}(\widetilde{v}^{(t+1)}, s_{t+1})$ would equal $\sum_{i=0}^{t+1} \delta^i u_0(c_i) + \delta^{t+1} V^{(t)}(s_{t+1})$. In this specification the exponent $t+1$ of the discount factor for valuing savings is the same as the exponent of the discount factor for instantaneous utility in period $t+1$, which is not in line with the functional structure of ad hoc utility.[19] ■

[19] Still, in principle this last way (and even other ways) to separate ad hoc utility (and compare instantaneous utility levels) *could* be used here, but then to end up with an efficient learning procedure, later on in the framework adjustments would have to be made to account for time discounting.

In the more general setting, however, commodity sets may be shaped very differently in different periods, and comparisons of commodity bundles across periods may not at all be possible in such a straightforward way. Therefore it needs to be investigated how comparisons between the levels of the instantaneous preferences across periods could be established, and how an appropriate scaling for $v^{(t+1)}$ could be constructed.

All functions $\hat{v}^{(t+1)}$ and $\bar{v}^{(t+1)}$ that would represent $\succsim_{s_{t+1}}^{(t+1)}$, would be such that $\hat{v}^{(t+1)}(w_t, x_{t+1}) = \hat{v}^{(t+1)}(w'_t, x'_{t+1})$ if and only if $\bar{v}^{(t+1)}(w_t, x_{t+1}) = \bar{v}^{(t+1)}(w'_t, x'_{t+1})$ for all pairs (w_t, x_{t+1}) and (w'_t, x'_{t+1}) from $W_t \times X_{t+1}$. Hence all utility functions $\hat{v}^{(t+1)}$ that represent $\succsim_{s_{t+1}}^{(t+1)}$ will have exactly the same level sets $\{(w_t, x_{t+1}) \in W_t \times X_{t+1} : \hat{v}^{(t+1)}(w_t, x_{t+1}) = \upsilon\}$ (for some $\upsilon \in \mathbb{R}$). Thus determining a particular scaling for a function representing $\succsim_{s_{t+1}}^{(t+1)}$ would simply consist of attaching real numbers to all of these level sets. (It is easy to see that for any utility function that represents a certain preference relation, the level sets of the utility function exactly correspond to the indifference classes of the preference relation. Of course the above attaching of numbers should be such that higher numbers would correspond to more desirable level sets, or indifference classes.) The given utility function $v^{(t)}$ also attaches real numbers to indifference classes on the set $W_{t-1} \times X_t$.

By an appropriate scaling for new instantaneous utility $v^{(t+1)}$ we would mean that the levels of $v^{(t+1)}$ could be compared to the levels of old instantaneous utility $v^{(t)}$. This would mean that for any bundles $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ and $(w_t, x_{t+1}) \in W_t \times X_{t+1}$ statements such as $v^{(t)}(w_{t-1}, x_t) \geq v^{(t+1)}(w_t, x_{t+1})$ or $v^{(t)}(w_{t-1}, x_t) \leq v^{(t+1)}(w_t, x_{t+1})$ would be meaningful, in the sense that these 'bigger than' or 'smaller than' relations would indeed reflect the relative desirability of the underlying bundles. Thus we see that finding an appropriate scaling for $v^{(t+1)}$ would require being able to state preferences over the union of the sets $W_{t-1} \times X_t$ and $W_t \times X_{t+1}$.

If an appropriately chosen scaling of $v^{(t+1)}$ would attach the same number to a certain level set on $W_t \times X_{t+1}$ as $v^{(t)}$ attaches to some level set on $W_{t-1} \times X_t$, this would have to mean that the two level sets should be regarded as consisting of equally desirable bundles. Therefore, an appropriate scaling for $v^{(t+1)}$ would also join level sets on $W_{t-1} \times X_t$ with level sets on $W_t \times X_{t+1}$ into larger level sets, or indifference classes on $W_{t-1} \times X_t$ with indifference classes on $W_t \times X_{t+1}$.

The following subsections will make the above reasoning more precise.

6.3.2 Comparing instantaneous preferences

The two instantaneous preference relations $\succsim_{s_t}^{(t)}$ on $W_{t-1} \times X_t$ and $\succsim_{s_{t+1}}^{(t+1)}$ on $W_{t-1} \times X_t \times X_{t+1}$ are given. As noted in section 6.2, the relation $\succsim_{s_t}^{(t)}$ defines a partition on $W_{t-1} \times X_t$ into a number of indifference classes, and $\succsim_{s_t}^{(t)}$ defines a strict ordering of the elements (indifference classes) in this partition. Of course, $\succsim_{s_{t+1}}^{(t+1)}$ does the same on $W_{t-1} \times X_t \times X_{t+1}$.

What would now be needed is a way to compare the instantaneous preferences of elements from $W_{t-1} \times X_t$ to the instantaneous preferences of elements from $W_{t-1} \times X_t \times X_{t+1}$. That is, a consumer would need to be able to associate certain indifference classes as defined by $\succsim_{s_{t+1}}^{(t+1)}$ on $W_{t-1}\times X_t\times X_{t+1}$, to certain indifference classes as defined by $\succsim_{s_t}^{(t)}$ on $W_{t-1} \times X_t$. Such an association would have the meaning that a certain indifference class as defined by $\succsim_{s_{t+1}}^{(t+1)}$ on $W_{t-1} \times X_t \times X_{t+1}$, is judged to be indifferent to another indifference class as defined by $\succsim_{s_t}^{(t)}$ on $W_{t-1} \times X_t$. So what is actually needed is an ordering of all the indifference classes (this time not a strict ordering), both the ones determined by $\succsim_{s_{t+1}}^{(t+1)}$ on $W_{t-1}\times X_t\times X_{t+1}$, and the ones determined by $\succsim_{s_t}^{(t)}$ on $W_{t-1}\times X_t$.

Thus what is formally needed is some sort of an ordering or a preference relation on the set of all indifference classes on $W_{t-1}\times X_t\times X_{t+1}$ and on $W_{t-1}\times X_t$. But since these indifference classes are in turn defined by preference relations on $W_{t-1} \times X_t \times X_{t+1}$ and on $W_{t-1} \times X_t$, we could also let such an ordering of indifference classes be represented by one big ordering or preference relation on the set $(W_{t-1} \times X_t) \cup (W_{t-1} \times X_t \times X_{t+1})$ directly.

Definition 6.3.1 *Given instantaneous preference relations $\succsim_{s_t}^{(t)}$ on $W_{t-1}\times X_t$ for period t, and $\succsim_{s_{t+1}}^{(t+1)}$ on $W_{t-1} \times X_t \times X_{t+1}$ for period $t+1$, an* ***intertemporal instantaneous preference relation*** *is a preference relation on the set $(W_{t-1}\times X_t)\cup(W_{t-1}\times X_t\times X_{t+1})$ that agrees with $\succsim_{s_t}^{(t)}$ on $W_{t-1} \times X_t$, and with $\succsim_{s_{t+1}}^{(t+1)}$ on $W_{t-1} \times X_t \times X_{t+1}$.*

Such an intertemporal instantaneous preference relation on the set $(W_{t-1} \times X_t) \cup (W_{t-1} \times X_t \times X_{t+1})$ will be denoted by $\succsim^{\cup}$.

Before specifying what the extra conditions of $\succsim^{\cup}$ agreeing with $\succsim_{s_t}^{(t)}$ on $W_{t-1} \times X_t$, and with $\succsim_{s_{t+1}}^{(t+1)}$ on $W_{t-1} \times X_t \times X_{t+1}$ exactly entail, we will first take a look at what these conditions are needed for. Note that any preference relation on the set $(W_{t-1}\times X_t)\cup(W_{t-1}\times X_t\times X_{t+1})$ also divides this union set into a collection of indifference classes, and that it provides a strict ordering on this collection of indifference classes. Now, this new preference relation defined on the union set, was needed to join certain indifference classes as defined by $\succsim_{s_{t+1}}^{(t+1)}$ on $W_{t-1} \times X_t \times X_{t+1}$, to certain indifference classes as defined by $\succsim_{s_t}^{(t)}$ on $W_{t-1} \times X_t$. Therefore, to ensure that this is what such an intertemporal instantaneous preference relation actually does, the new preference relation on the union set should leave the indifference classes as defined by $\succsim_{s_{t+1}}^{(t+1)}$ on $W_{t-1} \times X_t \times X_{t+1}$, and by $\succsim_{s_t}^{(t)}$ on $W_{t-1} \times X_t$ intact. The following definition formally specifies this property.

Definition 6.3.2 *Given a preference relation $\succsim_1$ defined on the set S_1, we say that the preference relation $\succsim_2$ defined on the set $S_1 \cup S_2$* ***agrees*** *with $\succsim_1$ on S_1 if for all $s_1, s_1' \in S_1$ it holds that $s_1 \succsim_1 s_1'$ if and only if $s_1 \succsim_2 s_1'$.*

Applying this definition, the relation $\succsim^{\cup}$ on $(W_{t-1}\times X_t)\cup(W_{t-1}\times X_t\times X_{t+1})$ agrees with $\succsim_{s_t}^{(t)}$ on $W_{t-1} \times X_t$ if for all (w_{t-1}, x_t) and (w'_{t-1}, x'_t) in $W_{t-1} \times X_t$ we have that

$$(w_{t-1}, x_t) \succsim^{\cup} (w'_{t-1}, x'_t) \Longleftrightarrow (w_{t-1}, x_t) \succsim_{s_t}^{(t)} (w'_{t-1}, x'_t),$$

and similarly $\succsim^{\cup}$ agrees with $\succsim_{s_{t+1}}^{(t+1)}$ on $W_{t-1}\times X_t\times X_{t+1}$ if for all (w_t, x_{t+1}) and (w'_t, x'_{t+1}) in $W_{t-1} \times X_t \times X_{t+1}$ it holds that

$$(w_t, x_{t+1}) \succsim^{\cup} (w'_t, x'_{t+1}) \Longleftrightarrow (w_t, x_{t+1}) \succsim^{(t+1)}_{s_{t+1}} (w'_t, x'_{t+1}).$$

Thus, since by definition $\succsim^{\cup}$ agrees with $\succsim^{(t)}_{s_t}$ and with $\succsim^{(t+1)}_{s_{t+1}}$, we see that $\succsim^{\cup}$ indeed does not change the partitions of $W_{t-1} \times X_t$ and $W_{t-1} \times X_t \times X_{t+1}$ into indifference classes, as defined by $\succsim^{(t)}_{s_t}$ and $\succsim^{(t+1)}_{s_{t+1}}$, respectively. And $\succsim^{\cup}$ also does not change the ordering of all of the indifference classes within $W_{t-1} \times X_t$ (or within $W_{t-1} \times X_t \times X_{t+1}$). But since the relation $\succsim^{\cup}$ is defined on $(W_{t-1} \times X_t) \cup (W_{t-1} \times X_t \times X_{t+1})$, this now also gives the possibility to compare elements $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ with elements $(w'_{t-1}, x'_t, x'_{t+1}) \in W_{t-1} \times X_t \times X_{t+1}$, and to join indifference classes as defined by $\succsim^{(t)}_{s_t}$ with indifference classes as defined by $\succsim^{(t+1)}_{s_{t+1}}$.

Note that although it is not reflected in the notation $\succsim^{\cup}$, intertemporal instantaneous preference relation *will* generally depend on the relevant period t. The instantaneous preferences $\succsim^{(t)}_{s_t}$ and $\succsim^{(t+1)}_{s_{t+1}}$ are independent of s_t and s_{t+1}, respectively, and as the intertemporal instantaneous preferences $\succsim^{\cup}$ agree with $\succsim^{(t)}_{s_t}$ and $\succsim^{(t+1)}_{s_{t+1}}$ on the relevant subsets, $\succsim^{\cup}$ will similarly be independent of s_t and s_{t+1}.

6.3.3 Comparable scalings for instantaneous utility

At time t the instantaneous utility function $v^{(t)}$ was given, and in time $t+1$ the relations $\succsim^{(t+1)}_{s_{t+1}}$ and $\succsim^{\cup}$ are exogenously given. It was already given that $\succsim^{(t+1)}_{s_{t+1}}$ can be represented by some instantaneous utility function(s), and the problem that remained was whether there exists an instantaneous utility function $v^{(t+1)}$ that has a scaling that is comparable to that of $v^{(t)}$. Here we will specify what exactly is meant by $v^{(t+1)}$ having a comparable scaling as $v^{(t)}$, and we will establish when, given $v^{(t)}$, there exists such a function $v^{(t+1)}$ that has a comparable scaling.

Intertemporal instantaneous preference relations can now be used to specify and achieve such a comparable scaling for new instantaneous utility $v^{(t+1)}$. Earlier in this section we noted that the given utility function $v^{(t)}$ attaches real numbers to indifference classes on the set $W_{t-1} \times X_t$, and that a scaling for a particular function $v^{(t+1)}$ representing $\succsim^{(t+1)}_{s_{t+1}}$, would simply attach real numbers to all of the indifference classes on $W_t \times X_{t+1}$. Then, if according to the intertemporal instantaneous preference relation $\succsim^{\cup}$ a certain indifference class on $W_t \times X_{t+1}$ is equally desirable as an indifference class on $W_{t-1} \times X_t$, it seems that the appropriate scaling for $v^{(t+1)}$ would have to attach to this indifference class on $W_t \times X_{t+1}$ exactly the same real number as $v^{(t)}$ attaches to the indifference class on $W_{t-1} \times X_t$. That is, given $\succsim^{\cup}$, instantaneous utilities $v^{(t)}$ and $v^{(t+1)}$ would indeed be comparable if for all $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ and all $(w_t, x_{t+1}) \in W_t \times X_{t+1}$ it holds that $v^{(t)}(w_{t-1}, x_t) = v^{(t+1)}(w_t, x_{t+1})$ if and only if $(w_{t-1}, x_t) \sim^{\cup} (w_t, x_{t+1})$.

The above properties can be formalized in the following definition and axiom.

Definition 6.3.3 *Suppose given instantaneous preference relations $\succsim^{(t)}_{s_t}$ on $W_{t-1} \times X_t$ for period t, and $\succsim^{(t+1)}_{s_{t+1}}$ on $W_{t-1} \times X_t \times X_{t+1}$ for period $t+1$, and an intertemporal* *instantaneous preference relation* $\succsim^{\cup}$. *Also suppose given instantaneous utility functions* $v^{(t)}$ *and* $v^{(t+1)}$ *that represent* $\succsim^{(t)}_{s_t}$ *and* $\succsim^{(t+1)}_{s_{t+1}}$. *Then the scaling of* $v^{(t+1)}$ *is* ***comparable*** *to the scaling of* $v^{(t)}$ *if the function* $v^{\cup} : (W_{t-1} \times X_t) \cup (W_{t-1} \times X_t \times X_{t+1}) \to \mathbb{R}$ *which is defined so that its restriction to the set* $W_{t-1} \times X_t$ *equals* $v^{(t)}$, *and so that its restriction to the set* $W_{t-1} \times X_t \times X_{t+1}$ *equals* $v^{(t+1)}$, *represents* $\succsim^{\cup}$.

We will also call a function $v^{\cup}$ as in the above definition, that satisfies $v^{\cup}(w_{t-1}, x_t) = v^{(t)}(w_{t-1}, x_t)$ for all $(w_{t-1}, x_t) \in W_{t-1} \times X_t$, and $v^{\cup}(w_t, x_{t+1}) = v^{(t+1)}(w_t, x_{t+1})$ for all $(w_t, x_{t+1}) \in W_t \times X_{t+1}$, an **intertemporal instantaneous utility function**. If the scalings of $v^{(t)}$ and $v^{(t+1)}$ are indeed comparable, then the intertemporal instantaneous utility function $v^{\cup}$, would represent the intertemporal instantaneous preference relation $\succsim^{\cup}$. Thus, for all $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ and all $(w_t, x_{t+1}) \in W_t \times X_{t+1}$ that satisfy $(w_{t-1}, x_t) \sim^{\cup} (w_t, x_{t+1})$ it would indeed hold that $v^{\cup}(w_{t-1}, x_t) = v^{\cup}(w_t, x_{t+1})$, and thus that $v^{(t)}(w_{t-1}, x_t) = v^{(t+1)}(w_t, x_{t+1})$.

But does such an intertemporal instantaneous utility function that represents a given intertemporal instantaneous preference relation always exist? Or in other words, for $v^{(t)}$, $\succsim^{(t+1)}_{s_{t+1}}$ and $\succsim^{\cup}$given, does there always exist a function $v^{(t+1)}$ that has a comparable scaling as $v^{(t)}$? If the intertemporal instantaneous preference relation $\succsim^{\cup}$ is continuous[20], then theorem 2.2.1 can be applied to show that there exist continuous[21] functions on $(W_{t-1} \times X_t) \cup (W_{t-1} \times X_t \times X_{t+1})$ that represent $\succsim^{\cup}$. Now we can also construct a new instantaneous utility function $v^{(t+1)}$ that will have a comparable scaling as $v^{(t)}$, using $\succsim^{\cup}$ and old instantaneous utility $v^{(t)}$. We define the set Ψ as

$$\Psi := \{(w_t, x_{t+1}) \in W_t \times X_{t+1} : \exists (w_{t-1}, x_t) \in W_{t-1} \times X_t \text{ s.t. } (w_{t-1}, x_t) \sim^{\cup} (w_t, x_{t+1})\}.$$

Thus for all $(w_t, x_{t+1}) \in \Psi$, there exists some $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ for which $(w_{t-1}, x_t) \sim^{\cup} (w_t, x_{t+1})$, and we define $v^{(t+1)}$ on Ψ by $v^{(t+1)}(w_t, x_{t+1}) := v^{(t)}(w_{t-1}, x_t)$. By continuity of $\succsim^{\cup}$ it can be shown that for all $(w_t, x_{t+1}) \in \Psi^c = (W_t \times X_{t+1}) \setminus \Psi$ it must hold that $(w_t, x_{t+1}) \succ^{\cup} (w_{t-1}, x_t)$ for all $(w_{t-1}, x_t) \in W_{t-1} \times X_t$, or that $(w_t, x_{t+1}) \prec^{\cup} (w_{t-1}, x_t)$ for all $(w_{t-1}, x_t) \in W_{t-1} \times X_t$.[22] Therefore the function $v^{(t+1)}$ can simply be extended from Ψ to the whole of $W_t \times X_{t+1}$, by choosing some scaling on Ψ^c that simply attaches higher numbers to more desirable indifference classes or level sets (as defined by $\succsim^{(t+1)}_{s_{t+1}}$, or equivalently by $\succsim^{\cup}$).

[20]With respect to the topology $\mathcal{T}^{\cup} = \{S_t \cup S_{t+1} : S_t \in \mathcal{T}_t, S_{t+1} \in \mathcal{T}_{t+1}\}$, where $\mathcal{T}_t$ denotes the Euclidian topology on $W_{t-1} \times X_t$, and $\mathcal{T}_{t+1}$ denotes the Euclidian topology on $W_{t-1} \times X_t \times X_{t+1}$.

[21]With respect to $\mathcal{T}^{\cup}$. It can be shown that if such a function on $(W_{t-1} \times X_t) \cup (W_t \times X_{t+1})$ is continuous with respect to $\mathcal{T}^{\cup}$, its restriction to $W_{t-1} \times X_t$ is continuous with respect to $\mathcal{T}_t$, and its restriction to $W_t \times X_{t+1}$ is continuous with respect to $\mathcal{T}_{t+1}$.

[22]For $(w_t, x_{t+1}) \in \Psi^c$ it will hold that the sets $\{(w_{t-1}, x_t) \in W_{t-1} \times X_t : (w_{t-1}, x_t) \succsim^{\cup} (w_t, x_{t+1})\}$ and $\{(w_{t-1}, x_t) \in W_{t-1} \times X_t : (w_{t-1}, x_t) \precsim^{\cup} (w_t, x_{t+1})\}$ are each others complements. By continuity both sets should always be closed, which would imply that each of the above sets should either be the empty set, or $W_{t-1} \times X_t$.

Then, given $v^{(t)}$ and the newly defined function $v^{(t+1)}$, the accompanying intertemporal instantaneous utility function $v^{\cup}$ would indeed represent $\succsim^{\cup}$. To see this, we will distinguish a few cases.

The restriction of $v^{\cup}$ to $W_{t-1} \times X_t$ equals $v^{(t)}$, which represents $\succsim^{(t)}$, which in turn agrees with $\succsim^{\cup}$ on $W_{t-1} \times X_t$. Thus for all (w_{t-1}, x_t) and all $(w'_{t-1}, x'_t) \in W_{t-1} \times X_t$ it holds that $v^{\cup}(w_{t-1}, x_t) \geq v^{\cup}(w'_{t-1}, x'_t)$ if and only if $(w_{t-1}, x_t) \succsim^{\cup} (w'_{t-1}, x'_t)$.

Similarly, the restriction of $v^{\cup}$ to $W_t \times X_{t+1}$ equals $v^{(t+1)}$, and because of the way that $v^{(t+1)}$ is defined on $W_t \times X_{t+1}$, it is easy to see that for all $(w_t, x_{t+1}) \in \Psi^c$ and all $(w'_t, x'_{t+1}) \in W_t \times X_{t+1}$ we get that $v^{\cup}(w_t, x_{t+1}) \geq v^{\cup}(w'_t, x'_{t+1})$ if and only if $(w_t, x_{t+1}) \succsim^{\cup} (w'_t, x'_{t+1})$. Also, for all $(w_t, x_{t+1}), (w'_t, x'_{t+1}) \in \Psi$ we get that

$$v^{\cup}(w_t, x_{t+1}) \geq v^{\cup}(w'_t, x'_{t+1}) \Leftrightarrow v^{(t+1)}(w_t, x_{t+1}) \geq v^{(t+1)}(w'_t, x'_{t+1}) \Leftrightarrow$$

$$v^{(t)}(w_{t-1}, x_t) \geq v^{(t)}(w'_{t-1}, x'_t) \Leftrightarrow (w_{t-1}, x_t) \succsim^{(t)} (w'_{t-1}, x'_t) \Leftrightarrow$$

$$(w_{t-1}, x_t) \succsim^{\cup} (w'_{t-1}, x'_t) \Leftrightarrow (w_t, x_{t+1}) \succsim^{\cup} (w'_t, x'_{t+1}).$$

Here the bundles (w_{t-1}, x_t) and (w'_{t-1}, x'_t) are such that $(w_{t-1}, x_t) \sim^{\cup} (w_t, x_{t+1})$, and such that $(w'_{t-1}, x'_t) \sim^{\cup} (w_t, x_{t+1})$, respectively. Thus $v^{\cup}$ does represent $\succsim^{\cup}$ on $W_t \times X_{t+1}$.

Also, for all $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ and all $(w_t, x_{t+1}) \in \Psi$ it will hold that

$$v^{\cup}(w_{t-1}, x_t) \geq v^{\cup}(w_t, x_{t+1}) \Leftrightarrow v^{(t)}(w_{t-1}, x_t) \geq v^{(t+1)}(w_t, x_{t+1}) \Leftrightarrow$$

$$v^{(t)}(w_{t-1}, x_t) \geq v^{(t)}(w'_{t-1}, x'_t) \Leftrightarrow (w_{t-1}, x_t) \succsim^{(t)} (w'_{t-1}, x'_t) \Leftrightarrow$$

$$(w_{t-1}, x_t) \succsim^{\cup} (w'_{t-1}, x'_t) \Leftrightarrow (w_{t-1}, x_t) \succsim^{\cup} (w_t, x_{t+1}).$$

Here the bundle (w'_{t-1}, x'_t) is such that $(w'_{t-1}, x'_t) \sim^{\cup} (w_t, x_{t+1})$.

And lastly, because of how the function $v^{(t+1)}$ was defined on Ψ^c, it can be seen that for all $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ and all $(w_t, x_{t+1}) \in \Psi^c$ it will hold that $v^{\cup}(w_{t-1}, x_t) \geq v^{\cup}(w_t, x_{t+1})$ if and only if $(w_{t-1}, x_t) \succsim^{\cup} (w_t, x_{t+1})$.

Axiom 6.3.1 *For any time $t \in \mathbb{N}_0$, and any instantaneous preference relations $\succsim^{(t)}_{s_t}$ on $W_{t-1} \times X_t$, and $\succsim^{(t+1)}_{s_{t+1}}$ on $W_{t-1} \times X_t \times X_{t+1}$, there exists a continuous*[23] *intertemporal instantaneous preference relation $\succsim^{\cup}$. Moreover, given $\succsim^{(t)}_{s_t}$, $\succsim^{(t+1)}_{s_{t+1}}$ and $\succsim^{\cup}$, the instantaneous utility functions $v^{(t)}$ and $v^{(t+1)}$ that are used, represent $\succsim^{(t)}_{s_t}$ and $\succsim^{(t+1)}_{s_{t+1}}$, and have comparable scalings.*

Thus, for all t we assume that the scalings of the instantaneous utility functions $v^{(t)} : W_{t-1} \times X_t \to \mathbb{R}$ and $v^{(t+1)} : W_{t-1} \times X_t \times X_{t+1} \to \mathbb{R}$ that are used in the ad hoc framework, are indeed comparable.

One possible way in which such level comparisons of $\succsim^{(t)}_{s_t}$ and $\succsim^{(t+1)}_{s_{t+1}}$ through $\succsim^{\cup}$ could be made specific is by assuming that $(w_{t-1}, x_t) \sim^{\cup} (w_{t-1}, x_t, 0)$, (for $0 \in X_{t+1}$) for every $(w_{t-1}, x_t) \in W_{t-1} \times X_t$. This way to compare elements of $W_{t-1} \times X_t$ with elements

[23] With respect to the topology $\mathcal{T}^{\cup}$.

of $W_{t-1} \times X_t \times X_{t+1}$ was already referred to at the beginning of this section. From this specification we would obtain, for every bundle $(w_{t-1}, x_t, x_{t+1}) \in W_{t-1} \times X_t \times X_{t+1}$ for which there is a $(w'_{t-1}, x'_t) \in W_{t-1} \times X_t$ such that $(w_{t-1}, x_t, x_{t+1}) \sim^{(t+1)}_{s_{t+1}} (w'_{t-1}, x'_t, 0)$, that $(w_{t-1}, x_t, x_{t+1}) \sim^{\cup} (w'_{t-1}, x'_t)$ must also hold. If for every bundle $(w_{t-1}, x_t, x_{t+1}) \in W_{t-1} \times X_t \times X_{t+1}$, there would indeed exist a bundle $(w'_{t-1}, x'_t) \in W_{t-1} \times X_t$ such that $(w_{t-1}, x_t, x_{t+1}) \sim^{(t+1)}_{s_{t+1}} (w'_{t-1}, x'_t, 0)$, then $\succsim^{\cup}$ would be completely specified by this relation. In this case we could indeed make explicit how indifference classes from $W_{t-1} \times X_t$ should be paired with indifference classes from $W_{t-1} \times X_t \times X_{t+1}$, and can be joined to obtain indifference classes on $(W_{t-1} \times X_t) \cup (W_{t-1} \times X_t \times X_{t+1})$.

In terms of utility functions the above specification of $\succsim^{\cup}$ would give rise to a function $v^{\cup} : (W_{t-1} \times X_t) \cup (W_{t-1} \times X_t \times X_{t+1}) \to \mathbb{R}$, that satisfies $v^{\cup}(w_{t-1}, x_t) = v^{\cup}(w_{t-1}, x_t, 0)$, for every $(w_{t-1}, x_t) \in W_{t-1} \times X_t$. We would also have that $v^{\cup}(w_{t-1}, x_t, x_{t+1}) = v^{\cup}(w'_{t-1}, x'_t)$, for every bundle $(w_{t-1}, x_t, x_{t+1}) \in W_{t-1} \times X_t \times X_{t+1}$, and every $(w'_{t-1}, x'_t) \in W_{t-1} \times X_t$ such that $(w_{t-1}, x_t, x_{t+1}) \sim^{\cup} (w'_{t-1}, x'_t, 0)$, (or $v^{\cup}(w_{t-1}, x_t, x_{t+1}) = v^{\cup}(w'_{t-1}, x'_t, 0)$). This procedure would indeed exactly specify a way in which numbers (utility levels) can be attached to elements from $W_{t-1} \times X_t \times X_{t+1}$, that are in agreement with the numbers attached by $v^{(t)}$ on $W_{t-1} \times X_t$.

And although this particular way of comparing $\succsim^{(t)}_{s_t}$ with $\succsim^{(t+1)}_{s_{t+1}}$ seems to have some intuitive appeal, it is by no means the only way in which such a comparison can be established. In fact, in some circumstances other ways to compare might be more appropriate. For example, suppose that instantaneous utility would be of a Cobb-Douglas form. If we would have that $n_i = 1$ for every period i, so that $X_i = \mathbb{R}_+$, then instantaneous utility in period $t+1$ could be written as $v^{(t+1)}(w_t, x_{t+1}) = \prod_{i=0}^{t+1} x_i^{\alpha_i}$, for some $\alpha_i \geq 0$, for all $i \leq t+1$. Then for any $(w_{t-1}, x_t) \in W_{t-1} \times X_t$ the specification $v^{\cup}(w_{t-1}, x_t) = v^{\cup}(w_{t-1}, x_t, 0)$ would imply that

$$v^{(t)}(w_{t-1}, x_t) = v^{\cup}(w_{t-1}, x_t) = v^{\cup}(w_{t-1}, x_t, 0) = v^{(t+1)}(w_{t-1}, x_t, 0) = 0.$$

Therefore in a Cobb-Douglas case, this way to compare instantaneous utilities only works in the degenerate case where $v^{(t)}(w_{t-1}, x_t) = 0$, for all $(w_{t-1}, x_t) \in W_{t-1} \times X_t$.

More generally, it seems that this particular way of comparing instantaneous utilities, with $v^{\cup}(w_{t-1}, x_t) = v^{\cup}(w_{t-1}, x_t, 0)$, would be more appropriate if instantaneous consumptions are substitutes across periods, such as in the cases of linear and of additive utility. And it seems less appropriate if instantaneous consumptions are complements across periods, such as in the cases of Cobb-Douglas and Leontief utility functions.

In what follows, we will no longer explicitly make use of intertemporal instantaneous preference relations $\succsim^{\cup}$ or intertemporal instantaneous utility functions $v^{\cup}$. We will simply use instantaneous preference relations $\succsim^{(t)}_{s_t}$ and instantaneous utility functions $v^{(t)}$ to model preferences on the period-t commodity space $W_{t-1} \times X_t$, and similarly we will use $\succsim^{(t+1)}_{s_{t+1}}$ and $v^{(t+1)}$ to model preferences on $W_{t-1} \times X_t \times X_{t+1}$. Axiom 6.3.1 would then ensure that these preferences are comparable, and it would justify using last period's trade-off between instantaneous utility and money as a basis for the current period's trade-off between instantaneous utility and money.

6.4 Adjusting time preference

Recall that in the new period $t+1$ new ad hoc preferences are needed. With instantaneous preferences exogenously given, new time preferences are needed to complete these ad hoc preferences. We assume that the new period's time preferences would be obtained by adjusting the old period's time preferences, and that the new period's instantaneous preferences are used in updating time preference.

The previous subsection established a procedure for comparing instantaneous preferences across periods, which ensures that new instantaneous utility $v^{(t+1)}$ can be entered into the old time preference function $U^{(t)}$, so that the new function $U^{(t)}(v^{(t+1)}(w_{t-1}, x_t, x_{t+1}), s_{t+1})$ is well-defined, and so that it can be used as a basis for the new period's ad hoc preferences.

However, $U^{(t)}$ is a time-t guess at what the time preference function should look like, determining a time-t estimate of how to value saving relative to instantaneous consumption, which might at time $t+1$ have become obsolete. Since some time has passed between time t when time preference was last established and the present moment, within this time interval the outlook on the future may have changed, which may be a reason to evaluate, and possibly adjust, this estimate as represented by time preference.

Therefore, we proceed by assuming that the process of adjusting time preferences from old to new, is derived from an evaluation of old time preferences, using the changed outlook to assess the performance of these old time preferences in retrospect. Thus, our consumer would assess whether a new perspective necessitates him to adjust his time preference, and if so, how.

Before providing exact specifications of the procedure of updating time preferences, here we will first present the basic idea more informally. The idea with which we will proceed, is that instead of starting from scratch in determining new time preferences, our consumer would use old time preferences as a starting point, and adjusts these, thereby incorporating the new perspective. That is, the structure of the old time preference function $U^{(t)}$ may be kept more or less intact, while modifying the exact way in which the variables instantaneous utility and savings are weighted. Such a modification could be established by shifting one (or both) of the variables of the time preference function, before making the trade-offs. In mathematical terms such a shift could be realized by an adjustment factor a that is 'inserted' into U by multiplying instantaneous utility with it. Instead of the unadjusted function $U^{(t)}(v^{(t+1)}, s_{t+1})$, this would specify new ad hoc preferences by $U^{(t+1)}(v^{(t+1)}, s_{t+1}) = U^{(t)}(a \cdot v^{(t+1)}, s_{t+1})$, for some $a \in \mathbb{R}_{++}$. Thus, such an adjustment factor a would change the relative weights of instantaneous utility and savings.

Similarly, a scalar $b \in \mathbb{R}_{++}$ could shift the savings variable: $U^{(t)}(v^{(t+1)}, b \cdot s_{t+1})$. As such an adjustment factor would change the relative weights for both arguments of the time preference function, the basic idea of this way of making adjustments would be the same.

More generally, instead of inserting positive real numbers into the time preference function, we could alternatively insert a (strictly increasing) function $\alpha : \mathbb{R} \to \mathbb{R}$ into it:

$$U^{(t+1)}(v^{(t+1)}, s_{t+1}) = U^{(t)}(\alpha(v^{(t+1)}), s_{t+1}).$$

Or similarly a strictly increasing adjustment function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ could shift the savings variable $U^{(t)}(v^{(t+1)}, \beta(s_{t+1}))$. Or both variables could be shifted simultaneously by an adjustment function $\gamma : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \times \mathbb{R}_+$, that would be monotonely increasing in both arguments, giving $U^{(t)}(\gamma(v^{(t+1)}, s_{t+1}))$. Here we proceed with the simplest choice of multiplicative, uniform adjustment by means of a simple adjustment factor $a \in \mathbb{R}_{++}$ that shifts the instantaneous utility variable.

6.4.1 Consumption/savings

In a consumption/savings models an ad hoc utility function in period t was supposed to be of the following form:

$$u^{(t)}(w_{t-1}, c_t, s_t) = \sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1} V^{(t)}(s_t).$$

Here u_0 denotes the instantaneous utility function, and $V^{(t)}$ denotes a value function that collapses the whole future into one dimension. Instantaneous preferences could be represented by $v^{(t)}(w_{t-1}, c_t) = \sum_{i=0}^{t} \delta^{i-t} u_0(c_i)$, and time preferences could be represented by the function $U^{(t)}(v^{(t)}, s_t) = \delta^t(v^{(t)} + \delta V^{(t)}(s_t))$.

If in period $t+1$ a new ad hoc utility function is needed, given exogenous instantaneous utility $v^{(t+1)}(w_t, c_{t+1}) = \sum_{i=0}^{t+1} \delta^{i-t-1} u_0(c_i)$, then new time preferences are needed, which would be represented by a function of the form $\delta^{t+1}(v^{(t+1)} + \delta V^{(t+1)}(s_{t+1}))$. As this specific form and the discount factor δ are given, a new value function $V^{(t+1)}$ is basically what is needed.

Now, in the above uniform, multiplicative way to make adjustments the time preference function $U^{(t)}(v^{(t)}, s_t)$ is adjusted into $U^{(t+1)}(v^{(t+1)}, s_{t+1}) = U^{(t)}(a \cdot v^{(t+1)}, s_{t+1})$. In the present setting, a new time preference function would thus be given by $U^{(t+1)}(v^{(t+1)}, s_{t+1}) = \delta^t(a \cdot v^{(t+1)} + \delta V^{(t)}(s_{t+1}))$. Without loss of generality we may now multiply the whole right-hand-side by the scalar δa^{-1}. Thus, without loss of generality the new time preference function could be written as

$$U^{(t+1)}(v^{(t+1)}, s_{t+1}) = \delta^{t+1}(v^{(t+1)} + \delta a^{-1} V^{(t)}(s_{t+1})).$$

Therefore, the new value function could be written as $V^{(t+1)}(s_{t+1}) = a^{-1} V^{(t)}(s_{t+1})$. Indeed, in this way, the notation would be more in line with the above story of instantaneous utility being exogenous, and the value function being endogenous.

If we had instead made adjustments by shifting the savings variable through a scalar $b \in \mathbb{R}_{++}$ so that $U^{(t+1)}(v^{(t+1)}, s_{t+1}) = U^{(t)}(v^{(t+1)}, b \cdot s_{t+1})$, then we would alternatively have gotten that $V^{(t+1)}(s_{t+1}) = \delta V^{(t)}(b \cdot s_{t+1})$. In this case, the adjustment factor b would enter *inside* the old value function, rather than *outside* the old value function as we have in our chosen specification. Although this would conceptually not make much of a difference, mathematically it would. ■

Of course, the question that would naturally arise then, is what adjustment factor $a \in \mathbb{R}_{++}$ should be used. It seems only natural that the answer to this question of what adjustment factor seems appropriate would depend on the specifics of the economic environment. That is, it would (or could) depend on what has happened and on what is known about what may still happen. Thus what would be needed is some adjustment *function* that would return a strictly positive real number for every such possible environment. However, it seems quite plausible (and maybe even desirable) that such a function would be dependent on only a few selected features from such an environment. Here we will not let adjustment functions depend on complete descriptions of the economic environment as a whole, but rather we will let such functions depend on only one specific selected feature of such an economic environment. Two different specifications for such features will be introduced in the next subsection.

6.4.2 Excess expenditure

Here we will specify two features of an economic environment that an adjustment function may depend on, namely a feature called 'regular excess expenditure', and a feature called 'expected excess expenditure'. As these names suggest, both features try to measure excess expenditure. In both cases, the measure of excess expenditure in a period t is determined in the next period $t+1$, and signifies the difference between the actual expenditure from period t, and the expenditure that should have been made in period t, from a period-$(t+1)$ point of view.

Thus, we argue that the changing perspective between periods t and $t+1$ may indicate that the choice made in period t was suboptimal, more specifically that in retrospect too much or too little was spent. Then, the idea behind using excess expenditure to update preferences, would be that if excess expenditure in period t turns out to be positive (negative), then in period $t+1$ the adjustment function would have to give more (less) weight to instantaneous utility in time preferences.

In either of the two measures, excess expenditure is determined as a difference $e_t^* - e_t^\circ$. Here the first determinant e_t^* of excess expenditure in period $t+1$ is the actual expenditure from the previous period t. As the choice made in period t and the prices for the commodities in period t are known in period $t+1$, clearly this quantity is unequivocally defined, and it will be the same for both measures of excess expenditure. The second determinant e_t° of excess expenditure in period $t+1$ denotes the expenditure that should have been done in period t, from a period-$(t+1)$ point of view. However, it does not seem equally clear how this last quantity e_t° should precisely be determined, and the two excess expenditure measures approach this question in different ways. In fact, these two measures of excess expenditure are precisely distinguished by two different specifications of e_t°.

The main difficulty in defining the last quantity e_t° of how much *should* have been spent, is that such a retrospective evaluation seems to require being able to make trade-offs between instantaneous utility and savings in the new period $t+1$. But of course these trade-offs for the new period, were exactly what needed to be explained endogenously from excess expenditures (and e_t°) in the first place. Now, for both specifications of excess expenditure, our way to deal with this difficulty is by assuming that old time preferences (from the previous period) are used to make these trade-offs.

That is, in order to be able to state that the period-t choice x_t^* was indeed chosen suboptimally, we need some time-$(t+1)$ way to compare elements from $W_{t-1} \times X_t \times X_{t+1} \times \mathbb{R}_+$. Of course, new instantaneous preferences $v^{(t+1)}$ are given, and are defined on the set $W_t \times X_{t+1} = W_{t-1} \times X_t \times X_{t+1}$, so this gives a way to assess the instantaneous desirability of x_t^*. But, in order to be able to judge x_t^* as being too expensive or too cheap, it is necessary to compare levels of instantaneous utility with levels of money. That is, time preferences in period $t+1$ would be needed, which is exactly what was needed to make choices in the new period. Thus to avoid a circle reasoning, we will simply assume that old time preferences as found in $U^{(t)}$ are used to make the trade-offs needed to re-evaluate x_t^*.

At first sight it may seem somewhat strange to again use $U^{(t)}$ to evaluate x_t^*, since indeed x_t^* was determined by $U^{(t)}$. After all, if according to $U^{(t)}$ the bundle x_t^* was thought to be optimal, how can the same bundle x_t^* now in turn be used to judge whether $U^{(t)}$ was optimal? Here the new time-$(t+1)$ outlook may shed a different light on $U^{(t)}$. The time preference function $U^{(t)}$ was only used to determine an increment x_t^* of the total consumption bundle, given the previous composition of the consumption bundle $\bar{w}_{t-1}$, and the wider perspective of the next period may already make things look different, as these time preferences need not necessarily be time-consistent.

Regular excess expenditure The first measure of excess expenditure is presented here. As the prefix 'regular' seems to suggest, in what follows we will treat this measure of excess expenditure as the most basic one. This is also due to the fact that, in contrast to the second measure, this first measure of excess expenditure can be defined and determined in all models that fit the general ad hoc framework as presented thus far. We will also sometimes use the abbreviation **REE** to refer to this regular excess expenditure measure.

Definition 6.4.1 *At time $t+1$, the period-t* ***regular excess expenditure*** *is determined by $E_t := p_t \cdot x_t^* - p_t \cdot x_t^\triangleleft$. Here x_t^* is part of a pair (x_t^*, s_t^*) that was chosen in time t, as it solved*

$$\max_{(x_t,s_t)} U^{(t)}(v^{(t)}(\bar{w}_{t-1}, x_t), s_t) \text{ s.t. } p_t \cdot x_t + s_t \leq m_t,$$

where a profile of past choices $\bar{w}_{t-1} \in W_{t-1}$, the set X_t, prices p_t, the budget m_t, instantaneous utility $v^{(t)} : W_{t-1} \times X_t \to \mathbb{R}$, and the period-$t$ time preference function $U^{(t)} : \mathbb{R} \times X_t \to \mathbb{R}$ were all given. And $x_t^\triangleleft$ is part of a solution $(x_t^\triangleleft, x_{t+1}^\triangleleft, s_{t+1}^\triangleleft)$ to

$$\max_{(x_t,x_{t+1},s_{t+1})} U^{(t)}(v^{(t+1)}(\bar{w}_{t-1}, x_t, x_{t+1}), s_{t+1}),$$

sub to $p_t \cdot x_t + s_t \leq m_t$ and $p_{t+1} \cdot x_{t+1} + s_{t+1} \leq s_t + I_{t+1}$, where all time-$t$ information (the set X_{t+1}, prices p_{t+1}, the budget $m_{t+1} = m_t - p_t \cdot x_t^* + I_{t+1}$) and the instantaneous utility function $v^{(t+1)} : W_{t-1} \times X_t \times X_{t+1} \to \mathbb{R}$ are given.

Here the auxiliary function $U^{(t)}(v^{(t+1)}(\bar{w}_{t-1}, x_t, x_{t+1}), s_{t+1})$ is used to determine $x_t^{\lhd}$ and therefore to assess x_t^*. This last maximization problem over x_t and x_{t+1} simultaneously (instead of over x_t only) is a hypothetical problem, in the sense that at time $t+1$ when this problem is considered, the choice of x_t^* is already made, so the solution $x_t^{\lhd}$ found in this new problem can not anymore be actually implemented, but it will only be used to asses x_t^*, and indirectly the efficacy of time preference.

Expected excess expenditure Expected excess expenditure is distinguished from regular excess expenditure by the prefix 'expected'. This prefix reflects the use of expected utility to re-evaluate the previous period's expenditure. Here it is important to note that whereas regular excess expenditure (REE) can be defined and determined in all models that fit the ad hoc framework, the next measure of expected excess expenditure can only be defined and determined in expected utility models. More specifically, information is needed about the realizations that the additional income random variable I can take, and about the probabilities each of these possible realizations will occur with.

The expected excess expenditure measure is similar to the measure of regular excess expenditure, it is still the difference of actual expenditure in the previous period and some measure of how much should have been spent in the previous period, from the perspective of the new, current period. However, this measure of how much should have been spent in the previous period is different from the measure used for REE in the previous subsection. We will also sometimes use the abbreviation **EEE** to refer to the expected excess expenditure measure.

Definition 6.4.2 *At time $t+1$, the period-t* ***expected excess expenditure*** *is determined by $F_t := p_t \cdot x_t^* - p_t \cdot x_t^{\diamond}$. Here x_t^* is part of a pair (x_t^*, s_t^*) that was chosen in time t, as it solved*

$$\max_{(x_t,s_t)} U^{(t)}(v^{(t)}(\bar{w}_{t-1}, x_t), s_t) \text{ s.t. } p_t \cdot x_t + s_t \leq m_t.$$

And $x_t^{\diamond}$ is part of a plan that solves

$$\max_{(x_t,s_t):p_t\cdot x_t+s_t\leq m_t} E_{I_{t+1}}[\max_{(x_{t+1},s_{t+1}):p_{t+1}\cdot x_{t+1}+s_{t+1}\leq s_t+I_{t+1}} U^{(t)}(v^{(t+1)}(\bar{w}_{t-1}, x_t, x_{t+1}), s_{t+1})],$$

where all time-t information (the set X_{t+1}, prices p_{t+1}, the budget $m_{t+1} = m_t - p_t \cdot x_t^* + I_{t+1}$) and the instantaneous utility function $v^{(t+1)} : W_{t-1} \times X_t \times X_{t+1} \to \mathbb{R}$ are given.

The bundle $x_t^{\diamond}$ is part of a contingency plan that specifies a bundle $x_{t+1}^r \in X_{t+1}$ and an amount of savings $s_{t+1}^r \in \mathbb{R}_+$ for every realization I^r that the additional income random variable can take. That is, if the number of realizations that I can take equals $R+1 \in \mathbb{N}$, then the last maximization problem from the definition is solved by a plan $(x_t^{\diamond}, (x_{t+1}^r, s_{t+1}^r)_{r=0}^R)$, that satisfies $p_t \cdot x_t^{\diamond} + s_t^{\diamond} \leq m_t$ and $p_{t+1} \cdot x_{t+1}^r + s_{t+1}^r \leq s_t^{\diamond} + I^r$, for all realizations r.

While the realization of I_{t+1} is known at time $t+1$, in the determination of expected excess expenditure this information is not used. When compared with regular excess expenditure, the idea behind expected excess expenditure is basically to disregard the information known at time $t+1$ about the realization of the additional income random variable I_{t+1}, and instead use the information known at time t about I_{t+1}. This can only be done if there is some ex ante information available about the values that additional income can take. And indeed, in an expected utility model this information is given, and the additional income I can be treated as a random variable.

Besides the additional income realization, the realizations of all other uncertain variables are not disregarded in determining EEE. Of course, these other uncertain variables could in principle be treated in the same way as additional income, but for more (notational) simplicity the above specification is chosen. Moreover, in what follows we will see that such an alternative specification would not make a difference.

Although at this point it may seem strange to disregard available information in the process of re-evaluating past choices (and adjusting time preferences), this EEE measure will later on serve as a useful benchmark, and even prove to be more efficient[24] than the REE measure in some instances. We will come back to these questions in chapter 9.

Note that F_t does not actually specify an expectation of excess expenditure, so do not be misled by the term 'expected excess expenditure'. Rather, F_t gives the excess expenditure as determined by the actual previous expenditure level minus the expenditure level that would maximize the ex ante expected utility.

Also note that in models of certainty, the two measures REE and EEE coincide. In fact, if there is no uncertainty, then additional income can only take one realization, so the random variable specifying additional income is a degenerate one. Then the expectation drops out of the last formula in the EEE definition, and expected utility simplifies to regular utility. Thus the last formula in the EEE definition that $x_t^{\diamond}$ is supposed to maximize, reduces to the last formula in the REE definition that $x_t^{\triangleleft}$ maximizes. ■

The specific adjustment functions, that will be specified shortly, will deal with the two measures of excess expenditure E and F in exactly the same way. Therefore to have notation that deals with both measures of excess expenditure at the same time, we use $\mathcal{E}$ to denote either of the measures of excess expenditure, so $\mathcal{E} \in \{E, F\}$, and $\mathcal{E}_t \in \{E_t, F_t\}$.

For either of the two measures of excess expenditure that may be used, we suppose that this measure will be determined in every period. Thus in period $t+1$ the regular excess expenditures $\mathcal{E}_0, \mathcal{E}_1, ..., \mathcal{E}_t$ are known.[25]

Definition 6.4.3 *Given a measure $\mathcal{E}$ of excess expenditure, at time $t+1$, a* ***history of excess expenditures*** *is a vector $\eta_{t+1} := (\mathcal{E}_0, \mathcal{E}_1, ..., \mathcal{E}_t)$ of excess expenditures.*

While a history of excess expenditures η_{t+1} may represent both measures of excess expenditure, we also want to have more specific notation. At time $t+1$, a **history of**

[24] For some measure of efficiency that will be specified in later chapters.

[25] While excess expenditure will be determined in every period, each of these past excess expenditures only looks one period back, as in any period i, $\mathcal{E}_{i-1}$ is determined.

expected excess expenditures will be denoted by $\phi_{t+1} := (F_0, F_1, ..., F_t)$. At time $t+1$, a **history of regular excess expenditures** will be denoted $\varepsilon_{t+1} := (E_0, E_1, ..., E_t)$. So η_{t+1} can represent both ε_{t+1} and ϕ_{t+1}. Since for either measure of excess expenditure, $\mathcal{E}_t$ is an element of $\mathbb{R}$, a history η_{t+1} of excess expenditures is an element of $\mathbb{R}^{t+1}$.

6.4.3 Adjustment functions

As argued in the previous subsections, what adjustment factor seems appropriate would depend on the economic environment, or rather on some specifics of such an economic environment. Here an adjustment function may depend on histories of excess expenditures: either histories of regular excess expenditures, or histories of expected excess expenditures. Thus an adjustment function should return positive real numbers for such histories.

Definition 6.4.4 *An* ***adjustment function*** *is a function from histories η_{t+1} into the strictly positive real numbers: $a : \cup_{t=0}^{\infty}\mathbb{R}^{t+1} \to \mathbb{R}_{++}$.*

An adjustment function returns for every history η_{t+1} a strictly positive real number $a(\eta_{t+1}) \in \mathbb{R}_{++}$. For notational convenience we will also let $a(\eta_{t+1})$ be denoted as $a_{\eta_{t+1}}$. The above definition of an adjustment function has remained completely general, only the domain and the range of such a function have been specified. Before providing a specific example of an adjustment function, here we will first look at the rationale for letting an adjustment function depend on some measure of excess expenditure (whether it be REE or EEE), and explore *how* an adjustment function should depend on excess expenditure.

Recall that the last period's ad hoc utility $u^{(t)}(\bar{w}_{t-1}, x_t, s_t)$ must have somehow represented something of an aggregate estimate of the desirability of all affordable future consumption opportunities after period t. In the process of maximizing last period's ad hoc utility $u^{(t)}$, time preferences gave a cut-off point that determined not so much the proportions of the various commodities that were purchased from X_t, but rather how much was spent on the commodities in X_t as a whole.

Next, at time $t+1$, the choice x_t^* from period t is re-evaluated, using new ad hoc preferences $v^{(t+1)}$ and old time preferences $U^{(t)}$. Then, the wider perspective of an additional period may put x_t^* in a different light, the changing perspective at time $t+1$ may mean that the choice of (x_t^*, s_t^*) that was based on previous time preferences $U^{(t)}$ may now turn out to have been suboptimal.

Time preference basically determines the trade-offs between spending and saving. Therefore if a consumer should think that he has been spending too much in the previous period (if excess expenditure is positive), this seems to imply that the prevailing time preference from stage t should be judged as being suboptimal, more specifically that old time preferences weighted instantaneous consumption too heavily. Conversely, if our consumer should think that he should have spent more in the previous period (if excess expenditure is negative), this should mean that old time preferences weighted savings too heavily. Thus, if time preference $U^{(t)}$ is considered suboptimal, so that the trade-offs between instantaneous consumption and money could be improved upon, this can be seen as an indication that time preferences might have to be adjusted.

Of course the idea would then be that finding that the actual choice made in the previous period turned out to be too expensive, would mean that the previous period's time preferences put too much emphasis on instantaneous utility, so that time preferences will have to be updated such as to give less weight to instantaneous utility, and more weight to saving. That is, when x_t^* turns out to have been too expensive (when $\mathcal{E}_t > 0$), then the adjustment factor a should be smaller than one. Conversely, finding that more should have been spent in the previous period, would have to mean that the previous period's time preferences did not put enough emphasis on instantaneous utility, so that time preferences will have to be updated such as to give more weight to instantaneous utility, and less weight to saving. Thus, when it turns out that x_t^* was actually too cheap (when $\mathcal{E}_t < 0$), then the adjustment factor a should be larger than one.

And when it would incidentally turn out that x_t^* *was* judged to be optimal from a period-$(t+1)$ perspective (when $\mathcal{E}_t = 0$), then it was judged to be neither too expensive, nor too cheap. In that case there does not seem to be any reason to change the weights of instantaneous utility or savings in any direction, so $U^{(t)}$ would not have to be adjusted and the adjustment factor a would equal one. In that case $U^{(t)}(v^{(t+1)}(\bar{w}_{t-1}, x_t, x_{t+1}), s_{t+1})$ *would* simply suffice to describe the new ad hoc preferences $u^{(t+1)}((\bar{w}_{t-1}, x_t, x_{t+1}), s_{t+1})$.

Next, we provide two different specifications of adjustment functions that achieve all of the dynamic implications mentioned in this subsection. Both specifications of adjustment functions may depend both on regular excess expenditure histories, and on expected excess expenditure histories.

Value-based adjustment Although formally adjustment functions were defined to be dependent on histories of excess expenditures, such an adjustment function need not necessarily use all the information in such a history. The first adjustment function that we specify here only depends on excess expenditure in the most recent period.

In the previous subsection we saw that an adjustment function should be such that a positive excess expenditure in the previous period would lead to an adjustment factor that is smaller than one, and thus that the weighting for instantaneous utility in time preference is decreased. And conversely, negative excess expenditure in the previous period should lead to an adjustment factor that is larger than one, so that the weighting for instantaneous utility in time preference would be increased. Any positive-valued adjustment function $a_v : \mathbb{R} \to \mathbb{R}_{++}$ that would depend on, and decrease in, last period's excess expenditure $\mathcal{E}_t$ and satisfy $a_v(0) = 1$, would suffice to achieve this. We will call adjustment functions of this form **value-based** adjustment functions, as they will be contrasted with sign-based adjustment functions in the next subsection.

For value-based adjustment functions, it holds that the higher (the value of) excess expenditure, the smaller the adjustment factor, and the further the weighting for instantaneous utility is decreased. And the smaller excess expenditure, the higher the adjustment factor, and the further the weighting for instantaneous utility is increased.

As an example of such a value-based adjustment function, we could define $\tilde{a}_v : \mathbb{R} \to \mathbb{R}_{++}$ by $\tilde{a}_v(\mathcal{E}_t) = e^{-\zeta \mathcal{E}_t}$, for some $\zeta > 0$, and for all $\mathcal{E}_t \in \mathbb{R}$.

In what follows we will not make much use of value-based adjustment functions, the second type of adjustment functions will be used throughout. The reason for this has to do with tractability. We will come back to this (and to the differences between value-based adjustment functions and sign-based adjustment functions) in the last section of the present chapter, and in the last chapter.

Min-max adjustment Before we will be ready to present the second specification of an adjustment function, we first need some notation such as to summarize what happened in the past. The specification for the second, min-max adjustment function will be defined recursively. We take a period-$(t+1)$ perspective, where it is assumed that in the previous periods $1 \leq i \leq t$ adjustments were already made.

The process started with an initial time preference function $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. And, in period 1, excess expenditure $\mathcal{E}_0 = \eta_1$ and an adjustment factor a_{η_1} were determined, and the initial time preferences $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ were updated according to $U^{(1)}(v, s_1) = U^{(0)}(a_{\eta_1} \cdot v, s_1)$. Similarly, in every period $1 \leq i \leq t$ excess expenditure $\mathcal{E}_{i-1}$, the history $\eta_i = (\eta_{i-1}, \mathcal{E}_{i-1})$ and an adjustment factor a_{η_i} were determined, and previous time preferences $U^{(i-1)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ were updated according to $U^{(i)}(v, s_i) = U^{(i-1)}(a_{\eta_i} \cdot v, s_i)$. This happened in every period i before $t+1$, so that we can write the period-t time preference function $U^{(t)}(v, s)$ also as

$$U^{(t-1)}(a_{\eta_t} \cdot v, s) = U^{(t-2)}(a_{\eta_t} \cdot a_{\eta_{t-1}} \cdot v, s) = ... = U^{(0)}(a_{\eta_t} \cdot a_{\eta_{t-1}} \cdot ... \cdot a_{\eta_1} \cdot v, s).$$

Note that η_{t+1} and all of the information about the sequence $(a_{\eta_1}, a_{\eta_2}, ..., a_{\eta_t})$ of adjustment factors is known at time $t+1$.

Definition 6.4.5 *Given a history η_{t+1} at period $t+1$, from which information about all the past adjustment factors $a_{\eta_1}, a_{\eta_2}, ..., a_{\eta_t}$ can be induced, an **adjustment product** θ_t is defined as the product of the adjustment factors of all periods up to period t: $\theta_t := a_{\eta_t} \cdot a_{\eta_{t-1}} \cdot ... \cdot a_{\eta_1}$.*

Note that by definition we have that $\theta_t = a_{\eta_t} \cdot \theta_{t-1}$. By using the adjustment product θ_t, we can write the function $U^{(t)}(v, s)$ more conveniently as: $U^{(t)}(v, s) = U^{(0)}(\theta_t v, s)$. Instead of considering a whole sequence of time preference functions $\{U^{(i)}(.,.)\}_{i=0}^{t}$, this same information can also more conveniently be represented by an initial time preference function $U^{(0)}(.,.)$ and a sequence of adjustment products $(\theta_1, \theta_2, ..., \theta_t)$. Of course, we could also write $\theta_0 := 1$, but we do not include it here in the sequence.

Given a history η_{t+1}, the corresponding set of past adjustment products will be denoted by

$$\Theta(\eta_{t+1}) := \{\theta_i \in \mathbb{R}_{++} : 1 \leq i \leq t, \theta_i = a_{\eta_i} \cdot a_{\eta_{i-1}} \cdot ... \cdot a_{\eta_0}\}.$$

Such a set would thus give all the adjustment products θ_i that were used in periods i before time $t+1$, to determine period-i time preferences: $U^{(i)}(v,s) = U^{(0)}(\theta_i v, s)$.

For any time $t+1$, the set of adjustment products $\Theta(\eta_{t+1})$ can be partitioned into two subsets $\Theta^{\mathcal{E}+}(\eta_{t+1})$ and $\Theta^{\mathcal{E}-}(\eta_{t+1})$, according to the sign of the excess expenditure (either regular or expected excess expenditure) in the corresponding period. That is, we define $\Theta^{\mathcal{E}+}(\eta_{t+1}) := \{\theta_i \in \Theta(\eta_{t+1}) : \mathcal{E}_i \geq 0\}$ to denote the set of all past adjustment products θ_i from $\Theta(\eta_{t+1})$ that yielded a positive excess expenditure in the corresponding period i, and we define $\Theta^{\mathcal{E}-}(\eta_{t+1}) := \{\theta_i \in \Theta(\eta_{t+1}) : \mathcal{E}_i \leq 0\}$ to denote the set of all adjustment products θ_i from $\Theta(\eta_{t+1})$ that yielded a negative excess expenditure in the corresponding period.

Then, we may define the new variable

$$\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) := \min_{\theta_i \in \Theta^{\mathcal{E}+}(\eta_{t+1})} \theta_i$$

to denote smallest element of the set $\Theta^{\mathcal{E}+}(\eta_{t+1})$ of past adjustment products that yielded a positive excess expenditure, and thus as the smallest element of $\Theta^{\mathcal{E}+}(\eta_{t+1})$ that (still) led to over-spending. Similarly, we define

$$\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) := \max_{\theta_i \in \Theta^{\mathcal{E}-}(\eta_{t+1})} \theta_i$$

to denote the largest past adjustment product that yielded a negative excess expenditure, and thus as the largest element in $\Theta^{\mathcal{E}-}(\eta_{t+1})$ that (still) led to under-spending. For any period $t+1$, the set $\Theta^{\mathcal{E}+}(\eta_{t+1})$ only has a finite number of elements, so if $\Theta^{\mathcal{E}+}(\eta_{t+1}) \neq \emptyset$, the variable $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$ is always well-defined. Similarly, if $\Theta^{\mathcal{E}-}(\eta_{t+1}) \neq \emptyset$, the variable $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1})$ is well-defined.

All of this information about adjustment factors and adjustment products of previous periods, and the signs of the corresponding excess expenditures, is known at time $t+1$. With this notation we are ready to specify min-max adjustment function.

Definition 6.4.6 *Given a measure $\mathcal{E}$ of excess expenditure, the* ***min-max adjustment function*** $a : \cup_{t=0}^{\infty}\mathbb{R}^{t+1} \to \mathbb{R}_{++}$ *is given by*

$$a(\eta_{t+1}) = \begin{cases} \rho & \text{if } \Theta^{\mathcal{E}+}(\eta_{t+1}) = \emptyset \\ \sigma & \text{if } \Theta^{\mathcal{E}-}(\eta_{t+1}) = \emptyset \\ (\lambda \cdot \theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) + (1-\lambda) \cdot \theta^{\mathcal{E}-}_{\max}(\eta_{t+1}))/\theta_t & \text{otherwise.} \end{cases}$$

for some $\rho > 1$, some $\sigma < 1$, and some $\lambda \in (0,1)$, and for all $\eta_{t+1} \in \cup_{t=0}^{\infty}\mathbb{R}^{t+1}$.

Again, the excess expenditure variable $\mathcal{E}$ can denote both regular excess expenditure E and expected excess expenditure F, and a history η can denote both an REE history ε and an EEE history ϕ. Whenever we want to express in our notation which specific measure of excess expenditure is used, we will denote the adjustment factors by different subscripts that reflect either REE or EEE adjustment: $a(\varepsilon)$ or a_ε, and $a(\phi)$ or a_ϕ.

The above definition may need some explaining, as the workings and the rationale for min-max adjustment may not be immediately clear from the definition. However, both the workings and the rationale can better be explained in terms of the implied evolutionary properties, so in a context where whole lifetimes of repeated min-max adjustment are considered, instead of zooming in on only two subsequent periods. For instance, note that the min-max adjustment function treats all previous periods similarly. The min-max adjustment function deals with information from the recent past in exactly the same way as with information concerning a more distant past.

Hence we will provide some explanations later in this chapter, after having completed the framework by specifying a learning algorithm. First we will make some remarks on the above definition, and rewrite it in terms of adjustment products.

First note that for any measure $\mathcal{E}$ of excess expenditure, and any period $t+1 \geq 1$, the set $\Theta(\eta_{t+1})$ is never empty, so that it is impossible that both $\Theta^{\mathcal{E}+}(\eta_{t+1}) = \emptyset$ and $\Theta^{\mathcal{E}-}(\eta_{t+1}) = \emptyset$ occur together. Hence the three cases as distinguished in the definition are exhaustive, and the definition is well-defined.

If in a certain period the prevailing history is such that the first case as distinguished in the above definition holds, where all previous excess expenditures have been strictly negative, then this min-max adjustment function will return an adjustment factor $a(\eta_{t+1})$ (or $a_{\eta_{t+1}}$), that equals a constant strictly larger than one. Similarly, in the second case as distinguished in the definition, where all previous excess expenditures were strictly positive, this min-max adjustment function will return a constant strictly smaller than one. In the third case from the above definition, the new adjustment product $\theta_{t+1} = a(\eta_{t+1}) \cdot \theta_t$ is a convex combination of the smallest past adjustment product that yielded a positive excess expenditure, and the largest past adjustment product that yielded a negative excess expenditure. Indeed, the adjustments implied by the min-max adjustment function can similarly be stated in terms of adjustment products, rather than adjustment factors. This new formulation may even be more illuminating.

In terms of adjustment products, min-max adjustment would give that

$$\theta_{t+1} = a(\eta_{t+1}) \cdot \theta_t = \begin{cases} \rho \cdot \theta_t & \text{if } \Theta^{\mathcal{E}+}(\eta_{t+1}) = \emptyset \\ \sigma \cdot \theta_t & \text{if } \Theta^{\mathcal{E}-}(\eta_{t+1}) = \emptyset \\ \lambda \cdot \theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) + (1-\lambda) \cdot \theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) & \text{otherwise.} \end{cases}$$

A rather straightforward choice for the scalar $\lambda \in (0,1)$ that determines the weights in the convex combination from the definition, would simply be to set $\lambda = 1/2$. However, other scalars could also be used, and it seems possible that a decision-maker would, for instance, be more worried about spending too much than about spending too little, so he might want to set new time preference cautiously, by choosing a λ that is smaller than one half.

6.5 The learning algorithm

To close the model, we suppose that in every period the consumer solves the corresponding basic ad hoc consumer problem, where ad hoc preferences consist of exogenous instantaneous preferences and of endogenous time preferences. The initial time preferences are given, and in all subsequent periods min-max adjustment is applied to generate new time preferences.

The ideas mentioned informally above, about linking stages can now be summarized and made precise in a learning algorithm.

Algorithm 6.5.1 *Given axioms 4.1.1, 4.4.1, 6.2.1, 6.2.2 and 6.3.1, and given the min-max adjustment function* $a : \cup_{t=1}^{\infty}\mathbb{R}^t \to \mathbb{R}_+$,[26] *an initial time preference function* $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *and an initial budget* m_0*, the learning process is specified by:*

▲ *Time* $t = 0$*. A pair* (x_0^*, s_0^*) *is chosen, that solves the basic ad hoc consumer problem corresponding to the initial time preference function* $U^{(0)}$ *and* m_0*.*

▲ *Time* $t \geq 1$*. A history* η_t *and a budget* m_t *are given. The time preference function* $U^{(t-1)} : v^{(t-1)}(X_{t-1}) \times \mathbb{R}_+ \to \mathbb{R}$ *from the previous period is adjusted according to* $U^{(t)}(v, s_t) = U^{(t-1)}(a_{\eta_t} v, s_t)$*. A pair* (x_t^*, s_t^*) *is chosen, that solves the basic ad hoc consumer problem corresponding to* $U^{(t)}$ *and* m_t*.*

Here we will first expand on the workings of the learning algorithm (using min-max adjustment), and then we will investigate some implications for the generated system dynamics.

Suppose that the sequences of adjustment factors $(a_{\eta_1}, a_{\eta_2}, ...)$ and adjustment products $(\theta_1, \theta_2, ..., \theta_t)$ are generated by the learning algorithm. Note that in period 1, the history η_1 only consists of a single excess expenditure $\mathcal{E}_0$, so that the process will typically (unless $\mathcal{E}_0 = 0$) find itself in one of the first two cases, as distinguished in definition 6.4.6. Then, usually the process will eventually be absorbed in the third case.

First suppose that at time 1 the process is in the first case of definition 6.4.6. That is, at time 1 it turned out that $\theta_0 = 1$ yielded a strictly negative excess expenditure: $\mathcal{E}_0 < 0$. In this case, we would have that $\Theta^{\mathcal{E}+}(\eta_1) = \emptyset$ and $\Theta^{\mathcal{E}-}(\eta_1) \neq \emptyset$. Then the adjustment factors $\theta_t = \rho^t\theta_0 = \rho^t$ will increase exponentially ($\rho > 1$) as long as the process remains in this first case. Usually, excess expenditure will eventually become positive in some period, and the third case from the definition, where both positive and negative excess expenditures have occurred in the past, will be entered.

Similarly, if at time 1 the process is in the second case of definition 6.4.6, so that $\mathcal{E}_0 > 0$, $\Theta^{\mathcal{E}+}(\eta_1) \neq \emptyset$ and $\Theta^{\mathcal{E}-}(\eta_1) = \emptyset$, then the adjustment factors $\theta_t = \sigma^t$ will decrease exponentially as long as the process remains in this second case, and usually we will get that eventually excess expenditure will become negative, and the third case from the definition will be entered.

Also note that it is never possible to go back from the third case to either the first or the second case, as $\Theta^{\mathcal{E}+}(\eta_t) \neq \emptyset$ implies that $\Theta^{\mathcal{E}+}(\eta_{t+\tau}) \neq \emptyset$, for all $\tau \geq 0$, and similarly for $\Theta^{\mathcal{E}-}(\eta_t)$.

Thus, typically the learning process will eventually be absorbed in the third case of definition 6.4.6, and in fact this last case is the most interesting one.

[26] For any $\rho > 1$, $\sigma < 1$ and $\lambda \in (0, 1)$.

From the specifications of the learning algorithm and the min-max adjustment function we can already derive some dynamic implications. But first we need a lemma. The proof of this lemma is quite instructive as to the workings of repeated min-max adjustment.

Lemma 6.5.1 *Suppose given a sequence of adjustment products* $(\theta_0, \theta_1, \theta_2, ...)$*, that is generated by the learning algorithm. Then for every period* t *for which both variables* $\theta^{\mathcal{E}-}_{\max}(\eta_t)$ *and* $\theta^{\mathcal{E}+}_{\min}(\eta_t)$ *are well-defined (i.e. if* $\Theta^{\mathcal{E}+}(\eta_t) \neq \emptyset$ *and* $\Theta^{\mathcal{E}-}(\eta_t) \neq \emptyset$*), it will hold that* $\theta^{\mathcal{E}-}_{\max}(\eta_t) \leq \theta^{\mathcal{E}+}_{\min}(\eta_t)$*.*

Proof. Recall that $\theta^{\mathcal{E}+}_{\min}(\eta_t) = \min_{\theta_i \in \Theta^{\mathcal{E}+}(\eta_t)} \theta_i$ denotes the smallest element of the set $\Theta^{\mathcal{E}+}(\eta_t)$ of adjustment products that yielded a positive excess expenditure, and similarly that $\theta^{\mathcal{E}-}_{\max}(\eta_t) = \max_{\theta_i \in \Theta^{\mathcal{E}-}(\eta_t)} \theta_i$. Here we will prove the statement "$\theta^{\mathcal{E}-}_{\max}(\eta_t) \leq \theta^{\mathcal{E}+}_{\min}(\eta_t)$ if both $\theta^{\mathcal{E}-}_{\max}(\eta_t)$ and $\theta^{\mathcal{E}+}_{\min}(\eta_t)$ are well-defined" by induction.

For $t = 1$, the statement clearly holds. If both the variables $\theta^{\mathcal{E}-}_{\max}(\eta_1)$ and $\theta^{\mathcal{E}+}_{\min}(\eta_1)$ are well-defined, then apparently $\theta_0 = 1$ yielded an excess expenditure equal to zero: $\mathcal{E}_0 = 0$, and we know that $\Theta^{\mathcal{E}+}(\eta_t) = \{\theta_0\} = \Theta^{\mathcal{E}-}(\eta_t)$, and that $\theta^{\mathcal{E}-}_{\max}(\eta_1) = \theta^{\mathcal{E}+}_{\min}(\eta_1) = \theta_0$.

Then suppose that the statement holds for period $t \geq 1$: $\theta^{\mathcal{E}-}_{\max}(\eta_t) \leq \theta^{\mathcal{E}+}_{\min}(\eta_t)$ holds if both variables $\theta^{\mathcal{E}-}_{\max}(\eta_t)$ and $\theta^{\mathcal{E}+}_{\min}(\eta_t)$ are well-defined. Then we must prove that the statement will also hold for $t + 1$. We need to consider three cases:

(A) in period t the variable $\theta^{\mathcal{E}-}_{\max}(\eta_t)$ was not well-defined,
(B) in period t the variable $\theta^{\mathcal{E}+}_{\min}(\eta_t)$ was not well-defined, and
(C) in period t both variables $\theta^{\mathcal{E}-}_{\max}(\eta_t)$ and $\theta^{\mathcal{E}+}_{\min}(\eta_t)$ were well-defined.

In **case (A)**, we assume that at time $t + 1$ the process has just (for the first time) entered the third case of definition 6.4.6 (there is nothing to prove if $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1})$ still does not exist). That is, in period $t + 1$ both sets $\Theta^{\mathcal{E}-}(\eta_{t+1})$ and $\Theta^{\mathcal{E}+}(\eta_{t+1})$ are non-empty, but we know that $\Theta^{\mathcal{E}-}(\eta_t) = \emptyset$. Then, since $t+1$ is finite, both $\Theta^{\mathcal{E}-}(\eta_{t+1})$ and $\Theta^{\mathcal{E}+}(\eta_{t+1})$ have a finite number of elements, and both $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$ and $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1})$ are well-defined. By $\Theta^{\mathcal{E}-}(\eta_t) = \emptyset$ we know that $a_{\eta_t} = \sigma < 1$ (and $a_{\eta_i} = \sigma < 1$, for all $i \leq t$). Also, $\Theta^{\mathcal{E}-}(\eta_{t+1}) \neq \emptyset$ so apparently $\mathcal{E}_t \leq 0$, and we see that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta_t = \sigma\theta_{t-1}$. We also know that $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) = \theta_{t-1}$ if $\mathcal{E}_t < 0$, or $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) = \theta_t$ if $\mathcal{E}_t = 0$. In any case we see that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) \leq \theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$, and the statement holds.

Similarly, in **case (B)** both sets $\Theta^{\mathcal{E}-}(\eta_{t+1})$ and $\Theta^{\mathcal{E}+}(\eta_{t+1})$ are non-empty, but $\Theta^{\mathcal{E}+}(\eta_t)$ *was* empty. Then $a_{\eta_{t-1}} = \rho > 1$, and by $\Theta^{\mathcal{E}+}(\eta_{t+1}) \neq \emptyset$ we know that $\mathcal{E}_t \geq 0$, and that $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) = \theta_t = \rho\theta_{t-1}$. We also know that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta_{t-1}$ if $\mathcal{E}_t > 0$, or $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta_t$ if $\mathcal{E}_t = 0$. Thus indeed we see that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) \leq \theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$, and the statement holds.

Finally, in **case (C)**, at time t the process was already in the third case of definition 6.4.6, so that all of the sets $\Theta^{\mathcal{E}-}(\eta_t)$, $\Theta^{\mathcal{E}+}(\eta_t)$, $\Theta^{\mathcal{E}-}(\eta_{t+1})$ and $\Theta^{\mathcal{E}+}(\eta_{t+1})$ are non-empty. Then both variables $\theta^{\mathcal{E}-}_{\max}(\eta_t)$ and $\theta^{\mathcal{E}+}_{\min}(\eta_t)$ were well-defined, and by the induction hypothesis they satisfied $\theta^{\mathcal{E}-}_{\max}(\eta_t) \leq \theta^{\mathcal{E}+}_{\min}(\eta_t)$. In period t the new adjustment product θ_t was determined as a convex combination of the minimal adjustment product with a positive excess expenditure and the maximal adjustment product with a negative excess expenditure from the past:

$$\theta_t = a_{\eta_t}\theta_{t-1} = \lambda \cdot \theta^{\mathcal{E}+}_{\min}(\eta_t) + (1-\lambda) \cdot \theta^{\mathcal{E}-}_{\max}(\eta_t).$$

This implies that $\theta^{\mathcal{E}-}_{\max}(\eta_t) \leq \theta_t \leq \theta^{\mathcal{E}+}_{\min}(\eta_t)$. Then in period $t+1$, from θ_t the excess expenditure $\mathcal{E}_t$ is determined, and we can distinguish three cases, depending on the sign of $\mathcal{E}_t$.

(I) Excess expenditure is strictly negative: $\mathcal{E}_t < 0$, in this case it will hold that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta_t$ (since $\theta_t \geq \theta^{\mathcal{E}-}_{\max}(\eta_t)$) and $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) = \theta^{\mathcal{E}+}_{\min}(\eta_t)$. Then we see that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta_t \leq \theta^{\mathcal{E}+}_{\min}(\eta_t) = \theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$.

(II) Excess expenditure is strictly positive: $\mathcal{E}_t > 0$, so that $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) = \theta_t$ and $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta^{\mathcal{E}-}_{\max}(\eta_t)$, which implies that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta^{\mathcal{E}-}_{\max}(\eta_t) \leq \theta_t = \theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$.

(III) Excess expenditure equals zero: $\mathcal{E}_t = 0$, and we find that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) = \theta_t$.

Hence indeed, in all cases we find that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) \leq \theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$ if both $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1})$ and $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$ are well-defined, which proves the lemma. ■

Making adjustment factors depend on excess expenditure was motivated by the idea that finding a positive (negative) excess expenditure in the previous period should be followed by decreasing (increasing) the adjustment product, thereby giving less (more) weight to instantaneous utility, and more (less) weight to savings in time preferences.

From the above lemma we can now show that repeated min-max adjustment complies with this idea that finding a positive (negative) excess expenditure in a certain period will be followed by giving less (more) weight to instantaneous utility in the next period. Moreover, with this lemma we can also show that the same impact will be felt in all subsequent periods: finding a positive (negative) excess expenditure in a certain period will lead to smaller (larger) weights for instantaneous utility in all subsequent periods.

To see this, suppose that the consumer finds that the excess expenditure in period t' turned out to be positive: $\mathcal{E}_{t'} \geq 0$. We then want to show that all subsequent adjustment products will be smaller than the one used in period t': $\theta_{t'+\tau} \leq \theta_{t'}$, for all $\tau \geq 0$.

We know that $\mathcal{E}_{t'} \geq 0$ so the process cannot be in the first case of definition 6.4.6 in which $\Theta^{\mathcal{E}+}(\eta_{t'+1}) = \emptyset$.

If the process is in the second case, so that $\Theta^{\mathcal{E}-}(\eta_{t'+1})$ is empty, then $\theta_{t'+\tau} = \sigma^{\tau}\theta_{t'}$ (with $\sigma < 1$) will hold as long as $\Theta^{\mathcal{E}-}(\eta_{t'+\tau})$ remains empty. Therefore, $\theta_{t'+\tau} \leq \theta_{t'}$ will certainly hold until the third case is entered.

Then, suppose that the process is in the third case of definition 6.4.6. Note that the sets $\Theta^{\mathcal{E}+}(.)$ never get smaller over time, so that $\theta^{\mathcal{E}+}_{\min}(\eta_{t'+1}) \geq \theta^{\mathcal{E}+}_{\min}(\eta_{t'+\tau})$, for all $\tau \geq 1$. And since the excess expenditure in period t' was positive $\mathcal{E}_{t'} \geq 0$, we know that $\theta_{t'} \in \Theta^{\mathcal{E}+}(\eta_{t'+1})$, and by definition we have the inequality $\theta_{t'} \geq \theta^{\mathcal{E}+}_{\min}(\eta_{t'+1})$.

Then for every period $t'+\tau > t'$, we see that

$$\theta_{t'+\tau} = \lambda \cdot \theta^{\mathcal{E}+}_{\min}(\eta_{t'+\tau}) + (1-\lambda) \cdot \theta^{\mathcal{E}-}_{\max}(\eta_{t'+\tau}) \leq$$

$$\lambda \cdot \theta^{\mathcal{E}+}_{\min}(\eta_{t'+\tau}) + (1-\lambda) \cdot \theta^{\mathcal{E}+}_{\min}(\eta_{t'+\tau}) = \theta^{\mathcal{E}+}_{\min}(\eta_{t'+\tau}) \leq \theta^{\mathcal{E}+}_{\min}(\eta_{t'+1}) \leq \theta_{t'}.$$

Here the first inequality follows from the lemma. Thus indeed we find that $\theta_{t'+\tau} \leq \theta_{t'}$, for all $\tau \geq 0$.

Similarly, $\mathcal{E}_{t'} \leq 0$ implies that $\theta_{t'} \in \Theta^{\mathcal{E}-}(\eta_{t'+1})$, and that $\theta_{t'} \leq \theta^{\mathcal{E}-}_{\max}(\eta_{t'+1})$. And with a similar reasoning as above, $\theta^{\mathcal{E}-}_{\max}(\eta_{t'+1}) \leq \theta^{\mathcal{E}-}_{\max}(\eta_{t'+\tau})$ will hold for all $\tau \geq 1$. Then for every period $t'+\tau > t'$, it holds that

$$\theta_{t'+\tau} = \lambda \cdot \theta^{\mathcal{E}+}_{\min}(\eta_{t'+\tau}) + (1-\lambda) \cdot \theta^{\mathcal{E}-}_{\max}(\eta_{t'+\tau}) \geq$$

$$\lambda \cdot \theta^{\mathcal{E}-}_{\max}(\eta_{t'+\tau}) + (1-\lambda) \cdot \theta^{\mathcal{E}-}_{\max}(\eta_{t'+\tau}) = \theta^{\mathcal{E}-}_{\max}(\eta_{t'+\tau}) \geq \theta^{\mathcal{E}-}_{\max}(\eta_{t'+1}) \geq \theta_{t'}.$$

Thus if excess expenditure is negative in period t, then in all periods after t adjustment products will be larger than the one used in period t.

Combining these two results also gives a third implication that if at time t' the prevailing adjustment product $\theta_{t'}$ yielded an excess expenditure equal to zero $\mathcal{E}_{t'} = 0$, then in every later period $t'+\tau \geq t'$ the adjustment product will equal the time-t' adjustment product: $\theta_{t'+\tau} = \theta_{t'}$, as after period t' essentially no adjustments are made anymore: $a(\eta_{t'+\tau}) = 1$.

6.6 Motivations

In this chapter the ad hoc framework was closed. We assumed that all ad hoc utility functions were separable in past and present consumption, and that instantaneous utility for past and present consumption is exogenous. Time preference functions are endogenously determined (except for in the first period) by adjusting the previous period's time preference function. These adjustments are based on excess expenditure in the previous periods, and excess expenditure is determined by means of the previous period's time preference function. Whereas we tried to motivate the assumptions underlying this learning procedure, these assumptions could still be questioned. It may not seem completely clear why such a learning procedure would be reasonable, or that it would work (in the sense that it does lead to improvements over time of the time preference functions that are being used).

Of course then, the proof is in the pudding. As we are trying to devise a learning model in which time preferences are improved over time, the reasonability of the intermediary steps in devising the learning procedure may ultimately be justified if this procedure will lead to time preferences being improved (or if at least it can in some cases). The next chapters study these questions, and they will show that under some circumstances time preference functions *will* improve over time, even to the extent that convergence towards optimal (or consistent) time preferences may occur.

However, the convergence results in the next chapters are based on repeated min-max adjustment of time preference functions. As promised earlier in this chapter, here we return to the question of why min-max adjustment is used.

In section 6.4 two types of adjustment functions were specified. The alternative value-based adjustment functions may seem a bit simpler and more straightforward than min-max adjustment, for instance because it would seem more in line with (a straightforward interpretation of) error-correction models (see subsection 3.4.2). Whereas value-based adjustment only depends on excess expenditure in the very last period, min-max adjustment depends on excess expenditures in all previous periods. Moreover, min-max adjustment treats all previous periods similarly; it deals with information from the recent past in exactly the same way as with information concerning a more distant past. As will become apparent in the next chapter, min-max adjustment is quite rational and (under some circumstances) it will prove quite efficient in reaching optimality. Therefore, using min-max adjustment requires quite a bit of rationality on the part of decision-makers, which may descriptively seem questionable. So why then have we chosen to use min-max adjustment in our learning algorithm?

As mentioned in section 6.4, the reasons for choosing min-max adjustment have to do with tractability. Remember one of the ultimate aims of this research: we wanted to investigate the possibility that (near-)optimal behaviour (defined as behaviour of a rational utility maximizer) could be learned in the context of consumer choice. Therefore, we set up a learning model of consumer choice, and we want to investigate the more normative question of whether it would be possible that over time the behaviour of such a learning consumer would converge to the benchmark-case of optimality and rationality. Thus, we want to investigate under what conditions a learning ad hoc utility maximizer's ad hoc preferences would converge to ad hoc preferences that would be consistent with total preferences in the standard framework.

And as the next chapter will show, sequential min-max adjustment will yield quite convenient convergence properties. In contrast, dealing with questions of convergence is very demanding if value-based adjustment is used. Under value-based adjustment it is very hard to establish convergence to any limit, let alone convergence to a limit representing rational behaviour.

Therefore min-max adjustment is just a more convenient choice for studying convergence to optimality. Thus, the reasons for choosing min-max adjustment stem much more from tractability, than from considerations of behavioural or descriptive plausibility. We will come back to these issues of min-max versus value-based adjustment in the final chapter.

7 Evolution and convergence

After having closed the model in the previous chapter by specifying a learning algorithm, in the present chapter we turn our attention to the evolutionary dynamics that would result from the learning algorithm. In this chapter (and in the following chapters) we change the scope of our analysis from the fine levels of very limited time intervals, to the more crude levels corresponding to consumers' lifetimes, and we investigate what types of evolutionary and asymptotic properties would result from the interplay of large sequences of min-max adjustments of ad hoc preferences.

More specifically, some convergence properties of generated sequences of ad hoc preferences will be investigated in this chapter. Here convergence will mean convergence of a generated sequence of time preferences, or equivalently, convergence of the generated sequence of adjustment products. And we will specifically be looking for convergence of adjustment products towards an adjustment product as implied by ad hoc preferences that would be consistent with total preferences. Thus, we will investigate convergence in the ad hoc framework towards the benchmark of the standard microeconomic framework for consumer choice.

This chapter is organized as follows. The first section will show that under repeated min-max adjustment, as will be the case in the learning algorithm, convergence will always occur to some limit. In the second section we will establish when such a limit would represent rationality or optimality, and we will see that convergence does not necessarily mean convergence to such a limit, so that convergence towards the standard benchmark does not always occur. In the third section we will formally define convergence towards optimality, and identify some conditions needed to establish that (or when) it *will* occur.

7.1 Convergence

In this chapter we start investigating convergence of the learning procedure. We will do these investigations within the general setting of the ad hoc framework that includes models of certainty, uncertainty and structural ignorance. First we present a formal account of the evolutionary dynamics, as generated by the learning algorithm, and we will specify what exactly we mean by convergence in this setting.

The learning algorithm models a procedure for updating time preferences. Thus each of these generated time preference functions (except for the very first) is obtained by adjusting the time preference function from the period before that, and these adjustments depend on excess expenditure. Any two subsequent time preference functions from any generated sequence of time preference functions are related according to $U^{(t)}(v,s) = U^{(t-1)}(a_{\eta_t} \cdot v, s)$, where a_{η_t} is the adjustment factor that is determined by the min-max adjustment function. Then, by induction any time preference function from such a sequence can also be written as

$$U^{(t)}(v,s) = U^{(0)}(a_{\eta_t} \cdot a_{\eta_{t-1}} \cdot \ldots \cdot a_{\eta_1} \cdot v, s) = U^{(0)}(\theta_t \cdot v, s).$$

Here θ_t denotes the adjustment product. Instead of considering sequences of time preference functions $(U^{(t)})_{t=0}^{\infty}$, essentially the same information can also more conveniently be represented by the initial time preference function $U^{(0)}(.,.)$ and the sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$.

As the learning algorithm would generate sequences of time preference functions $(U^{(t)})_{t=0}^{\infty}$, convergence of the learning procedure should mean convergence of a learning ad hoc utility maximizer's preferences. And when we are studying convergence of a sequence of time preference functions $(U^{(t)})_{t=0}^{\infty}$, we can alternatively and more conveniently study convergence of the corresponding sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$.

Definition 7.1.1 *A sequence of time preference functions $(U^{(t)}(v,s))_{t=0}^{\infty}$ that is of the form $(U^{(0)}(\theta_t v,s))_{t=0}^{\infty}$ is said to* ***converge***[27] *if the associated sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$ converges in $\overline{\mathbb{R}}_+$.*

A first condition needed for convergence was already noted in chapter 4, before the introduction of the ad hoc framework, and this condition is in fact already implicit in the learning algorithm and in the above definition. This condition is that our models should be such that the number of periods is (countably) infinite. In general we cannot expect to get convergence in a finite number of steps, and this condition is required for investigating convergence. If indeed the number of periods is infinite, we will see that under repeated min-max adjustment convergence of preferences will naturally occur. That is, if the number of periods is infinite and if the learning algorithm with min-max adjustment is used, then this is already sufficient to show that convergence will always occur to some limit.

Proposition 7.1.1 *Any sequence of time preference functions $(U^{(t)}(v,s))_{t=0}^{\infty} = (U^{(0)}(\theta_t v,s))_{t=0}^{\infty}$ that is generated by the learning algorithm, will converge.*

Proof. For this proof we distinguish three cases, depending on whether the adjustment process will ultimately be absorbed in the first, second or third case of definition 6.4.6. In all three cases we will see that convergence to some $\theta_\infty \in \overline{\mathbb{R}}_+$ takes place.

▲ Firstly, suppose that the corresponding sequence of adjustment products $(\theta_0, \theta_1, \theta_2, ...)$ is such that every θ_t yields $\mathcal{E}_t < 0$, so that $\Theta^{\mathcal{E}+}(\eta_{t+1}) = \emptyset$, for all t. Then the process will stay absorbed in the first case of definition 6.4.6. Thus the sequence of adjustment products $(\theta_0, \theta_1, \theta_2, ...)$, that is generated by min-max adjustment from $\theta_0 = 1$, will be such that $\theta_t = \rho\theta_{t-1} = \rho^t\theta_0 = \rho^t$, for all t. And since $\rho > 1$, we see that the sequence of adjustment products will indeed converge to $\theta_\infty = \infty$. (In this case, $\theta^{\mathcal{E}+}_{\min}(\eta_t) = \min_{\theta_i \in \Theta^{\mathcal{E}+}(\eta_t)} \theta_i$ will never be well-defined, and $\theta^{\mathcal{E}-}_{\max}(\eta_t) = \max_{\theta_i \in \Theta^{\mathcal{E}-}(\eta_t)} \theta_i$ will converge towards $\theta_\infty = \infty$.)

▲ Secondly, suppose that the sequence $(\theta_0, \theta_1, \theta_2, ...)$ is such that every θ_t yields $\mathcal{E}_t > 0$, so that $\Theta^{\mathcal{E}-}(\eta_{t+1}) = \emptyset$, for all t. Then a similar reasoning shows that the

[27] By convergence of a sequence of functions $(f_t)_{t=0}^{\infty}$ (with $f_t : S \to \mathbb{R}$, for all t), here we mean pointwise convergence: for all $s \in S$, the sequence $f_t(s)$ converges in $\mathbb{R} \cup \{-\infty, +\infty\}$.

process stays absorbed in the second case, and that the sequence $(\theta_0, \theta_1, \theta_2, ...)$ will be such that $\theta_t = \sigma^t$ for all t. Since $\sigma < 1$, we see that the sequence of adjustment products will converge to $\theta_\infty = 0$. (In this case, $\theta^{\mathcal{E}+}_{\min}$ will converge towards $\theta_\infty = 0$, and $\theta^{\mathcal{E}-}_{\max}(\eta_t)$ will never be well-defined.)

▲ Thirdly, we suppose that the sequence $(\theta_0, \theta_1, \theta_2, ...)$ is such that in some period t, the process enters the third case as distinguished in definition 6.4.6. That is, in period t both sets $\Theta^{\mathcal{E}-}(\eta_t)$ and $\Theta^{\mathcal{E}+}(\eta_t)$ are non-empty. Then from the lemma in section 6.5, we know that $\theta^{\mathcal{E}-}_{\max}(\eta_t) \leq \theta^{\mathcal{E}+}_{\min}(\eta_t)$ will hold. Therefore we must have that $\chi_t := \theta^{\mathcal{E}+}_{\min}(\eta_t) - \theta^{\mathcal{E}-}_{\max}(\eta_t) \geq 0$.

In period t the new adjustment product θ_t is determined as a convex combination of the minimal adjustment product with a positive excess expenditure and the maximal adjustment product with a negative excess expenditure from the past:

$$\theta_t = a_{\eta_t}\theta_{t-1} = \lambda \cdot \theta^{\mathcal{E}+}_{\min}(\eta_t) + (1-\lambda) \cdot \theta^{\mathcal{E}-}_{\max}(\eta_t).$$

This implies that $\theta^{\mathcal{E}-}_{\max}(\eta_t) \leq \theta_t \leq \theta^{\mathcal{E}+}_{\min}(\eta_t)$.

Then, in period $t+1$ the excess expenditure $\mathcal{E}_t$ is determined from θ_t, and we can distinguish three cases, depending on the sign of $\mathcal{E}_t$.

(I) Excess expenditure is strictly negative: $\mathcal{E}_t < 0$, so in this case it will hold that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta_t$ (since $\theta_t \geq \theta^{\mathcal{E}-}_{\max}(\eta_t)$) and $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) = \theta^{\mathcal{E}+}_{\min}(\eta_t)$. This implies that $\chi_{t+1} = (1-\lambda)\chi_t$.

(II) Excess expenditure is strictly positive: $\mathcal{E}_t > 0$, so that $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) = \theta_t$ and $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta^{\mathcal{E}-}_{\max}(\eta_t)$, which implies that $\chi_{t+1} = \lambda\chi_t$.

(III) Excess expenditure equals zero: $\mathcal{E}_t = 0$, and we find that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) = \theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) = \theta_t$, so that $\chi_{t+1} = 0$.[28]

In each of these cases it will hold that $0 \leq \chi_{t+1} = \theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) - \theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) \leq \tilde{\lambda}\chi_t$, for $\tilde{\lambda} = \max\{\lambda, 1-\lambda\}$. And we had that $\theta_t = \theta^{\mathcal{E}-}_{\max}(\eta_{t+1})$, or that $\theta_t = \theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$ (or both if and only if $\mathcal{E}_t = 0$), so that $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) \leq \theta_t \leq \theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$ must hold. Moreover, in the following stage $t+1$ min-max adjustment will again be applied, and $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) \leq \theta_{t+1} \leq \theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$ will hold. Therefore we also see that $|\theta_{t+1} - \theta_t| \leq \chi_{t+1} \leq \tilde{\lambda}\chi_t$.

The same procedure will be applied in all subsequent periods, and the same reasoning will show that

$$\chi_{t+\tau} = \theta^{\mathcal{E}+}_{\min}(\eta_{t+\tau}) - \theta^{\mathcal{E}-}_{\max}(\eta_{t+\tau}) \leq \tilde{\lambda}\chi_{t+\tau-1} \leq ... \leq \tilde{\lambda}^{\tau}\chi_t,$$

and that $|\theta_{t+\tau} - \theta_{t+\tau-1}| \leq \tilde{\lambda}^{\tau}\chi_t$, for every period $t+\tau \geq t$. Now, since $\tilde{\lambda} = \max\{\lambda, 1-\lambda\} < 1$ and $\chi_t \in \mathbb{R}_+$, we see that $\tilde{\lambda}^{\tau}\chi_t$ will converge to 0 as τ goes to infinity. Thus indeed, $|\theta_{t+\tau} - \theta_{t+\tau-1}|$ will also converge to zero, and the sequence of adjustment products $(\theta_0, \theta_1, \theta_2, ...)$ must converge to some $\theta_\infty \in \mathbb{R}_{++}$. (In this case, both $\theta^{\mathcal{E}+}_{\min}$ and $\theta^{\mathcal{E}-}_{\max}(\eta_t)$ will converge towards θ_∞.) ■

[28] We then get that $a(\eta_{t+1}) = 1$, so that $\theta_{t+1} = a(\eta_{t+1}) \cdot \theta_t = \theta_t$. Subsequently, min-max adjustment is applied in all periods, and the same procedure applies to yield $\theta^{\mathcal{E}+}_{\min}(\eta_{t'}) = \theta_t = \theta^{\mathcal{E}-}_{\max}(\eta_{t'})$, and $a(\eta_{t'}) = 1$ for all $t' \geq t+1$. Thus all subsequent adjustment products $\theta_{t'}$ will equal θ_t.

7.2 Consistency-inducing scalars

In the previous section we saw that under repeated min-max adjustment convergence will always occur. However, recall that here we were not so much interested in convergence per se, but that we were particularly interested in convergence towards rationality or optimality, as determined by the benchmark case of (total) utility maximization in the standard framework of consumer choice. In this and in the following section we will formalize what exactly we mean by convergence towards optimality, and we will investigate when it will occur.

In the previous section we found that the learning algorithm always yields convergence of a generated sequence of adjustment products $(\theta_1, \theta_2, ...)$ to some limit θ_∞. Therefore, convergence towards optimality should mean that such a limit θ_∞ would correspond to consistency with rationality or optimality. Recall that consistency of ad hoc preferences with total preferences (representing optimality and rationality) was formally defined in section 5.1. Here we will link the property of convergence towards optimality to the property of consistency.

The convergence result from the previous section implies that the learning algorithm will always yield convergence of a generated sequence of adjustment products $(\theta_1, \theta_2, ...)$ to a single limit scalar θ_∞, and never towards a variable or moving limit pattern. Thus, convergence towards optimality would require that a single scalar θ_∞ would yield consistency in all periods. The next definition makes this property precise, both in models of expected utility and in models of certainty. Here rationality is only properly defined in models of certainty or of expected utility, and likewise, consistency can only be defined in those settings. And since models of certainty are special (degenerate) cases of expected utility models, it would suffice to only define consistency in expected utility models. For the reader's convenience we will also present a distinct formal definition of consistency in models of certainty.

Definition 7.2.1 *Given a state space Ω, a probability distribution $\pi : \Omega \to [0,1]$, a (total) Bernouilli utility function $u : \tilde{X} \to \mathbb{R}$, and an initial time preference function $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, the scalar $\tilde{\theta} \in \mathbb{R}_{++}$ is called a* ***consistency-inducing scalar*** *if for every period t, for every past state $\omega_0^t = (\omega_0, ..., \omega_t)$ and every future state space Ω_{t+1}^∞, it holds that the ad hoc utility function $U^{(0)}(\tilde{\theta} v^{(t)}(w_{t-1}, x_t), s_t)$ is consistent with u, given ω_0^t, Ω_{t+1}^∞ and π.*

Definition 7.2.2 *Given a price vector $q_0 = (p_0, p_1, ...)$ and an income stream $J_0 = (m_0, I_1, ...)$, a total utility function $u : X \to \mathbb{R}$ and an initial time preference function $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, the scalar $\tilde{\theta} \in \mathbb{R}_{++}$ is called a* ***consistency-inducing scalar*** *if for every period t, it holds that the ad hoc utility function $U^{(0)}(\tilde{\theta} v^{(t)}(w_{t-1}, x_t), s_t)$ is consistent with u, given $q_{t+1} = (p_{t+1}, p_{t+2}, ...)$ and $J_{t+1} = (I_{t+1}, I_{t+2}, ...)$.*

In what follows, we will also use the abbreviation **CIS** to denote a consistency-inducing scalar. From the above definition it may seem clear that, given an initial time preference function $U^{(0)}(.,.)$, such a CIS need not generally exist. Staying within the general ad hoc framework, here we will present some necessary conditions that models will have to satisfy for a CIS to exist. Convergence towards such an optimal scalar is yet another question, which will be dealt with in the next section. Here we will first present and explain these necessary conditions that models will have to satisfy in order for such a CIS $\tilde{\theta}$ to exist (given initial time preferences).

- Firstly, models should be set within the standard framework. The above definition refers back to the notion of consistency, so existence of a consistency-inducing scalar in a certain model presupposes that consistency is well-defined within such a model. The notion of consistency was defined as a relation between ad hoc utility and total utility; ad hoc utility was defined to be consistent with total utility if ad hoc utility for savings would always correspond to total utility of optimally chosen future consumption plans, given savings. Hence a total (Bernouilli) utility function is required for defining consistency, and total utility functions are only specified in the setting of the standard framework.

- Secondly, consistent ad hoc preferences should always be such that present and past consumption are independent of savings. In the above definition we see that when a consistency-inducing scalar is entered into the initial time preference function, the resulting ad hoc utility function is consistent with total utility. Obviously, the consistent ad hoc utility function from this definition is separable in past and present consumption. Hence, by theorem 2.4.1 consistent preferences should always be such that present and past consumption are independent of savings, and such that instantaneous preferences and time preferences can be distinguished.

- Thirdly, there should be a consistent time preference function that is stationary or time-invariant. In the above definition a (single) consistency-inducing scalar is defined to yield consistency in all periods when inserted into the initial time preference function. Thus, a single time preference function should be consistent in all periods.

- And fourthly, the initial time preference function should, except for an adjustment product that is to be inserted into it, already have a functional structure similar to that of a consistent time preference function. In the above definition, what is entered into the initial time preference function in order to obtain consistency, is a scalar and not a function. That is, this scalar that should be entered into the initial time preference function to obtain consistency, will not only have to be the same across periods, but it will also have to be the same for all different combinations of instantaneous utility and savings that are possible. Thus, the initial time preference function should already be quite similar to some consistent time preference function.

These conditions, and probably especially the last two, are very restrictive. As for the third condition, in principle it would be possible to investigate convergence towards a moving limit pattern, but obviously, this would greatly complicate our analyses. And as for the fourth condition, do note that it suffices that the initial time preference function has, except for an adjustment product, the same functional structure as *some* consistent time preference function. In choosing consistent ad hoc utility (and consequently in choosing consistent time preference functions) there is still some freedom, as consistent ad hoc utility is unique only up to a strictly increasing transformation. Still, this last condition seems quite stringent. In the last chapter we will come back to the restrictiveness of these conditions.

The next two chapters will be set within the more specific setting of consumption/savings models. There we will be able to precisely express when a CIS exists in terms of a model's primitives, and we will see that (in some cases) the above four conditions will hold, and that a consistency-inducing scalar will indeed exist.

7.3 Convergence to optimality

Having defined convergence and consistency-inducing scalars in the previous sections, in this section we will define convergence towards optimality. Also, we will specify three conditions that will help determine when convergence towards optimality will take place. The first of these three conditions is a necessary condition for convergence towards optimality to occur. The second and third conditions are such that the three conditions combined are sufficient for convergence towards optimality to take place.

But first we present the definition of convergence towards optimality.

Definition 7.3.1 *A sequence of time preference functions $(U^{(t)}(v,s))_{t=0}^{\infty}$ that is of the form $(U^{(0)}(\theta_t v,s))_{t=0}^{\infty}$ is said to* ***converge towards optimality*** *if the corresponding sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$ converges towards a consistency-inducing scalar.*

From the above definition it may be clear that convergence towards optimality need not always occur. The learning algorithm and the above definition are set in the ad hoc framework in its completely general form, and not in all models that could be set up within the ad hoc framework convergence towards optimality will necessarily occur. For instance, from the above definition we see that existence of a consistency-inducing scalar is a necessary condition for convergence towards optimality to occur.

Condition 7.3.1 *Given the initial time preference function $U^{(0)}$, a consistency-inducing scalar $\widetilde{\theta} \in \mathbb{R}_{++}$ exists.*

As noted in the previous section, a CIS need not always exist in all models that are set in the general ad hoc framework. The four conditions that were listed in the previous section are necessary for the existence of a consistency-inducing scalar. And since the existence of a CIS is necessary for convergence towards optimality to occur, so are the four conditions from the previous section.

By the previous proposition we know that convergence will always occur. However, even if in a certain model we know that a CIS exists, we still cannot be sure that convergence towards this specific scalar will indeed occur. To be able to prove that convergence towards optimality will occur, two more conditions are needed. Together with condition 7.3.1, these conditions are sufficient for obtaining convergence towards optimality.

The second condition entails that the consistency-inducing scalar should be stable. That is, as we are trying to establish convergence of adjustment products towards a single scalar (adjustment product) that will induce consistency in all periods, it seems desirable that this CIS would be a 'fixed point' of the learning algorithm and of the adjustment process. We would like to have that if in a certain period the consistency-inducing scalar is used, it will not be adjusted, so that it will also be used in the following period (and hence in all subsequent periods). If $\theta_t = \tilde{\theta}$ for any t, we should also have that $\theta_{t+1} = \tilde{\theta}$ will hold, and we require a unit adjustment factor $a(\eta_{t+1}) = 1$. And since $a(\eta_{t+1}) = 1$ will only happen if $\mathcal{E}_t = 0$, we require here that setting $\theta_t = \tilde{\theta}$ will always (for all budgets m_t available in this period) yield a zero excess expenditure in the corresponding period.

Condition 7.3.2 *If, given an initial time preference function $U^{(0)}$ and a CIS $\widetilde{\theta}$, in some period t the prevailing time preference function is of the form $U^{(t)}(v,s) = U^{(0)}(\theta_t v, s)$, then $\theta_t = \widetilde{\theta}$ will always (for all available budgets m_t) yield $\mathcal{E}_t = 0$.*

And thirdly, the model should be such that adjustment products that are larger than the CIS will yield positive excess expenditures, and such that adjustment products that are smaller than the CIS will yield negative excess expenditures. To see why this is needed, recall from section 6.5 that positive excess expenditures will always be followed by a decrease in the adjustment product, and that negative excess expenditures will always be followed by an increase in the adjustment product. Therefore, if the above implications hold, then if the prevailing adjustment product is larger than the CIS the adjustment product will be decreased, and if the prevailing adjustment product is smaller than the CIS the adjustment product will be increased. This would clearly facilitate convergence towards the CIS.

Condition 7.3.3 *If, given an initial time preference function $U^{(0)}$ and a CIS $\widetilde{\theta}$, in some period t the prevailing time preference function is of the form $U^{(t)}(v,s) = U^{(0)}(\theta_t v, s)$, then $\theta_t > \tilde{\theta}$ will always (for all available budgets m_t) yield $\mathcal{E}_t > 0$, and $\theta_t < \tilde{\theta}$ will always yield $\mathcal{E}_t < 0$.*

Note that if, given an initial time preference function, conditions 7.3.2 and 7.3.3 hold, then the consistency-inducing scalar $\tilde{\theta}$ will be unique.

If the above three conditions are met, then the learning algorithm will indeed yield convergence towards optimality, and indeed conditions 7.3.1, 7.3.2 and 7.3.3 are sufficient for convergence towards optimality to occur.

Proposition 7.3.1 *Suppose that in a certain model, given an initial time preference function* $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$*, conditions 7.3.1, 7.3.2 and 7.3.3 are satisfied. Then any sequence of time preference functions* $(U^{(t)})_{t=0}^{\infty}$ *that is generated by the learning algorithm, will converge towards optimality.*

Proof. The generated sequence of time preference functions is of the form $U^{(t)}(v,s) = U^{(0)}(\theta_t v, s)$, and from conditions 7.3.1 (the existence of a CIS $\tilde{\theta} \in \mathbb{R}_{++}$) and 7.3.3, we can first show that from some period onwards, the adjustment process will remain absorbed in the third case of definition 6.4.6, in which both sets $\Theta^{\mathcal{E}-}(.)$ and $\Theta^{\mathcal{E}+}(.)$ are non-empty. To see this, we will distinguish three cases, depending on the sign of the excess expenditure $\mathcal{E}_0$ for the first period.

Firstly, suppose that θ_0 was such that $\mathcal{E}_0 < 0$, so that the process started in the first case of definition 6.4.6. Min-max adjustment is applied every period, so the adjustment factors $\theta_t = \rho^t \theta_0$ will increase exponentially as long as the process remains in this first case. Then for some period t' it will hold that $\theta_{t'} = \rho^{t'} \theta_0 > \tilde{\theta}$, and by condition 7.3.3 we get that excess expenditure will be strictly positive in this period, and the third case from the definition, where both positive and negative excess expenditures have occurred in the past, will be entered.

Secondly, if θ_0 was such that $\mathcal{E}_0 > 0$, the process started in the second case, and similarly min-max adjustment is applied every period, so that $\theta_t = \sigma^t \theta_0$ as long as the process remains in this second case. For some period t' it will hold that $\theta_{t'} = \sigma^{t'} \theta_0 < \widetilde{\theta}$, and we get that excess expenditure will become strictly negative in this period, and again the third case from the definition will be entered.

And thirdly, if θ_0 was such that $\mathcal{E}_0 = 0$, then we would have that $\theta_0 \in \Theta^{\mathcal{E}-}(\eta_1)$ and $\theta_0 \in \Theta^{\mathcal{E}+}(\eta_1)$ so neither of the sets are empty, and the process starts in the third case.

Also remember from the discussion following the learning algorithm that it is never possible to go back from the third case to either the first or the second case.

Thus suppose that the sequence $(\theta_0, \theta_1, \theta_2, ...)$ is such that in some period t, the process has entered the third case as distinguished in definition 6.4.6, where both sets $\Theta^{\mathcal{E}-}(\eta_t)$ and $\Theta^{\mathcal{E}+}(\eta_t)$ are non-empty. From the lemma in section 6.5, we know that $\theta^{\mathcal{E}+}_{\min}(\eta_t) \geq \theta^{\mathcal{E}-}_{\max}(\eta_t)$ must hold, so that $\chi_t := \theta^{\mathcal{E}+}_{\min}(\eta_t) - \theta^{\mathcal{E}-}_{\max}(\eta_t) \geq 0$. By the existence of $\tilde{\theta}$ and its associated properties (condition 7.3.3), we know that $\theta^{\mathcal{E}+}_{\min}(\eta_t) \geq \tilde{\theta} \geq \theta^{\mathcal{E}-}_{\max}(\eta_t)$. Then, in period t the new adjustment product θ_t is determined by

$$\theta_t = a_{\eta_t} \theta_{t-1} = \lambda \cdot \theta^{\mathcal{E}+}_{\min}(\eta_t) + (1 - \lambda) \cdot \theta^{\mathcal{E}-}_{\max}(\eta_t),$$

which implies that $\theta^{\mathcal{E}+}_{\min}(\eta_t) \geq \theta_t \geq \theta^{\mathcal{E}-}_{\max}(\eta_t)$. Therefore we see that $|\theta_t - \tilde{\theta}| \leq \chi_t$.

Similarly, by the lemma in section 6.5 we know that in period $t+1$ it will hold that $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) \geq \theta^{\mathcal{E}-}_{\max}(\eta_{t+1})$, and by the properties associated with $\widetilde{\theta}$ we know that $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) \geq \tilde{\theta} \geq \theta^{\mathcal{E}-}_{\max}(\eta_{t+1})$. In period $t+1$, min-max adjustment will again be applied, so that $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) \geq \theta_{t+1} \geq \theta^{\mathcal{E}-}_{\max}(\eta_{t+1})$. Therefore, again we see that $|\theta_{t+1} - \tilde{\theta}| \leq \chi_{t+1}$. Moreover, from the proof of the previous proposition we know that $\chi_{t+1} = \theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) - \theta^{\mathcal{E}-}_{\max}(\eta_{t+1}) \leq \tilde{\lambda} \chi_t$, for $\tilde{\lambda} = \max\{\lambda, 1 - \lambda\}$.

The same procedure will be applied in all subsequent periods, and this will yield that $|\theta_{t+\tau} - \tilde{\theta}| \leq \chi_{t+1}$, and that $\chi_{t+\tau} = \theta^{\mathcal{E}+}_{\min}(\eta_{t+\tau}) - \theta^{\mathcal{E}-}_{\max}(\eta_{t+\tau}) \leq \tilde{\lambda}^{\tau}\chi_t$ for every period $t + \tau \geq t$. Since $\tilde{\lambda} = \max\{\lambda, 1 - \lambda\} < 1$ and $\chi_t \in \mathbb{R}_+$, we see that $\tilde{\lambda}^{\tau}\chi_t$ will converge to 0 as τ goes to infinity. Thus indeed, $|\theta_{t+\tau} - \tilde{\theta}|$ will converge to zero, and the sequence of adjustment factors $(\theta_0, \theta_1, \theta_2, ...)$ converges to $\tilde{\theta}$. ■

The above three conditions that are sufficient for convergence towards optimality to occur, are still stated within the general ad hoc framework. Within this general ad hoc framework, models that satisfy these conditions will be such that convergence towards optimality will always occur. The next two chapters will be set within the more specific setting of models of consumption/savings, and in these chapters we will be able to express the three conditions that were presented in this section in terms of a model's primitives. There we will see that (in some cases) the above conditions will indeed be satisfied, and that convergence towards optimality will indeed occur.

8 Convergence under certainty

In what follows we will investigate convergence of preferences and behaviour, as generated by the learning algorithm, towards optimality. In this and in the next chapter, we will do these investigations in a class of models that is particularly convenient and tractable: we will use stationary models of consumption/savings decisions. In the previous chapter, we stated a number of conditions needed for convergence (of preferences) towards optimality in a general setting. In the more specific settings of this and the next chapter we will see that under some (rather specific) circumstances, these conditions will be met, and convergence towards optimality will occur.

As noted in the previous chapter, given initial time preferences, convergence (of preferences) towards optimality presupposes the existence of a consistency-inducing scalar. In the present and in the next chapter we will see that for all stationary consumption/savings models, a consistency-inducing scalar exists if and only if the initial time preferences are of a certain (rather specific) form. Furthermore, the present chapter will establish that if indeed the initial time preferences are of this specific form, then in some cases convergence towards optimality will occur, and in some cases convergence towards optimality will not occur.

Finally, this chapter will consider convergence of choices towards optimality and the relations between convergence towards optimality of preferences and convergence towards optimality of choices.

The present chapter will deal with consumption/savings models under certainty, the next chapter will consider consumption/savings models with expected utility.

This chapter consists of five sections. The first section will specify the setting of stationary models of consumption/savings under certainty, and it will specify how the ad hoc framework can be fit into this setting. In the second section we will see that in this setting convergence of preferences will always take place. The third section will consider when a consistency-inducing scalar exists, and it provides a necessary condition (with respect to the initial time preferences with which the learning algorithm starts) for convergence towards optimality to occur. The fourth section will establish conditions under which convergence towards optimality does occur, there we will see that in models without additional income convergence towards optimality will occur (in some circumstances), and that in models with additional income convergence towards optimality will generally not occur. Finally, while sections 2, 3 and 4 deal with convergence of preferences, section 5 deals with convergence of choices. The fifth section will show that convergence towards optimality in terms of preferences implies convergence towards optimality in terms of choice functions, and it investigates when convergence towards optimality in terms of choices does occur.

8.1 Models of consumption/savings under certainty

In this chapter we will consider behaviour and preferences as governed by the learning algorithm in a model of certainty. Although the existence of uncertainty played an important part in the motivation for devising a learning model for consumer choice, another important consideration was that of computational complexity. In the present setting the problems of the standard framework under certainty would still involve infinitely many periods and infinitely many variables. Therefore, even in the absence of uncertainty, the maximization problems that would result from the standard approach are still computationally very complex, which may justify making a bounded rationality assumption.

Thus here we model a consumer who has no uncertainty about the components of the economic environment that he faces. Then, we suppose that this consumer *does* know total preferences on $X = \times_{t=0}^{\infty} X_t$, and that he *does* know all his instantaneous preferences, but that he *does not* know ad hoc preferences that are consistent with total preferences.

While we assume that the consumer would know all his instantaneous preferences and his total preferences on the total commodity space X, we will also assume that our consumer lacks the cognitive sophistication to solve the associated problem of maximizing his total preferences over a budget set at once, and similarly we assume that our consumer would lack the cognitive sophistication to derive consistent ad hoc preferences from the total preferences, as in chapter 5. Therefore here we assume that our consumer would try to tackle his lifetime consumption problem in the piecemeal way as in the learning algorithm, using ad hoc preferences to decide in all periods, and updating time preferences after any period as specified in chapter 6.

8.1.1 The setting

For tractability both this and the next chapter will deal exclusively with stationary models of consumption/savings. In this class of models we will see that (under some circumstances) convergence towards optimality will occur.

As before, we start from axiom 4.1.1, so there is an infinite number of periods in which consumption decisions have to be made. Accordingly, the commodity space is subdivided into distinct period-t commodity spaces X_t, and the total commodity space corresponds to their Cartesian product: $X = \times_t X_t$. Since in consumption/savings models in any period t only a consumption level $c_t \in \mathbb{R}_+$ has to be chosen, each of the sets X_t is one-dimensional, and here we simply set $X_t = \mathbb{R}_+$ for all t.

The total utility function on $X = \mathbb{R}_+^{\infty}$ is additively separable with respect to time, and satisfies exponential time discounting, so for all $c = (c_0, c_{1,} c_{2,} \ldots)$ preferences can be expressed by $U(c) = \sum_{t=0}^{\infty} \delta^t u_0(c_t)$, for some discount factor $0 < \delta < 1$, and some instantaneous utility function $u_0 : \mathbb{R}_+ \to \mathbb{R}$. Here it is also assumed that the instantaneous utility function u_0 satisfies one of the next two axioms.[29]

Axiom 8.1.1 *A function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies axiom 8.1.1 if it is strictly increasing, strictly concave and continuously differentiable.*

[29] If u_0 satisfies axiom 8.1.2, then $u_0'(0)$ should be well-defined. The term $u_0'(0)$ should denote the derivative of the differentiable function u_0, evaluated in the point 0. However, such a function is defined on $\mathbb{R}_+$, which is a closed set (in $\mathbb{R}$). And mathematically the derivative is only properly defined on the internal of this closed set, in this case $\mathbb{R}_{++}$. Hence the term $u_0'(0)$ is mathematically not well-defined in its usual sense.

Axiom 8.1.2 *A function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies axiom 8.1.2 if it satisfies axiom 8.1.1 and if $f'(0) = \infty$.*

There is no uncertainty, so at time 0 the consumer knows for any period $t \in \mathbb{N}_0$ the relevant components of the economic environment. In the (sub)sections of chapter 6 dealing with consumption/savings, we saw that commodity spaces $X_t = \mathbb{R}_+$ and prices $p_t = 1$ (the interest rate is assumed to equal zero) are by assumption already known. In the present chapter all additional incomes $I_t \geq 0$ also need to be known at time 0.

Here we also assume that models are stationary. In the present setting, the fact that X_t and p_t are the same for all periods, and the use of a uniform $0 < \delta < 1$ and a single function u_0 are consistent with stationarity, but here stationarity also requires that the additional incomes are the same for all periods after the first, so that one of the next two axioms will hold.

Axiom 8.1.3 *No additional income is obtained after time 0: $I_t = 0$, for all $t \geq 1$.*

Axiom 8.1.4 *A constant periodical income is obtained every period after time 0: $I_t = I > 0$, for all $t \geq 1$.*

Saving is possible at a zero interest rate, and borrowing is not possible. This implies that the budget available in period t equals the savings brought over from the previous period plus the additional income received in that period, and that this budget also equals the sum of all that was obtained in periods up to time t minus the sum of what was spent in periods prior to time t:

$$m_t = s_{t-1} + I = m_{t-1} - c_{t-1} + I = ... = m_0 - \sum_{i=0}^{t-1} c_i + tI.$$

Still, the right-sided derivative

$$u'_{0+}(0) := \lim_{h \downarrow 0} (u_0(h) - u_0(0))/h,$$

is well-defined. Here $\lim_{h\downarrow 0}$ denotes the limit of h approaching the point 0 from above. Mathematically the derivative $u'_0(c)$ at a strictly positive $c > 0$ is defined as:

$$u'_0(c) = \lim_{h \to 0} (u_0(c + h) - u_0(c))/h,$$

where $\lim_{h\to 0}$ denotes the limit of h approaching the point 0 (from anywhere). And since u_0 is differentiable on $(0, \infty)$, $u'_{0+}(0)$ also satisfies $u'_{0+}(0) = \lim_{c\downarrow 0} u'_0(c)$. In what follows, we will keep using the notation $u'_0(0)$ as shorthand for the well-defined $u'_{0+}(0)$.

The term $V'(0)$ will also appear in this work. Value functions are also defined on $\mathbb{R}_+$, and similarly $V'(0)$ should be thought of as short-hand for the right-sided derivative $V'_+(0)$.

8.1.2 Dynamic programming

Within the standard framework for consumer choice, the basic consumer problem for the consumption/savings setting as described above, would be given by the following sequence problem

$$\max_{(c_0,c_1,c_2,....)} \sum_{t=0}^{\infty} \delta^t u_0(c_t) \ s.t. \ \sum_{i=0}^{t} c_i \leq m_0 + tI, \ \forall t \geq 0.$$

Corresponding to this sequence problem is the following functional equation

$$V^*(m) = \max_{(c,s)}[u_0(c) + \delta V^*(s+I)] \ s.t. \ c+s \leq m.$$

In section 2.5 we saw that the theory of dynamic programming shows that the value function $V^* : \mathbb{R}_+ \to \mathbb{R}$ that solves this functional equation, is also exactly the function that returns for every budget level m available at period 0, the maximum discounted lifetime utility value that can be attained from m, that is:

$$V^*(m) = \max_{(c_0,c_1,c_2,...)} \{\sum_{t=0}^{\infty} \delta^t u_0(c_t) : \sum_{i=0}^{t} c_i \leq m_0 + tI, \forall t \geq 0\}.$$

Thus we found a link between sequence problems and the corresponding functional equations. It can also be shown (e.g. Stokey and Lucas ([43], chapter 4)) that if instantaneous utility u_0 satisfies axiom 8.1.1, then the corresponding value function V^* will also satisfy axiom 8.1.1.

8.1.3 The ad hoc framework

However, here we model the decision-making of a boundedly rational individual who somehow has trouble dealing with the complexity of the above choice problem. We assume that our consumer cannot solve the above problem at once. Hence the standard framework cannot be used to model this decision-maker's behaviour, and we will use the ad hoc framework.

We suppose that axiom 4.4.1 holds, so our consumer would cut up his lifetime consumption choice problem into smaller ad hoc choice problems, where in each of these he uses ad hoc preferences to solve them. Recall from the (sub)sections of chapter 4 dealing with consumption/savings models, that in these models ad hoc utility functions were still supposed to be of an additively separable form and to exhibit exponential discounting, so that we can write:

$$u^{(t)}(w_{t-1}, c_t, s_t) = \sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1} \widetilde{V}^{(t)}(s_t).$$

Here w_{t-1} denotes the vector of past consumption choices $(c_0, c_1, ..., c_{t-1})$. In order to stay closer to the notation and interpretation of value functions in dynamic programming, and without loss of generality, the function $\widetilde{V}^{(t)}(s_t)$ can be replaced with

$V^{(t)}(s_t + I)$, where the function $V^{(t)} : [I, \infty) \to \mathbb{R}$ denotes a value function that values the next period's budget. Again without loss of generality the above specification of ad hoc utility can be divided by δ^t, such as to arrive at the more convenient form of the next axiom.

Axiom 8.1.5 *For any time t, the ad hoc utility function $u^{(t)}(w_{t-1}, c_t, s_t)$ can be written as*

$$u^{(t)}(c_1, c_2, ..., c_t, s_t) = \sum_{i=1}^{t} \delta^{i-t} u_0(c_i) + \delta V^{(t)}(s_t + I).$$

Here $V^{(t)}$ is called a ***value function****.*

As in section 6.2, ad hoc preferences can be separated into instantaneous preferences, as represented by

$$v^{(t)}(w_{t-1}, c_t) = \sum_{i=0}^{t} \delta^{i-t} u_0(c_i),$$

and into time preferences, as represented by

$$U^{(t)}(v^{(t)}, s_t) = v^{(t)} + \delta V^{(t)}(s_t + I).$$

We assume that axiom 6.2.2 holds, so that instantaneous preferences (and thus $v^{(t)}$) are thought to be exogenous, while time preferences (and thus the value function $V^{(t)}$) are endogenous. Our decision-maker does not know what his consistent ad hoc preferences, and therefore the optimal value function V^*, would be. To make decisions, good value functions $V^{(t)}$ remain to be found. The decision-maker is assumed to approach this problem by using an initial guess at such a value function in the first period, and by updating value functions in all subsequent periods.

Time preferences (and thus the value function) are adjusted by means of min-max adjustment, as dependent on regular excess expenditure (REE)[30] in the previous periods. Given the budget m_t in period t, the (regular) excess expenditure $E_t = c_t^* - c_t^{\triangleleft}$ in period t is determined as the difference of the actual expenditure in period t and the ex-post optimal expenditure in period t, as determined in period $t + 1$.

Here c_t^* denotes the actual expenditure in period t, which is determined as part of a pair (c_t^*, s_t^*) that solves the corresponding basic ad hoc consumer problem

$$\max_{(c_t, s_t): c_t + s_t \leq m_t} U^{(t)}(v^{(t)}(w_{t-1}, c_t), s_t) =$$

$$\max_{(c_t, s_t): c_t + s_t \leq m_t} \sum_{i=1}^{t} \delta^{i-t} u_0(c_i) + \delta V^{(t)}(s_t + I).$$

[30] Recall from subsection 6.4.2 that in models of certainty (and thus also in the present setting) the REE and EEE measures coincide. As in models of certainty it seems strange to use the EEE measure, in this chapter we will let the adjustment function depend on REE excess expenditure.

Since at time t the vector $w_{t-1} = (c_0, c_1, ..., c_{t-1})$ is given, by subtracting the constant $\sum_{i=1}^{t-1} \delta^{i-t} u_0(c_i)$, we see that c_t^* is also part of a solution to the simpler looking problem

$$\max_{(c_t, s_t): c_t + s_t \leq m_t} u_0(c_t) + \delta V^{(t)}(s_t + I).$$

Then, given the budget m_t and the additional income I, $c_t^{\triangleleft}$ denotes the (regular) ex-post optimal expenditure in period t, as determined in period $t+1$. That is, $c_t^{\triangleleft}$ is part of a tuple $(c_t^{\triangleleft}, c_{t+1}^{\triangleleft}, s_{t+1}^{\triangleleft})$ that solves the following hypothetical problem over c_t, c_{t+1}, and s_{t+1} simultaneously

$$\max_{(c_t, c_{t+1}, s_{t+1})} U^{(t)}(v^{(t+1)}(w_{t-1}, c_t, c_{t+1}), s_{t+1}) =$$

$$\max_{(c_t, c_{t+1}, s_{t+1})} \sum_{i=1}^{t+1} \delta^{i-t-1} u_0(c_i) + \delta V^{(t)}(s_{t+1} + I)$$

sub to $c_t + s_t \leq m_t$ and $c_{t+1} + s_{t+1} \leq s_t + I$. And, subtracting a constant $\sum_{i=1}^{t-1} \delta^{i-t-1} u_0(c_i)$ and multiplying with the constant δ gives that $(c_t^{\triangleleft}, c_{t+1}^{\triangleleft}, s_{t+1}^{\triangleleft})$ also solves the simpler looking problem

$$\max_{(c_t, c_{t+1}, s_{t+1})} u_0(c_t) + \delta u_0(c_{t+1}) + \delta^2 V^{(t)}(s_{t+1} + I)$$

sub to $c_t + s_t \leq m_t$ and $c_{t+1} + s_{t+1} \leq s_t + I$.

Instantaneous utility u_0 was assumed to be strictly concave (by axiom 8.1.1), and we also assume that the initial value function $V^{(0)}$ is strictly concave. The specific form of the adjustment process is such that any subsequent value function $V^{(t)}$ will then be strictly concave as well. Therefore, all of the above maximization problems have unique solutions.

In period 0 an initial time preference function $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is given. From the above axiom we see that this initial time preference function will take the following form: $U^{(0)}(v, s) = v + \delta V^{(0)}(s + I)$. Here we also assume that the initial value function $V^{(0)} : [I, \infty) \to \mathbb{R}$ satisfies axiom 8.1.1.

In any later period $t+1$, given a time preference function $U^{(t)}(v, s) = v + \delta V^{(t)}(s + I)$ from the previous period, our consumer adjusts old time preferences such as to arrive at new time preferences $U^{(t+1)}$. That is, given the vector $\varepsilon_{t+1} := (E_0, E_1, ..., E_t)$ of past excess expenditures, the min-max adjustment function yields an adjustment factor $a_{\varepsilon_{t+1}}$, and in period $t+1$ time preferences will be adjusted according to $U^{(t+1)}(v, s) = U^{(t)}(a_{\varepsilon_{t+1}} \cdot v, s)$.

In the present setting, the term $U^{(t)}(a_{\varepsilon_{t+1}} \cdot v, s)$ can be rewritten as $a_{\varepsilon_{t+1}} \cdot v + \delta V^{(t)}(s + I)$, which would specify a new time preference function $U^{(t+1)}(v, s)$. Without loss of generality this specification can be divided by the scalar $a_{\varepsilon_{t+1}}$, and we write

$$U^{(t+1)}(v, s) = v + \delta a_{\varepsilon_{t+1}}^{-1} V^{(t)}(s + I).$$

As argued in subsection 6.4.1, this rewriting does not change anything essential, but this notation does seem to reflect the fact that instantaneous preferences and thus v are exogenous while time preferences and thus V are supposed endogenous. Moreover, this notation is in line with the above axiom.

Thus, because of the specific form of ad hoc utility functions as imposed in the above axiom, the procedure of adjusting time preferences can be shortened to a procedure where the value functions are adjusted directly.

Axiom 8.1.6 *In period* 0 *an initial value function* $V^{(0)} : \mathbb{R}_+ \to \mathbb{R}$*, that satisfies axiom 8.1.1, is exogenously given. In any later period* $t+1$*, a new value function* $V^{(t+1)}$ *is obtained from the old value function by:*

$$V^{(t+1)}(.) := a_{\varepsilon_{t+1}}^{-1} V^{(t)}(.),$$

where the adjustment factors $a_{\varepsilon_{t+1}}$ *are determined by min-max adjustment.*

8.2 Convergence of the value function

Given an initial value function $V^{(0)}$, the learning algorithm will give rise to a sequence of value functions $(V^{(t)})_{t=0}^{\infty}$. Then, we may wonder if such a sequence of value functions will converge.[31]

The relation $V^{(t+1)} = a_{\varepsilon_{t+1}}^{-1} V^{(t)}$ holds for every period t. Therefore by a repeated argument any value function that is generated by the learning algorithm from the initial value function $V^{(0)}$, can also be written as

$$V^{(t)}(.) = a_{\varepsilon_t}^{-1} \cdot a_{\varepsilon_{t-1}}^{-1} \cdot ... \cdot a_{\varepsilon_1}^{-1} \cdot V^{(0)}(.) = \theta_t^{-1} V^{(0)}(.),$$

where θ_t denotes the adjustment product. Like in the previous chapter, instead of considering sequences of value functions $(V^{(t)})_{t=0}^{\infty}$, here we can consider the corresponding sequences of adjustment products $(\theta_t)_{t=0}^{\infty}$.

Since in the present setting min-max adjustment is applied in all periods, proposition 7.1.1 applies. That is, we know that any sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$ as generated by the learning algorithm, will converge to some limit $\theta_\infty \in \overline{\mathbb{R}}_+$. Consequently, any sequence of value functions $(V^{(t)})_{t=0}^{\infty}$, as generated by the learning algorithm, will converge to some limit function $V^{(\infty)}(.) = \theta_\infty V^{(0)}(.)$.

8.3 Existence of a consistency-inducing scalar

However, more than in the question of convergence per se, here we are interested in the question of convergence towards optimality. In section 7.3 we saw that convergence towards optimality will take place if three conditions are satisfied. In this section we will establish when the first of these conditions, condition 7.3.1 (which says that, given the initial time preference function $U^{(0)}$, there exists a consistency-inducing scalar $\tilde{\theta}$), will hold. The other conditions will be dealt with in the following sections.

[31] By convergence of a sequence of functions we mean pointwise convergence.

First we will take a look at what consistency would entail in the present setting. The optimal value function V^* (that solves the functional equation) returns for every budget m_0 available at time 0 the maximum discounted lifetime utility that can be attained from m_0. Suppose that after time t the choices $(w_{t-1}, c_t) = (c_0, c_1, ..., c_t)$ and an amount of savings s_t are given. These past choices yielded the (provisional) utility $\sum_{i=0}^{t} \delta^i u_0(c_i)$, and the period-$(t+1)$ budget will equal $m_{t+1} = s_t + I$. Then, given m_{t+1}, the maximally attainable additional utility from period $t+1$ onwards would be given by

$$\max_{(c_{t+1}, c_{t+2}, \ldots)} \sum_{i=t+1}^{\infty} \delta^i u_0(c_i) \; s.t. \; \sum_{\tau=t+1}^{i} c_\tau \leq m_{t+1} + (i-t-1)I, \; \forall i \geq t+1.$$

This problem faced at time $t+1$ is an exact copy of the problem that is faced at time 0 (as in the sequence problem in section 8.1.2), and by a change of variables it can be seen that this maximal additional utility that can be attained in and after period $t+1$, is given by $\delta^{t+1} V^*(m_{t+1})$. Therefore, by definition of consistency, for any time t any consistent ad hoc utility function $\widetilde{u}^{(t)}(w_{t-1}, c_t, s_t)$ must be of the form

$$\widetilde{u}^{(t)}(w_{t-1}, c_t, s_t) = \widetilde{f}_t(\sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1} V^*(s_t + I)),$$

for some strictly increasing function $\widetilde{f}_t : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

For any time t, instantaneous preferences were supposed to be exogenous, and represented by $v^{(t)}(w_{t-1}, c_t) = \sum_{i=0}^{t} \delta^{i-t} u_0(c_i)$. Therefore, the above condition can be rewritten as the condition that for any time t, any consistent ad hoc utility function $\widetilde{u}^{(t)}(w_{t-1}, c_t, s_t)$ should be of the form

$$\widetilde{u}^{(t)}(w_{t-1}, c_t, s_t) = f_t(v^{(t)}(w_{t-1}, c_t) + \delta V^*(s_t + I)),$$

for some strictly increasing function $f_t : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ (here with $f_t(x) = \widetilde{f}_t(\delta^t x)$).

Thus we also see that for any time t, given the exogenous instantaneous utility, any consistent time preference function $\widetilde{U}^{(t)}(v^{(t)}, s_t)$ must be of the form

$$\widetilde{U}^{(t)}(v^{(t)}, s_t) = f_t(v^{(t)} + \delta V^*(s_t + I)),$$

for some strictly increasing function $f_t : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

Now we can investigate when condition 7.3.1, that a consistency-inducing scalar exists, will hold. The initial time preference function was given by $U^{(0)}(v^{(0)}, s_0) = v^{(0)} + \delta V^{(0)}(s_0 + I)$. Therefore, if given $U^{(0)}$ a CIS $\tilde{\theta}$ exists, then for any period t the function $U^{(t)}(v^{(t)}, s_t) := v^{(t)} + \delta \tilde{\theta}^{-1} V^{(0)}(s_t + I)$ should be a consistent time preference function. That is, if a consistency-inducing scalar $\tilde{\theta}$ exists, then for every period t there should be a strictly increasing function f_t such that $v^{(t)} + \delta \tilde{\theta}^{-1} V^{(0)}(s_t + I)$ equals $f_t(v^{(t)} + \delta V^*(s_t + I))$.

The next proposition provides a condition on the initial value function $V^{(0)}$, that is both sufficient and necessary for the existence of a consistency-inducing scalar in the present context, and thus for condition 7.3.1 to hold.

Proposition 8.3.1 *Assume that axioms 4.1.1, 4.4.1, 8.1.5 and 8.1.6, and 8.1.3 or 8.1.4 hold. Then given an initial time preference function* $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *that is of the form* $U^{(0)}(v^{(0)}, s_0) = v^{(0)} + \delta V^{(0)}(s_0 + I)$*, there exists a consistency-inducing scalar* $\tilde{\theta}$ *if and only if the initial value function* $V^{(0)}$ *is of the form* $V^{(0)}(m) = \tilde{\theta} V^*(m) + \alpha$*, for some* $\alpha \in \mathbb{R}$ *and all* $m \geq I$*, where* V^* *denotes the optimal value function that solves the corresponding functional equation.*

Proof. ▲ The 'if'-part is quite straightforward. Suppose that the initial time preference function $U^{(0)}(v^{(0)}, s_0) = v^{(0)} + \delta V^{(0)}(s_0 + I)$ is such that the initial value function $V^{(0)}$ satisfies $V^{(0)}(m) = \tilde{\theta} V^*(m) + \alpha$ for some $\alpha \in \mathbb{R}$, some $\tilde{\theta} > 0$, and all $m \geq I$. Then, for the function $f_t(x) = x + \delta \tilde{\theta}^{-1} \alpha$ it will hold that

$$v^{(t)} + \delta \tilde{\theta}^{-1} V^{(0)}(s_t + I) = v^{(t)} + \delta \tilde{\theta}^{-1} (\tilde{\theta} V^*(s_t + I) + \alpha) =$$

$$v^{(t)} + \delta V^*(s_t + I) + \delta \tilde{\theta}^{-1} \alpha = f_t(v^{(t)} + \delta V^*(s_t + I)),$$

for all $s_t \geq 0$. Hence there exists a strictly increasing function f_t such that $v^{(t)} + \delta \tilde{\theta}^{-1} V^{(0)}(s_t + I)$ equals $f_t(v^{(t)} + \delta V^*(s_t + I))$, and $\tilde{\theta}$ is indeed a consistency-inducing scalar.

▲ Conversely, suppose that, given the initial time preference function $U^{(0)}(v^{(0)}, s_0) = v^{(0)} + \delta V^{(0)}(s_0 + I)$, there exists a CIS $\tilde{\theta}$. Then for all t there must be some strictly increasing function $f_t : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that

$$v^{(t)} + \delta \tilde{\theta}^{-1} V^{(0)}(s_t + I) = f_t(v^{(t)} + \delta V^*(s_t + I)).$$

The partial derivative of the right-hand side with respect to $v^{(t)}$ equals $f'_t(v^{(t)} + \delta V^*(s_t + I))$, and the partial derivative of the left-hand side with respect to $v^{(t)}$ equals one. By the above equality, these derivatives should also equal, so that $f'_t(x) = 1$ must hold for all relevant x. This implies that the function f_t must be of the form $f_t(x) = x + \beta_t$, for some $\beta_t \in \mathbb{R}$ (and all relevant x). Therefore the above equality gives that $v^{(t)} + \delta \tilde{\theta}^{-1} V^{(0)}(s_t + I)$ must equal $v^{(t)} + \delta V^*(s_t + I) + \beta_t$, so that

$$\delta \tilde{\theta}^{-1} V^{(0)}(s_t + I) = \delta V^*(s_t + I) + \beta_t.$$

Thus indeed we see that the initial time preference function $V^{(0)}$ must satisfy: $V^{(0)}(m) = \tilde{\theta} V^*(m) + \alpha$, for all $m \geq I$ and some $\alpha \in \mathbb{R}$. ■

Note that here convergence of a sequence of adjustment products towards a consistency-inducing scalar $\tilde{\theta}$ does not necessarily imply convergence of the corresponding sequence of value functions towards the optimal value function that solves the functional equation. To see this, suppose given some initial value function that satisfies $V^{(0)}(m) = \tilde{\theta} V^*(m) + \alpha$, for some $\alpha \in \mathbb{R}$, so that by the above proposition the scalar $\tilde{\theta}$ is a CIS. Then convergence of the sequence of adjustment products towards $\tilde{\theta}$ implies that the corresponding sequence of value functions will converge towards

$$\tilde{\theta}^{-1} V^{(0)}(m) = \tilde{\theta}^{-1} (\tilde{\theta} V^*(m) + \alpha) = V^*(m) + \alpha \tilde{\theta}^{-1}.$$

Thus, convergence of the sequence of adjustment products towards the CIS $\tilde{\theta}$ does imply that the corresponding sequence of value functions converges towards the optimal value function, *up to a constant*. Note however, that such a constant $\alpha\tilde{\theta}^{-1}$ is essentially irrelevant, as the choices that will be made are determined by marginal instantaneous utility and marginal value for savings. Therefore, we *will* keep referring to this kind of convergence as convergence towards optimality.

8.4 Convergence towards optimality

In section 7.3 we identified three conditions that (together) are sufficient for convergence towards optimality to occur. The first of these, condition 7.3.1, was that a consistency-inducing scalar exists. In the previous section we established when this condition holds. The second of these conditions, condition 7.3.2, entailed that the consistency-inducing scalar should be a fixed point of the adjustment procedure, and this condition will be dealt with next.

The following proposition is preliminary to establishing that this second condition will hold. It shows that if in some period the optimal value function V^* (that solves the functional equation) is used, this will always yield a zero excess expenditure.

Proposition 8.4.1 *Let a model be given that satisfies axioms 4.4.1 and 8.1.5, and 8.1.3 or 8.1.4, and with an instantaneous utility function that satisfies axiom 8.1.1. Then if at time t the value function $V^{(t)}$ that prevails is identical to the optimal value function V^*, the corresponding (regular) excess expenditure will be zero: $E_t = 0$.*

Proof. Given is a model with an instantaneous utility function u_0 that satisfies axiom 8.1.1, a discount factor $0 < \delta < 1$ and an equal additional income incurred every period after the first: $I_t = I \geq 0$, for all $t \geq 1$. The value function that is used in period t is identical to the optimal value function that solves the functional equation corresponding to this model: $V^{(t)}(m) = V^*(m)$, for all $m \in \mathbb{R}_+$. We will now show that, for any period-t budget $m_t \in \mathbb{R}_+$ it will always hold that $c_t^* = c_t^{\triangleleft}$, so that $E_t = 0$.

To see this, for $w_{t-1} = (c_0, ..., c_{t-1}) \in W_{t-1}$ given, $c_t^{\triangleleft}$ is part of a tuple $(c_t^{\triangleleft}, c_{t+1}^{\triangleleft}, s_{t+1}^{\triangleleft})$ that solves

$$\max_{c_t+s_t \leq m_t} \max_{c_{t+1}+s_{t+1} \leq s_t+I} u_0(c_t) + \delta u_0(c_{t+1}) + \delta^2 V^*(s_{t+1} + I).$$

This maximization problem can alternatively be written as

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta \cdot \max_{c_{t+1}+s_{t+1} \leq s_t+I} u_0(c_{t+1}) + \delta V^*(s_{t+1} + I). \tag{1}$$

Now, since the function V^* solves the functional equation

$$V^*(m_{t+1}) = \max_{c_{t+1}+s_{t+1} \leq m_{t+1}} u_0(c_{t+1}) + \delta V^*(s_{t+1} + I),$$

we see that (1) could also be written as

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta V^*(s_t + I). \tag{2}$$

And, the triple $(c_t^{\triangleleft}, c_{t+1}^{\triangleleft}, s_{t+1}^{\triangleleft})$ maximizes the problem we started with, so if we write $s_t^{\triangleleft} := m_t - c_t^{\triangleleft}$ $(= c_{t+1}^{\triangleleft} + s_{t+1}^{\triangleleft} - I$, by strict increasingness), we see that the pair $(c_t^{\triangleleft}, s_t^{\triangleleft})$ must also solve (2).

Of course, problem (2) is also exactly the one that c_t^* was supposed to maximize in the first place. And since instantaneous utility and the optimal value function are strictly concave, so is their sum, and we see that solutions are unique. Hence indeed we find that $c_t^* = c_t^{\triangleleft}$ so that $E_t = 0$. ■

For a zero excess expenditure in period t, the (min-max) adjustment function will always yield a unit adjustment factor $a_{\varepsilon_{t+1}} = 1$. Therefore, with the above proposition we see that if at time t the prevailing value function $V^{(t)}$ is identical to the optimal value function V^*, then in period $t+1$ the resulting adjusted value function $V^{(t+1)} = a_{\varepsilon_t}^{-1} V^{(t)}$ will also equal the optimal value function $V^{(t+1)} = V^*$.

The next proposition shows that adding a constant to a value function does not influence the resulting excess expenditure. As noted in the previous section, the idea is that such a constant has no effect on the choices (both actual and ex-post optimal) that will be made, as these choices rely only on marginal instantaneous utility and on marginal value.

Proposition 8.4.2 *Let a model be given that satisfies axioms 4.4.1 and 8.1.5, and 8.1.3 or 8.1.4, and with an instantaneous utility function that satisfies axiom 8.1.1. And suppose we are given two period-t value functions* $\overline{V}^{(t)} : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $\underline{V}^{(t)} : \mathbb{R}_+ \to \mathbb{R}_+$, *that satisfy axiom 8.1.1 and are such that* $\overline{V}^{(t)}(m) = \underline{V}^{(t)}(m) + \alpha$, *for all* $m \in \mathbb{R}_+$ *and some constant* $\alpha \in \mathbb{R}$. *Then for any budget* $m_t \in \mathbb{R}_+$, *the corresponding (regular) excess expenditures* $\overline{E}_t$ *and* $\underline{E}_t$, *will always satisfy* $\overline{E}_t = \underline{E}_t$.

Proof. Given is a model with an instantaneous utility function u_0 that satisfies axiom 8.1.1, and a discount factor $0 < \delta < 1$. Also given are two period-t value functions $\overline{V}^{(t)} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\underline{V}^{(t)} : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy axiom 8.1.1, and that differ by a constant: $\overline{V}^{(t)}(m) = \underline{V}^{(t)}(m) + \alpha$ for all $m \in \mathbb{R}_+$ and some scalar $\alpha \in \mathbb{R}$. Then, given any initial budget m_t, and the additional income $I \geq 0$ that will be obtained in period $t+1$, excess expenditure can be determined for both value functions, and we want to show that $\overline{E}_t = \overline{c}_t^* - \overline{c}_t^{\triangleleft}$ (corresponding to $\overline{V}^{(t)}$) will equal $\underline{E}_t = \underline{c}_t^* - \underline{c}_t^{\triangleleft}$ (corresponding to $\underline{V}^{(t)}$).

Now, the choice $(\underline{c}_t^*, \underline{s}_t^*)$ that will be made in period t, given past choices $w_{t-1} = (c_0, ..., c_{t-1}) \in W_{t-1}$, and the prevailing value function $\underline{V}^{(t)}(.)$ will solve

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \underline{V}^{(t)}(s_t + I).$$

It is mathematically obvious that the solution to this last problem will be the same as the solution to the problem

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \underline{V}^{(t)}(s_t + I) + \delta\alpha =$$

$$\max_{c_t+s_t\leq m_t} u_0(c_t)+\delta\overline{V}^{(t)}(s_t+I).$$

And $(\underline{c}_t^*, \underline{s}_t^*)$ was supposed to maximize the first problem, so it must also solve the last maximization problem, which is exactly the one that $(\overline{c}_t^*, \overline{s}_t^*)$ is supposed to maximize. Then by strict concavity we know that solutions are unique, and we see that $\underline{c}_t^* = \overline{c}_t^*$.

Then, for $w_{t-1} \in W_{t-1}$ given, $\underline{c}_t^{\triangleleft}$ will be part of a tuple $(\underline{c}_t^{\triangleleft}, \underline{c}_{t+1}^{\triangleleft}, \underline{s}_{t+1}^{\triangleleft})$ that solves the hypothetical maximization problem of

$$\max_{c_t+s_t\leq m_t}\ \max_{c_{t+1}+s_{t+1}\leq s_t+I} u_0(c_t)+\delta u_0(c_{t+1})+\delta^2\underline{V}^{(t)}(s_{t+1}+I).$$

Again, $(\underline{c}_t^{\triangleleft}, \underline{c}_{t+1}^{\triangleleft}, \underline{s}_{t+1}^{\triangleleft})$ must also solve

$$\max_{c_t+s_t\leq m_t}\ \max_{c_{t+1}+s_{t+1}\leq s_t+I} u_0(c_t)+\delta u_0(c_{t+1})+\delta^2\underline{V}^{(t)}(s_{t+1}+I)+\delta^2\alpha =$$

$$\max_{c_t+s_t\leq m_t}\ \max_{c_{t+1}+s_{t+1}\leq s_t+I} u_0(c_t)+\delta u_0(c_{t+1})+\delta^2\overline{V}^{(t)}(s_{t+1}+I).$$

And $(\underline{c}_t^{\triangleleft}, \underline{c}_{t+1}^{\triangleleft}, \underline{s}_{t+1}^{\triangleleft})$ solves the first problem, so it must also solve the last, which is exactly the one that $(\overline{c}_t^{\triangleleft}, \overline{c}_{t+1}^{\triangleleft}, \overline{s}_{t+1}^{\triangleleft})$ is supposed to maximize. By strict concavity solutions are unique, and we see that $\underline{c}_t^{\triangleleft} = \overline{c}_t^{\triangleleft}$.

Therefore as desired, the excess expenditures $\overline{E}_t$ and $\underline{E}_t$ will satisfy

$$\overline{E}_t = \overline{c}_t^* - \overline{c}_t^{\triangleleft} = \underline{c}_t^* - \underline{c}_t^{\triangleleft} = \underline{E}_t.$$

■

With this proposition we know that, if we are given two period-0 value functions $\overline{V}^{(0)}: \mathbb{R}_+ \to \mathbb{R}_+$ and $\underline{V}^{(0)}: \mathbb{R}_+ \to \mathbb{R}_+$ that differ by a constant: $\overline{V}^{(0)}(m) = \underline{V}^{(0)}(m)+\alpha_0$, for all $m \in \mathbb{R}_+$ and some $\alpha_0 \in \mathbb{R}$, then the two corresponding excess expenditures will always be the same. Therefore both value functions would always be adjusted with the same factor a. Thus, if $\overline{V}^{(0)}$ is adjusted to $\overline{V}^{(1)}: \mathbb{R}_+ \to \mathbb{R}_+$, and $\underline{V}^{(0)}$ is adjusted to $\underline{V}^{(1)}: \mathbb{R}_+ \to \mathbb{R}_+$, we will get that

$$\overline{V}^{(1)}(m) = a^{-1}\overline{V}^{(0)}(m) = a^{-1}(\underline{V}^{(0)}(m)+\alpha_0) = a^{-1}\underline{V}^{(0)}(m)+\alpha_1 = \underline{V}^{(1)}(m)+\alpha_1,$$

for all $m \in \mathbb{R}_+$, with $\alpha_1 = a^{-1}\alpha_0$. Therefore $\overline{V}^{(1)}$ and $\underline{V}^{(0)}$ will again differ by a constant $\alpha_1 \in \mathbb{R}$.

A repeated argument will show that, given that $\overline{V}^{(0)}(m) = \underline{V}^{(0)}(m)+\alpha_0$, in every subsequent period t the generated value functions $\overline{V}^{(t)}$ and $\underline{V}^{(t)}$ will differ by a constant: $\overline{V}^{(t)}(m) = \underline{V}^{(t)}(m)+\alpha_t$, for all $m \in \mathbb{R}_+$ and some $\alpha_t \in \mathbb{R}$.

With the above two propositions we now get that in the present setting condition 7.3.2 (that a consistency-inducing scalar is a fixed point of the adjustment procedure) will always hold.

Proposition 8.4.3 *Suppose that axioms 4.1.1, 4.4.1, 8.1.5 and 8.1.6, and 8.1.3 or 8.1.4 hold. And suppose that, given an initial time preference function $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ that is of the form $U^{(0)}(v^{(0)}, s_0) = v^{(0)} + \delta V^{(0)}(s_0 + I)$, a consistency-inducing scalar $\tilde{\theta}$ exists. Then, if at time t the prevailing adjustment product θ_t equals $\widetilde{\theta}$, the corresponding excess expenditure will equal zero: $E_t = 0$.*

Proof. Given an initial time preference function $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ that is of the form $U^{(0)}(v^{(0)}, s_0) = v^{(0)} + \delta V^{(0)}(s_0 + I)$, a consistency-inducing scalar $\tilde{\theta}$ exists. Therefore by proposition 8.3.1 we know that $V^{(0)}$ must satisfy $V^{(0)}(m) = \tilde{\theta} V^*(m) + \alpha$, for some $\alpha \in \mathbb{R}$.

Suppose that in some period t the prevailing adjustment product θ_t equals $\widetilde{\theta}$, and thus that the prevailing value function $V^{(t)}$ equals $\tilde{\theta}^{-1} V^{(0)} = \tilde{\theta}^{-1}(\tilde{\theta} V^* + \alpha) = V^* + \alpha\tilde{\theta}^{-1}$. From the previous proposition (8.4.2) we know that this prevailing value function will yield the same excess expenditure as the optimal value function V^*. And from proposition 8.4.1 we know that the excess expenditure will equal zero in this case: $E_t = 0$. ■

By this proposition we see that if at time t the prevailing adjustment product θ_t equals the CIS $\widetilde{\theta}$, then excess expenditure will always equal zero, so that the next period's adjustment factor $a_{\varepsilon_{t+1}}$ will equal one, and the subsequent adjustment product θ_{t+1} will also equal $\tilde{\theta}$. Thus indeed the CIS is a fixed point of the adjustment procedure.

In the previous section we already established when condition 7.3.1, the first of the three conditions needed for convergence towards optimality, will hold. The above proposition shows that the second condition, condition 7.3.2, will always hold in the present setting.

The last of these conditions, condition 7.3.3, entailed that adjustment products larger than the CIS yield positive excess expenditures, and that adjustment products smaller than the CIS yield negative excess expenditures. Next, we will deal with this last condition, which is much less straightforward to do than for the first two conditions. First we need a number of technical lemmas. These lemmas do not have straightforward economic interpretations that are very important by themselves, still they are needed to help establish when condition 7.3.3 will hold, and when convergence towards optimality will occur.

Condition 7.3.3 says that adjustment products larger (smaller) than the CIS will yield positive (negative) excess expenditures. Establishing that this condition holds, would entail showing that actual choices c^* are larger (smaller) than ex-post optimal choices $c^{\triangleleft}$ in certain situations. We will ultimately show that (or when) such inequalities hold from the next lemma.

This next lemma will consider solutions c° to problems such as

$$\max_{0 \leq c \leq m'} u_0(c) + \delta V(m' - c + I'), \tag{†}$$

and it will show that steeper V-functions will yield smaller choices. Here obviously the V-functions in the lemma could apply to value functions, however we will also apply this lemma to more general functions. In fact, for more generality we write c° to be able to capture both actual choices c^* and ex-post optimal choices $c^\triangleleft$ in one notation. It may seem clear that actual choices c_t^* would be derived from a maximization problem as in (†) (with $m' = m_t$, $V = V^{(t)}$ and $I' = I$). In proofs of later lemmas we will see that ex-post optimal choices $c_t^\triangleleft$ could also be derived from maximization problems as in (†), although with somewhat less straightforward V-functions. After the next lemma we will specify what these V-functions should look like in that case. Thus, in the next lemma we do not refer to V as a value function, but instead $V : \mathbb{R}_+ \to \mathbb{R}$ is simply a function that satisfies axiom 8.1.1.

Lemma 8.4.1 *Let a $0 < \delta < 1$ be given, and let the functions $u_0 : \mathbb{R}_+ \to \mathbb{R}$, $\underline{V} : \mathbb{R}_+ \to \mathbb{R}$ and $\overline{V} : \mathbb{R}_+ \to \mathbb{R}$ satisfy axiom 8.1.1, with $\underline{V}$ and $\overline{V}$ such that $\underline{V}'(m) \leq \overline{V}'(m)$, for all $m \in \mathbb{R}_+$. Then for every $m' \in \mathbb{R}_+$ and every $I' \in \mathbb{R}_+$, the choices $\underline{c}^\circ$ and $\overline{c}^\circ$ that maximize the problems*

$$c^\circ \in \arg \max_{0 \leq c \leq m'} u_0(c) + \delta V(m' - c + I'),$$

corresponding to $\underline{V}$ and $\overline{V}$ respectively, will be such that $\underline{c}^\circ \geq \overline{c}^\circ$.

Moreover, additionally suppose that $\underline{V}$ and $\overline{V}$ are such that $\underline{V}'(m) < \overline{V}'(m)$ for all $m \in \mathbb{R}_{++}$, and that for $m' \in \mathbb{R}_{++}$ and $I' \in \mathbb{R}_+$ it holds that $\underline{V}'(m' + I') < u_0'(0)/\delta$, and that $\overline{V}'(I') > u_0'(m')/\delta$. Then the choices $\underline{c}^\circ$ and $\overline{c}^\circ$ will be such that $\underline{c}^\circ > \overline{c}^\circ$.

Proof. Given are a function $u_0 : \mathbb{R}_+ \to \mathbb{R}$ that satisfies axiom 8.1.1, and a scalar $0 < \delta < 1$. Also given are two functions $\underline{V} : \mathbb{R}_+ \to \mathbb{R}$ and $\overline{V} : \mathbb{R}_+ \to \mathbb{R}$ that satisfy axiom 8.1.1, and are such that $\underline{V}'(m) \leq \overline{V}'(m)$, $\forall\, m \in \mathbb{R}_+$.

Now, for any given $m' \in \mathbb{R}_+$, and any $I' \in \mathbb{R}_+$, the choice c° will maximize $u_0(c) + \delta V(s + I')$ sub to $c + s \leq m'$ (here c° may represent $\underline{c}^\circ$ and $\overline{c}^\circ$, V may represent $\underline{V}$ and $\overline{V}$). By strict increasingness of the utility function and of the value function, this problem is the same as $\max_{0 \leq c \leq m'} u_0(c) + \delta V(m' - c + I')$. Now, we will distinguish three cases depending on whether c° is an internal or a boundary solution.

(I) $0 < c^\circ < m'$, in which case $u_0'(c^\circ) = \delta V'(m' - c^\circ + I')$,
(II) $c^\circ = 0$, in which case $u_0'(c^\circ) \leq \delta V'(m' - c^\circ + I')$, and
(III) $c^\circ = m'$, in which case $u_0'(c^\circ) \geq \delta V'(m' - c^\circ + I')$.

By strict concavity of u_0 we know that $u_0'(c)$ is strictly decreasing in c. And similarly, by strict concavity of V, $V'(m' - c + I')$ is strictly increasing in c.

Now, $\underline{c}^\circ$ can satisfy case (I), case (II) or case (III), and we will show that in each case it will hold that $\overline{c}^\circ \leq \underline{c}^\circ$.

First we assume that $\underline{c}^\circ$ satisfies case (III): $\underline{c}^\circ$ is a boundary solution with $\underline{c}^\circ = m'$, then obviously any solution $\overline{c}^\circ$ to the problem corresponding to $\overline{V}$, must satisfy $0 \leq \overline{c}^\circ \leq m'$. Thus indeed we find $\underline{c}^\circ \geq \overline{c}^\circ$.

Secondly, we assume that $\underline{c}^\circ$ satisfies case (II): $\underline{c}^\circ$ is a boundary solution with $\underline{c}^\circ = 0$. Then we see that $u_0'(\underline{c}^\circ) \leq \delta \underline{V}'(m' - \underline{c}^\circ + I')$ must hold. Then if we would suppose that $0 < \overline{c}^\circ$, it must hold that $u_0'(\overline{c}^\circ) \geq \delta \overline{V}'(m' - \overline{c}^\circ + I')$. But we know that

$$u_0'(0) \leq \delta \underline{V}'(m' - 0 + I') \leq \delta \overline{V}'(m' - 0 + I'),$$

and by concavity we know that for c larger than 0, $u_0'(c)$ will be smaller than $u_0'(0)$, and $\delta \overline{V}'(m' - c + I')$ will be larger than $\delta \overline{V}'(m' - 0 + I')$, thereby only aggravating the inequality in marginal utilities, so that $u_0'(c) < \delta \overline{V}'(m' - c + I')$ must hold for all $c > 0$. Therefore assuming $0 < \overline{c}^\circ$ leads to a contradiction, and indeed $\underline{c}^\circ \geq \overline{c}^\circ$.

Thirdly, we assume that $\underline{c}^\circ$ satisfies case (I): $0 < \underline{c}^\circ < m'$. Then $\underline{c}^\circ$ will satisfy $u_0'(\underline{c}^\circ) = \delta \underline{V}'(m' - \underline{c}^\circ + I')$. Now, since $\underline{V}'(m) \leq \overline{V}'(m)$, for all $m \in \mathbb{R}_+$, we see that $u_0'(\underline{c}^\circ) \leq \delta \overline{V}'(m' - \underline{c}^\circ + I')$ must hold. Now, similarly as in the second case above, by strict increasingness of $\overline{V}'(m' - c + I')$ in c, and strict decreasingness of u_0' in c, we see that $u_0'(c) < \delta \overline{V}'(m' - c + I')$ must hold for all $c > \underline{c}^\circ$. However, for any solution $\overline{c}^\circ > 0$ to the problem corresponding to $\overline{V}$, from the above three cases we see that it must hold that $u_0'(\overline{c}^\circ) \geq \delta \overline{V}'(m' - \overline{c}^\circ + I')$. Hence, assuming $\overline{c}^\circ > \underline{c}^\circ$ yields a contradiction, and it must hold that $\underline{c}^\circ \geq \overline{c}^\circ$.

▲ For the second part of the proposition we additionally impose that $m' \in \mathbb{R}_{++}$ (rather than $m' \in \mathbb{R}_+$), and we additionally impose stronger requirements on the two value functions: $\underline{V}$ and $\overline{V}$ should now be such that $\underline{V}'(m) < \overline{V}'(m)$, for all $m \in \mathbb{R}_{++}$ (the strict inequality need not necessarily hold for $m = 0$), and $\underline{V}'(m' + I') < u_0'(0)/\delta$, and $\overline{V}'(I') > u_0'(m')/\delta$ should hold. Again, by c° we denote the unique (by strict concavity) solution to $\max_{0 \leq c \leq m'} u_0(c) + \delta V_i(m' - c + I')$.

Now, we want to prove that $\underline{c}^\circ > \overline{c}^\circ$. The first part of the lemma still applies, so we already know that it must hold that $\underline{c}^\circ \geq \overline{c}^\circ$. Thus to prove the second part of the lemma it suffices to additionally show that $\underline{c}^\circ \neq \overline{c}^\circ$.

By the additional condition that $\underline{V}'(m' + I') < u_0'(0)/\delta$ we see that $\underline{c}^\circ$ cannot equal zero. Similarly, the additional condition $\overline{V}'(I') > u_0'(m')/\delta$ implies that $\overline{c}^\circ$ cannot equal m'. Therefore we only need to show that it can never happen that $0 < \underline{c}^\circ = \overline{c}^\circ < m'$.

If $\underline{c}^\circ$ is an internal solution $0 < \underline{c}^\circ < m'$, then it must hold that $u_0'(\underline{c}^\circ) = \delta \underline{V}'(m' - \underline{c}^\circ + I')$. Then $\underline{c}^\circ = \overline{c}^\circ$ would imply that it should also hold that $u_0'(\underline{c}^\circ) = \delta \overline{V}'(m' - \underline{c}^\circ + I')$. However, by $\underline{V}' < \overline{V}'$ this cannot hold, and we see that $\underline{c}^\circ \neq \overline{c}^\circ$ must be true. ■

From the proof of the previous lemma we can see that the additional condition $\underline{V}'(m' + I') < u_0'(0)/\delta$ that is required for the second part of the lemma, is only needed to make sure that $\underline{c}^\circ = 0$ does not occur. If $\underline{c}^\circ = 0$ would occur, then by the first part of the lemma $\overline{c}^\circ = 0$ would also hold, and we would get that $\underline{c}^\circ = \overline{c}^\circ$. Similarly, from the proof of the second part of the lemma we see that the additional requirement $\overline{V}'(I') > u_0'(m')/\delta$ is only needed to rule out the possibility that $\overline{c}^\circ = m'$ occurs, as this would also imply that $\underline{c}^\circ = m'$, and that $\underline{c}^\circ = \overline{c}^\circ$.

The next lemma is a result that we will make much use of in later proofs. For two functions u_0 and V and a scalar γ given, it defines a new function $W_\gamma(m)$ as $\max_{c+s\leq m} u_0(c) + \delta\gamma V(s+I)$, and it relates derivatives of W_γ to derivatives of u_0 and V.

As stated before the previous lemma, ex-post optimal choices $c_t^{\triangleleft}$ would also solve maximization problems as in (†). In proofs of later lemmas we will see that this is indeed the case, if the V-functions from (†) would be of the form of W_γ.

Lemma 8.4.2 *Let two functions $u_0 : \mathbb{R}_+ \to \mathbb{R}$ and $V : \mathbb{R}_+ \to \mathbb{R}$ that satisfy axiom 8.1.1, and some scalars $0 < \delta < 1$ and $I \geq 0$ be given. Then, for any $\gamma \in \mathbb{R}_{++}$, we can define the function $W_\gamma : \mathbb{R}_+ \to \mathbb{R}$ so that for all $m \in \mathbb{R}_+$:*

$$W_\gamma(m) := \max_{c+s\leq m} u_0(c) + \delta\gamma V(s+I). \tag{*}$$

Then the function W_γ is well-defined and it satisfies axiom 8.1.1. Moreover, its derivative is given by

$$W'_\gamma(m) = \max\{\delta\gamma V'(m - c_\gamma(m) + I), u'_0(m)\},$$

where $c_\gamma(m)$ is such that $(c_\gamma(m), m - c_\gamma(m))$ solves (*).

Proof. Given are the functions $u_0 : \mathbb{R}_+ \to \mathbb{R}$ and $V : \mathbb{R}_+ \to \mathbb{R}$ that satisfy axiom 8.1.1, and the scalars $0 < \delta < 1$ and $I \geq 0$. Then, for any $\gamma \in \mathbb{R}_{++}$, the function $W_\gamma : \mathbb{R}_+ \to \mathbb{R}$ is defined by:

$$W_\gamma(m) := \max_{c+s\leq m} u_0(c) + \delta\gamma V(s+I),$$

for all $m \in \mathbb{R}_+$.

Thus the function W_γ is defined as the maximum of the function $u_0(c) + \delta\gamma V(s+I)$ over the set $\{(c,s) \in \mathbb{R}^2_+ : c + s \leq m\}$. Since by axiom 8.1.1 the functions $u_0 : \mathbb{R}_+ \to \mathbb{R}$ and $V : \mathbb{R}_+ \to \mathbb{R}$ are strictly increasing, we can alternatively write W_γ as the optimum of an unconstrained maximization problem:

$$W_\gamma(m) = \max_{0\leq c\leq m} u_0(c) + \delta\gamma V(m - c + I).$$

And since by axiom 8.1.1 the functions u_0 and V are continuous, so is the function $u_0(c) + \delta\gamma V(m - c + I)$. This function is maximized over the compact set $\{c \in \mathbb{R} : 0 \leq c \leq m\}$. Hence, this maximum is attained, and the function W_γ is indeed a well-defined function for any $I \in \mathbb{R}_+$, any $\gamma \in \mathbb{R}_{++}$, and all $m \in \mathbb{R}_+$.

We now also want to show that W_γ satisfies axiom 8.1.1, that is, we want to show that W_γ is a strictly increasing, strictly concave and continuously differentiable function. First note that since, for any $I \in \mathbb{R}_+$ given, for any $\gamma \in \mathbb{R}_{++}$ and for all $m \in \mathbb{R}_+$ the above maximum is attained, there must exist a $0 \leq c_\gamma(m) \leq m$, that satisfies

$$u_0(c_\gamma(m)) + \delta\gamma V(m - c_\gamma(m) + I) = W_\gamma(m).$$

▲ For strict increasingness of W_γ, assume given m' and m'', that satisfy $m' < m''$. We should now show that $W_\gamma(m') < W_\gamma(m'')$. We know that there exists a $c_\gamma(m') \in [0, m']$ that attains the maximum in (*), given m'. Then $0 \le c_\gamma(m') \le m''$ also holds, so that

$$W_\gamma(m') = u_0(c_\gamma(m')) + \delta\gamma V(m' - c_\gamma(m') + I) <$$

$$u_0(c_\gamma(m')) + \delta\gamma V(m'' - c_\gamma(m') + I) \le W_\gamma(m'').$$

The first inequality follows by strict increasingness of V, the last by definition of W_γ. Hence indeed $W_\gamma(m') < W_\gamma(m'')$ holds.

▲ For strict concavity of W_γ, we assume given m' and m'', that satisfy $m' \neq m''$. We should now show that

$$W_\gamma(\lambda m' + (1-\lambda)m'') > \lambda W_\gamma(m') + (1-\lambda)W_\gamma(m''),$$

for any $\lambda \in (0,1)$. We know that there are $c_\gamma(m') \in [0, m']$ and $c_\gamma(m'') \in [0, m'']$ such that

$$W_\gamma(m') = u_0(c_\gamma(m')) + \delta\gamma V(m' - c_\gamma(m') + I)$$

and that

$$W_\gamma(m'') = u_0(c_\gamma(m'')) + \delta\gamma V(m'' - c_\gamma(m'') + I).$$

Here we will introduce two new variables μ and κ by $\mu := \lambda m' + (1-\lambda)m''$, and by $\kappa := \lambda c_\gamma(m') + (1-\lambda)c_\gamma(m'')$. Since $0 \le c_\gamma(m') \le m'$, and $0 \le c_\gamma(m'') \le m''$, we also know that $0 \le \kappa \le \mu$. By definition of W_γ, we know that

$$W_\gamma(\mu) \ge u_0(\kappa) + \delta\gamma V(\mu - \kappa + I).$$

And by strict concavity of u_0 we know that

$$u_0(\kappa) > \lambda u_0(c_\gamma(m')) + (1-\lambda)u_0(c_\gamma(m'')). \tag{1}$$

Similarly, we know that

$$\mu - \kappa + I = \lambda(m' - c_\gamma(m') + I) + (1-\lambda)(m'' - c_\gamma(m'') + I),$$

so that by strict concavity of V we see that

$$V(\mu - \kappa + I) > \lambda V(m' - c_\gamma(m') + I) + (1-\lambda)V(m'' - c_\gamma(m'') + I). \tag{2}$$

Therefore, combining (1) and (2), we get that

$$W_\gamma(\mu) \ge u_0(\kappa) + \delta\gamma V(\mu - \kappa + I) >$$

$$\lambda u_0(c_\gamma(m')) + (1-\lambda)u_0(c_\gamma(m''))+$$

$$\lambda\delta\gamma V(m' - c_\gamma(m') + I) + (1-\lambda)\delta\gamma V(m'' - c_\gamma(m'') + I) =$$

$$\lambda W_\gamma(m') + (1-\lambda)W_\gamma(m'').$$

Hence indeed the desired inequality holds.

▲ For continuous differentiability of $W_\gamma(m)$, we need to show that the function $W_\gamma(m)$ is continuous and that the derivative function $W'_\gamma(m)$ is well-defined and continuous. Here we will first show that $W_\gamma(m)$ is continuous.

Above we saw that for any $I \in \mathbb{R}_+$, any $\gamma \in \mathbb{R}_{++}$ and any $m \in \mathbb{R}_+$, there exists a $0 \le c_\gamma(m) \le m$, that satisfies (*). By continuity of u_0 and V it can be shown that the function $c_\gamma(.)$ will also be continuous in m (see e.g. Luenberger [28], p. 462). Thus, continuity of u_0, V and c_γ also implies continuity of W_γ.

▲ Next, we will show that the derivative function $W'_\gamma(m)$ is well-defined, or in other words that the function $W_\gamma(m)$ is differentiable. It is known that for any function f on $\mathbb{R}_+$ that is continuous in a point $x \in \mathbb{R}_{++}$, this function will also be differentiable in x if both the left-sided derivative $f'_-(x) := \lim_{h\uparrow 0}(f(x+h) - f(x))/h$ and the right-sided derivative $f'_+(x) := \lim_{h\downarrow 0}(f(x+h) - f(x))/h$ in the point x exist, and if they are equal: $f'_-(x) = f'_+(x)$. This property will be used to establish differentiability.

We already saw that the function $c_\gamma(m)$ is continuous. Here we denote the derivative of $c_\gamma(m)$ with respect to m by $c'_\gamma(m)$. If indeed $c'_\gamma(.)$ is well-defined in m, then the derivative $W'_\gamma(.)$ of the function W_γ in the point m can be expressed as

$$W'_\gamma(m) = u'_0(c_\gamma(m))c'_\gamma(m) + \delta\gamma V'(m - c_\gamma(m) + I)(1 - c'_\gamma(m)) =$$

$$\delta\gamma V'(m - c_\gamma(m) + I) + c'_\gamma(m)[u'_0(c_\gamma(m)) - \delta\gamma V'(m - c_\gamma(m) + I)]. \tag{3}$$

Thus if $c'_\gamma(.)$ is well-defined in m, then $W'_\gamma(.)$ is well-defined in m. Note that it still remains to be seen whether this function $c'_\gamma(m)$ is well-defined everywhere. For any given $m > 0$, we will now distinguish five cases with respect to whether $c_\gamma(m) \in [0, m]$ is an internal solution and whether $c_\gamma(m)$ satisfies the first order condition.

(I) $c_\gamma(m) = 0$, and $u'_0(c_\gamma(m)) < \delta\gamma V'(m - c_\gamma(m) + I)$,

(II) $c_\gamma(m) = 0$, and $u'_0(c_\gamma(m)) = \delta\gamma V'(m - c_\gamma(m) + I)$,

(III) $0 < c_\gamma(m) < m$, and $u'_0(c_\gamma(m)) = \delta\gamma V'(m - c_\gamma(m) + I)$,

(IV) $c_\gamma(m) = m$, and $u'_0(c_\gamma(m)) = \delta\gamma V'(m - c_\gamma(m) + I)$,

(V) $c_\gamma(m) = m$, and $u'_0(c_\gamma(m)) > \delta\gamma V'(m - c_\gamma(m) + I)$.

We will see that although $c'_\gamma(m)$ may not be well-defined in cases (II) and (IV), these cases only correspond to single points. And we will see that nevertheless $W'_\gamma(m)$ *is* well-defined in all cases.

If **case (III)** holds for m, then we can directly apply the envelope theorem, which will show that $W'_\gamma(m)$ is well-defined in m, and that

$$W'_\gamma(m) = \left.\frac{\partial[u_0(c) + \delta\gamma V(m - c + I)]}{\partial m}\right|_{c=c_\gamma(m)} = \delta\gamma V'(m - c_\gamma(m) + I).$$

If **case (I)** holds for m, then by continuity of $c_\gamma(.)$, of $u'_0(.)$ and of $V'(.)$, it will also hold that

$$u'_0(c_\gamma(m')) < \delta\gamma V'(m' - c_\gamma(m') + I)$$

for all m' sufficiently close to m. Therefore $c_\gamma(m') = 0$ will also hold for all m' sufficiently close to m, and we see that $c'_\gamma(m) = 0$. Thus, in this case the function $c'_\gamma(.)$ is indeed well-defined in the point m. This also implies that $W'_\gamma(.)$ is well-defined in m, and by (3) we see that $W'_\gamma(m) = \delta\gamma V'(m - c_\gamma(m) + I)$.

If **case (V)** holds for m, then like in case (I), by continuity of $c_\gamma(.)$, of $u'_0(.)$ and of $V'(.)$,

$$u'_0(c_\gamma(m')) > \delta\gamma V'(m' - c_\gamma(m') + I)$$

will also hold for all m' sufficiently close to m. Therefore $c_\gamma(m') = m'$ will hold for all m' sufficiently close to m, and we see that $c'_\gamma(m) = 1$. Thus, the function $c'_\gamma(.)$ is well-defined in the point m, which also implies that $W'_\gamma(.)$ is well-defined in m, and by (3) we see that $W'_\gamma(m) = u'_0(c_\gamma(m))$.

If **case (II)** holds for m, then things get more complicated, as $c'_\gamma(m)$ may not be well-defined. To see this, note that in this case it holds that $u'_0(0) = \delta\gamma V'(m+I)$, so for any $m' < m$, it will hold that $u'_0(0) < \delta\gamma V'(m' + I)$. This implies that $c_\gamma(m') = 0$ and that $c_\gamma(m')$ will be in case (I), for any $m' < m$. Thus the left-sided derivative of c_γ will equal zero. However, there is no reason why the same should hold for the right-sided derivative of c_γ. To see this, for any $m'' > m$, it will hold that $u'_0(0) > \delta\gamma V'(m'' + I)$, which implies that $c_\gamma(m'') > 0$ must be true. Thus, $c_\gamma(m'')$ must belong to one of the last three cases. However, the last two cases require that $u'_0(m'') \geq \delta\gamma V'(I)$, while

$$u'_0(m'') < u'_0(0) = \delta\gamma V'(m + I) \leq \delta\gamma V'(I),$$

so that, for $m'' > m$, $c_\gamma(m'')$ will be in case (III).

Still, in case (II) $W'_\gamma(.)$ *will* be well-defined in m. To see this, note that case (I) will prevail to the left of m, where the function W_γ is differentiable with derivatives $W'_\gamma(m') = \delta\gamma V'(m' - c_\gamma(m') + I)$ for $m' < m$. Therefore as m' tends to m, these derivatives $W'_\gamma(m')$ will converge to the left-sided derivative in m. Similarly, case (III) will prevail to the right of m, and the function W_γ is differentiable to the right of m, with derivatives $W'_\gamma(m'') = \delta\gamma V'(m'' - c_\gamma(m'') + I)$ for $m'' > m$. And as m'' tends to m, these derivatives $W'_\gamma(m'')$ will converge to the right-sided derivative in m. Now, by continuity of $c_\gamma(.)$ and of $V'(.)$ we see that the left-sided derivative equals the right-sided derivative ($= \delta\gamma V'(m - c_\gamma(m) + I)$). Thus indeed, $W'_\gamma(.)$ is well-defined in m, with $W'_\gamma(m) = \delta\gamma V'(m - c_\gamma(m) + I)$.

If **case (IV)** holds for m, then similarly to case (II), $c'_\gamma(m)$ may not be well-defined. In this case it must hold that $u'_0(m) = \delta\gamma V'(I)$, so for any $m' < m$, it will hold that $u'_0(m') > \delta\gamma V'(I)$. This implies that $c_\gamma(m') = m'$ and that $c_\gamma(m')$ will be in case (V), for $m' < m$. Thus the left-sided derivative of c_γ will equal one. However, the right-sided derivative of c_γ need not equal one, as for any $m'' > m$, it will hold that $u'_0(m'') < \delta\gamma V'(I)$, which implies that $c_\gamma(m'') < m''$. Thus, $c_\gamma(m'')$ must belong to one of the first three cases. However, the first two cases require that $u'_0(0) \leq \delta\gamma V'(m'' + I)$, while

$$u'_0(0) \geq u'_0(m) = \delta\gamma V'(I) > \delta\gamma V'(m'' + I),$$

so that, for $m'' > m$, $c_\gamma(m'')$ will be in case (III).

Still, $W'_\gamma(m)$ *is* well-defined in case (IV). Case (V) will prevail to the left of m, where the function W_γ is differentiable, with derivatives $W'_\gamma(m') = u'_0(c_\gamma(m'))$, for $m' < m$.

As m' tends to m, these derivatives $W'_\gamma(m')$ will converge to the left-sided derivative in m. Similarly, case (III) will prevail to the right of m, where W_γ is differentiable, with derivatives $W'_\gamma(m'') = \delta\gamma V'(m'' - c_\gamma(m'') + I)$, for $m'' > m$. As m'' tends to m, these derivatives $W'_\gamma(m'')$ will converge to the right-sided derivative in m. Now, by continuity of $c_\gamma(.)$, of $u'_0(.)$ and of $V'(.)$, the left-sided derivative will equal $u'_0(c_\gamma(m))$, and the right-sided derivative will equal $\delta\gamma V'(m - c_\gamma(m) + I)$. In case (IV) these quantities are known to be equal. Thus, $W'_\gamma(.)$ is well-defined in m, with

$$W'_\gamma(m) = u'_0(c_\gamma(m)) = \delta\gamma V'(m - c_\gamma(m) + I).$$

Hence indeed $W'_\gamma(m)$ is well-defined everywhere (in all cases).

▲ Next, we will show that

$$W'_\gamma(m) = \max\{\delta\gamma V'(m - c_\gamma(m) + I), u'_0(m)\}.$$

We can summarize the above implications for $W'_\gamma(m)$ into

$$W'_\gamma(m) = \begin{cases} \delta\gamma V'(m - c_\gamma(m) + I) & \text{in cases (I), (II), (III) and (IV)} \\ u'_0(m) & \text{in cases (IV) and (V)} \end{cases}$$

Recall that $u'_0(m) = \delta\gamma V'(m - c_\gamma(m) + I)$ holds in case (IV). Therefore, we can also write the derivative as:

$$W'_\gamma(m) = \begin{cases} \delta\gamma V'(m - c_\gamma(m) + I) & \text{if } 0 \leq c_\gamma(m) < m \\ u'_0(m) & \text{if } c_\gamma(m) = m \end{cases} \tag{4}$$

If $0 \leq c_\gamma(m) < m$, then it will also hold that

$$\delta\gamma V'(m - c_\gamma(m) + I) \geq u'_0(c_\gamma(m)) \geq u'_0(m),$$

where the last inequality holds by concavity of u_0.

If $c_\gamma(m) = m$ it will hold that

$$u'_0(m) = u'_0(c_\gamma(m)) \geq \delta\gamma V'(m - c_\gamma(m) + I).$$

Therefore, indeed we get that $W'_\gamma(m) = \max\{\delta\gamma V'(m - c_\gamma(m) + I), u'_0(m)\}$.

▲ Finally then, we need to show that the derivative function $W'_\gamma(m)$ is continuous. This can now easily be seen from the previous formulation as $c_\gamma(.)$, $u'_0(.)$ and $V'(.)$ are also known to be continuous. ■

The next lemma follows up on the previous. It says that for any γ smaller than one the function W_γ (as defined in the above lemma) will be steeper than the function

$$\gamma W_1 = \gamma \cdot \max_{c+s \leq m} u_0(c) + \delta V(s + I),$$

and conversely it says that γW_1 will be steeper than W_γ for γ larger than one.

As mentioned above, we will ultimately use lemma 8.4.1 to show that in certain situations actual choices c^* are larger (smaller) than ex-post optimal choices $c^{\triangleleft}$, so that excess expenditures will be positive (negative), and so that condition 7.3.3 will hold. However, for applying lemma 8.4.1 it is needed that certain V-functions are steeper than others. To prove that (or when) this will hold, the next lemma will be used. We will see shortly that (in certain situations) actual choices c_t^* will solve maximization problems such as $\max_{c_t+s_t\leq m_t} u_0(c_t) + \delta\gamma_t W_1(s_t + I)$, and that ex-post optimal choices $c^{\triangleleft}$ will solve maximization problems such as $\max_{c_t+s_t\leq m_t} u_0(c_t) + \delta W_{\gamma_t}(s_t + I)$. Here the functions W_1 and W_{γ_t} are as in the above lemma. The next lemma shows that for $\gamma_t \leq 1$ it will hold that $W'_{\gamma_t}(.) \geq \gamma_t W'_1(.)$. Then lemma 8.4.1 can in turn be used to show that $c_t^* \geq c_t^{\triangleleft}$, and thus that $E_t \geq 0$. Similarly, for $\gamma \geq 1$ we will get that $E_t \leq 0$.

But first it needs to be established that $\gamma_t \leq 1$ indeed implies that $W'_{\gamma_t}(.) \geq \gamma_t W'_1(.)$, and that $\gamma_t \geq 1$ implies that $W'_{\gamma_t}(.) \leq \gamma_t W'_1(.)$.

Lemma 8.4.3 *Let two functions $u_0 : \mathbb{R}_+ \to \mathbb{R}$ and $V : \mathbb{R}_+ \to \mathbb{R}$ that satisfy axiom 8.1.1, and some scalars $0 < \delta < 1$ and $I \geq 0$ be given. Then, let the functions W_1 and W_γ, for any $\gamma \in \mathbb{R}_{++}$, be defined as in lemma 8.4.2. Then if $\gamma \leq 1$ it will hold that $W'_\gamma(m) \geq \gamma W'_1(m)$, for all $m \in \mathbb{R}_+$, and if $\gamma \geq 1$ it will hold that $W'_\gamma(m) \leq \gamma W'_1(m)$, for all $m \in \mathbb{R}_+$.*

Moreover, we may additionally assume that $\delta V'(I) \leq u'_0(0)$. Then if $\gamma < 1$ it will hold that $W'_\gamma(m) > \gamma W'_1(m)$, for all $m \in \mathbb{R}_{++}$, and if $\gamma > 1$ it will hold that $W'_\gamma(m) < \gamma W'_1(m)$, for all $m \in \mathbb{R}_{++}$.

Proof. Two functions $u_0 : \mathbb{R}_+ \to \mathbb{R}$ and $V : \mathbb{R}_+ \to \mathbb{R}$ that satisfy axiom 8.1.1, and some scalars $0 < \delta < 1$ and $I \geq 0$ are given. The functions W_1 and W_γ, for any $\gamma \in \mathbb{R}_{++}$, are defined as in lemma 8.4.2. By this lemma we know that W_1 and W_γ are well-defined and that they satisfy axiom 8.1.1. This lemma also shows that for any $\gamma \in \mathbb{R}_{++}$, the derivative of W_γ is given by

$$W'_\gamma(m) = \max\{\delta\gamma V'(m - c_\gamma(m) + I), u'_0(m)\},$$

where $c_\gamma(m)$ solves problem (*) from the previous lemma, for any given γ. And, the derivative of γW_1 is given by

$$\gamma W'_1(m) = \max\{\delta\gamma V'(m - c_1(m) + I), \gamma u'_0(m)\},$$

where $c_1(m)$ solves problem (*) from the previous lemma, for $\gamma = 1$.

▲ Now we want to prove that $W'_\gamma(m) \geq \gamma W'_1(m)$ if $\gamma \leq 1$. Recall that $c_\gamma(m)$ maximizes $u_0(c)+\delta\gamma V(m-c+I)$, and that $c_1(m)$ maximizes $u_0(c)+\delta V(m-c+I)$ ($c_1(m)$ also maximizes $\gamma u_0(c)+\delta\gamma V(m-c+I)$) over the same budget set $\{c \in \mathbb{R} : 0 \leq c \leq m\}$. Thus, by lemma 8.4.1 we find that $c_\gamma(m) \geq c_1(m)$ for $\gamma \leq 1$. Therefore we can now distinguish three cases, and we will show that $W'_\gamma(m) \geq \gamma W'_1(m)$ will hold in any case:

(A) $c_1(m) \leq c_\gamma(m) < m$,

(B) $c_1(m) < c_\gamma(m) = m$, and

(C) $c_1(m) = c_\gamma(m) = m$.

In case (A) we have $c_1(m) < m$ and $c_\gamma(m) < m$. Recall from (4) in the proof of the previous lemma that this implies that $W'_\gamma(m) = \delta\gamma V'(m - c_\gamma(m) + I)$ and that $\gamma W'_1(m) = \delta\gamma V'(m - c_1(m) + I)$. Now, we know that $c_\gamma(m) \geq c_1(m)$, so by strict concavity of V we get that

$$\delta\gamma V'(m - c_\gamma(m) + I) \geq \delta\gamma V'(m - c_1(m) + I),$$

and thus that the desired inequality $W'_\gamma(m) \geq \gamma W'_1(m)$ holds.

In case (B) we get that $W'_\gamma(m) = u'_0(m)$ and that $\gamma W'_1(m) = \delta\gamma V'(m - c_1(m) + I)$. Now, by $c_\gamma(m) = m$ we know that $u'_0(c_\gamma(m)) \geq \delta\gamma V'(m - c_\gamma(m) + I)$, and $c_1(m) < c_\gamma(m)$ so like in case (A) we get that

$$u'_0(c_\gamma(m)) \geq \delta\gamma V'(m - c_\gamma(m) + I) \geq \delta\gamma V'(m - c_1(m) + I)$$

which again shows that $W'_\gamma(m) \geq \gamma W'_1(m)$.

In case (C) we have that $W'_\gamma(m) = u'_0(m)$, and that $\gamma W'_1(m) = \gamma u'_0(m)$. Then from $u'_0(.) > 0$ and from the assumption that $\gamma \leq 1$ it immediately follows that $u'_0(m) \geq \gamma u'_0(m)$, and the desired inequality $W'_\gamma(m) \geq \gamma W'_1(m)$ again holds.

Thus indeed inequality $W'_\gamma(m) \geq \gamma W'_1(m)$ holds if $\gamma \leq 1$.

▲ Then to establish that $W'_\gamma(m) \leq \gamma W'_1(m)$ if $\gamma \geq 1$, the proof is almost the same. Again, lemma 8.4.1 shows that $c_\gamma(m) \leq c_1(m)$ for $\gamma \geq 1$, and we distinguish three cases:

(A') $c_\gamma(m) \leq c_1(m) < m$,

(B') $c_\gamma(m) < c_1(m) = m$, and

(C') $c_\gamma(m) = c_1(m) = m$.

In cases (A') and (C') it can be shown that $W'_\gamma(m) \leq \gamma W'_1(m)$ will hold, analogously to cases (A) and (C) above.

In case (B') we have that $W'_\gamma(m) = \delta\gamma V'(m - c_\gamma(m) + I)$ and that $\gamma W'_1(m) = \gamma u'_0(m)$. Then, similarly to case (B) above, we get

$$\delta\gamma V'(m - c_\gamma(m) + I) \leq \delta\gamma V'(m - c_1(m) + I) \leq \gamma u'_0(c_1(m)),$$

which again shows that $W'_\gamma(m) \leq \gamma W'_1(m)$.

Thus, in either case $W'_\gamma(m) \leq \gamma W'_1(m)$ holds if $\gamma \geq 1$, which concludes the proof of the first part of the lemma.

▲ For the second part of the lemma, we additionally assume that $\delta V'(I) \leq u'_0(0)$. Then if $\gamma < 1$, we want to show that $W'_\gamma(m) > \gamma W'_1(m)$, for all $m \in \mathbb{R}_{++}$.

Here we will first show that for all $m > 0$, we will now either get that $c_\gamma(m) > c_1(m)$, or that $c_\gamma(m) = c_1(m) = m$. We will prove this from the last part of lemma 8.4.1 which, in the present lemma's notation, can be applied under the conditions that $\gamma V' < V'$, that $\gamma V'(m + I) < u'_0(0)/\delta$, and that $V'(I) > u'_0(m)/\delta$.

The first condition follows easily from $\gamma < 1$.

The second condition holds as

$$\gamma V'(m + I) < V'(m + I) < V'(I) \leq u'_0(0)/\delta.$$

Here the first inequality follows from $\gamma < 1$, the second from strict concavity of V, and the third from the additional assumption that $\delta V'(I) \leq u_0'(0)$.

However, the third condition $V'(I) > u_0'(m)/\delta$ needed to apply lemma 8.4.1 does generally not hold. Still, from the discussion after lemma 8.4.1 we know that this condition was only needed to exclude the possibility that $c_\gamma(m) = c_1(m) = m$. Therefore, in the present context we see that it will indeed hold that $c_\gamma(m) > c_1(m)$, or that $c_\gamma(m) = c_1(m) = m$.

Now, we will show that the desired inequality $W_\gamma'(m) > \gamma W_1'(m)$ will hold in each of the three possible cases:

(a) $c_1(m) < c_\gamma(m) < m$,

(b) $c_1(m) < c_\gamma(m) = m$, and

(c) $c_1(m) = c_\gamma(m) = m$.

In case (a) we get that $W_\gamma'(m) = \delta\gamma V'(m - c_\gamma(m) + I)$ and similarly that $\gamma W_1'(m) = \delta\gamma V'(m - c_1(m) + I)$, and as in (A) above, by strict concavity we get that

$$W_\gamma'(m) = \delta\gamma V'(m - c_\gamma(m) + I) > \delta\gamma V'(m - c_1(m) + I) = \gamma W_1'(m)$$

follows from the given that $c_\gamma(m) > c_1(m)$.

In case (b) we get that $W_\gamma'(m) = u_0'(m)$ and that $\gamma W_1'(m) = \delta\gamma V'(m - c_1(m) + I)$, and similarly to (B) above, we get that

$$u_0'(m) \geq \delta\gamma V'(m - c_\gamma(m) + I) > \delta\gamma V'(m - c_1(m) + I).$$

Here the last inequality again follows from $c_\gamma(m) = m > c_1(m)$. This again shows that $W_\gamma'(m) > \gamma W_1'(m)$.

In case (c) we know that $W_\gamma'(m) = u_0'(m)$ and that $\gamma W_1'(m) = \gamma u_0'(m)$, so $\gamma < 1$ implies that $u_0'(m) > \gamma u_0'(m)$, and the desired inequality $W_\gamma'(m) > \gamma W_1'(m)$ holds.

Thus indeed the property that $W_\gamma'(m) > \gamma W_1'(m)$ holds if $\gamma < 1$ and if $\delta V'(I) \leq u_0'(0)$.

▲ To conclude the proof, we want to show that $W_\gamma'(m) < \gamma W_1'(m)$, for all $m \in \mathbb{R}_{++}$, under the assumptions that $\gamma > 1$ and that $\delta V'(I) \leq u_0'(0)$.

Similarly to above, applying lemma 8.4.1 would require the conditions $\gamma V' > V'$, $V'(m + I) < u_0'(0)/\delta$, and $\gamma V'(I) > u_0'(m)/\delta$. Again, the first two conditions can easily be seen to hold, while the third generally does not hold, and this will then imply that $c_\gamma(m) < c_1(m)$, or that $c_\gamma(m) = c_1(m) = m$. Then we can again distinguish three cases:

(a') $c_\gamma(m) < c_1(m) < m$,

(b') $c_\gamma(m) < c_1(m) = m$, and

(c') $c_\gamma(m) = c_1(m) = m$.

In cases (a') and (c') it can be shown that $W_\gamma'(m) < \gamma W_1'(m)$ will hold, analogously to cases (a) and (c) above.

In case (b'), we have that $W_\gamma'(m) = \delta\gamma V'(m - c_\gamma(m) + I)$ and that $\gamma W_1'(m) = \gamma u_0'(m)$. Then, we get that

$$\delta\gamma V'(m - c_\gamma(m) + I) < \delta\gamma V'(m - c_1(m) + I) \leq \gamma u_0'(m),$$

which again shows that $W'_\gamma(m) \leq \gamma W'_1(m)$.

Thus, in either case $W'_\gamma(m) < \gamma W'_1(m)$ holds if $\gamma > 1$ and if $\delta V'(I) \leq u'_0(0)$, which concludes the proof. ■

The next lemma will formalize the reasoning that was outlined between the above lemmas, about how we will prove that condition 7.3.3 will hold. However, unlike condition 7.3.3, which is set in terms of adjustment products, the next lemma is set in terms of value functions. It will show that adjustment does work in the right direction. If a value function $V^{(t)} = \gamma_t V^*$ equals a scalar $\gamma_t \in \mathbb{R}_{++}$ times the optimal value function, then for $\gamma_t \leq 1$ the excess expenditure will be non-negative, which will be followed by a heavier weight for savings, so that $\gamma_{t+1} \geq \gamma_t$. And conversely, if $\gamma_t \geq 1$, then the lemma will show that excess expenditure will be non-positive, which will be followed by a smaller weight for savings, so that $\gamma_{t+1} \leq \gamma_t$.

Lemma 8.4.4 *Assume given a model that satisfies axioms 4.4.1 and 8.1.5, and 8.1.3 or 8.1.4, and with an instantaneous utility function that satisfies axiom 8.1.1. Then, for some period t, suppose given a value function $V^{(t)} = \gamma_t V^*$ that equals a scalar $\gamma_t \in \mathbb{R}_{++}$ times the optimal value function V^*. Then for any budget $m_t \geq 0$ it will hold that $\gamma_t \leq 1$ implies that the corresponding (regular) excess expenditure will satisfy $E_t \geq 0$, and $\gamma_t \geq 1$ implies that $E_t \leq 0$.*

Proof. Given is a model with an instantaneous utility function u_0 that satisfies axiom 8.1.1, and a discount factor $0 < \delta < 1$. A constant additional income will be obtained in every period after period 0: $I_t = I \geq 0$, for all $t \geq 1$. Then, in some period $t \geq 0$, given is a value function $V^{(t)} : \mathbb{R}_+ \to \mathbb{R}$ that equals a scalar $\gamma_t \in (0, 1]$ times the optimal value function that solves the functional equation: $V^{(t)}(m) = \gamma_t V^*(m)$, for all $m \in \mathbb{R}_+$. And suppose given some budget $m_t \geq 0$ in period t.

We now want to establish that the (regular) excess expenditure associated with this value function $V^{(t)}$ will be non-negative: $E_t = c^*_t - c^\triangleleft_t \geq 0$. Here c^*_t is part of a pair (c^*_t, s^*_t) that solves

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta\gamma_t V^*(s_t + I). \tag{1}$$

The second term that the excess expenditure depends on is $c^\triangleleft_t$, which is part of a tuple $(c^\triangleleft_t, c^\triangleleft_{t+1}, s^\triangleleft_{t+1})$ that solves

$$\max_{c_t+s_t \leq m_t} \max_{c_{t+1}+s_{t+1} \leq s_t+I} u_0(c_t) + \delta u_0(c_{t+1}) + \delta^2\gamma_t V^*(s_{t+1} + I). \tag{2}$$

The maximization problem in (2) can be rewritten as

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta \cdot \max_{c_{t+1}+s_{t+1} \leq s_t+I} u_0(c_{t+1}) + \delta\gamma_t V^*(s_{t+1} + I). \tag{2'}$$

We now define the function $W_{\gamma_t} : \mathbb{R}_+ \to \mathbb{R}$ as the last part of (2'):

$$W_{\gamma_t}(m_{t+1}) = \max_{c_{t+1}+s_{t+1} \leq m_{t+1}} u_0(c_{t+1}) + \delta\gamma_t V^*(s_{t+1} + I). \tag{3}$$

By lemma 8.4.2 the function W_{γ_t} is indeed a well-defined function, and we can now also write (2') in analogy to (1) as

$$\max_{c_t+s_t\leq m_t} u_0(c_t)+\delta W_{\gamma_t}(s_t+I). \tag{2''}$$

We know that $(c_t^{\triangleleft}, c_{t+1}^{\triangleleft}, s_{t+1}^{\triangleleft})$ solves (2), and now $(c_t^{\triangleleft}, s_t^{\triangleleft})$, with $s_t^{\triangleleft} := m_t - c_t^{\triangleleft} = c_{t+1}^{\triangleleft} + s_{t+1}^{\triangleleft} - I$ (by strict increasingness), must also solve (2"). (By strict concavity all of the above maximization problems have unique solutions.)

From lemma 8.4.2 we know that the function W_{γ_t} satisfies axiom 8.1.1, and note that is an example of a function W_γ, as defined in lemma 8.4.3. Also note that since the function V^* solves the functional equation as in subsection 8.1.2, $\gamma_t V^*$ satisfies

$$\gamma_t V^*(m_{t+1}) = \max_{c_{t+1}+s_{t+1}\leq m_{t+1}} \gamma_t u_0(c_{t+1}) + \delta\gamma_t V^*(s_{t+1}+I), \tag{4}$$

so that it is an example of a function γW_1 as defined in lemma 8.4.3.

Since $\gamma_t \leq 1$ we can apply the previous lemma to obtain that

$$W'_{\gamma_t}(m_{t+1}) \geq \gamma_t W'_1(m_{t+1}) = \gamma_t V^{*\prime}(m_{t+1}),$$

for all $m_{t+1} \in \mathbb{R}_+$.

Then because of the above inequality, and because c_t^* solves (1), and $c_t^{\triangleleft}$ solves (2"), we can apply lemma 8.4.1 to obtain that $c_t^* \geq c_t^{\triangleleft}$. Hence indeed we find that excess expenditure $E_t = c_t^* - c_t^{\triangleleft} \geq 0$ is non-negative.

▲ For $\gamma_t \in [1,\infty)$ the proof follows the same pattern, while reversing the inequalities. Then, with $\gamma_t \geq 1$ the preceding lemma will show that $W'_{\gamma_t}(m_{t+1}) \leq \gamma_t V^{*\prime}(m_{t+1})$, for all $m_{t+1} \in \mathbb{R}_+$. Subsequently, we can again apply lemma 8.4.1 to obtain that $c_t^* \leq c_t^{\triangleleft}$. Therefore, excess expenditure $E_t = c_t^* - c_t^{\triangleleft} \leq 0$ will indeed be non-positive. ■

If there exists a consistency-inducing scalar $\widetilde{\theta}$, then with the above lemma we also get that $\theta_t \geq \widetilde{\theta}$ implies that $E_t \geq 0$, and that $\theta_t \leq \widetilde{\theta}$ implies that $E_t \leq 0$.

To see this, suppose that a CIS $\widetilde{\theta}$ exists, so that by proposition 8.3.1 the initial value function $V^{(0)}$ must be of the form: $V^{(0)}(m) = \widetilde{\theta} V^*(m) + \alpha$, for some $\alpha \in \mathbb{R}$. If the value function $V^{(t)}$ is generated by the learning algorithm from $V^{(0)}$, it will satisfy $V^{(t)} = \theta_t^{-1} V^{(0)} = \theta_t^{-1}(\widetilde{\theta} V^* + \alpha)$. If we now define $\gamma_t := \theta_t^{-1}\widetilde{\theta}$ and $\alpha_t := \theta_t^{-1}\alpha$, then this generated value function can also be written as $V^{(t)} = \gamma_t V^* + \alpha_t$.

Then suppose that $\theta_t \geq \widetilde{\theta}$ holds, which implies that $\gamma_t \leq 1$. By proposition 8.4.2 the constant α_t does not influence the excess expenditure that will result from $V^{(t)}$, and by the previous lemma we see that $\gamma_t \leq 1$ implies that $E_t \geq 0$. Hence indeed $\theta_t \geq \widetilde{\theta}$ implies that $E_t \geq 0$. Reversing the inequalities will similarly show that $\theta_t \leq \widetilde{\theta}$ implies that $E_t \leq 0$.

Note that the above implications almost specify condition 7.3.3, except that all the above inequalities would have to be replaced by strict inequalities. Hence, while the previous lemma is useful, it is not sufficient for establishing convergence towards optimality. What is needed is a strengthening of this lemma, where the inequalities can be made strict: so that $\gamma_t > 1$ would imply that $E_t < 0$ (and thus that $\gamma_{t+1} < \gamma_t$), and so that $\gamma_t < 1$ would imply that $E_t > 0$ (and thus that $\gamma_{t+1} > \gamma_t$). In that case condition 7.3.3 *would* hold, and convergence towards optimality *would* occur.

Thus far in this chapter we have treated all models under certainty simultaneously. All results apply for both models *with* and models *without* an income steam. To provide such a strengthening of the above lemma, and thus to establish if and when convergence towards optimality will take place, we will consider the two cases of models with and without income streams separately, as the implications of the two cases are quite different.

8.4.1 Models with no income stream

We start with the simpler case of models where axiom 8.1.3 holds, so that no additional income is received after the first period. In that setting, the additional assumption that instantaneous utility u_0 satisfies axiom 8.1.2 instead of axiom 8.1.1 (so that $u_0'(0) = \infty$ additionally holds) is sufficient for proving that the next lemma holds. This lemma provides the required strengthening of the previous lemma, where all the inequalities are strict, which will imply that condition 7.3.3 will hold.

Lemma 8.4.5 *Let a model be given that satisfies axioms 4.4.1, 8.1.3 and 8.1.5, and with an instantaneous utility function that satisfies axiom 8.1.2. Then, for some period t, suppose given a value function $V^{(t)} = \gamma_t V^*$ that equals a scalar $\gamma_t \in \mathbb{R}_{++}$ times the optimal value function V^*. Then for any budget $m_t > 0$ it will hold that $\gamma_t < 1$ implies that the corresponding (regular) excess expenditure will satisfy $E_t > 0$, and $\gamma_t > 1$ implies that $E_t < 0$.*

Proof. Given is a model with an instantaneous utility function u_0 that satisfies axiom 8.1.1 and $u_0'(0) = \infty$, a discount factor $0 < \delta < 1$, and where no additional income is obtained after period 0: $I_t = 0$, for all $t \geq 1$. Then, for some period t, suppose given a period-t value function $V^{(t)} : \mathbb{R}_+ \to \mathbb{R}$ that equals a scalar $\gamma_t \in \mathbb{R}_{++}$ times the optimal value function that solves the functional equation: $V^{(t)}(m) = \gamma_t V^*(m)$, for all $m \in \mathbb{R}_+$. And suppose given some budget $m_t > 0$ in period t.

▲ First suppose that $\gamma_t \in (0, 1)$. We now want to determine that the excess expenditure associated with this value function $V^{(t)}$ will be strictly positive. This proof will follow the same pattern as that of the previous lemma. Like in that proof we will use lemmas 8.4.3 and 8.4.1 to arrive at the desired result. Here, however, we are interested in strict inequalities, so we will need to use the second parts of these lemmas.

Excess expenditure is determined by $E_t = c_t^* - c_t^{\triangleleft}$, where c_t^* is part of a pair (c_t^*, s_t^*) that solves

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \gamma_t V^*(s_t). \tag{1}$$

Then, $c_t^{\triangleleft}$ is determined as part of a tuple $(c_t^{\triangleleft}, c_{t+1}^{\triangleleft}, s_{t+1}^{\triangleleft})$ that solves

$$\max_{c_t+c_{t+1}+s_{t+1}\leq m_t} u_0(c_t) + \delta u_0(c_{t+1}) + \delta^2\gamma_t V^*(s_{t+1}). \tag{2}$$

This maximization problem can be rewritten as

$$\max_{c_t+s_t\leq m_t} u_0(c_t) + \delta \cdot \max_{c_{t+1}+s_{t+1}\leq s_t} u_0(c_{t+1}) + \delta\gamma_t V^*(s_{t+1}). \tag{2'}$$

If we write the function $W_{\gamma_t} : \mathbb{R}_+ \to \mathbb{R}$ as the last part of (2'),

$$W_{\gamma_t}(m_{t+1}) = \max_{c_{t+1}+s_{t+1}\leq m_{t+1}} u_0(c_{t+1}) + \delta\gamma_t V^*(s_{t+1}), \tag{3}$$

then we can also write (2') in analogy to (1) as

$$\max_{c_t+s_t\leq m_t} u_0(c_t) + \delta W_{\gamma_t}(s_t). \tag{2''}$$

Like in the proof of the previous lemma, we know that W_{γ_t} is well-defined, and that it satisfies axiom 8.1.1. And since $(c_t^{\triangleleft}, c_{t+1}^{\triangleleft}, s_{t+1}^{\triangleleft})$ solves (2), the pair $(c_t^{\triangleleft}, s_t^{\triangleleft})$, with $s_t^{\triangleleft} := m_t - c_t^{\triangleleft} = c_{t+1}^{\triangleleft} + s_{t+1}^{\triangleleft}$ (by strict increasingness), will also solve (2"). (By strict concavity all of the above maximization problems have unique solutions.)

Now, we want to show that it holds that $E_t = c_t^* - c_t^{\triangleleft} > 0$. Recall that c_t^* solves (1), and that $c_t^{\triangleleft}$ solves (2"). Therefore, we can apply the second part of lemma 8.4.1 to obtain that $c_t^* > c_t^{\triangleleft}$, if the functions $\gamma_t V^*$ and W_{γ_t} satisfy the following conditions:

(I) $\gamma_t V^{*\prime}(m_{t+1}) < W_{\gamma_t}'(m_{t+1}), \forall m_{t+1}$,

(II) $\delta\gamma_t V^{*\prime}(m_t) < u_0'(0)$,

(III) $\delta W_{\gamma_t}'(0) > u_0'(m_t)$.

(III) For the last condition needed to apply the lemma, recall from lemma 8.4.2 that $W_{\gamma_t}'(0) \geq u_0'(0)$. Thus $u_0'(0) = \infty$ also implies that $W_{\gamma_t}'(0) = \infty$. The function $u_0(.)$ is differentiable on the interval $(0, \infty)$, which means that the derivative function $u_0'(.)$ is real-valued on this interval. Thus by $m_t > 0$ we get that $u_0'(m_t) \in \mathbb{R}$, and indeed we see that $u_0'(m_t) < \delta W_{\gamma_t}'(0) = \infty$, so that this condition is satisfied.

(II) The second of the conditions needed to apply the lemma reads $\delta\gamma_t V^{*\prime}(m_t) < u_0'(0)$. In subsection 8.1.2 we saw that if u_0 satisfies axiom 8.1.1, so will the value function V^*, which implies that $V^*(.)$ will be differentiable on the interval $(0, \infty)$. Therefore by $m_t > 0$ we get that $V^{*\prime}(m_t) \in \mathbb{R}$, and indeed we see that $\delta\gamma_t V^{*\prime}(m_t) < u_0'(0) = \infty$, and this second condition is satisfied.

(I) The first condition needed for the lemma reads $\gamma_t V^{*\prime}(m_{t+1}) < W_{\gamma_t}'(m_{t+1})$, for all $m_{t+1} \in \mathbb{R}_+$. To show that this condition will indeed be satisfied, we want to apply the second part of lemma 8.4.3. And as in the proof of the preceding lemma, the function W_{γ_t} is an example of a function W_γ, as defined in lemma 8.4.3, and the function $\gamma_t V^*$ satisfies

$$\gamma_t V^*(m_{t+1}) = \max_{c_{t+1}+s_{t+1}\leq m_{t+1}} \gamma_t u_0(c_{t+1}) + \delta\gamma_t V^*(s_{t+1}), \tag{4}$$

so that it is an example of a function γW_1, as defined in lemma 8.4.3. Since $\gamma_t < 1$, the second part of lemma 8.4.3 can be applied if the additional assumption that $\delta V'(0) \leq u_0'(0)$ is satisfied. This additional assumption is simply implied by $u_0'(0) = \infty$. Thus, lemma 8.4.3 applies to show that

$$W_{\gamma_t}'(m_{t+1}) > \gamma_t V^{*\prime}(m_{t+1}),$$

for all $m_{t+1} \in \mathbb{R}_+$.

Thus indeed all three conditions needed for lemma 8.4.1 are satisfied, and this lemma shows that $E_t = c_t^* - c_t^{\triangleleft} > 0$.

▲ For $\gamma_t \in (1, \infty)$ the proof follows a similar pattern. We then want to use lemma 8.4.1 to show that $c_t^* < c_t^{\triangleleft}$. This lemma can be applied if the functions $\gamma_t V^*$ and W_{γ_t} satisfy the following conditions:

(I') $\gamma_t V^{*\prime}(m_{t+1}) > W_{\gamma_t}'(m_{t+1}),\ \forall m_{t+1}$,

(II') $\delta W_{\gamma_t}'(m_t) < u_0'(0)$,

(III') $\delta \gamma_t V^{*\prime}(0) > u_0'(m_t)$.

(III') For the third condition, since V^* solves the functional equation, from lemma 8.4.2 we know that $V^{*\prime}(0) \geq u_0'(0) = \infty$. And by differentiability of u_0, $m_t > 0$ implies that $u_0'(m_t) \in \mathbb{R}$, so that this condition holds.

(II') For the second condition, by lemma 8.4.2 we know that $W_{\gamma_t}'(m_t)$ satisfies axiom 8.1.1, so that it is differentiable on $\mathbb{R}_{++}$. Thus, by $m_t > 0$ we see that $\delta W_{\gamma_t}'(m_t) < \infty = u_0'(0)$, and the second condition is satisfied.

(I') To show that the first condition will indeed be satisfied, we want to apply the second part of lemma 8.4.3. Since $\gamma_t < 1$, this second part of lemma 8.4.3 can be applied if the additional assumption that $\delta V'(0) \leq u_0'(0)$ is satisfied. This additional assumption was already shown to hold in (I), so lemma 8.4.3 does show that $W_{\gamma_t}'(m_{t+1}) < \gamma_t V^{*\prime}(m_{t+1})$, for all $m_{t+1} \in \mathbb{R}_+$.

Indeed, all three conditions needed for lemma 8.4.1 are satisfied, and this lemma then shows that $E_t = c_t^* - c_t^{\triangleleft} < 0$. ■

One of the conditions that are needed to prove this lemma is that marginal instantaneous utility becomes infinitely large near zero: $u_0'(0) = \infty$. This condition ensures that the actual expenditures c_t^* and the ex-post optimal expenditures $c_t^{\triangleleft}$ will always be strictly positive: $c_t^* > 0$ and $c_t^{\triangleleft} > 0$ (see the definition of regular excess expenditure in subsection 6.4.2). And the optimal value function solves the functional equation, so by lemma 8.4.2 $u_0'(0) = \infty$ also implies that $V^{*\prime}(0) = \infty$. Then if $V^{(t)} = \gamma_t V^*$, it will also hold that $V^{(t)\prime}(0) = \infty$, which would imply that $c_t^* < m_t$. Similarly, with lemma 8.4.2 we see that $u_0'(0) = \infty$ implies that $W_{\gamma_t}'(0) = \max\{\delta \gamma_t V^{*\prime}(0), u_0'(0)\} = \infty$, and thus that $c_t^{\triangleleft} < m_t$. Thus in the present case (with $I = 0$), from $u_0'(0) = \infty$ we get that c_t^* and $c_t^{\triangleleft}$ will always be internal solutions, which essentially enables us to establish the above result.

With the previous lemma we can now establish that convergence towards optimality will occur, if the initial value function is an affine transformation of the optimal value function that solves the functional equation: $V^{(0)} = \gamma_0 V^* + \alpha_0$, for some γ_0 and some α_0.

Proposition 8.4.4 *Let a model be given that satisfies axioms 4.1.1, 4.4.1, 8.1.3, 8.1.5 and 8.1.6, and with an instantaneous utility function that satisfies axiom 8.1.2. Suppose that the initial budget m_0 is strictly positive, and that the initial value function is an affine transformation of the optimal value function: $V^{(0)} = \gamma_0 V^* + \alpha_0$, for some $\gamma_0 \in \mathbb{R}_{++}$ and some $\alpha_0 \in \mathbb{R}$. If subsequently the learning algorithm is used, then convergence towards optimality will occur*

Proof. Given is a model with a discount factor $0 < \delta < 1$, and an instantaneous utility function u_0 that satisfies axiom 8.1.1 and $u_0'(0) = \infty$. No additional income is obtained after period 0: $I_t = 0$, for all $t \geq 1$. Also given are an initial budget $m_0 > 0$ and an initial value function $V^{(0)}$, that is an affine transformation of the optimal value function that solves the functional equation: $V^{(0)} = \gamma_0 V^* + \alpha_0$, for some $\gamma_0 \in \mathbb{R}_{++}$, and some $\alpha_0 \in \mathbb{R}$. Subsequently, the learning algorithm (with min-max adjustment[32]) is applied, and this learning algorithm will generate a sequence of value functions $(V^{(t)})_{t=0}^{\infty}$ from the initial value function $V^{(0)}$.

Any two subsequent value functions from such a sequence are related according to $V^{(t)}(m) = a_{\varepsilon_t}^{-1} V^{(t-1)}(m)$, where a_{ε_t} is the adjustment factor as determined by min-max adjustment. Consequently, any value function from such a sequence can be written as

$$V^{(t)}(m) = a_{\varepsilon_t}^{-1} \cdot a_{\varepsilon_{t-1}}^{-1} \cdot \ldots \cdot a_{\varepsilon_1}^{-1} \cdot V^{(0)}(m) = \theta_t^{-1} V^{(0)}(m) = \theta_t^{-1}(\gamma_0 V^*(m) + \alpha_0),$$

where θ_t denotes the adjustment product. If we define γ_t and α_t by $\gamma_t := \theta_t^{-1}\gamma_0$ and by $\alpha_t := \theta_t^{-1}\alpha_0$, then we see that any value function $V^{(t)}$ will also be an affine transformation of the optimal value function that solves the functional equation: $V^{(t)} = \gamma_t V^* + \alpha_t$.

By proposition 7.1.1 we know that convergence will always occur. Here we want to apply proposition 7.3.1 in order to show that convergence towards optimality will occur, so we need to show that the conditions 7.3.1, 7.3.2 and 7.3.3 are satisfied. By proposition 8.3.1 we know that in the present setting (with $V^{(0)} = \gamma_0 V^* + \alpha_0$) γ_0 is a consistency-inducing scalar, so that condition 7.3.1 holds. Also, from proposition 8.4.3 we know that in the present setting condition 7.3.2 (that setting $\theta_t = \gamma_0$ will yield $E_t = 0$) holds. Thus what remains to be shown is that condition 7.3.3 is satisfied.

That is, we need to establish that for any time t, setting $\theta_t > \gamma_0$ will yield $E_t > 0$ for all m_t, and that setting $\theta_t < \gamma_0$ yield $E_t < 0$ for all m_t.

Now, $\theta_t > \gamma_0$ would imply that $\gamma_t = \theta_t^{-1}\gamma_0 < 1$, so with proposition 8.4.2 and with the previous lemma we know that the excess expenditure corresponding to γ_t will be strictly positive, as long as $m_t > 0$ is satisfied. Similarly, $\theta_t < \gamma_0$ would imply

[32]For any $\rho > 1$, $\sigma < 1$ and $\lambda \in (0, 1)$.

that $\gamma_t = \theta_t^{-1}\gamma_0 > 1$, and the previous lemma shows that the excess expenditure corresponding to γ_t will be strictly negative, as long as $m_t > 0$ is satisfied.[33]

Therefore, here it only remains to be shown that for any given $m_0 > 0$, the sequence of budgets $(m_t)_{t=0}^{\infty}$ that will result from the choices as made by a decision-maker who follows the learning algorithm, will be such that $m_t > 0$ for all t. To show that this holds, we will use an induction argument. For $t = 0$, it was already given that $m_0 > 0$. Thus what is left to prove, is that $m_t > 0$ implies that $m_{t+1} > 0$. Suppose that $m_t > 0$. Then in period t, the consumption choice c_t^* that will be chosen, will solve

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta V^{(t)}(s_t).$$

Now, $V^{(t)\prime}(0) = \theta_t^{-1}\gamma_0 V^{*\prime}(0)$, and from the proof of the previous lemma, we know that $V^{*\prime}(0) = \infty$. Thus $V^{(t)\prime}(0) = \infty$, which implies that $c_t^* < m_t$ must hold. Therefore we get that $m_{t+1} = s_t^* = m_t - c_t^* > 0$. Thus we have established that $m_t > 0$ for all t.

Therefore, conditions 7.3.1, 7.3.2 and 7.3.3 are indeed satisfied, and proposition 7.3.1 shows that the sequence of adjustment products $(\theta_t)_{t=1}^{\infty}$ converges to $\tilde{\theta} = \gamma_0$. Hence convergence towards optimality occurs. ■

8.4.2 Models with an income stream

In this section we will investigate convergence towards optimality in situations where axiom 8.1.4 holds, so where an initial endowment $m_0 \geq 0$ is given and where in all later periods the same additional income $I_t = I > 0$ is obtained. In this situation results will be very different from those in situations without an income stream.

In models like these, we will see that convergence towards optimality will generally not occur. However, if the initial value function is an affine transformation of the optimal value function, then any sequence of value functions that is generated by the learning algorithm from this initial value function, will converge to a limit function that will lie in some 'neighbourhood' of the optimal value function.

To show this, first we need a lemma. This lemma is somewhat like lemma 8.4.4 from the previous subsection, although the implications of the next lemma are not as strong as those of lemma 8.4.4.

Lemma 8.4.6 *Assume given a model that satisfies axioms 4.4.1, 8.1.4 and 8.1.5, and with an instantaneous utility function that satisfies axiom 8.1.2. Then, in some period t, suppose given a budget $m_t \geq I$, and a value function $V^{(t)} = \gamma_t V^*$ that equals a scalar $\gamma_t > \delta^{-1}$ times the optimal value function V^*. Then (regular) excess expenditure will be strictly negative $E_t < 0$.*

Proof. Given is a model with an instantaneous utility function u_0 that satisfies axiom 8.1.1 and $u_0'(0) = \infty$, and a discount factor $0 < \delta < 1$. In every period after period

[33] For $m_t = 0$ we will always get that excess expenditure equals zero, irrespective of θ_t (or of γ_t). Therefore, if we can show that $m_t = 0$ will never occur in the present context, then $E_t = 0$ will only occur for $\theta_t = \gamma_0$, so that $\theta_t > \gamma_0$ will always yield $E_t > 0$, and $\theta_t < \gamma_0$ will always yield $E_t < 0$.

0 a strictly positive constant additional income is obtained: $I_t = I > 0$, for all $t \geq 1$. Then, in some period t, a budget $m_t \geq I$ is given, and the period-t value function $V^{(t)} : \mathbb{R}_+ \to \mathbb{R}$ equals a scalar $\gamma_t > \delta^{-1}$ times the optimal value function that solves the functional equation: $V^{(t)} = \gamma_t V^*$.

We now want to show that excess expenditure will be strictly negative: $E_t = c_t^* - c_t^{\triangleleft} < 0$. Again, the proof of this lemma follows a familiar pattern. The variable c_t^* denotes the actual period-t choice as derived from

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \gamma_t V^*(s_t + I). \tag{1}$$

Next, in period $t+1$, $c_t^{\triangleleft}$ is determined as part of a tuple $(c_t^{\triangleleft}, c_{t+1}^{\triangleleft}, s_{t+1}^{\triangleleft})$ that solves

$$\max_{c_t + s_t \leq m_t} \max_{c_{t+1} + s_{t+1} \leq s_t + I} u_0(c_t) + \delta u_0(c_{t+1}) + \delta^2 \gamma_t V^*(s_{t+1} + I) = \tag{2}$$

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \cdot \max_{c_{t+1} + s_{t+1} \leq s_t + I} u_0(c_{t+1}) + \delta \gamma_t V^*(s_{t+1} + I). \tag{2'}$$

If we now write the function $W_{\gamma_t} : \mathbb{R}_+ \to \mathbb{R}$ as the last part of (2'):

$$W_{\gamma_t}(m_{t+1}) = \max_{c_{t+1} + s_{t+1} \leq m_{t+1}} u_0(c_{t+1}) + \delta \gamma_t V^*(s_{t+1} + I), \tag{3}$$

then we can also write (2') in analogy to (1) as

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta W_{\gamma_t}(s_t + I). \tag{2''}$$

Since $(c_t^{\triangleleft}, c_{t+1}^{\triangleleft}, s_{t+1}^{\triangleleft})$ solves (2), the pair $(c_t^{\triangleleft}, s_t^{\triangleleft})$, with $s_t^{\triangleleft} := m_t - c_t^{\triangleleft} = c_{t+1}^{\triangleleft} + s_{t+1}^{\triangleleft}$ will also solve (2"). (By strict concavity the above maximization problems have unique solutions.)

We now want to show that $c_t^* < c_t^{\triangleleft}$ by showing that the functions $\gamma_t V^*$ and W_{γ_t} satisfy the conditions required for the second part of lemma 8.4.1. That is, we need that

(I) $\gamma_t V^{*\prime}(m_{t+1}) > W'_{\gamma_t}(m_{t+1}), \forall m_{t+1}$,
(II) $\delta W'_{\gamma_t}(m_t + I) < u'_0(0)$,
(III) $\delta \gamma_t V^{*\prime}(I) > u'_0(m_t)$.

(III) For the third condition, recall that V^* solves the functional equation (such as in subsection 8.1.2), so that by lemma 8.4.2 we know that $V^{*\prime}(I) \geq u'_0(I)$. Thus, we see that $\delta \gamma_t V^{*\prime}(I) > V^{*\prime}(I) \geq u'_0(I) \geq u'_0(m_t)$. Here the first inequality follows from the assumption that $\gamma_t > \delta^{-1}$, and the last inequality follows from strict concavity and from $I \leq m_t$. Therefore this third condition is satisfied.

(II) For the second condition, since both u_0 and V^* satisfy lemma 8.1.1, by lemma 8.4.2 we know that the function W_{γ_t} also satisfies lemma 8.1.1. Therefore W_{γ_t} is differentiable on $\mathbb{R}_{++}$, and by $m_t + I > 0$ we get that $W'_{\gamma_t}(m_t + I) < \infty = u'_0(0)$, and we see that the second condition is always satisfied.

(I) The first condition needed for the lemma reads $\gamma_t V^{*\prime}(m_{t+1}) > W'_{\gamma_t}(m_{t+1})$, for all $m_{t+1} \in \mathbb{R}_{++}$. To show that this condition will indeed be satisfied, we want to apply the second part of lemma 8.4.3. And as in previous proofs, the function W_{γ_t} is an example of a function W_γ, as defined in lemma 8.4.3, and V^* solves a functional equation, so that $\gamma_t V^*$ is an example of a function γW_1, as defined in lemma 8.4.3. Now, since $\gamma_t > \delta^{-1} > 1$, the second part of lemma 8.4.3 can be applied if the additional assumption $\delta V'(I) \leq u'_0(0)$ is satisfied. This additional assumption is simply implied by $u'_0(0) = \infty$. Thus, lemma 8.4.3 does apply to show that

$$W'_{\gamma_t}(m_{t+1}) < \gamma_t V^{*\prime}(m_{t+1}),$$

for all $m_{t+1} \in \mathbb{R}_+$.

And indeed all three conditions needed for lemma 8.4.1 are satisfied, and as desired, this lemma then shows that excess expenditure is strictly negative: $E_t = c_t^* - c_t^{\triangleleft} < 0$. ■

In the previous section, where we dealt with models where no additional income is obtained, we saw that the condition $u'_0(0) = \infty$ ensures that actual expenditures and ex-post optimal expenditures will be strictly positive $c_t^* > 0$ and $c_t^{\triangleleft} > 0$. Also, $u'_0(0) = \infty$ implied that $V^{(t)\prime}(0) = \infty$ and $W'_{\gamma_t}(0) = \infty$, so that $c_t^* < m_t$ and $c_t^{\triangleleft} < m_t$ would always hold. Thus, in settings without additional income, the condition $u'_0(0) = \infty$ ensured that c_t^* and $c_t^{\triangleleft}$ would always be internal solutions.

In the present setting, where additional income is strictly positive, if $u'_0(0) = \infty$ then by a similar reasoning the first conclusion that $c_t^* > 0$ and $c_t^{\triangleleft} > 0$, will still hold. However, $c_t^* < m_t$ and $c_t^{\triangleleft} < m_t$ can no longer be guaranteed to hold if $I > 0$. This is also the reason why in the present setting a converse of the statement in the previous lemma, where it would always hold that excess expenditure would be strictly positive $E_t > 0$ for all γ_t small enough and for all m_t, will not hold.

To see that c_t^* and $c_t^{\triangleleft}$ need not be internal solutions here, suppose that the function $V^{(t)}$ can indeed be written as $V^{(t)} = \gamma_t V^*$, for some $\gamma_t < 1$. Then, $E_t > 0$ can only be shown to always hold, if the possibility of $c_t^* = c_t^{\triangleleft} = m_t$ can be ruled out. However, $c_t^* = c_t^{\triangleleft} = m_t$ will hold if $c_t^{\triangleleft} = m_t$, which will hold if $\delta W'_{\gamma_t}(I) \leq u'_0(m_t)$. By lemma 8.4.2 this inequality can be expanded to

$$\delta \cdot \max\{\delta \gamma_t V^{(t)\prime}(I - c_{\gamma_t}(I) + I), u'_0(I)\} \leq u'_0(m_t).$$

Now, $c_\gamma(I) \in [0, I]$, so that $I - c_\gamma(I) + I \geq I > 0$. Like $V^{(0)}$, the function $V^{(t)}$ satisfies axiom 8.1.1, so it is differentiable, and $V^{(t)\prime}(I - c_\gamma(I) + I) < \infty$. And $I > 0$ similarly implies that $u'_0(I) < \infty$, so we see that $\delta W'_{\gamma_t}(I) < \infty$. Therefore, $u'_0(0) = \infty$ implies that for small enough budgets, $\delta W'_{\gamma_t}(I) \leq u'_0(m_t)$ will indeed hold, and $c_t^* = c_t^{\triangleleft} = m_t$ will occur. Thus indeed here the problem arises from $I > 0$, which implies that $\delta W'_{\gamma_t}(I) < \infty$.

By proposition 7.1.1 we know that any sequence of value functions that is generated by the learning algorithm from an initial value function, will converge. However, since in the present setting a converse of the statement in the previous lemma will not hold, convergence towards optimality will generally not occur. The next proposition shows that if the initial value function is an affine transformation of the optimal value function, then the limit function, towards which convergence will occur, will lie in some 'neighbourhood' of the optimal value function that solves the functional equation. That is, convergence will occur to some value function $V^\infty = \gamma_\infty V^* + \alpha_\infty$, where the coefficient γ_∞ will belong to a neighbourhood $[0, \delta^{-1}]$ of one. Still, a proposition in the next section will show that in this case the corresponding sequence of choices that a learning ad hoc utility maximizer would make, *will* converge to the same limit as the sequence of choices that would be made by a rational utility maximizer.

Proposition 8.4.5 *Let a model be given that satisfies axioms 4.1.1, 4.4.1, 8.1.4, 8.1.5 and 8.1.6, and with an instantaneous utility function that satisfies axiom 8.1.2. Suppose that the initial budget is large enough $m_0 \geq I$, and that the initial value function is an affine transformation of the optimal value function: $V^{(0)} = \gamma_0 V^* + \alpha_0$, for some $\gamma_0 \in \mathbb{R}_{++}$ and some $\alpha_0 \in \mathbb{R}$. If subsequently the learning algorithm is used, then the generated sequence of value functions $(V^{(t)})_{t=0}^\infty$ will converge to some function $V^\infty = \gamma_\infty V^* + \alpha_\infty$, with $\gamma_\infty \in [0, \delta^{-1}]$ and $\alpha_\infty = \alpha_0 \gamma_\infty / \gamma_0$.*

Proof. A model is given with a discount factor $0 < \delta < 1$, and an instantaneous utility function u_0 that satisfies axiom 8.1.1 and $u_0'(0) = \infty$. A strictly positive constant additional income is obtained every period after time 0: $I_t = I > 0$, for all $t \geq 1$, and the initial budget is sufficiently large: $m_0 \geq I$.

The initial value function $V^{(0)}$ is such that $V^{(0)} = \gamma_0 V^* + \alpha_0$, for some $\gamma_0 \in \mathbb{R}_{++}$, and some $\alpha_0 \in \mathbb{R}$, and subsequently the learning algorithm with min-max adjustment[34] is used. By construction of the learning algorithm and the adjustment function, any resulting value function $V^{(t)}$ would satisfy $V^{(t)} = \theta_t^{-1} V^{(0)}$, where θ_t denotes the adjustment product. If we define γ_t and α_t by $\gamma_t := \theta_t^{-1}\gamma_0$ and by $\alpha_t := \theta_t^{-1}\alpha_0$, then we see that each of the generated value functions $V^{(t)}$ will also be an affine transformation of the optimal value function: $V^{(t)} = \gamma_t V^* + \alpha_t$.

By proposition 7.1.1 the sequence of adjustment products $(\theta_t)_{t=0}^\infty$ will always converge to some $\theta_\infty \in \overline{\mathbb{R}}_+$. Consequently, the sequence $(\gamma_t)_{t=0}^\infty$ will converge to some $\gamma_\infty := \theta_\infty^{-1}\gamma_0 \in \overline{\mathbb{R}}_+$ (we suppose that $\gamma_\infty = \infty$ if $\theta_\infty = 0$, and that $\gamma_\infty = 0$ if $\theta_\infty = \infty$), and the sequence of value functions $V^{(t)} = \gamma_t V^* + \alpha_t$ will converge to some function $V^\infty = \gamma_\infty V^* + \alpha_\infty$, with $\alpha_\infty := \theta_\infty^{-1}\alpha_0$.

What remains to be shown here is that $\theta_\infty \geq \delta\gamma_0$, and thus that $0 \leq \gamma_\infty \leq \delta^{-1}$ will now hold. We will prove this in a way that is somewhat similar to the proof of proposition 7.1.1, where we derived convergence of adjustment products θ_t from convergence of the variables $\theta_{\min}^{\mathcal{E}+}(\eta_{t+1}) = \min_{\theta_i \in \Theta^{\mathcal{E}+}(\eta_{t+1})} \theta_i$ and $\theta_{\max}^{\mathcal{E}-}(\eta_{t+1}) = \max_{\theta_i \in \Theta^{\mathcal{E}-}(\eta_{t+1})} \theta_i$. For any $\theta_t < \delta\gamma_0$ it will hold that $\gamma_t > \delta^{-1}$, and the previous lemma shows that in this case the corresponding excess expenditure will be strictly negative. (Since $m_0 \geq I$, and since I will be obtained in every subsequent period, all budgets m_t that will be faced

[34] For some $\rho > 1$, $\sigma < 1$ and $0 < \lambda < 1$.

during the learning process will also satisfy $m_t \geq I$, so that the previous lemma can indeed be applied.) Therefore we see that the variable $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1}) \geq \delta\gamma_0$ must hold for all t. This has two implications. Firstly, that convergence of $(\theta_t)_{t=0}^{\infty}$ towards $\theta_\infty = 0$ is impossible, as by the proof of proposition 7.1.1 we know that this would require that the variable $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$ would converge towards zero. And secondly, that convergence towards a $\theta_\infty \in (0, \delta\gamma_0)$ cannot happen, since by the proof of proposition 7.1.1 we know that this would require that the variables $\theta^{\mathcal{E}+}_{\min}(\eta_{t+1})$ and $\theta^{\mathcal{E}-}_{\max}(\eta_{t+1})$ would both converge towards θ_∞.

Hence indeed $\theta_\infty \geq \delta\gamma_0$ must hold, and we get that $0 \leq \gamma_\infty \leq \delta^{-1}$. Also, $\gamma_\infty = \theta_\infty^{-1}\gamma_0$ implies that $\theta_\infty^{-1} = \gamma_\infty/\gamma_0$, and indeed we see that $\alpha_\infty = \theta_\infty^{-1}\alpha_0 = \alpha_0\gamma_\infty/\gamma_0$. ■

8.5 Convergence of choices

Thus far in this chapter we considered sequences of value functions as generated by the learning algorithm. We investigated convergence of such sequences of value functions, and more specifically we investigated convergence towards optimality. Thus, we have considered convergence of ad hoc preferences.

We may also wonder whether the choices that would be made by an individual whose behaviour would be generated by the learning algorithm, would converge. That is, given a sequence of value functions generated by the learning algorithm, in this section we focus merely on sequences of choices, each of which will maximize the ad hoc utility as determined by the corresponding value function. And we investigate whether such sequences of choices will converge, and more specifically, we investigate whether these choices will converge towards choices that would be made by a rational utility maximizer in the same setting.

The next proposition shows that convergence towards optimality in terms of ad hoc preferences also implies convergence towards optimality of the corresponding sequences of consumption functions.

Proposition 8.5.1 *Let a model be given that satisfies axioms 4.1.1, 4.4.1, 8.1.5 and 8.1.6, and 8.1.3 or 8.1.4, and with an instantaneous utility function u_0 that satisfies 8.1.1. Suppose that given an initial value function $V^{(0)}$ that satisfies axiom 8.1.1, the learning algorithm is used, and that preferences do converge to optimality. Then, the sequence of choice functions $(c_t^*(m))_{t=0}^{\infty}$, as defined by*

$$c_t^*(m) := \arg\max_{0 \leq c_t \leq m} u_0(c_t) + \delta V^{(t)}(m - c_t + I),$$

that corresponds to the sequence of value functions $(V^{(t)})_{t=0}^{\infty}$ as generated by the learning algorithm from $V^{(0)}$, will converge to the optimal choice function $\tilde{c}(m)$, as defined by

$$\tilde{c}(m) := \arg\max_{0 \leq c \leq m} u_0(c) + \delta V^*(m - c + I).$$

Proof. Let a model be given with an instantaneous utility function u_0 that satisfies axiom 8.1.1, a discount factor $0 < \delta < 1$, and an constant additional periodical income $I_t = I \geq 0$, for all $t \geq 1$. And suppose that, given an initial value function

$V^{(0)}$ that satisfies axiom 8.1.1, the learning algorithm is used, and that preferences converge to optimality. This means that for $(V^{(t)})_{t=0}^{\infty} = (\theta_t^{-1}V^{(0)})_{t=0}^{\infty}$ the sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$ converges to some consistency-inducing scalar $\tilde{\theta}$. By proposition 8.3.1 we know that if such a CIS $\tilde{\theta}$ exists, the initial value function must be an affine transformation of the optimal value function that solves the functional equation: $V^{(0)} = \tilde{\theta}V^* + \alpha$.

For any period t, the choice function $c_t^* : \mathbb{R}_+ \to \mathbb{R}_+$ will return for any budget m that is available in period t, the choice $c_t^*(m)$ that solves

$$\max_{0 \leq c_t \leq m} u_0(c_t) + \delta\theta_t^{-1}V^{(0)}(m - c_t + I).$$

We can also capture such a sequence of choice functions by the single function $f : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+$, as defined by

$$f(m, \theta) := \arg\max_{0 \leq c \leq m} u_0(c) + \delta\theta^{-1}V^{(0)}(m - c + I),$$

for all $m \in \mathbb{R}_+$ and all $\theta \in \mathbb{R}_{++}$. Then we see that $c_t^*(m) = f(m, \theta_t)$. By Luenberger ([28], p.462) we know that for any given $m \in \mathbb{R}_+$, the function $f(m, \theta)$ must be continuous in θ. Since the sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$ converges to the scalar $\tilde{\theta}$, we see that the sequence of choice functions $(c_t^*(m))_{t=0}^{\infty} = (f(m, \theta_t))_{t=0}^{\infty}$ will converge[35] to $f(m, \tilde{\theta})$.

For any given $m \in \mathbb{R}_+$, this limit quantity $f(m, \tilde{\theta})$ solves

$$\max_{0 \leq c \leq m} u_0(c) + \delta\tilde{\theta}^{-1}V^{(0)}(m - c + I) = \max_{0 \leq c \leq m} u_0(c) + \delta V^*(m - c + I) + \delta\tilde{\theta}^{-1}\alpha.$$

And, for any given $m \in \mathbb{R}_+$, by proposition 8.4.2 we see that $\tilde{c}(m)$ must also maximize this last maximization problem. Since u_0 and $V^{(0)}$ are strictly concave, the solutions to all of the above maximization problems are unique, and we see that $\tilde{c}(m) = f(m, \tilde{\theta})$ must hold. Thus indeed we see that the consumption functions $c_t^*(m)$ will converge to $\tilde{c}(m)$. ■

Note that the above proposition proves that *consumption functions* will converge towards the optimal consumption function. The rest of this chapter will investigate convergence of actual consumption patterns. That is, we will investigate convergence of sequences of choices $c_t^*(m_t^*)$, as determined by the above choice functions $c_t^*(.)$, when evaluated at the budgets $m_t^* = m_0 + tI - \sum_{i=0}^{t-1} c_i^*$ that actually result from the learning algorithm.

8.5.1 Models with no income stream

We model a decision-maker who cannot borrow and who can save, but who would receive no interest for saving. Here the decision-maker would face a situation with a finite initial budget of m_0 and without additional income after the first period. Hence in this

[35] Pointwise.

situation there is a single budget constraint, which reads $\sum_{t+1=0}^{\infty} c_t \leq m_0$. Then, for any sequence of choices $(c_0, c_1, ...)$ that satisfies this budget constraint, we will necessarily get that $\lim_{t\to\infty} c_t = 0$.

Hence we see that the choices made by a decision-maker who behaves as postulated in the previous chapters, do converge. Similarly, the choices that would be made by a rational utility maximizer would converge, and we get that the two limits are the same.

Note however, that this convergence to zero is already implied by the budget constraint; it is completely independent of the value functions that the particular choices c_t are derived from. Therefore, no matter how sophisticated or primitive a decision-maker would be in approaching this problem, behaviour in the limit will always be the same.

In the setting of proposition 8.4.4, without additional income and with an infinite marginal instantaneous utility in the point 0, in every period the consumption and savings choices will be internal solutions. This will not only be the case when these choices are made by a rational utility maximizer, but also when they are made by a learning ad hoc utility maximizer whose initial value function has an infinite marginal value for saving in the point 0 (see the discussion before proposition 8.4.4). Therefore each of the elements of a generated sequence of savings will be strictly positive, and the same holds for the generated elements of a sequence of consumption choices. Thus convergence within each of these generated sequences of choices and savings will never be achieved in finite time.

In models with an income stream, this will be very different, in that case convergence of choices *will* generally happen in finite time.

8.5.2 Models with an income stream

In this section we will deal with models with an income stream. Here we will get convergence of choices in finite time, both in the case where these choices would be made by a rational utility maximizer, and in the case where they would be made by a learning ad hoc utility maximizer.

It can be shown that in consumption/savings models where a constant additional income I is obtained in all periods after the first, the choices that a rational utility maximizer would make would converge in finite time. In fact, within the standard framework an optimal policy would consist of always consuming strictly more than the constant additional income: $\tilde{c}_t > I$ (if of course this is possible, if $m_t > I$) so that $m_{t+1} < m_t$ (unless $m_{t+1} = m_t = I$), and of exhausting the initial endowment m_0 in finite time, and consuming the fixed additional income I forever thereafter. Here the number of periods in which this initial endowment is exhausted increases in the initial endowment. More specifically, for m_0 small enough, the optimal policy consists of consuming $\tilde{c}_0 = m_0$ (and saving nothing) in period 0, and consuming the fixed income thereafter: $\tilde{c}_t = I$, for all $t > 0$. Thus, for these small values of the initial endowment, the optimal value function would be given by $V^*(m_0) = u_0(m_0) + \frac{\delta}{(1-\delta)} u_0(I)$ (for m_0 small enough). (See Stokey and Lucas ([43], section 5.17).)

The next proposition will show that convergence of choices towards the limit I will also occur in finite time if these choices are made by an ad hoc utility maximizer whose initial value function would be an affine transformation of the optimal value function.

Proposition 8.5.2 *Let a model be given that satisfies axioms 4.1.1, 4.4.1, 8.1.4, 8.1.5 and 8.1.6, and with an instantaneous utility function that satisfies axiom 8.1.2. Suppose that the initial budget is larger than the constant additional income:* $m_0 \geq I$*, and that the initial value function is an affine transformation of the optimal value function:* $V^{(0)} = \gamma_0 V^* + \alpha_0$*, for some* $\gamma_0 \in \mathbb{R}_{++}$ *and some* $\alpha_0 \in \mathbb{R}$*. If subsequently the learning algorithm is used, then the resulting sequence of choices* $(c_t^*)_{t=0}^{\infty}$ *will converge in finite time to the limit* I*.*

Proof. A model with a discount factor $0 < \delta < 1$, and an instantaneous utility function u_0 that satisfies axiom 8.1.1 and $u_0'(0) = \infty$, is given. A strictly positive constant additional income is obtained every period after time 0: $I_t = I > 0$, for all $t \geq 1$, and the initial budget is sufficiently large: $m_0 \geq I$.

We know (see Stokey and Lucas [43]) that within the standard framework a rational utility maximizer's optimal policy would entail always (if possible) consuming strictly more than the constant additional income: $\tilde{c}_t(m_t) > I$ if $m_t > I$, and $\tilde{c}_t(m_t) = I$ if $m_t = I$. This also implies that the occurring budgets $\tilde{m}_{t+1} = \tilde{m}_t - \tilde{c}_t(m_t) + I$ will satisfy $\tilde{m}_{t+1} < \tilde{m}_t$ (unless $\tilde{m}_{t+1} = \tilde{m}_t = I$). Moreover, a rational utility maximizer's optimal policy would consist of exhausting the initial endowment m_0 in finite time, and consuming the fixed additional income I forever thereafter, so that the choices $\tilde{c}_t$ that a rational utility maximizer would make will converge in finite time to I. Similarly, the resulting budgets $\tilde{m}_t$ will converge in finite time to I.

The initial value function $V^{(0)}$ is such that $V^{(0)} = \gamma_0 V^* + \alpha_0$, for some $\gamma_0 \in \mathbb{R}_{++}$ and some $\alpha_0 \in \mathbb{R}$. Subsequently the learning algorithm with min-max adjustment[36] (as dependent on regular excess expenditure) is used, and this will generate a sequence of value functions $(V^{(t)})_{t=0}^{\infty}$ from the initial value function $V^{(0)}$. As before each of these value functions $V^{(t)}$ will satisfy $V^{(t)} = \theta_t^{-1} V^{(0)}$, where θ_t denotes the adjustment product. If we define γ_t and α_t by $\gamma_t := \theta_t^{-1}\gamma_0$ and by $\alpha_t := \theta_t^{-1}\alpha_0$, then we see that each of the generated value functions $V^{(t)}$ will also be an affine transformation of the optimal value function: $V^{(t)} = \gamma_t V^* + \alpha_t$.

Here we focus on sequences of choices $(c_t^*)_{t=0}^{\infty}$ that will result from the learning algorithm. That is, the learning algorithm will generate a sequence of value functions $(V^{(t)})_{t=0}^{\infty}$, and for any period t the choice c_t^* will maximize the ad hoc utility corresponding to the value function $V^{(t)}$, given the period-t budget m_t^*:

$$c_t^* := \arg \max_{0 \leq c_t \leq m_t^*} u_0(c_t) + \delta V^{(t)}(m_t^* - c_t + I).$$

This period-t budget m_t^* is determined endogenously by what happened before time t: $m_t^* = m_0 + tI - \sum_{i=0}^{t-1} c_i^*$.

[36] For some $\rho > 1$, $\sigma < 1$ and $0 < \lambda < 1$.

By proposition 8.4.5 we know that converge will take place towards some limit function $V^\infty = \gamma_\infty V^* + \alpha_\infty$, with $\gamma_\infty \in [0, \delta^{-1}]$ and with $\alpha_\infty = \alpha_0 \gamma_\infty / \gamma_0$. We now want to show that the corresponding sequence of choices $(c_t^*)_{t=0}^\infty$ will converge in finite time to the limit I.

If $\gamma_\infty \in [0, 1)$, then there must be some period $\hat{t}$ for which it will hold that $\gamma_t < 1$, for all $t \geq \hat{t}$. By lemma 8.4.1 we know that for any budget m_t the expenditure c_t^* as derived from the value function $\gamma_t V^*$ (with $\gamma_t < 1$) will never be smaller than the expenditure $\tilde{c}_t$ that would correspond to the optimal value function V^*, given the same budget m_t. And we know that the choices $\tilde{c}_t$ and the budgets $\tilde{m}_t$ that would be derived from V^* would converge in finite time to I. Therefore, we see that in the present case budgets will decrease even faster than in the case of optimality, and convergence of the sequence of consumption choices c_t^* to the limit I will also occur in finite time.

If $\gamma_\infty \in [1, \delta^{-1}]$, then it will hold that $0 < \delta\gamma_0 \leq \theta_\infty \leq \gamma_0 < \infty$. By the proof of proposition 7.1.1 we know that the two variables $\theta_{\min}^{\mathcal{E}+}(\varepsilon_{t+1}) = \min_{\theta_i \in \Theta^{\mathcal{E}+}(\varepsilon_{t+1})} \theta_i$ and $\theta_{\max}^{\mathcal{E}-}(\varepsilon_{t+1}) = \max_{\theta_i \in \Theta^{\mathcal{E}-}(\varepsilon_{t+1})} \theta_i$ must then both converge to θ_∞, and one of the following two cases must hold:

(I) the variables $\theta_{\min}^{\mathcal{E}+}(\varepsilon_{t+1})$ and $\theta_{\max}^{\mathcal{E}-}(\varepsilon_{t+1})$ are equal from some period t' onwards,

(II) the variables $\theta_{\min}^{\mathcal{E}+}(\varepsilon_{t+1})$ and $\theta_{\max}^{\mathcal{E}-}(\varepsilon_{t+1})$ become arbitrarily close as t goes to infinity.

Suppose that **case (I)** holds: that $\theta_{\min}^{\mathcal{E}+}(\varepsilon_{t'+1}) = \theta_{\max}^{\mathcal{E}-}(\varepsilon_{t'+1})$ will hold for some t'. With lemma 6.5.1 it can be shown that this equality can only hold if $E_{t'} = c_{t'}^* - c_{t'}^{\triangleleft} = 0$. By the assumption that $u_0'(0) = \infty$, we know that the possibility of boundary solutions with $c_{t'}^* = c_{t'}^{\triangleleft} = 0$ is excluded. Thus there remain two possibilities as to how this zero excess expenditure came about.

(A) $c_{t'}^* = c_{t'}^{\triangleleft} = m_{t'}^*$. Recall from the discussion after lemma 8.4.1 that we could rule out boundary solutions where all is spent for both c_t^* and $c_t^{\triangleleft}$, if the additional assumption $\delta\gamma_t V^{*\prime}(I) > u_0'(m_t^*)$ would hold. Now however, for small $\gamma_{t'}$ and small budgets this condition may be violated. If indeed $\gamma_{t'}$ and $m_{t'}^*$ are such that the inverse condition $\delta\gamma_{t'} V^{*\prime}(I) \leq u_0'(m_{t'}^*)$ holds, then we will get boundary solutions with $c_{t'}^* = c_{t'}^{\triangleleft} = m_{t'}^*$.

This would also imply that

$$m_{t'+1}^* = s_{t'}^* + I = 0 + I \leq s_{t'-1}^* + I = m_{t'}^*.$$

And $E_{t'} = 0$ will yield an adjustment factor equal to one, so that no adjustment takes place in period $t' + 1$: $\theta_{t'+1} = \theta_{t'}$ and $\gamma_{t'+1} = \gamma_{t'}$. This again implies that

$$\delta\gamma_{t'+1} V^{*\prime}(I) = \delta\gamma_{t'} V^{*\prime}(I) \leq u_0'(m_{t'}^*) \leq u_0'(m_{t'+1}^*).$$

Hence in the next period $t' + 1$ we will again get that $c_{t'+1}^* = c_{t'+1}^{\triangleleft} = m_{t'+1}^* = I$, and that excess expenditure will equal zero, so that the value function will still remain constant. Of course, by repeating the above argument we will then get that the equalities $m_{t'+\tau}^* = I$, $\gamma_{t'+\tau} = \gamma_{t'}$, $c_{t'+\tau}^* = c_{t'+\tau}^{\triangleleft} = I$ and $E_{t'+\tau} = 0$ will hold in all subsequent periods $t' + \tau > t'$. Thus indeed, we see that in this case $(c_t^*)_{t=0}^\infty$ converges to I in finite time.

(B) $0 < c^*_{t'} = c^{\lhd}_{t'} < m^*_{t'}$. From the proof of lemma 8.4.1 we can see that this case can only happen if $\gamma_{t'} = 1$ holds, which would imply that $V^{(t')} = V^* + \alpha_{t'}$, for some $\alpha_{t'} \in \mathbb{R}$. In this case, we would also get that $\gamma_{t'+1} = \gamma_{t'} = 1$, and by lemma 8.4.4 that $E_{t'+1} = 0$, and etcetera for all later periods. Thus, from period t' onwards, the ad hoc utility maximizer's behaviour would be the same as a rational utility maximizer's behaviour. Therefore the ad hoc utility maximizer's choices $(c^*_t)^{\infty}_{t=0}$ would converge in a finite number of periods after time t' to the limit I.

Then suppose that **case (II)** holds: the variables $\theta^{\mathcal{E}+}_{\min}(\varepsilon_{t+1})$ and $\theta^{\mathcal{E}-}_{\max}(\varepsilon_{t+1})$ never equal, but become arbitrarily close to each other (and to θ_∞) as t goes to infinity. This can only occur for $\theta_\infty = \gamma_0$ and thus for $\gamma_\infty = 1$. To see this, first note that in case (II) only strictly positive and strictly negative excess expenditures occur along the learning path. After all, from the last paragraphs of section 6.5 we know that $E_t = 0$ would imply that $\theta^{\mathcal{E}+}_{\min}(\varepsilon_{t+1}) = \theta^{\mathcal{E}-}_{\max}(\varepsilon_{t+1})$, which would contradict case (II).

Now, for $\theta_t \geq \gamma_0$ it will hold that $\gamma_t = \theta_t^{-1}\gamma_0 \leq 1$, and by lemma 8.4.4 the excess expenditure corresponding to $\gamma_t V^*$ will be non-negative. Similarly, for $\theta_t \leq \gamma_0$ it will hold that $\gamma_t \geq 1$, and the excess expenditure will be non-positive. For $\theta_t = \gamma_0$ it will hold that $\gamma_t = 1$, and from lemma 8.4.4 we know that in this case the excess expenditure corresponding to $\gamma_t V^*$ will equal zero. Thus, excess expenditure can only be strictly negative for $\gamma_t > 1$ (so for $\theta_t < \gamma_0$), and strictly positive for $\gamma_t < 1$ (for $\theta_t > \gamma_0$). Therefore, as zero excess expenditures do not occur, we see that $\theta^{\mathcal{E}+}_{\min}(\varepsilon_{t+1}) > \gamma_0$ and that $\theta^{\mathcal{E}-}_{\max}(\varepsilon_{t+1}) < \gamma_0$ must hold for all t, and convergence where the two variables $\theta^{\mathcal{E}+}_{\min}(\varepsilon_{t+1})$ and $\theta^{\mathcal{E}-}_{\max}(\varepsilon_{t+1})$ become arbitrarily close, but never equal, can indeed only occur for $\theta_\infty = \gamma_0$ (so for $\gamma_\infty = 1$).

We already saw that excess expenditure will never equal zero, and we can now also see that there can never be a last adjustment product that gives a positive (or a negative) excess expenditure. For suppose not, suppose that time $\bar{t}$ is such that $E_{\bar{t}} > 0$ (so that $\gamma_{\bar{t}} < 1$, and $\theta_{\bar{t}} > \gamma_0$), and such that $E_{\bar{t}+\tau} < 0$ for all $\bar{t} + \tau > \bar{t}$. Then it must hold that $\theta^{\mathcal{E}+}_{\min}(\varepsilon_{\bar{t}+\tau}) = \theta_{\bar{t}}$ for all $\bar{t} + \tau > \bar{t}$, and that $\theta^{\mathcal{E}-}_{\max}(\varepsilon_{\bar{t}+\tau})$ must converge to $\theta_{\bar{t}} > \gamma_0$ as $\tau \to \infty$. However, this would imply that from some period onwards, $\theta_{\bar{t}+\tau}$ would become larger than γ_0, and thus that excess expenditure would become positive. This contradicts the assumption that $\bar{t}$ was the last period in which a positive excess expenditure occurred. Similarly it can be shown that there can never be a last period in which a negative excess expenditure occurs.

As γ_t converges towards one, there must be a period $\breve{t}$ such that for all periods $t \geq \breve{t}$ it will hold that $\gamma_t \leq \delta^{-1}$. Now we will show that for $\gamma_t \leq \delta^{-1}$ it will hold that the choice c^*_t corresponding to $\gamma_t V^*$, will always satisfy $c^*_t \geq I$. To show this, it suffices to show that $c^*_t \geq I$ will always hold for $\gamma_t = \delta^{-1}$. After all, for any given budget m_t, by lemma 8.4.1 the expenditure corresponding to $\gamma_t < \delta^{-1}$ will never be smaller than the expenditure corresponding to $\gamma_t = \delta^{-1}$.

Thus, here we want to show that $\gamma_t = \delta^{-1}$ implies that $c^*_t \geq I$. Remember that the choice c^*_t that the ad hoc utility maximizer would make in period t, would maximize the function $u_0(c_t) + \delta\gamma_t V^*(s_t + I)$, which now becomes $u_0(c_t) + V^*(s_t + I)$. Thus c^*_t

will be such that

$$u_0(c_t^*) + V^*(m_t^* - c_t^* + I) = \max_{0 \le c_t \le m_t^*} u_0(c_t) + V^*(m_t^* - c_t + I).$$

By $u_0'(0) = \infty$ we know that $c_t^* = 0$ cannot happen. Also, $c_t^* = m_t^*$ cannot happen, as by lemma 8.4.4 this would also imply that $c_t^{\triangleleft} = m_t^*$, and thus that $E_t = 0$, which we just concluded cannot occur in this case. Therefore c_t^* will be internal, and satisfy $u_0'(c_t^*) = V^{*\prime}(m_t^* - c_t^* + I)$.

Now, for any $m \in \mathbb{R}_+$ it will hold that $V^{*\prime}(m) = u_0'(\tilde{c}(m))$, where $\tilde{c}(m)$ denotes the choice that would solve the maximization problem inside the functional equation: $u_0(c) + \delta V^*(m - c + I)$ subject to $0 \le c \le m$. To see this, the assumption that $u_0'(0) = \infty$ implies that $\tilde{c}(m) > 0$ will always hold. Therefore from the proof of lemma 8.4.2 (cases (I) and (II) of the proof of lemma 8.4.2 cannot occur here) we can see that $V^{*\prime}(m) = u_0'(\tilde{c}(m))$ will now always hold.

Thus $V^{*\prime}(m_t^* - c_t^* + I) = u_0'(\tilde{c}(m_t^* - c_t^* + I))$, and we see that c_t^* will satisfy

$$u_0'(c_t^*) = V^{*\prime}(m_t^* - c_t^* + I) = u_0'(\tilde{c}(m_t^* - c_t^* + I)),$$

and by strict concavity of u_0 we get that $c_t^* = \tilde{c}(m_t^* - c_t^* + I)$. The quantity $\tilde{c}(m_t^* - c_t^* + I)$ is derived from the maximization of $u_0 + \delta V^*$, and would thus correspond to the behaviour of a rational utility maximizer, which we saw satisfied $\tilde{c}(m_t^* - c_t^* + I) > I$ if $m_t^* - c_t^* + I > I$. By $c_t^* < m_t^*$ both these inequalities are indeed satisfied. Thus we see that $\gamma_t = \delta^{-1}$ implies that $c_t^* > I$, so that no money accumulation will take place. And indeed, more generally for $\gamma_t \le \delta^{-1}$, the expenditure c_t^* corresponding to $\gamma_t V^*$ is never smaller than the constant additional income I, so that $m_{t+1}^* = m_t^* - c_t^* + I < m_t^*$, and no capital accumulation will occur.

For all $\theta_t > \gamma_0$ it will hold that $\gamma_t < 1$, and as in the case with $\gamma_\infty \in [0, 1)$ above, we know that for any budget m_t, the expenditure c_t^* that corresponds to the value function $\gamma_t V^*$, will never be smaller than the expenditure $\tilde{c}_t$ that would correspond to the optimal value function V^*. And choices and budgets corresponding to V^* would always be such that $\tilde{c}_t \ge I$ and such that $\tilde{m}_{t+1} < \tilde{m}_t$ (unless $\tilde{m}_t = I$), and both would converge in finite time to I. Therefore if only γ_t's smaller than one would occur, the budgets m_t^* would certainly decrease in a finite number of steps towards I, and the choices c_t^* would convergence in finite time towards I. However, in the case we are considering here, the γ_t's will vary between being smaller than one and larger than one. Therefore, the decrease of the budgets m_t^* in finite time towards I due to the γ_t's smaller than one, could in principle be undone by capital accumulation due to the γ_t's larger than one. Still, we saw that for $\gamma_t \le \delta^{-1}$ no capital accumulation will occur, and it must be the case that the budgets will indeed decrease towards I in a finite number of steps after period $\check{t}$. Thus indeed, convergence of the sequence of consumption choices c_t^* to the limit I will occur in finite time. The γ_t's larger than one are close enough to one so that the exhausting of resources by the γ_t's smaller than one is not undone by the γ_t's larger than one, and the adjustment factors smaller than one ensure convergence to the constant additional income in finite time. ■

In this chapter we considered the properties of convergence and of convergence towards optimality, in stationary consumption/savings models under certainty. We saw that sequences of value functions as generated by the learning algorithm always converge. We saw that a necessary condition for convergence towards optimality is that the initial value function is an affine transformation of the optimal value function. In cases without additional income after the first period, we found that convergence towards optimality will occur if indeed the initial value function is an affine transformation of the optimal value function. In cases with additional income we found that convergence towards optimality will generally not occur. Finally, we considered convergence of choices. We saw that convergence towards optimality of preferences implies convergence towards optimality of choice functions, and we found that both in models with and without income streams, choices will converge towards optimality if the initial value function is an affine transformation of the optimal value function.

9 Convergence under expected utility

Like the previous chapter, this chapter follows up on chapter 7. We will investigate convergence of ad hoc preferences, as generated by the learning algorithm, towards a limit that would represent optimality. We will again do these investigations within the class of stationary consumption/savings models. The previous chapter dealt with consumption/savings models under certainty, and in this previous chapter we saw that under some circumstances convergence towards optimality does occur. The present chapter is similar to the last; here we will establish when convergence towards optimality will occur, and when it will not occur, in stationary consumption/savings models with expected utility.

The present chapter has a structure that is similar to that of the previous chapter. This chapter consists of six sections. The first section will specify the setting of stationary consumption/savings models under expected utility, and it will specify how the ad hoc framework can be fit into this setting, both in the case of regular excess expenditure adjustment and in the case of expected excess expenditure adjustment. In the second section we will see that learned ad hoc preferences do always converge. The third section will consider when a consistency-inducing scalar exists, and thus it provides a necessary condition (with respect to the initial time preferences with which the learning algorithm starts) for convergence towards optimality to occur. The fourth and fifth sections will establish if or when convergence towards optimality does occur. In the fourth section we will see that convergence towards optimality will generally not occur in the case of regular excess expenditure adjustment. In the fifth section we will see that in the case of expected excess expenditure adjustment, convergence towards optimality *will* occur under some (rather specific) circumstances. Finally, the sixth section investigates the relation between convergence towards optimality in terms of actual choices, and convergence towards optimality of preferences.

9.1 Models of consumption/savings under expected utility

In this chapter we will consider the asymptotic properties of lifetimes of behaviour and preferences, as governed by the learning algorithm in stationary models of expected utility. While the learning algorithm and the min-max adjustment function are defined unequivocally, in models of expected utility the adjustment function may either depend on regular excess expenditure (REE), or on expected excess expenditure (EEE). Since in models of certainty the two excess expenditure measures coincide, in the previous chapter we only considered adjustments based on the more basic REE measure. In the present chapter with expected utility models, the two measures of excess expenditure generally do not coincide. Therefore the dynamic properties of the learning algorithm may be very different for the two excess expenditure measures, and here we will need to consider the cases of REE adjustment and EEE adjustment separately. As mentioned in subsection 6.4.2, REE adjustment should probably be thought of as more basic, while the more complicated EEE adjustment can be thought of as a benchmark (that will prove to be more efficient).

As in the previous chapter, we model a consumer who *does* know all his instantaneous preferences and who *does* know his total preferences. However, we assume that our consumer lacks the cognitive sophistication to solve the problem of maximizing his total preferences over a budget set at once, and that he lacks the cognitive sophistication to derive consistent ad hoc preferences from the total preferences. Therefore we assume that our consumer *does not* know ad hoc preferences that are consistent with total preferences, and we suppose that our consumer would try to tackle his lifetime consumption problem by using the learning algorithm.

9.1.1 The setting

Here we consider models of expected utility; we model a consumer who does not know all features of the economic environment that he will be facing, but who does know all the realizations of these features that could possibly occur, and all the probabilities each of these realizations will occur with.

As in the previous chapter, we suppose that axiom 4.1.1 applies so that the number of periods is infinite, and that in every period t the consumer has to choose a consumption level $c_t \in \mathbb{R}_+$. The total (Bernouilli) utility function u on $X = \mathbb{R}_+^\infty$ is additively separable with respect to time, and satisfies exponential time discounting, so for every $c = (c_0, c_1, c_2, ...)$, preferences can be expressed as $u(c) = \sum_{t=0}^{\infty} \delta^t u_0(c_t)$, where δ is a discount factor in $(0, 1)$, and where $u_0 : \mathbb{R}_+ \to \mathbb{R}$ is an instantaneous utility function, which is assumed to satisfy axiom 8.1.1.

Our decision-maker is faced with an uncertain economic environment. However, in the subsections of chapter 6 dealing with consumption/savings models, we saw that in this setting commodity spaces $X_t = \mathbb{R}_+$ and prices $p_t = 1$ are by assumption known. Hence there isn't any uncertainty about commodity spaces and prices.

Still, consumption/savings models do allow for budgetary uncertainty. We suppose that for any period $t \geq 1$ the additional income $I_t \geq 0$ becomes known at time t. Where in the previous chapter these additional incomes were fixed numbers, now they are random variables.

Like the previous chapter, this chapter deals with stationary consumption/savings models. In the present context, stationarity requires that the additional income random variables would be independently and identically distributed (IID). More specifically, here we assume that for any period t, the additional income random variable I_t will take its realizations from the finite set $\{I^0, I^1, ..., I^R\}$ (that does not depend on t) with $I^r \in \mathbb{R}_+$ for all $0 \leq r \leq R$, and $R \in \mathbb{N}$. We suppose that the indexation of these realizations reflects their sizes: $I^0 \leq I^1 \leq ... \leq I^{R-1} \leq I^R$. We denote the probability that the random variable I_t, for any $t \geq 1$, will take the value $I^r \in \{I^0, I^1, ..., I^R\}$ by π_r. These probabilities $\{\pi_0, \pi_1, ..., \pi_R\}$ (that also do not depend on t) satisfy $\pi_r > 0$, for all $0 \leq r \leq R$, and $\sum_{r=0}^{R} \pi_r = 1$, so this indeed specifies a well-defined random variable. We suppose that one of the following two axioms is satisfied.

Axiom 9.1.1 *The additional periodical incomes I_t, for $t \geq 1$, are IID distributed and each can take a finite number of realizations.*

Axiom 9.1.2 *The additional periodical incomes satisfy axiom 9.1.1, and the smallest realization equals $I^0 = 0$.*

Saving is possible at a zero interest rate, borrowing is not possible, so that the budget available in period t equals the savings brought over from the previous period plus the additional income received in that period: $m_t = s_{t-1} + I_t = m_{t-1} - c_{t-1} + I_t$.

9.1.2 Dynamic programming

In consumption/savings models with income uncertainty, the period-t consumption choice c_t may depend on the information that is known at time t about the uncertain income stream, so we can write $c_t(m_0, I_1, ..., I_t)$. Such consumption choices should also satisfy the budget constraints $0 \leq c_t(m_0, I_1, ..., I_t) \leq m_t$. Saving is possible and borrowing is not, so the implicit time-t budgets m_t would be specified by

$$m_t = m_0 + \sum_{\tau=1}^{t} I_\tau - \sum_{\tau=0}^{t-1} c_\tau(m_0, I_1, ..., I_\tau).$$

Hence, the period-t budget constraint $c_t(m_0, I_1, ..., I_t) \leq m_t$ can also be written as

$$\sum_{\tau=0}^{t} c_\tau(m_0, I_1, ..., I_\tau) \leq m_0 + \sum_{\tau=1}^{t} I_\tau,$$

which simply says that what was spent should at no point in time exceed what was incurred.

Within the standard framework for consumer choice, the basic consumer problem in the present context would thus be given by the following sequence problem:

$$\max E_{I_1} E_{I_2} ... [\sum_{t=0}^{\infty} \delta^t u_0(c_t(m_0, I_1, ..., I_t))] \tag{1}$$

sub to

$$\sum_{\tau=0}^{t} c_\tau(m_0, I_1, ..., I_\tau) \leq m_0 + \sum_{\tau=1}^{t} I_\tau, \forall t \geq 0. \tag{2}$$

Corresponding to this sequence problem is the following functional equation

$$V^*(m) = \max_{(c,s):c+s\leq m} \{u_0(c) + \delta E_I[V^*(s + I)]\} =$$

$$\max_{(c,s):c+s\leq m} \{u_0(c) + \delta \sum_{r=0}^{R} \pi_r V^*(s + I^r)\}.$$

In subsection 2.6.5 we saw that the theory of dynamic programming shows that the value function $V^* : \mathbb{R}_+ \to \mathbb{R}$ that solves this functional equation, is also exactly the function that returns for every budget level m available in period 0, the maximum discounted lifetime expected utility that can be attained from the budget m. That is, for any budget level m, the value $V^*(m)$ exactly equals the maximum of (1) sub to (2).

Thus we found a link between sequence problems and their corresponding functional equations. It can also be shown (Stokey and Lucas [43]) that if instantaneous utility u_0 satisfies axiom 8.1.1, then the optimal value function V^* will also satisfy axiom 8.1.1.

Given the optimal value function V^*, we can define the **optimal expected value function** $EV^* : \mathbb{R}_+ \to \mathbb{R}$ by

$$EV^*(s) := E_I[V^*(s+I)] = \sum_{r=0}^{R} \pi_r V^*(s+I^r),$$

as the expectation of the optimal value that will derived from a certain amount of savings s. Note that the arguments that enter the value function V^* are actual budgets m that can be spent in a certain period, while the arguments that enter the expected value function EV^* are amounts of savings s that are brought over from the previous period, and will yet have to be augmented with an additional income to arrive at an actual budget. Then the functional equation can alternatively be written as

$$V^*(m) = \max_{(c,s):c+s\leq m} \{u_0(c) + \delta EV^*(s)\}.$$

Above we saw that if instantaneous utility u_0 satisfies axiom 8.1.1, then the optimal value function V^* will also satisfy axiom 8.1.1. It is easy to see that in that case the optimal expected value function EV^* will consequently also satisfy axiom 8.1.1.

9.1.3 The ad hoc framework

As before, here we model a boundedly rational individual who has trouble solving the above problem at once, so that the standard framework for consumer choice cannot be used to model this decision-maker's behaviour, and we use the ad hoc framework. Axiom 4.4.1 is assumed to hold, so our consumer would cut up his lifetime consumption choice into smaller ad hoc choice problems, where in each of these he uses ad hoc preferences $u^{(t)}(w_{t-1}, c_t, s_t)$ to reach a decision.

As in the previous chapter, we suppose that ad hoc utility functions are of an additively separable form and exhibit exponential discounting, and we write:

$$u^{(t)}(w_{t-1}, c_t, s_t) = \sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1} \widetilde{V}^{(t)}(s_t).$$

Here w_{t-1} denotes the vector of past consumption choices $(c_0, c_1, ..., c_{t-1})$.

The function $\widetilde{V}^{(t)} : \mathbb{R}_+ \to \mathbb{R}$ denotes some function that values saving. In the previous chapter, that dealt with models of certainty, without loss of generality the functions $\widetilde{V}^{(t)}(s_t)$ were replaced by $V^{(t)}(s_t + I)$, where I was the known per-period additional income. Thus, in order to stay closer to the notation and the interpretation of value functions as in dynamic programming, the arguments of the $V^{(t)}$-functions that would be generated by the learning algorithm were chosen to be next period's budgets, rather than current savings.

Here we will do something similar, we will again use value functions $V^{(t)}$ that depend on budgets $m_{t+1} = s_t + I_{t+1}$ rather than on amounts of savings s_t. However, in the setting of this chapter, the additional incomes I_{t+1} and consequently the value of next period's budget $V^{(t)}(s_t + I_{t+1})$ are not known at time t. Still, similar to how EV^* was defined in the previous section, given a value function $V^{(t)} : \mathbb{R}_+ \to \mathbb{R}$ we can define a corresponding **expected value function** $EV^{(t)} : \mathbb{R}_+ \to \mathbb{R}$ as the expectation of the next period's value $V^{(t)}$ that will derived from a certain amount of savings s_t, by

$$EV^{(t)}(s_t) := E_{I_{t+1}}[V^{(t)}(s_t + I_{t+1})] = \sum_{r=0}^{R} \pi_r V^{(t)}(s_t + I^r).$$

Then, to remain in line with the notation and interpretation of value functions as in the previous chapter and in dynamic programming, here we replace $\widetilde{V}^{(t)}(s_t)$ by $EV^{(t)}(s_t)$. Without loss of generality the above specification of ad hoc utility can be divided by δ^t, and with the above notation we arrive at the specification of the next axiom.

Axiom 9.1.3 *For any time t, the ad hoc utility function $u^{(t)}(w_{t-1}, c_t, s_t)$ can be written as*

$$u^{(t)}(c_1, c_2, ..., c_t, s_s) = \sum_{i=1}^{t} \delta^{i-t} u_0(c_i) + \delta EV^{(t)}(s_t).$$

Here $EV^{(t)}$ is called an expected value function.

As in chapter 6, ad hoc preferences can be separated into instantaneous preferences, which can be represented by

$$v^{(t)}(w_{t-1}, c_t) = \sum_{i=0}^{t} \delta^{i-t} u_0(c_i),$$

and into time preferences, which can be represented by

$$U^{(t)}(v^{(t)}, s_t) = v^{(t)} + \delta EV^{(t)}(s_t).$$

Axiom 6.2.2 is assumed to apply, so instantaneous utility is thought to be exogenous, while time preferences, and thus the functions $EV^{(t)}(.)$, are endogenous.

As in the previous chapter, our decision-maker does not know what his consistent ad hoc preferences, and therefore the optimal expected value function EV^*, would be. The decision-maker is assumed to form an initial guess at such an expected value function in the first period, and to adjust expected value functions in all subsequent periods.

Adjustments can either depend on REE or on EEE excess expenditure. Since the present setting is one of expected utility models, the EEE measure is well-defined, and (except in the degenerate case where $R = 0$) REE will not coincide with EEE.

Then, in both cases of REE and EEE adjustment, in period 0 an initial time preference function $U^{(0)} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is given. From the above axiom we see that this initial time preference function will take the following form: $U^{(0)}(v, s) = v + \delta EV^{(0)}(s)$. Moreover, here we assume that the initial expected value function $EV^{(0)} : \mathbb{R}_+ \to \mathbb{R}$ satisfies axiom 8.1.1.

In any later period $t+1$, given a time preference function $U^{(t)}(v, s) = v + \delta EV^{(t)}(s)$ from the previous period, our consumer adjusts old time preferences into new time preferences $U^{(t+1)}(v, s) = U^{(t)}(a_{\eta_{t+1}} \cdot v, s)$. Here $a_{\eta_{t+1}}$ denotes an adjustment factor that is determined by min-max adjustment from the vector $\eta_{t+1} := (\mathcal{E}_0, \mathcal{E}_1, ..., \mathcal{E}_t)$ of past excess expenditures (REE or EEE).

As in the previous chapter, $U^{(t)}(a_{\eta_{t+1}} \cdot v, s)$ can be written as $a_{\eta_{t+1}} \cdot v + \delta EV^{(t)}(s)$, which can (without loss of generality) be divided by the scalar $a_{\eta_{t+1}}$, such as to arrive at

$$U^{(t+1)}(v, s) = v + \delta a_{\eta_{t+1}}^{-1} EV^{(t)}(s).$$

Thus, as in the previous chapter, both in the case of REE adjustment and in the case of EEE adjustment, the procedure of adjusting time preferences can be shortened to a procedure where the expected value functions are updated directly.

Axiom 9.1.4 *In period* 0 *an initial expected value function* $EV^{(0)} : \mathbb{R}_+ \to \mathbb{R}$ *is exogenously given, and assumed to satisfy axiom 8.1.1. In every later period, a new expected value function* $EV^{(t+1)}$ *is obtained from the old one by:*

$$EV^{(t+1)}(s) := a_{\eta_{t+1}}^{-1} EV^{(t)}(s),$$

where the adjustment factors $a_{\eta_{t+1}}$ *are determined by min-max adjustment from histories of excess expenditures* $\eta_{t+1} = (\mathcal{E}_0, ..., \mathcal{E}_t)$*.*

Regular excess expenditure adjustment In this subsection we specify how regular excess expenditure is determined in the present setting. This is the same as in the previous chapter (subsection 8.1.3); value functions would only have to be replaced by expected value functions. In period t, given a budget m_t and an additional income I_{t+1}, REE is still given by $E_t = c_t^* - c_t^{\triangleleft}$. Here c_t^* denotes the actual expenditure in period t that solves

$$\max_{(c_t, s_t): c_t + s_t \leq m_t} U^{(t)}(v^{(t)}(w_{t-1}, c_t), s_t) = \max_{(c_t, s_t): c_t + s_t \leq m_t} \sum_{i=1}^{t} \delta^{i-t} u_0(c_i) + \delta EV^{(t)}(s_t).$$

and that solves

$$\max_{(c_t, s_t): c_t + s_t \leq m_t} u_0(c_t) + \delta EV^{(t)}(s_t).$$

And, given m_t and I_{t+1}, the (regular) ex-post optimal expenditure $c_t^{\triangleleft}$ is determined as part of a solution $(c_t^{\triangleleft}, c_{t+1}^{\triangleleft}, s_{t+1}^{\triangleleft})$ to the following hypothetical maximization problems over c_t, c_{t+1}, and s_{t+1} simultaneously:

$$\max_{c_t+s_t\le m_t}\ \max_{c_{t+1}+s_{t+1}\le s_t+I_{t+1}} U^{(t)}(v^{(t+1)}(w_{t-1},c_t,c_{t+1}),s_{t+1}) =$$

$$\max_{c_t+s_t\le m_t}\ \max_{c_{t+1}+s_{t+1}\le s_t+I_{t+1}} \sum_{i=1}^{t+1}\delta^{i-t-1}u_0(c_i)+\delta EV^{(t)}(s_{t+1})$$

and

$$\max_{c_t+s_t\le m_t}\ \max_{c_{t+1}+s_{t+1}\le s_t+I_{t+1}} u_0(c_t)+\delta u_0(c_{t+1})+\delta^2 EV^{(t)}(s_{t+1}).$$

Instantaneous utility u_0 is assumed to be strictly concave, as was the initial expected value function $EV^{(0)}$. Therefore any subsequent expected value function will be strictly concave as well, and by strict concavity each of the above maximization problems has a unique solution.

Expected excess expenditure adjustment In this subsection we will specify how expected excess expenditure is determined in the present setting. This is similar to the case of REE. Given a budget m_t, the EEE excess expenditure in period t is defined as $F_t = c_t^* - c_t^\diamond$. Here c_t^* still denotes the actual expenditure in period t that is part of a solution to

$$\max_{(c_t,s_t):c_t+s_t\le m_t} u_0(c_t)+\delta EV^{(t)}(s_t).$$

(This is the same as for REE excess expenditure.)

The period-t ex-post optimal expenditure $c_t^\diamond$ (as determined in period $t+1$) is part of a plan $(c_t^\diamond,(c_{t+1}^r,s_{t+1}^r)_{r=0}^R)$ that solves the following hypothetical problem

$$\max_{c_t+s_t\le m_t} E_{I_{t+1}}[\max_{c_{t+1}+s_{t+1}\le s_t+I_{t+1}} U^{(t)}(v^{(t+1)}(w_{t-1},c_t,c_{t+1}),s_{t+1})] =$$

$$\max_{c_t+s_t\le m_t} E_{I_{t+1}}[\max_{c_{t+1}+s_{t+1}\le s_t+I_{t+1}} \sum_{i=1}^{t+1}\delta^{i-t-1}u_0(c_i)+\delta EV^{(t)}(s_{t+1})].$$

Again, $(c_t^\diamond,(c_{t+1}^r,s_{t+1}^r)_{r=0}^R)$ will also solve the simpler looking problem

$$\max_{c_t+s_t\le m_t} E_{I_{t+1}}[\max_{c_{t+1}+s_{t+1}\le s_t+I_{t+1}} u_0(c_t)+\delta u_0(c_{t+1})+\delta^2 EV^{(t)}(s_{t+1})] =$$

$$\max_{c_t+s_t\le m_t} u_0(c_t)+\delta E_{I_{t+1}}[\max_{c_{t+1}+s_{t+1}\le s_t+I_{t+1}} u_0(c_{t+1})+\delta EV^{(t)}(s_{t+1})].$$

That is, $(c_t^\diamond,(c_{t+1}^r,s_{t+1}^r)_{r=0}^R)$ solves

$$\max_{c_t+s_t\le m_t} u_0(c_t)+\delta\sum_{r=0}^{R}\pi_r[\max_{c_{t+1}+s_{t+1}\le s_t+I^r} u_0(c_{t+1})+\delta EV^{(t)}(s_{t+1})].$$

Again, this problem is hypothetical, and by strict concavity, each of the above maximization problems has a unique solution.

9.2 Convergence of the value function

Both in the cases of REE adjustment and of EEE adjustment, given an initial expected value function $EV^{(0)}$, the learning algorithm will give rise to a sequence of expected value functions $(EV^{(t)})_{t=0}^{\infty}$. Here we will investigate if such a sequence of expected value functions will converge.

As in the previous chapter, any expected value function that is generated by the learning algorithm from the initial expected value function $EV^{(0)}$, can be written as

$$EV^{(t)}(s) = a_{\eta_t}^{-1} \cdot a_{\eta_{t-1}}^{-1} \cdot ... \cdot a_{\eta_1}^{-1} \cdot EV^{(0)}(s) = \theta_t^{-1} EV^{(0)}(s),$$

where θ_t denotes the adjustment product, and we can consider sequences of adjustment products $(\theta_t)_{t=0}^{\infty}$.

We know that in the present setting, proposition 7.1.1 applies, so that any sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$ as generated by the learning algorithm, will converge to some limit $\theta_\infty \in \overline{\mathbb{R}}_+$. Consequently, any sequence of expected value functions $(EV^{(t)})_{t=0}^{\infty}$ that is generated by the learning algorithm, will converge to some limit function $EV^{(\infty)}(.) = \theta_\infty EV^{(0)}(.)$.

However, in the cases of REE adjustment and of EEE adjustment the measures of excess expenditure (E_t and F_t) do not coincide, so that the adjustment factors (a_{ε_t} and a_{ϕ_t}) will generally not be the same, and consequently the limit adjustment products θ_∞ towards which convergence occurs may not be the same. Therefore, although under both REE and EEE adjustment convergence of sequences of adjustment products will always occur, the limits θ_∞^E and θ_∞^F towards which convergence will occur will generally not be the same.

9.3 Existence of a consistency-inducing scalar

Of course, here we are mainly interested in the question of convergence towards optimality. In section 7.3 we saw that convergence towards optimality will take place if three conditions are satisfied. In this section we will establish when the first of these conditions, condition 7.3.1 (which says that, given the initial time preference function $U^{(0)}$, there exists a consistency-inducing scalar $\tilde{\theta}$), will hold.

First we will look at what consistency would entail in the present setting. The optimal value function V^* returns for every budget level m_0 available in period 0, the maximum discounted lifetime expected utility that can be attained from m_0.

Then, suppose that at time t the choices $(c_0, c_1, ..., c_t)$ and an amount of savings s_t are given. These past choices yielded the (provisional) utility $\sum_{i=0}^{t} \delta^i u_0(c_i)$, and the period-$(t+1)$ budget will equal $m_{t+1} = s_t + I_{t+1}$. If the next period's budget m_{t+1} were known, then the maximally attainable additional discounted expected utility from period $t+1$ onwards would be given by

$$\max E_{I_{t+2}} E_{I_{t+3}} ... \sum_{i=t+1}^{\infty} \delta^i u_0(c_i) \;\; s.t. \sum_{\tau=t+1}^{i} c_\tau \leq m_{t+1} + \sum_{\tau=t+2}^{i} I_\tau, \; \forall i \geq t+1.$$

This problem faced at time $t+1$ is an exact copy of the problem that is faced at time 0 (the problem of maximizing (1) sub to (2), as in section 9.1.2), and by a change of variables it can be seen that this maximal additional utility that can be attained in and after period $t+1$, is given by $\delta^{t+1}V^*(m_{t+1})$.

In the present setting at time t the amount of savings s_t is known, and the next period's budget $m_{t+1} = s_t + I_{t+1}$ is subject to uncertainty. Hence the maximal additional discounted expected utility that can be attained from s_t in and after period $t+1$, would be given by $E_{I_{t+1}}\delta^{t+1}V^*(s_t+I_{t+1})$ which equals $\delta^{t+1}EV^*(s_t)$ (the function EV^* is defined in section 9.1.2).

Then, as in the previous chapter, for any time t any consistent ad hoc utility function $\widetilde{u}^{(t)}(w_{t-1}, c_t, s_t)$ must be of the form

$$\widetilde{u}^{(t)}(w_{t-1}, c_t, s_t) = \widetilde{f}_t\left(\sum_{i=0}^{t} \delta^i u_0(c_i) + \delta^{t+1}EV^*(s_t)\right),$$

for some strictly increasing function $\widetilde{f}_t : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

For any time t the instantaneous utility $v^{(t)}(w_{t-1}, c_t) = \sum_{i=0}^{t} \delta^{i-t} u_0(c_i)$ is exogenous, so the above condition can be rewritten as the condition that any consistent ad hoc utility function $\widetilde{u}^{(t)}(w_{t-1}, c_t, s_t)$ should be of the form

$$\widetilde{u}^{(t)}(w_{t-1}, c_t, s_t) = f_t(v^{(t)}(w_{t-1}, c_t) + \delta EV^*(s_t)),$$

for some strictly increasing function $f_t : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ (here with $f_t(x) = \widetilde{f}_t(\delta^t x)$).

Thus we also see that, for any time t, any consistent time preference function $\widetilde{U}^{(t)}(v^{(t)}, s_t)$ must be of the form

$$\widetilde{U}^{(t)}(v^{(t)}, s_t) = f_t(v^{(t)} + \delta EV^*(s_t)),$$

for some strictly increasing function $f_t : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

Now we can investigate when condition 7.3.1, that a consistency-inducing scalar exists, will hold. The initial time preference function $U^{(0)}$ was given by $U^{(0)}(v^{(0)}, s_0) = v^{(0)} + \delta EV^{(0)}(s_0)$. Therefore, if given these initial time preferences a consistency-inducing scalar $\tilde{\theta}$ exists, then for any period t the function $v^{(t)} + \delta\tilde{\theta}^{-1}EV^{(0)}(s_t)$ should be a consistent time preference function, i.e. there should be a strictly increasing function f_t such that $v^{(t)} + \delta\tilde{\theta}^{-1}EV^{(0)}(s_t)$ equals $f_t(v^{(t)} + \delta EV^*(s_t))$.

Now, from proposition 8.3.1 it can be seen that, given an initial time preference function $U^{(0)}(v^{(0)}, s_0) = v^{(0)} + \delta EV^{(0)}(s_0)$, there exists a consistency-inducing scalar $\tilde{\theta}$ if and only if the initial expected value function $EV^{(0)}$ is an affine transformation of the optimal expected value function: $EV^{(0)}(m) = \tilde{\theta}EV^*(m) + \alpha$, for some $\alpha \in \mathbb{R}$.[37]

[37] While proposition 8.3.1 was set under certainty and dealt with ordinary value functions rather than expected value functions, the present setting can be fit into the setting of proposition 8.3.1. Both expected value functions and regular value functions are mathematically just functions mapping $\mathbb{R}_+$

Thus, like in the previous chapter, we have a condition on $EV^{(0)}$ that is both necessary and sufficient for the existence of a consistency-inducing scalar.

And like in the previous chapter, here convergence of a sequence of adjustment products towards a consistency-inducing scalar $\tilde{\theta}$ implies that the corresponding sequence of expected value functions will converge towards the optimal expected value function *up to a constant*. Again, such a constant has no effect on the choices that will be made, and we will keep referring to this kind of convergence as convergence towards optimality.

In general, given an initial time preference function, there exists at most one consistency-inducing scalar $\tilde{\theta}$. In the previous section we saw that in the cases of REE adjustment and of EEE adjustment, convergence always occurs, but that the limits towards which convergence occurs will generally not be the same for these two cases. Thus, if indeed these limits are not the same, then in at most one of the two cases this will mean that convergence towards optimality does take place. The subsequent sections of this chapter will investigate if and when convergence towards optimality will occur, first in the case of REE adjustment, and then in the case of EEE adjustment.

9.4 Convergence towards optimality with REE adjustment

Given the adjustment procedure for expected value functions as based on regular excess expenditure, we already saw that convergence will always occur, and here we investigate if this means convergence towards optimality.

That is, given an initial budget m_0 and an initial expected value function $EV^{(0)}$ that is used in period 0, and given that REE adjustment is applied in every period, will the resulting sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$ converge towards a consistency-inducing scalar $\tilde{\theta}$? Here we will see that in the present setting the answer to this question will generally be negative.

In comparison to the setting of the previous chapter, a complicating factor of the current setting is that now regular excess expenditure in any period does not only depend on the (expected) value function that is used and on the budget that is available in that particular period, but also on the additional income realization that occurs in the next period. The following lemma shows that, holding all other variables fixed, smaller additional income realizations will always yield larger (regular) excess expenditures. This makes sense, it seems quite intuitive that smaller additional income realizations would be more likely to lead to regret about having spent too much in the previous period.

Lemma 9.4.1 *Let a model be given that satisfies axioms 4.4.1 and 9.1.3, and with an instantaneous utility that satisfies axiom 8.1.1. Suppose that the additional income*

into $\mathbb{R}$. Therefore, in proposition 8.3.1 and in its proof, the term 'value function' and the notation $V^{(0)}$ and V^* can simply be replaced by the term 'expected value function' and the notation $EV^{(0)}$ and EV^* (when interpreted with the additional assumption that axiom 8.1.3 holds, so that $I = 0$), which yields the corresponding result.

random variables I_{t+1} satisfy 9.1.1, and that two realizations I^r and $I^{r'}$, with $I^{r'} < I^r$, are given. Then, if at time t a budget $m_t \in \mathbb{R}_+$ and an expected value function $EV^{(t)}$ that satisfies axiom 8.1.1 are given, $I_{t+1} = I^{r'}$ will yield a larger regular excess expenditure than $I_{t+1} = I^r$: $E_t(I^{r'}) \geq E_t(I^r)$.

Proof. Given is a model with an instantaneous utility function u_0 that satisfies axiom 8.1.1, and with a discount factor $0 < \delta < 1$. In every period $t \geq 1$ an uncertain additional income is obtained, that can be modelled by the random variable I_t. These random variables I_t follow an IID distribution that gives weight to only a finite number of realizations. In some specific period t a budget $m_t \in \mathbb{R}_+$, and an expected value function $EV^{(t)}$ that satisfies axiom 8.1.1 are given.

Here we will consider period-t regular excess expenditures for two different situations, corresponding to the different realizations I^r and $I^{r'}$ for the additional income random variable I_{t+1}. That is, excess expenditure in period t may depend on the additional income in period $t+1$, and we will use the notation $E_t(I_{t+1})$. Given the function $EV^{(t)}$ and the budget $m_t \in \mathbb{R}_+$, for an additional income I_{t+1} the regular excess expenditure $E_t(I_{t+1})$ is as before given by $E_t(I_{t+1}) = c_t^*(I_{t+1}) - c_t^{\triangleleft}(I_{t+1})$. Here $c_t^*(I_{t+1})$ is part of a solution to

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta EV^{(t)}(s_t).$$

Note that the additional income does not in any way influence $EV^{(t)}(s_t)$ or the decision problem that $c_t^*(I_{t+1})$ is supposed to maximize. Thus, the period-t choice $c_t^*(I_{t+1})$ will be the same for all additional income realizations, so that we may use the notation c_t^* instead of $c_t^*(I_{t+1})$.

Then, in period $t+1$, given the additional income I_{t+1}, $c_t^{\triangleleft}(I_{t+1})$ is determined as part of a tuple $(c_t^{\triangleleft}(I_{t+1}), c_{t+1}^{\triangleleft}(I_{t+1}), s_{t+1}^{\triangleleft}(I_{t+1}))$ that solves the hypothetical maximization problem

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta \max_{c_{t+1}+s_{t+1} \leq s_t+I_{t+1}} u_0(c_{t+1}) + \delta EV^{(t)}(s_{t+1}).$$

We may define the function $W_1 : \mathbb{R}_+ \to \mathbb{R}$ as the last part of the above formula, by

$$W_1(m_{t+1}) := \max_{c_{t+1}+s_{t+1} \leq m_{t+1}} u_0(c_{t+1}) + \delta EV^{(t)}(s_{t+1}).$$

Then $c_t^{\triangleleft}(I_{t+1})$ must also be part of a solution to

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta W_1(s_t + I_{t+1}). \tag{1}$$

Two realizations I^r and $I^{r'}$ for the additional income random variable I_{t+1} are given, with $0 \leq I^{r'} < I^r$. Then the quantities $c_t^{\triangleleft}(I^r)$ and $c_t^{\triangleleft}(I^{r'})$ are determined by the last maximization problem (1). Note that the function W_1 itself does not depend on the additional income realization. The only difference between the two maximization problems as in (1) that $c_t^{\triangleleft}(I^r)$ and $c_t^{\triangleleft}(I^{r'})$ are derived from, is that for each s_t the function W_1 will be evaluated at different points ($s_t + I^r$ or $s_t + I^{r'}$). Since $I^{r'} < I^r$, and by the fact that (by lemma 8.4.2) W_1 is a strictly concave function, we know that $W_1'(s_t + I^{r'}) > W_1'(s_t + I^r)$, for all $s_t \in \mathbb{R}_+$. Then by lemma 8.4.1 this implies that $c_t^{\triangleleft}(I^{r'}) \leq c_t^{\triangleleft}(I^r)$ must hold, so that

$$E_t(I^{r'}) = c_t^* - c_t^{\triangleleft}(I^{r'}) \geq c_t^* - c_t^{\triangleleft}(I^r) = E_t(I^r).$$

■

In the above proof we see that $W_1'(s_t+I^{r'}) > W_1'(s_t+I^r)$ holds for all $I^{r'} < I^r$. Then from the discussion after lemma 8.4.1 we see that $c_t^{\triangleleft}(I^{r'}) = c_t^{\triangleleft}(I^r)$ can only occur for boundary solutions. Thus, except for in boundary solutions we see that $E_t(I^{r'}) > E_t(I^r)$ will always hold for $I^{r'} < I^r$.

The dependence of regular excess expenditure on the next period's additional income realization from the previous lemma will also become apparent in the following proposition. This proposition shows that under REE adjustment the optimal expected value function EV^* is generally not a fixed point of the adjustment procedure.

Proposition 9.4.1 *Let a model be given that satisfies axioms 4.4.1 and 9.1.3, with instantaneous utility u_0 that satisfies axiom 8.1.1, and with additional income random variables I_{t+1} that satisfy 9.1.1. Suppose that at time t the prevailing expected value function $EV^{(t)}$ is identical to the optimal expected value function EV^*. Then, for any given budget $m_t \in \mathbb{R}_+$, the smallest realization I^0 for I_{t+1} will yield a non-negative regular excess expenditure: $E_t(m_t, I^0) \geq 0$, and the largest realization I^R for I_{t+1} will yield $E_t(m_t, I^R) \leq 0$. Moreover, if $\lim_{c\to\infty} u_0'(c) = 0$, then there is at most one additional income realization I^r such that the corresponding regular excess expenditure will always equal zero: $E_t(m_t, I^r) = 0$, for all m_t.*

Proof. Given is a model with a discount factor $0 < \delta < 1$, and an instantaneous utility function u_0 that satisfies axiom 8.1.1. In all periods after the first an uncertain additional income is obtained, which is represented by the random variable I_t. These random variables are IID distributed; in every period the corresponding random variable can take a finite number $R \geq 1$ of realizations $\{I^0, I^1, ..., I^R\}$ (where $I^r \in \mathbb{R}_+$ for all $0 \leq r \leq R$) with probabilities$\{\pi^0, \pi^1, ..., \pi^R\}$. Also given is that in some specific period t, the prevailing expected value function $EV^{(t)}$ is identical to the optimal expected value function: $EV^{(t)}(m) = EV^*(m)$, for all $m \in \mathbb{R}_+$.

Then given a budget $m_t \in \mathbb{R}_+$, and additional income I_{t+1}, regular excess expenditure $E_t(m_t, I_{t+1})$ is as before given by $E_t(m_t, I_{t+1}) = c_t^*(m_t) - c_t^{\triangleleft}(m_t, I_{t+1})$. Here $c_t^*(m_t)$ is part of a pair that solves

$$\max_{(c_t,s_t):c_t+s_t\leq m_t} u_0(c_t) + \delta EV^*(s_t) = \max_{c_t+s_t\leq m_t} u_0(c_t) + \delta \sum_{r=0}^{R} \pi_r V^*(s_t + I^r).$$

As in the previous lemma, additional income I_{t+1} does not influence the decision problem that $c_t^*(m_t)$ is supposed to maximize, so the period-t choice $c_t^*(m_t)$ does not depend on the period-$(t+1)$ additional income I_{t+1}.

Then, in period $t+1$, $c_t^{\triangleleft}(m_t, I_{t+1})$ is determined as part of a tuple $(c_t^{\triangleleft}(m_t, I_{t+1}), c_{t+1}^{\triangleleft}(m_t, I_{t+1}), s_{t+1}^{\triangleleft}(m_t, I_{t+1}))$ that solves the hypothetical maximization problem

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta \max_{c_{t+1}+s_{t+1} \leq s_t+I_{t+1}} u_0(c_{t+1}) + \delta EV^*(s_{t+1}).$$

And since the optimal value function V^* solves the functional equation

$$V^*(m_{t+1}) = \max_{c_{t+1}+s_{t+1} \leq m_{t+1}} u_0(c_{t+1}) + \delta EV^*(s_{t+1}), \tag{1}$$

we see that $c_t^{\triangleleft}(m_t, I_{t+1})$ is also part of a solution to

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta V^*(s_t + I_{t+1}).$$

▲ First suppose that in period $t+1$ the smallest realization I^0 for the additional income random variable I_{t+1} occurs. Then for all I^r we have that $I^r \geq I^0$, and by strict concavity of V^* we know that $V^{*\prime}(s_t + I^r) \leq V^{*\prime}(s_t + I^0)$ holds for all $s_t \in \mathbb{R}_+$. This also implies that

$$EV^{*\prime}(s_t) = \sum_{r=0}^{R} \pi_r V^{*\prime}(s_t + I^r) \leq V^{*\prime}(s_t + I^0)$$

holds for all $s_t \in \mathbb{R}_+$. Then by lemma 8.4.1 we see that $c_t^*(m_t) \geq c_t^{\triangleleft}(m_t, I^0)$ will always hold, and indeed we get that $E_t(m_t, I^0) = c_t^*(m_t) - c_t^{\triangleleft}(m_t, I^0) \geq 0$, for any budget m_t.

▲ Then suppose that the largest realization I^R for the random variable I_{t+1} occurs. Then similarly, for all I^r we have that $I^R \geq I^r$, so by strict decreasingness of $V^{*\prime}$ we see that $V^{*\prime}(s_t + I^r) \geq V^{*\prime}(s_t + I^R)$, and thus that $EV^{*\prime}(s_t) \geq V^{*\prime}(s_t + I^R)$ will hold for all $s_t \in \mathbb{R}_+$. With lemma 8.4.1 we get that $c_t^*(m_t) \leq c_t^{\triangleleft}(m_t, I^R)$, and indeed $E_t(m_t, I^R) = c_t^*(m_t) - c_t^{\triangleleft}(m_t, I^R) \leq 0$ is satisfied for all budgets $m_t \in \mathbb{R}_+$.

▲ Then additionally suppose that $\lim_{c\to\infty} u_0'(c) = 0$ holds, and suppose that the last part of the proposition is not true. That is, suppose that there are two realizations I^r and $I^{r'}$ (with $I^{r'} \neq I^r$) for the additional income random variable I_{t+1}, that will both yield that $E_t(m_t, I^{r'}) = E_t(m_t, I^r) = 0$ for all budgets $m_t \in \mathbb{R}_+$. Then for all m_t it needs to hold that $c_t^*(m_t) = c_t^{\triangleleft}(m_t, I^{r'})$, and that $c_t^*(m_t) = c_t^{\triangleleft}(m_t, I^r)$. As seen above, $c_t^*(m_t)$ does not depend on the additional income realization, so we must also have that $c_t^{\triangleleft}(m_t, I^{r'}) = c_t^{\triangleleft}(m_t, I^r)$ for all m_t.

If for some m_t it would hold that $0 < c_t^{\triangleleft}(m_t, I^{r'}) = c_t^{\triangleleft}(m_t, I^r) < m_t$, then we should have that

$$u_0'(c_t^{\triangleleft}(m_t, I^r)) = \delta V^{*\prime}(m_t - c_t^{\triangleleft}(m_t, I^r) + I^r),$$

and similarly that

$$u_0'(c_t^{\triangleleft}(m_t, I^{r'})) = \delta V^{*\prime}(m_t - c_t^{\triangleleft}(m_t, I^{r'}) + I^{r'}).$$

However, $u_0'(c_t^{\triangleleft}(m_t, I^{r'}))$ needs to equal $u_0'(c_t^{\triangleleft}(m_t, I^r))$, while by strict concavity of V^* we know that $V^{*\prime}(m_t - c_t^{\triangleleft}(m_t, I^r) + I^r)$ can never be equal to $V^{*\prime}(m_t - c_t^{\triangleleft}(m_t, I^{r'}) + I^{r'})$. Thus $0 < c_t^{\triangleleft}(m_t, I^{r'}) = c_t^{\triangleleft}(m_t, I^r) < m_t$ is not possible.

Therefore, for $c_t^*(m_t)$, $c_t^{\triangleleft}(m_t, I^{r'})$ and $c_t^{\triangleleft}(m_t, I^r)$, we should get boundary solutions for all budgets. And it can be shown (e.g. Luenberger [28], p. 462) that the function $c_t^*(m_t)$ will be continuous in m_t. Therefore one of the following two cases must occur, and we will show that both cases yield a contradiction.

(I) $c_t^*(m_t) = c_t^{\triangleleft}(m_t, I^r) = c_t^{\triangleleft}(m_t, I^{r'}) = 0$, for all m_t,

(II) $c_t^*(m_t) = c_t^{\triangleleft}(m_t, I^r) = c_t^{\triangleleft}(m_t, I^{r'}) = m_t$, for all m_t.

If case (I) holds, then we must have that $u_0'(0) \leq \delta V^{*\prime}(m_t + I^r)$ for all m_t. However, in subsection 9.1.2 we saw that if u_0 satisfies axiom 8.1.1, so will V^*, so that V^* is differentiable on $\mathbb{R}_{++}$. Thus for all $m_t > 0$ it will hold that $V^{*\prime}(m_t + I^r) < \infty$, and with $u_0'(0) = \infty$, we see that case (I) is impossible.

If case (II) holds, then we must have that $u_0'(m_t) \geq \delta V^{*\prime}(I^r)$ for all m_t. However, since V^* solves (1), by lemma 8.4.2 we know that $V^{*\prime}(I^r) \geq u_0'(I^r) > 0$. Therefore since $\lim_{c\to\infty} u_0'(c) = 0$ holds, for m_t large enough $u_0'(m_t)$ will drop below $\delta V^{*\prime}(I^{r'}) > 0$, so that case (II) is also impossible.

Hence, we indeed see that there cannot be two realizations for the additional income random variable that will both yield regular excess expenditures equal to zero for all budgets. ■

The above proposition shows that if $\lim_{c\to\infty} u_0'(c) = 0$ holds, there is *at most* one additional income realization $I^{r'}$ that will always yield a regular excess expenditure equal to zero, given $EV^{(t)} = EV^*$. Thus, such an additional income realization that always yields a zero REE need not exist. If there *would* exist one such additional income realization $I^{r'}$ that would always yield a zero REE, then obviously in all periods this realization $I^{r'}$ would only occur with probability $\pi_{r'} < 1$. For all other realizations the regular excess expenditure would not (always) equal zero. In fact, by the above lemma and the above proposition, for all $I^r > I^{r'}$ it would hold that $E_t(m_t, I^r) \leq 0$ for all $m_t \in \mathbb{R}_+$ and $E_t(m_t, I^r) < 0$ for some m_t, and for all $I^{r''} < I^{r'}$ it would hold that $E_t(m_t, I^{r''}) \geq 0$ for all $m_t \in \mathbb{R}_+$ and $E_t(m_t, I^r) > 0$ for some m_t.

If there is no realization $I^{r'}$ that always yields a regular excess expenditure equal to zero, then $E_t(m_t, I^r) = 0$ will only occur for specific combinations of I^r and m_t.

From the above proof we see that the condition $\lim_{c\to\infty} u_0'(c) = 0$ that is needed for this proposition is only used to rule out the possibility of boundary solutions where all is spent (for $c_t^*(m_t)$, $c_t^{\triangleleft}(m_t, I^r)$ and $c_t^{\triangleleft}(m_t, I^{r'})$). If this condition $\lim_{c\to\infty} u_0'(c) = 0$ is *not* satisfied, then there may exist multiple additional income realizations that always yield a zero REE. In fact, then it may even happen that REE will equal zero for all additional income realizations, given $EV^{(t)} = EV^*$. If indeed $\lim_{c\to\infty} u_0'(c) > 0$ holds, then from the last part of the above proof we can see that if the smallest additional income realization I^0 is sufficiently large (if $\lim_{c\to\infty} u_0'(c) \geq \delta V^{*\prime}(I^0)$), then it will hold that $u_0'(m_t) \geq \delta V^{*\prime}(I^0) \geq \delta V^{*\prime}(I^r)$, and thus that $c_t^{\triangleleft}(m_t, I^r) = m_t$ for all m_t and all realizations I^r. The above condition on I^0 would also imply that $\lim_{c\to\infty} u_0'(c) \geq \delta V^{*\prime}(I^0) \geq \delta EV^{*\prime}(0)$, which yields that $u_0'(m_t) \geq \delta EV^{*\prime}(0)$, and thus that $c_t^*(m_t) = m_t$ for all m_t. In this case it would indeed hold that $E_t(m_t, I^r) = 0$ for all m_t and all realizations I^r, as only boundary solutions where all is spent will occur.

More interestingly, though, if the condition $\lim_{c\to\infty} u_0'(c) = 0$ *is* satisfied, then with the above proposition we can see that in the present setting the conditions needed for convergence towards optimality will not be met, and in fact that convergence towards optimality will generally not occur. Conditions 7.3.1, 7.3.2 and 7.3.3 were needed to prove that convergence towards optimality will occur. In the previous section we saw that condition 7.3.1 (that a CIS exists) will hold if and only if the initial expected value function is an affine transformation of the optimal expected value function. However, even in situations where condition 7.3.1 does hold, condition 7.3.2 will not hold in the present setting.

Condition 7.3.2 entailed that the consistency-inducing scalar should be stable, i.e. that given an initial expected value function $EV^{(0)}$ and a CIS $\widetilde{\theta}$, setting $\theta_t = \tilde{\theta}$ would *always* yield $E_t = 0$. If given $EV^{(0)}$, $\tilde{\theta}$ is a consistency-inducing scalar, then $EV^{(0)}$ must be of the form $EV^{(0)}(m) = \tilde{\theta}EV^*(m) + \alpha$, for some $\alpha \in \mathbb{R}$. Then setting $\theta_t = \tilde{\theta}$ in some period t would give that the time-t expected value function $EV^{(t)} = \theta_t^{-1} EV^{(0)}$ would equal $\tilde{\theta}^{-1}(\tilde{\theta}EV^* + \alpha) = EV^* + \tilde{\theta}^{-1}\alpha$. As in proposition 8.4.2 the constant $\tilde{\theta}^{-1}\alpha$ does not influence (regular) excess expenditure, and from (the last part of) the previous proposition we know that the optimal expected value function EV^* will not always yield an excess expenditure equal to zero.

Thus condition 7.3.2 is violated, so that proposition 7.3.1 cannot be applied. Of course, conditions 7.3.2 and 7.3.3 were only sufficient for convergence towards optimality to occur, so just the fact that condition 7.3.2 is not satisfied does not prove that convergence towards optimality will not occur. In fact, in the present setting convergence towards optimality *can* occur, although it will only occur in very special circumstances.

For instance, if $\lim_{c\to\infty} u_0'(c) = 0$ holds, then setting $\theta_t = \tilde{\theta}$ will yield $E_t > 0$ for $I_{t+1} = I^0$ and some m_t.[38] Regular excess expenditure E_t is continuous in θ_t (see e.g. Luenberger ([28] p.462)). Therefore for any $\theta' < \tilde{\theta}$, but with θ' sufficiently close to $\widetilde{\theta}$,

[38] To see this, the above proposition shows that $E_t(m_t, I^0) \geq 0$ will always hold, and here we will see that I^0 cannot be the unique realization that always yields $E_t = 0$ for $\theta_t = \tilde{\theta}$.

Suppose that for some m_t it holds that $0 < c_t^*(m_t) = c_t^{\triangleleft}(m_t, I^0) < m_t$. Then it should hold that

$$u_0'(c_t^*(m_t)) = \delta EV^{*\prime}(m_t - c_t^*(m_t)),$$

and similarly that

$$u_0'(c_t^{\triangleleft}(m_t, I^0)) = \delta V^{*\prime}(m_t - c_t^{\triangleleft}(m_t, I^0) + I^0).$$

Thus, $c_t^*(m_t) = c_t^{\triangleleft}(m_t, I^0)$ implies that the two left-hand-sides equal, and for $s_t^* = m_t - c_t^*(m_t)$, the above equalities imply that $EV^{*\prime}(s_t^*) = V^{*\prime}(s_t^* + I^0)$. However, this cannot hold as by strict concavity of V^* we know that

$$EV^{*\prime}(s_t^*) = \sum_{r=0}^{R} \pi_r V^{*\prime}(s_t^* + I^r) < V^{*\prime}(s_t^* + I^0).$$

Moreover, as in case (I) from the last part of the above proof it can be seen that $c_t^*(m_t) = c_t^{\triangleleft}(m_t, I^0) = 0$ is not possible for any m_t.

Finally, as in case (II) from the last part of the above proof $c_t^*(m_t) = c_t^{\triangleleft}(m_t, I^0) = m_t$ can only occur for small m_t.

setting $\theta_t = \theta'$ would also yield $E_t > 0$ for m_t and I^0. (This would also violate condition 7.3.3.) From the last part of section 6.5 we know that in this case all subsequent adjustment products will be smaller than $\theta_t = \theta'$. In this case adjustment does not work in the right direction, and after this we can never get convergence of the adjustment products towards $\tilde{\theta}$.

Similarly, setting $\theta_t = \tilde{\theta}$ would yield $E_t \leq 0$ for $I_{t+1} = I^R$ and some m_t. And for any θ' larger than $\tilde{\theta}$ but sufficiently close to $\widetilde{\theta}$, setting $\theta_t = \theta'$ would also yield $E_t < 0$ for m_t and I^R, which would cause all subsequent adjustment products to be larger than θ_t, so that convergence towards $\tilde{\theta}$ will not occur.

We know that convergence towards optimality can only occur if $EV^{(0)} = \widetilde{\theta} EV^* + \alpha$. Then, convergence towards optimality that takes infinitely long (where the two variables $\theta_t^{+\min}$ and $\theta_t^{-\max}$ never equal, but become arbitrarily close to each other and to $\widetilde{\theta}$ as t goes to infinity) will only occur if $E_t > 0$ does always occur for all $\theta_t > \tilde{\theta}$, and if $E_t < 0$ does always occur for all $\theta_t < \tilde{\theta}$. However, as we have just seen, for θ_t close to $\widetilde{\theta}$, the signs of excess expenditure will also depend on whether large or small additional income realizations occur. Thus, adjustment may not always work in the right direction, and for θ_t close to $\widetilde{\theta}$ the probabilities $\pi(E_t > 0|\theta_t, m_t)$ and $\pi(E_t < 0|\theta_t, m_t)$ will both be strictly smaller than one. Hence, convergence towards optimality that takes infinitely long may only occur with probability zero.

And convergence towards optimality in finite time (where the two variables $\theta_t^{+\min}$ and $\theta_t^{-\max}$ are equal and equal to $\widetilde{\theta}$ in and from some period t' onwards) will only occur in the special case where in some period t' the prevailing expected value function would be an affine transformation of the optimal expected value function: $EV^{(t')} = EV^* + \alpha_{t'}$, and where the budget $m_{t'}$ and the additional income realization I^r would incidentally be such that $E_{t'}(m_{t'}, I^r) = 0$.

9.5 Convergence towards optimality with EEE adjustment

In the previous section we saw that in consumption/savings models with expected utility, convergence towards optimality will generally not occur in the more basic case of REE adjustment. In the present section we will consider the case of EEE adjustment, and here we will see that under some circumstances convergence towards optimality *will* occur.

The analysis in this section is similar to that of the previous chapter, and especially to that of the first part of section 8.4 and subsection 8.4.1. We start with a proposition that is a counterpart of proposition 8.4.1, it shows that expected excess expenditure will always equal zero in some period if in that period the prevailing expected value function equals the optimal expected value function. Recall from the previous section that this property did not hold for regular excess expenditure.

Proposition 9.5.1 *Let a model be given that satisfies axioms 4.4.1, 9.1.1 and 9.1.3, and with an instantaneous utility that satisfies axiom 8.1.1. Then, if the period-t expected value function $EV^{(t)}$ is identical to the optimal expected value function $EV^{(t)} = EV^*$, the corresponding expected excess expenditure will be zero: $F_t = 0$.*

Proof. Given is a model with an instantaneous utility function u_0 that satisfies axiom 8.1.1 and with a discount factor $0 < \delta < 1$. The additional periodical incomes I_t, for $t \geq 1$, are IID distributed and each can take a finite number of realizations $\{I^0, I^1, ..., I^R\}$ (where $I^r \in \mathbb{R}_+$ for all $0 \leq r \leq R$) with probabilities $\{\pi_0, \pi_1, ..., \pi_R\}$. The expected value function that is used in period t is identical to the optimal expected value function: $EV^{(t)}(m) = EV^*(m)$, for all $m \in \mathbb{R}_+$. We will now show that, for any period-t budget $m_t \in \mathbb{R}_+$ it will hold that $c_t^* = c_t^\diamond$, so that $F_t = 0$.

The quantity c_t^* is part of a solution to

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta EV^*(s_t) = \max_{c_t+s_t \leq m_t} u_0(c_t) + \delta \sum_{r=0}^{R} \pi_r V^*(s_t + I^r).$$

The optimal value function V^* solves the functional equation

$$V^*(m) = \max_{c+s \leq m} u_0(c) + \delta EV^*(s),$$

so that c_t^* will also be part of solution to the following problem:

$$\max_{c_t+s_t \leq m_t} u_0(c_t) + \delta \sum_{r=0}^{R} \pi_r [\max_{c_{t+1}+s_{t+1} \leq s_t + I^r} u_0(c_{t+1}) + \delta EV^*(s_{t+1})].$$

Of course, the quantity $c_t^\diamond$ would be determined as part of a plan that maximizes exactly this last problem. And since instantaneous utility and the value functions are strictly concave, so is their sum, and we see that solutions are unique. Thus indeed we find that $c_t^* = c_t^\diamond$, and that $F_t = 0$. ■

The previous proposition implies that the optimal expected value function EV^* is a fixed point of the adjustment procedure. The adjustment function satisfies $a_{\phi_t} = 1$ if $F_t = 0$, so this proposition implies that if at time t the prevailing expected value function $EV^{(t)}$ is identical to the optimal expected value function EV^*, then in period $t+1$ the adjusted expected value function $EV^{(t+1)} = a_{\phi_t}^{-1} EV^{(t)}$ will also equal the optimal expected value function EV^*.

The next proposition is a counterpart of proposition 8.4.2, and it shows that adding a constant to expected value functions does not change the resulting excess expenditures.

Proposition 9.5.2 *Let a model be given that satisfies axioms 4.4.1, 9.1.1 and 9.1.3, and with an instantaneous utility that satisfies axiom 8.1.1. Suppose we are given two period-t expected value functions* $\overline{EV}^{(t)} : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $\underline{EV}^{(t)} : \mathbb{R}_+ \to \mathbb{R}_+$ *that satisfy axiom 8.1.1, and that are such that* $\overline{EV}^{(t)}(m) = \underline{EV}^{(t)}(m) + \alpha$*, for all* $m \in \mathbb{R}_+$ *and some constant* $\alpha \in \mathbb{R}_+$*. Then for any budget* $m_t \in \mathbb{R}_+$ *the corresponding expected excess expenditures* $\overline{F}_t$ *and* $\underline{F}_t$*, will always satisfy* $\overline{F}_t = \underline{F}_t$*.*

Proof. Given is a model with an instantaneous utility function u_0 that satisfies axiom 8.1.1, and a discount factor $0 < \delta < 1$. The additional periodical incomes I_t, for $t \geq 1$, are IID distributed and each can take a finite number of realizations. Also given are two period-t expected value functions $\overline{EV}^{(t)} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\underline{EV}^{(t)} : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy axiom 8.1.1, and that differ by a constant: $\overline{EV}^{(t)}(m) = \underline{EV}^{(t)}(m) + \alpha$, for all $m \in \mathbb{R}_+$ and some scalar $\alpha \in \mathbb{R}_+$. Then, given any initial budget m_t, expected excess expenditure can be determined for both value functions, and we want to show that $\overline{F}_t = \overline{c}_t^* - \overline{c}_t^\diamond$ will equal $\underline{F}_t = \underline{c}_t^* - \underline{c}_t^\diamond$.

The choice $(\underline{c}_t^*, \underline{s}_t^*)$ that will be made in period t, given the prevailing expected value function $\underline{EV}^{(t)}(.)$ will attain

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \underline{EV}^{(t)}(s_t).$$

Now, it is mathematically obvious that a solution to this last problem will also solve

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \underline{EV}^{(t)}(s_t) + \delta\alpha = \max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \overline{EV}^{(t)}(s_t).$$

This last maximization problem is of course exactly the one that $(\overline{c}_t^*, \overline{s}_t^*)$ is supposed to solve. Therefore, by strict concavity (u_0, $\overline{EV}^{(t)}$ and $\underline{EV}^{(t)}$ satisfy axiom 8.1.1) solutions are unique, and we see that $\underline{c}_t^* = \overline{c}_t^*$.

The variable $\underline{c}_t^\diamond$ will be part of a plan that solves

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta E_I[\max_{c_{t+1} + s_{t+1} \leq s_t + I} u_0(c_{t+1}) + \delta \underline{EV}^{(t)}(s_{t+1})].$$

Then, $\underline{c}_t^\diamond$ will also be part of a plan that solves

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta E_I[\max_{c_{t+1} + s_{t+1} \leq s_t + I} u_0(c_{t+1}) + \delta \underline{EV}^{(t)}(s_{t+1}) + \delta\alpha] =$$

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta E_I[\max_{c_{t+1} + s_{t+1} \leq s_t + I} \delta u_0(c_{t+1}) + \delta \overline{EV}^{(t)}(s_{t+1})].$$

The quantity $\underline{c}_t^\diamond$ maximizes this last problem, and by strict concavity both $\underline{c}_t^\diamond$ and $\underline{c}_t^\diamond$ will be unique, so that $\underline{c}_t^\diamond = \overline{c}_t^\diamond$ must hold.

Hence the expected excess expenditures $\overline{F}_t$ and $\underline{F}_t$ will indeed satisfy

$$\overline{F}_t = \overline{c}_t^* - \overline{c}_t^\diamond = \underline{c}_t^* - \underline{c}_t^\diamond = \underline{F}_t.$$

■

As in proposition 8.3.1, with the above two propositions it can be shown that in the present setting condition 7.3.2 (that the CIS is stable) holds. This was one of three conditions that together were sufficient for establishing that convergence towards optimality will occur. Here we will not explicitly prove that this condition holds. Instead, a proof is implicit in the proof of the next proposition, which directly establishes that (or when) convergence towards optimality will occur.

But first we need a lemma. The next lemma is a counterpart of lemma 8.4.5. It shows that in the present setting EEE adjustment always works in the right direction.

If an expected value function equals a scalar times the optimal expected value function $EV^{(t)} = \gamma_t EV^*$, then for $\gamma_t < 1$ the expected excess expenditure will be strictly positive, which will be followed by a strictly heavier weight for savings, so that $EV^{(t+1)} = \gamma_{t+1} EV^*$ with $\gamma_{t+1} > \gamma_t$. And conversely, if $\gamma_t > 1$, then the lemma shows that expected excess expenditure will be strictly negative, which will be followed by a strictly smaller weight for savings, so that $\gamma_{t+1} < \gamma_t$. This lemma would also be sufficient for showing that in the present setting condition 7.3.3 (that adjustment products larger (smaller) than the CIS will yield positive (negative) excess expenditures) will hold. Again, a proof that this condition holds is implicit in the proof of the next proposition.

Lemma 9.5.1 *Let a model be given that satisfies axioms 4.4.1, 9.1.2 and 9.1.3, and with an instantaneous utility that satisfies axiom 8.1.2. Then, for some period t, suppose given a budget $m_t > 0$ and an expected value function $EV^{(t)} = \gamma_t EV^*$ that equals a scalar $\gamma_t \in \mathbb{R}_{++}$ times the optimal expected value function $EV^* : \mathbb{R}_+ \to \mathbb{R}_+$. Then $\gamma_t < 1$ implies that the corresponding expected excess expenditure will satisfy $F_t > 0$, and $\gamma_t > 1$ implies that $F_t < 0$.*

Proof. Given is a model with an instantaneous utility function u_0 that satisfies axiom 8.1.1 and $u_0'(0) = \infty$, and with a discount factor $0 < \delta < 1$. In every period after period 0 an uncertain additional income I_t will be obtained. The random variables I_t, for $t \geq 1$, are IID distributed and take only a finite number of realizations $\{I^0, I^1, ..., I^R\}$ with probabilities $\{\pi_0, \pi_1, ..., \pi_R\}$. The smallest additional income realization equals zero: $I^0 = 0$.

Then, for some period t, suppose given an expected value function $EV^{(t)} : \mathbb{R}_+ \to \mathbb{R}$ that equals a scalar $\gamma_t \in \mathbb{R}_{++}$ times the optimal expected value function $EV^* : \mathbb{R}_+ \to \mathbb{R}_+$ that solves the functional equation: $EV^{(t)}(m) = \gamma_t EV^*(m)$, for all $m \in \mathbb{R}_+$.

▲ First suppose that $\gamma_t \in (0, 1)$. We now want to establish that for any given budget $m_t > 0$, the expected excess expenditure $F_t = c_t^* - c_t^\diamond$ is strictly positive. The quantity c_t^* denotes the actual period-t choice corresponding to $EV^{(t)}$, given the budget m_t. That is, there is a $s_t^* \in \mathbb{R}_+$ such that (c_t^*, s_t^*) solves

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \gamma_t EV^*(s_t) = \tag{1}$$

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \sum_{r=0}^{R} \pi_r \gamma_t V^*(s_t + I^r). \tag{1'}$$

The second term that the excess expenditure depends on is $c_t^\diamond$, which is determined in period $t+1$ as part of a plan that solves

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta E_I[\max_{c_{t+1} + s_{t+1} \leq s_t + I} u_0(c_{t+1}) + \delta \gamma_t EV^*(s_{t+1})] = \tag{2}$$

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta \sum_{r=0}^{R} \pi_r [\max_{c_{t+1} + s_{t+1} \leq s_t + I^r} u_0(c_{t+1}) + \delta \gamma_t EV^*(s_{t+1})]. \tag{2'}$$

We now write the function $W_{\gamma_t} : \mathbb{R}_+ \to \mathbb{R}$ as the last part of this last formula within the brackets:

$$W_{\gamma_t}(m_{t+1}) := \max_{c_{t+1}+s_{t+1}\leq m_{t+1}} u_0(c_{t+1}) + \delta\gamma_t EV^*(s_{t+1}), \tag{3}$$

and we define the function $EW_{\gamma_t} : \mathbb{R}_+ \to \mathbb{R}$ as the expectation of W_{γ_t} with respect to additional income:

$$EW_{\gamma_t}(s_t) := \sum_{r=0}^{R} \pi_r W_{\gamma_t}(s_t + I^r). \tag{4}$$

Then we can also write (2') in analogy to (1) as

$$\max_{c_t+s_t\leq m_t} u_0(c_t) + \delta EW_{\gamma_t}(s_t). \tag{2”}$$

For the first part of the proposition, we want to show that $c_t^* > c_t^\diamond$, by showing that the functions $\gamma_t EV^*$ and EW_{γ_t} satisfy the conditions required for the second part of lemma 8.4.1. Recall that lemma 8.4.1 applies for functions $\underline{V} : \mathbb{R}_+ \to \mathbb{R}$ and $\overline{V} : \mathbb{R}_+ \to \mathbb{R}$ that satisfy axiom 8.1.1, and although thus far the lemma has only been applied to regular value functions, it can also be applied to expected value functions, as long as they satisfy axiom 8.1.1. Indeed, in subsection 9.1.2 we saw that if u_0 satisfies axiom 8.1.1, then so do V^* and EV^*. Then by lemma 8.4.2 the function W_{γ_t} will satisfy axiom 8.1.1, and consequently the function EW_{γ_t} will also satisfy axiom 8.1.1.

Then, to apply lemma 8.4.1 we require that $\gamma_t EV^*$ and EW_{γ_t} satisfy the following conditions:

(I) $\gamma_t EV^{*\prime}(s_t) < EW'_{\gamma_t}(s_t)$, for all $s_t \in \mathbb{R}_+$,
(II) $\delta\gamma_t EV^{*\prime}(m_t) < u_0'(0)$,
(III) $\delta EW'_{\gamma_t}(0) > u_0'(m_t)$.

(III) For the third condition, recall that $EW'_{\gamma_t}(0) = \sum_{r=0}^{R} \pi_r W'_{\gamma_t}(I^r)$, and that from lemma 8.4.2 we know that $W'_{\gamma_t}(I^r) \geq u_0'(I^r)$. The smallest additional income realization is $I^0 = 0$, and because of $u_0'(0) = \infty$ we see that $W'_{\gamma_t}(I^0) = \infty$. Therefore $\pi_0 > 0$ implies that $EW'_{\gamma_t}(0) = \infty$. On the other hand, $m_t > 0$ implies that $u_0'(m_t) \in \mathbb{R}$, and this condition is met.

(II) The second condition reads $\delta\gamma_t EV^{*\prime}(m_t) < u_0'(0)$. Again, $EV^{*\prime}(m_t) = \sum_{r=0}^{R} \pi_r V^{*\prime}(m_t+I^r)$, and V^* satisfies lemma 8.1.1, so $m_t > 0$ implies that $V^{*\prime}(m_t+I^r) \in \mathbb{R}$ for all I^r, and thus that $EV^{*\prime}(m_t) \in \mathbb{R}$. Hence by $u_0'(0) = \infty$ this condition is satisfied.

(I) The first condition reads $\gamma_t EV^{*\prime}(s_t) < EW'_{\gamma_t}(s_t)$, or

$$\sum_{r=0}^{R} \pi_r \gamma_t V^{*\prime}(s_t + I^r) < \sum_{r=0}^{R} \pi_r W'_{\gamma_t}(s_t + I^r),$$

for all $s_t \in \mathbb{R}_+$.

This requirement will be fulfilled if the functions $\gamma_t V^*$ and W_{γ_t} satisfy: $\gamma_t V^{*\prime}(m_{t+1}) < W'_{\gamma_t}(m_{t+1})$, for all $m_{t+1} \in \mathbb{R}_+$. To show that this is indeed the case, we want to apply the second part of lemma 8.4.3. As before, the function W_{γ_t} is defined by (3), so that it is an example of a function W_γ, as defined in lemma 8.4.3. And the function $\gamma_t V^*$ satisfies

$$\gamma_t V^*(m_{t+1}) = \max_{c_{t+1}+s_{t+1} \leq m_{t+1}} \gamma_t u_0(c_{t+1}) + \delta \gamma_t EV^*(s_{t+1})$$

so that it is an example of a function γW_1, as defined in lemma 8.4.3. Now, since $\gamma_t < 1$, the second part of lemma 8.4.3 can be applied if the additional assumption $\delta EV'(0) \leq u_0'(0)$ is satisfied. This additional assumption is simply implied by $u_0'(0) = \infty$. Thus, lemma 8.4.3 does apply to show that

$$W'_{\gamma_t}(m_{t+1}) > \gamma_t V^{*\prime}(m_{t+1}),$$

for all $m_{t+1} \in \mathbb{R}_+$. This in turn implies that $\gamma_t EV^{*\prime}(s_t) < EW'_{\gamma_t}(s_t)$, and the first condition is satisfied.

Thus indeed all conditions are satisfied, so that $c_t^* > c_t^\diamond$, and $F_t > 0$.

▲ A similar reasoning holds for the case that $\gamma_t > 1$. Then we want to show that $c_t^* < c_t^\diamond$, by using the second part of lemma 8.4.1. To be able to use this lemma, the functions $\gamma_t EV^*$ and EW_{γ_t} need to satisfy the conditions:

(I') $\gamma_t EV^{*\prime}(s_t) > EW'_{\gamma_t}(s_t)$, for all $s_t \in \mathbb{R}_+$,

(II') $\delta EW'_{\gamma_t}(m_t) < u_0'(0)$,

(III') $\delta \gamma_t EV^{*\prime}(0) > u_0'(m_t)$.

Condition (III') can be shown to hold in a way similar to (III) above. We know that $EV^{*\prime}(0) = \sum_{r=0}^R \pi_r V^{*\prime}(I^r)$, and V^* solves the functional equation, so by lemma 8.4.2 we know that $V^{*\prime}(I^r) \geq u_0'(I^r)$. Therefore, $I^0 = 0$ and $u_0'(0) = \infty$ imply that $V^{*\prime}(I^0) = \infty$, and that $EV^{*\prime}(0) = \infty$. And $m_t > 0$ implies that $u_0'(m_t) \in \mathbb{R}$, and this condition is met.

For condition (II'), recall that $EW'_{\gamma_t}(m_t) = \sum_{r=0}^R \pi_r W'_{\gamma_t}(m_t + I^r)$. The functions u_0 and EV^* satisfy axiom 8.1.1, and by lemma 8.4.2 so does W_{γ_t}. This means that W_{γ_t} is differentiable on $\mathbb{R}_{++}$, and by $m_t > 0$ we see that $W'_{\gamma_t}(m_t + I^r) < \infty$ for all I^r, and that $EW'_{\gamma_t}(m_t) < \infty$. Then by $u_0'(0) = \infty$ this condition is satisfied.

Condition (I') can be shown to hold like (I) above, by reversing the inequalities.

Thus indeed all conditions are satisfied, so that $c_t^* < c_t^\diamond$, and $F_t < 0$. ■

One of the conditions that are needed for this lemma (u_0 satisfying axiom 8.1.2) implies that marginal instantaneous utility becomes infinitely large near zero: $u_0'(0) = \infty$. As in the discussion before proposition 8.4.4, this condition ensures that all actual expenditures c_t^* and all ex-post optimal expenditures $c_t^\diamond$ will be strictly positive: $c_t^* > 0$ and $c_t^\diamond > 0$. Moreover, like in the discussion before proposition 8.4.4, the optimal value function V^* solves the functional equation (as in subsection 9.1.2) so by lemma 8.4.2 we know that $V^{*\prime}(0) \geq u_0'(0) = \infty$, and the derivative of the optimal value function will also become infinitely large near zero. And if $I^0 = 0$ (and axiom 8.1.2) holds, then consequently the derivative of the optimal expected value function will also become infinitely large near zero: $EV^{*\prime}(0) = \sum_{r=0}^{R} V^{*\prime}(I^r) = \infty$.

Then, $EV^{(t)} = \gamma_t EV^*$ implies that $EV^{(t)\prime}(0) = \infty$, so that $c_t^* < m_t$ will always hold. And similarly, $EV^{(t)} = \gamma_t EV^*$ implies that $W'_{\gamma_t}(0) = \max\{\delta\gamma_t EV^{*\prime}(0)), u_0'(0)\} = \infty$, and $c_t^\diamond < m_t$ will always hold. Thus, like in the discussion before proposition 8.4.4, $u_0'(0) = \infty$, $I^0 = 0$ and $EV^{(t)} = \gamma_t EV^*$ imply that c_t^* and $c_t^\diamond$ will always be internal solutions, which enables us to establish the above result.

With the previous lemma we can now establish that convergence towards optimality will occur, if the initial expected value function is an affine transformation of the optimal expected value function: $EV^{(0)} = \gamma_0 EV^* + \alpha_0$, for some γ_0 and some α_0. The following proposition is a counterpart of proposition 8.4.4.

Proposition 9.5.3 *Let a model be given that satisfies axioms 4.1.1, 4.4.1, 9.1.2, 9.1.3 and 9.1.4, and with an instantaneous utility that satisfies axiom 8.1.2. Suppose given an initial budget $m_0 \in \mathbb{R}_{++}$, and an initial expected value function $EV^{(0)}$ that is an affine transformation of the optimal expected value function: $EV^{(0)} = \gamma_0 EV^* + \alpha_0$, for some $\gamma_0 \in \mathbb{R}_{++}$ and some $\alpha_0 \in \mathbb{R}$. If subsequently the learning algorithm (based on EEE adjustment) is used, then convergence towards optimality will occur.*

Proof. This proof follows the same lines as the convergence proof under certainty and without additional income (proposition 8.4.4). Given is a model with a discount factor $0 < \delta < 1$, and an instantaneous utility function u_0 that satisfies axiom 8.1.1 and $u_0'(0) = \infty$. The additional periodical incomes I_t, for $t \geq 1$, are IID distributed and each can take a finite number of realizations, the smallest realization being $I^0 = 0$. Also given are an initial budget $m_0 \in \mathbb{R}_{++}$, and an initial expected value function $EV^{(0)}$ that is an affine transformation of the optimal expected value function: $EV^{(0)}(m) = \gamma_0 EV^*(m) + \alpha_0$, for some $\gamma_0 \in \mathbb{R}_{++}$, some $\alpha_0 \in \mathbb{R}$ and all $m \in \mathbb{R}_+$.

Subsequently, a sequence of expected value functions $(EV^{(t)})_{t+1}^{\infty}$ is generated from the initial expected value function $EV^{(0)}$ by the learning algorithm that uses min-max adjustment,[39] which depends on expected excess expenditure. Adjustments are made multiplicatively, so that any $EV^{(t)}$ in the sequence $(EV^{(t)})_{t=0}^{\infty}$ as generated by the learning algorithm from $EV^{(0)}$, will be such that

$$EV^{(t)} = \theta_t^{-1} EV^{(0)} = \theta_t^{-1}\gamma_0 EV^* + \theta_t^{-1}\alpha_0,$$

where θ_t denotes the adjustment product in period t. If we define γ_t and α_t by $\gamma_t := \theta_t^{-1}\gamma_0$ and by $\alpha_t := \theta_t^{-1}\alpha_0$, then we see that the expected value function $EV^{(t)}$ will also be an affine transformation of the optimal expected value function: $EV^{(t)} = \gamma_t EV^* + \alpha_t$.

By proposition 7.1.1 any sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$ will converge. Here however, we want to establish that convergence towards optimality will occur. To see that this will indeed occur, we need to verify that the model is such that the conditions needed for proposition 7.3.1 (conditions 7.3.1, 7.3.2 and 7.3.3) are satisfied.

[39]For any $\rho > 1$, $\sigma < 1$ and $\mu \in (0, 1)$.

As for condition 7.3.1, we know from section 9.3 that given $EV^{(0)}$, γ_0 is a consistency-inducing scalar, so that this condition is satisfied.

As for condition 7.3.2, we need to establish that for any time t, setting $\theta_t = \gamma_0$ will yield $F_t = 0$ for all m_t. If $\theta_t = \gamma_0$, then we will have that

$$EV^{(t)}(m) = \theta_t^{-1} EV^{(0)}(m) = \gamma_0^{-1}(\gamma_0 EV^*(m) + \alpha_0) = EV^*(m) + \gamma_0^{-1}\alpha_0,$$

so that by propositions 9.5.2 and 9.5.1 we know that excess expenditure will always equal zero: $F_t = 0$ for all $m_t \in \mathbb{R}_+$.

As for condition 7.3.3, we need to establish that for any time t, setting $\theta_t > \gamma_0$ will yield $F_t > 0$ for all m_t, and that setting $\theta_t < \gamma_0$ will yield $F_t < 0$ for all m_t. Now, $\theta_t > \gamma_0$ would imply that $\gamma_t = \theta_t^{-1}\gamma_0 < 1$, and by proposition 9.5.2 and by the previous lemma we know that the excess expenditure will be strictly positive, as long as $m_t > 0$ is satisfied. Similarly, $\theta_t < \gamma_0$ would imply that $\gamma_t = \theta_t^{-1}\gamma_0 > 1$, and with the previous lemma we see that the excess expenditure will be strictly negative, as long as $m_t > 0$ is satisfied.

Therefore, here it only remains to be shown that for any $m_0 > 0$ given, we will also get that $m_t > 0$ will hold for all elements m_t out of a sequence of budgets $(m_t)_{t=0}^{\infty}$ that will result from the choices of a decision-maker whose behaviour is generated by the learning algorithm, given the initial expected value function $EV^{(0)}$. To show that this holds, we will use an induction argument. For $t = 0$, it was already given that $m_0 > 0$. Thus, what is left to prove, is that $m_t > 0$ implies that $m_{t+1} > 0$. Suppose that $m_t > 0$. Then in period t, the consumption choice c_t^* that will be chosen, will be part of a solution to

$$\max_{c_t + s_t \leq m_t} u_0(c_t) + \delta EV^{(t)}(s_t).$$

Now, $EV^{(t)\prime}(0) = \theta_t^{-1}\gamma_0 EV^{*\prime}(0)$, and from the proof of the previous lemma, we know that $EV^{*\prime}(0) = \infty$. This implies that $c_t^* < m_t$ must hold, and we get that $m_{t+1} = s_t^* + I_{t+1} \geq s_t^* = m_t - c_t^* > 0$. Thus indeed we have established that $m_t > 0$ will hold for all t.

Hence, conditions 7.3.1, 7.3.2 and 7.3.3 are satisfied, and proposition 7.3.1 shows that the sequence of adjustment products $(\theta_t)_{t=1}^{\infty}$ converges to the CIS $\tilde{\theta} = \gamma_0$, so that convergence towards optimality occurs. ■

Recall from subsection 6.4.2 that, when compared with regular excess expenditure, the idea behind expected excess expenditure was to disregard the known information about the new additional income realization. In period $t+1$, when excess expenditure for period t is determined, the additional income realization I_{t+1} is known. Whereas REE for period t depends on the additional income realization I_{t+1}, this information is not used in the determination of EEE. Instead, EEE uses the information known at time t about I_{t+1}, which, in an expected utility model, is given by a probability distribution for I_{t+1}.

While at first sight it may have seemed strange to disregard readily available information in the process of re-evaluating past choices (and updating time preferences), from this and the previous section we may see why this might not seem such a bad idea after all. As already mentioned in subsection 6.4.2, from the previous sections we see that in expected utility models, the EEE measure is more efficient than the REE measure in terms of moving towards optimality. From this and the previous section we can see why. If a probability distribution is given for I_{t+1}, then adjustments conditioned on this probabilistic information will reliably yield improvements, while adjustments conditioned on the coincidental occurrences of small of large realizations may lead to moving away from optimality, rather than moving towards optimality.

9.6 Convergence of choices

Like in section 8.5, here we shift our attention from convergence of ad hoc preferences to convergence of choices. We investigate whether the choices of a consumer whose behaviour would be generated by the learning algorithm, will converge towards choices that would be made by a rational utility maximizer in the same setting.

The next proposition is a counterpart to proposition 8.5.1, it shows that convergence towards optimality in terms of ad hoc preferences, also implies convergence towards optimality in terms of the corresponding sequence of consumption functions.

Proposition 9.6.1 *Let a model be given that satisfies axioms 4.1.1, 4.4.1, 9.1.1, 9.1.3, and 9.1.4, and with an instantaneous utility u_0 that satisfies axiom 8.1.1. Suppose that given an initial expected value function $EV^{(0)}$ that satisfies axiom 8.1.1, the learning algorithm is used, and that preferences do converge to optimality. Then, the sequence of choice functions $(c_t^*(m))_{t=0}^{\infty}$, as defined by*

$$c_t^*(m) := \arg \max_{0 \leq c_t \leq m} u_0(c_t) + \delta EV^{(t)}(m - c_t),$$

that corresponds to the sequence of expected value functions $(EV^{(t)})_{t=0}^{\infty}$ as generated by the learning algorithm from $EV^{(0)}$, will converge to the optimal choice function $\tilde{c}(m)$, as defined by

$$\tilde{c}(m) := \arg \max_{0 \leq c \leq m} u_0(c) + \delta EV^*(m - c).$$

Proof. This proof will follow the same lines as that of proposition 8.5.1. Given is a model with an instantaneous utility function u_0 that satisfies axiom 8.1.1 and a discount factor $0 < \delta < 1$. The additional periodical incomes I_t, for $t \geq 1$, are IID distributed and each can take a finite number of realizations. Given an initial expected value function $EV^{(0)}$ that satisfies axiom 8.1.1, the learning algorithm is used and preferences converge to optimality. That is, for $(EV^{(t)})_{t=0}^{\infty} = (\theta_t^{-1} EV^{(0)})_{t=0}^{\infty}$, the corresponding sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$ converges to some consistency-inducing scalar $\tilde{\theta}$. The existence of such a consistency-inducing scalar $\tilde{\theta}$ implies that the initial expected value function must be an affine transformation of the optimal expected value function: $EV^{(0)} = \tilde{\theta} EV^* + \alpha$.

For any period t the choice function $c_t^* : \mathbb{R}_+ \to \mathbb{R}_+$ returns for any budget m available in period t, the choice $c_t^*(m)$ that solves

$$\max_{0 \leq c_t \leq m} u_0(c_t) + \delta \theta_t^{-1} EV^{(0)}(m - c_t).$$

We can capture such a sequence of choice functions by the single function $f : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+$, as defined by

$$f(m, \theta) := \arg \max_{0 \leq c \leq m} u_0(c) + \delta\theta^{-1} EV^{(0)}(m - c),$$

for all $m \in \mathbb{R}_+$ and all $\theta \in \mathbb{R}_{++}$. Then we see that $c_t^*(m) = f(m, \theta_t)$. By Luenberger ([28], p.462) the function $f(m, \theta)$ must be continuous in θ. And the sequence of adjustment products $(\theta_t)_{t=0}^{\infty}$ converges to the scalar $\tilde{\theta}$, so that the sequence of choice functions $(c_t^*(m))_{t=0}^{\infty} = (f(m, \theta_t))_{t=0}^{\infty}$ will converge[40] to $f(m, \tilde{\theta})$.

For any given $m \in \mathbb{R}_+$, this limit quantity $f(m, \tilde{\theta})$ solves

$$\max_{0 \leq c \leq m} u_0(c) + \delta\tilde{\theta}^{-1} EV^{(0)}(m - c) =$$

$$\max_{0 \leq c \leq m} u_0(c) + \delta EV^*(m - c) + \delta\tilde{\theta}^{-1}\alpha.$$

And, by proposition 9.5.2 we see that for any given $m \in \mathbb{R}_+$, the quantity $\tilde{c}(m)$ must also maximize this last maximization problem. By strict concavity, solutions are unique and we see that $\tilde{c}(m) = f(m, \tilde{\theta})$. Indeed, $(c_t^*(m))_{t=0}^{\infty}$ converges to $\tilde{c}(m)$. ■

In this chapter we have performed a similar analysis as in the previous chapter. We considered when convergence towards optimality will take place in stationary consumption/savings models of expected utility. We found that when adjustments are based on regular excess expenditure, convergence towards optimality will generally not occur. When adjustments are based on expected excess expenditure, we found that convergence towards optimality will occur under some (rather specific) circumstances. Finally, we considered the relation between convergence towards optimality of preferences and convergence towards optimality of choice functions.

[40] Pointwise.

10 Conclusions

In this concluding chapter we will look back at previous chapters and forward towards possible generalizations, extensions, implications and applications of the ad hoc framework. And we will discuss what the ad hoc framework can and cannot do.

As promised in previous chapters, here we will come back to some modelling choices that were made in setting up the ad hoc framework, we will consider some modelling choices that could alternatively have been made, and how these other choices might change the resulting framework, and its possible implications. In this chapter we will consider and interpret the implications of the current specification of the ad hoc framework. Most notably, we will try to answer whether optimal behaviour could be learned. Also, we will present some ideas as to how the ad hoc framework could be used to explain some behavioural irregularities, and we will consider the main shortcomings of the ad hoc framework.

This chapter consists of five sections. The first section considers possible extensions and generalizations of the learning approach taken in this work. The second section comes back to the question why the min-max adjustment function was used to complete the ad hoc framework instead of, for instance, a value-based adjustment function. The third section reviews and interprets the results of chapters 7, 8 and 9, that investigated convergence towards optimality. The fourth section will present some examples of how the ad hoc framework could be used to explain some seemingly suboptimal types of behaviour. Finally, the fifth section considers the main shortcomings of the current ad hoc framework.

10.1 Possible extensions

Here we will first consider a few ways in which the learning approach that is presented in this work can be extended. We will consider extensions of this learning approach to other areas of economics, and we will consider extending the learning framework in the regular consumer choice setting to allow for non-zero interest rates and borrowing. Also, we will consider extending the analysis of chapter 5 to expected utility models.

In this work a new learning approach was used in a consumer choice setting, and in a consumption/savings setting as a special case. The same, or a similar approach to learning could perhaps also be taken in other areas of economics where individuals (or institutions) face dynamic optimization problems. It certainly seems that the approach taken in this work, could be extended to those areas of economics where it is standard practice to model time-separable objective functions that satisfy exponential discounting, and where dynamic programming could be used to find and describe rational behaviour.

In chapter 2 time was explicitly modelled in the context of consumer choice. Initially this modelling of time included an interest rate $r \geq 0$ and an implicit interest rate of $R = 1 + r$ (r and R were supposed constant). At the end of chapter 2 this possibility of non-zero interest rates was dropped, and in the rest of this work we (implicitly) assumed that $r = 0$, so that $R = 1$.

The possibility of a (constant) non-zero interest rate $r > 0$ could also be included in the ad hoc framework as presented in chapters 4 and 6. Chapter 4 simply models how choices between present consumption and savings would be made in any ad hoc problem. In chapter 4 it is not yet specified how these amounts of savings would be related to next period's budgets. Therefore interest rates have no place in the story of chapter 4. The story of chapter 6 would almost be the same with non-zero interest rates; all ad hoc preferences should be separable with instantaneous preferences exogenous and comparable across periods, and time preferences would be updated by means of an adjustment function that depends on excess expenditure. The only place in chapter 6 where the relation between savings and next period's budgets plays a role in how the ad hoc framework is set up, is in section 6.4.2 where excess expenditure is defined. Excess expenditure (both for REE and EEE) is defined as the difference between actual expenditure and ex-post optimal expenditure. Actual expenditure is defined by a maximization problem over the single budget constraint $p_t \cdot x_t + s_t \leq m_t$, here interest rates do not seem relevant here as no relation between s_t and m_{t+1} is implicit in this specification. The interest rate *could* appear in the determination of ex-post optimal expenditure, as defined by a maximization problem over the twin budget constraints $p_t \cdot x_t + s_t \leq m_t$ and $p_{t+1} \cdot x_{t+1} + s_{t+1} \leq Rs_t + I_{t+1}$, here savings s_t would lead to a budget of $m_{t+1} = Rs_t + I_{t+1}$. Hence constant interest rates could rather straightforwardly be included in the ad hoc framework.

The ad hoc framework could even be extended to allow for uncertain (and thus non-constant) interest rates. Then, like all time-$(t + 1)$ information, the interest rate R_{t+1} (between periods t and $t + 1$) should be known at time $t + 1$, when the excess expenditure of period t is determined. Then, ex-post optimal expenditure for period t could simply depend on the realization of R_{t+1} as it does on the realizations of other uncertain variables, such as on the additional income realization I_{t+1}. Hence interest rates could simply be treated in the same way as other uncertain information about the future.

Thus allowing for non-zero interest rates would conceptually not make much of a difference; it would just change the way in which budgets would depend on savings in chapter 6. For simpler notation (especially in chapters 5, 8 and 9, where relations with the standard framework are considered) we opted for a zero interest rate. Thus, extending the ad hoc framework to include a non-zero interest rate would mainly be a question of bookkeeping.

In the current specification of the ad hoc framework saving is possible while borrowing is not possible. Somewhat similarly to the above ideas of including interest rates, the ad hoc framework could also be generalized to allow for the possibility of borrowing, by allowing for negative savings. In that case the lower bound for savings would not be zero but rather strictly negative. Then the story of chapter 4 would not change, except that ad hoc choice sets would not be of the form $X_t \times \mathbb{R}_+$, but rather of the form

$X_t \times [-\iota, \infty)$ (for some $\iota \in \mathbb{R}_{++}$) or $X_t \times \mathbb{R}_+$. Again, the story of chapter 6 would be more or less the same, except that budgets might be related to savings in a somewhat more complicated way than in the current specification of the ad hoc framework. Thus borrowing could also be incorporated into the ad hoc framework.

The analysis of chapter 5 could be extended to include models of expected utility. In chapter 5 a definition of consistency of an ad hoc utility function with a total Bernouilli utility function (given a probability structure) was given, since in chapter 7 we needed a formal definition of consistency under expected utility. However, besides this definition, chapter 5 is completely set under certainty. Still, similar conclusions could probably be drawn in expected utility models. For instance, it seems that the most important propositions in chapter 5, propositions 5.2.1, 5.3.1 and 5.3.2 could be extended to expected utility models.

10.2 Changing the adjustment function

As promised in chapter 6, here we come back to the question why the min-max adjustment function was used to complete the ad hoc framework. One alternative to the min-max adjustment procedure was already specified in Chapter 6, namely that of value-based adjustment functions. Such a value-based adjustment function would be given by a positive-valued function that would depend on, and decrease in, excess expenditure (either REE or EEE) in the corresponding period, and that would return an adjustment factor equal to one in case excess expenditure would equal zero. Like in the case of min-max adjustment, under value-based adjustment we would get that a positive (negative) excess expenditure in the previous period would lead to an adjustment factor that is smaller (larger) than one, so that the weighting for instantaneous utility in time preference is decreased (increased). And in case of value-based adjustment, the higher excess expenditure in the corresponding period, the smaller the adjustment factor, and the further the weighting for instantaneous utility is decreased. One example of such a value-based adjustment function that was given in chapter 6 was the exponential adjustment function $\tilde{a}(\mathcal{E}_t) = e^{-\zeta \mathcal{E}_t}$, for some $\zeta > 0$, and for all $\mathcal{E}_t \in \mathbb{R}$. Such value-based adjustment functions may seem quite straightforward, and in any case they would be more in line with error-correction models (see subsection 3.4.2) than min-max adjustment.

However, as noted in chapter 6, we opted to use the min-max adjustment function mainly for reasons of tractability; value-based adjustment functions would be much more difficult to work with. Under value-based adjustment, convergence propositions such as those in chapters 8 and 9 are much more difficult to prove, if in fact they can be proven.

To see this, recall that min-max adjustment is completely determined by the signs of excess expenditures, and not by excess expenditure values. Of course, under value-based adjustment these excess expenditure values *will* have to be taken into account. In models of certainty, lemma 8.4.4 (and similarly lemma 9.5.1 for EEE adjustment) shows that if the time-t value function equals a scalar times the optimal value function

$V^{(t)} = \gamma_t V^*$, then $\gamma_t \geq 1$ implies that $\mathcal{E}_t \leq 0$, and $\gamma_t \leq 1$ implies that $\mathcal{E}_t \geq 0$. Note that this result holds, irrespective of what adjustment function is used. Therefore, like with min-max adjustment, this lemma shows that under value-based adjustment for $V^{(t)} = \gamma_t V^*$ it will hold that $V^{(t+1)} = \gamma_{t+1} V^*$ with $\gamma_{t+1} \leq \gamma_t$ if $\gamma_t \geq 1$, and with $\gamma_{t+1} \geq \gamma_t$ if $\gamma_t \leq 1$. Thus, like min-max adjustment, value-based adjustment does work in the right direction.

How much smaller or larger γ_{t+1} will be than γ_t, is under min-max adjustment completely determined by the signs of previous excess expenditures. Under value-based adjustment the question of how much smaller or larger γ_{t+1} will be than γ_t, is determined by the value of excess expenditure in the corresponding period. And whereas the sign of $\mathcal{E}_t$ is completely determined by γ_t, the value of $\mathcal{E}_t$ is also influenced by the available budget m_t.

Under min-max adjustment strict versions of the above implications ($\gamma_{t+1} < \gamma_t$ if $\gamma_t > 1$ and $\gamma_{t+1} > \gamma_t$ if $\gamma_t < 1$) are sufficient for convergence of the sequence of γ_t's to one, and thus for convergence towards optimality to occur. Under value-based adjustment strict versions of these implications would imply that *if* the sequence of γ_t's converges, the limit *would* equal one, and convergence towards optimality *would* occur. Thus, under value-based adjustment it will additionally have to be shown that convergence *will* occur.

And under value-based adjustment the sequence of γ_t's need not necessarily converge. To see this, consider the case where the initial value function equals a scalar times the optimal value function $V^{(0)} = \gamma_0 V^*$, then $V^{(t)} = \gamma_t V^*$ where γ_t equals $\widetilde{a}(\mathcal{E}_{t-1}) \cdot \widetilde{a}(\mathcal{E}_{t-2}) \cdot ... \cdot \widetilde{a}(\mathcal{E}_0) \cdot \gamma_0$. Then convergence of a sequence of γ_t's can only happen if the sequence of $\widetilde{a}(\mathcal{E}_t)$'s would converge to one, which would in turn imply that the sequence of $\mathcal{E}_t$'s should converge to zero. While it might seem that as the γ_t's get closer to one, the $\mathcal{E}_t$'s would become closer to zero, this is complicated by the fact that excess expenditure $\mathcal{E}_t$ also depends on the available budget m_t.

Under min-max adjustment, investigating convergence only requires considering sequences $\{\gamma_t\}_{t=0}^{\infty}$, as $\gamma_{t+1} = a_{\eta_{t+1}} \gamma_t$ and the adjustment $a_{\eta_{t+1}}$ only depends on the signs of excess expenditures $\eta_{t+1} = (\mathcal{E}_0, ..., \mathcal{E}_t)$, which are completely determined by $(\gamma_0, ..., \gamma_t)$. Under value-based adjustment, the period-$(t+1)$ adjustment $\widetilde{a}_{\mathcal{E}_t}$ is determined by the value of $\mathcal{E}_t$, that is in turn determined by γ_t and by m_t. Therefore under value-based adjustment, establishing convergence requires considering the sequences $\{\gamma_t\}_{t=0}^{\infty}$, $\{\mathcal{E}_t\}_{t=0}^{\infty}$ and $\{m_t\}_{t=0}^{\infty}$ simultaneously, so that proofs will be much more complicated.

Maybe this problem that under value-based adjustment convergence need not occur, could be dealt with by assuming that the learning process would become more subtle over time, i.e. that the adjustment function would become less steep over time. This is a common assumption in learning models. In our example of exponential value-based adjustment this feature could be modelled by replacing the single fixed scalar ζ by a sequence of scalars $\{\zeta_t\}_{t=0}^{\infty}$ that decreases over time. However, this would also mean that investigating convergence would require keeping track of the sequences $\{\gamma_t\}_{t=0}^{\infty}$, $\{\mathcal{E}_t\}_{t=0}^{\infty}$, $\{m_t\}_{t=0}^{\infty}$ and $\{\zeta_t\}_{t=0}^{\infty}$ simultaneously. In any case, it seems clear that under value-based adjustment establishing convergence would be much more complicated than under min-max adjustment.

10.3 Convergence towards optimality

As noted in the introductory chapter, one of the reasons for trying to construct a learning model of consumption behaviour was to investigate whether (near-)optimal consumption behaviour could be learned over time. In chapters 7, 8 and 9 the ad hoc learning framework was used to try to answer this question.

We found that in the ad hoc framework convergence towards optimality *can* occur. There are circumstances under which sequences of ad hoc preferences as generated by the learning algorithm do converge towards consistent ad hoc preferences, and thus towards rationality. However, the requirements under which this would occur are very stringent. Not only the sufficient conditions for convergence towards optimality to occur are very restrictive, so are the necessary conditions. Convergence towards optimality requires the existence of a consistency-inducing scalar, and in section 7.2 a number of necessary conditions were specified for the existence of such a CIS.

One of these necessary conditions for the existence of a CIS was that models are stationary, in the sense that a single time preference function should be consistent in all periods. Learning entails using past experiences to determine current behaviour. Therefore we can only expect to see learned behaviour approach optimality if past economic conditions are a good predictor of future conditions. Thus stationarity in some form or another seems indispensable for learned behaviour to approach optimality.

However, in a consumer choice setting stationarity assumptions already seem quite strong. Stationarity is often assumed in economic modelling as it greatly enhances tractability. However, in reality it seems questionable whether consumers face stationary conditions. In fact, it seems that we cannot know for sure that we are facing stationary conditions, even if the economic environment has proven relatively stable in the past, this is no guarantee that the same will hold in the future. Only at the end of his lifetime can a consumer really determine whether his economic environment was stationary.

Another necessary condition is that the initial time preference function should already have a functional structure that is similar to that of some consistent time preference function. By inserting a single scalar into the initial time preference function, we should arrive at a consistent time preference function. In consumption/savings models this condition means that the initial (expected) value function should be an affine transformation of the optimal (expected) value function (proposition 8.3.1). If this condition is not met, then convergence towards optimality will *not* occur (at least not in terms of preferences).

It is a rather straightforward exercise to show that in the case of a consumption/savings model with logarithmic instantaneous utility ($u_0(c) = \ln(c)$) and without an income stream ($I_t = 0$, for all $t > 0$), the optimal value function would be an affine transformation of the instantaneous utility function.[41] In that specific case, if

[41] In this conveniently chosen example the sequence problem could simply be solved by means of the

the initial value function would be an affine transformation of the instantaneous utility function (which may not seem unreasonable), then the initial value function would also be an affine transformation of the optimal value function. Thus in this specific case, the affinity requirement seems less problematic. However, this case is really an exception, similar relations between instantaneous utility and value functions will generally certainly not hold. In fact, as noted in section 3.3, in many consumption/savings models the optimal value function does not even permit an analytical form; such an optimal value function could only be approximated.

Thus the severe requirement, that the initial time preference function should have a functional structure that is similar to that of some consistent time preference function, seems to very much constrain the relevance of the convergence result. This requirement is necessary because of the fact that we have opted for the simple way of uniform adjustment, where adjustments enter into old time preferences multiplicatively in order to arrive at new time preferences. Since this same updating procedure is applied in all periods, at any period the time preference function that will be used is equal to the initial time preference function, except for the adjustment product that is inserted into it. Thus, updating only occurs in this single parameter.

In principle this strong requirement could be relaxed by using other adjustment procedures. Here we will sketch two possible ways in which this could be done.

Firstly, we could model a situation in which in all periods a number of categories of commodities can be distinguished, and in which instantaneous utility is separable in each of these categories. That is, we could model a situation where in every period the available commodities could be divided into $N \in \mathbb{N}$ categories, and where these categories would be the same for all periods. Then in all periods excess expenditures could be determined for all categories separately. And in this situation in every period t instantaneous utility $v^{(t)}(w_{t-1}, x_t)$ should be separable in each category of goods, so that instantaneous utility could be written in a form in which each category of commodities x_t^j would have its own subutility function $v_t^j(x_t^j)$ that would determine instantaneous utility for this category of goods separately. That is, there should be a function $f_t : W_{t-1} \times \mathbb{R}^N \to \mathbb{R}$, with f_t strictly increasing in each of the last N arguments, such that $v^{(t)}(w_{t-1}, x_t) = f_t(w_{t-1}, v_t^1(x_t^1), ..., v_t^N(x_t^N))$. In this situation different adjustment factors could be used for different categories, where each of these adjustment factors would depend on the category-specific excess expenditures. This would allow for a more general learning process in which multiple category-specific parameters could be fine-tuned, in order to reach or approach optimality or consistency.

And secondly, instead of adjusting time preferences by inserting scalars into time preference functions, adjustments could be made by letting functions be inserted into time preferences functions. In that case adjustments would not anymore be uniform, in the sense that different instantaneous utility levels would no longer necessarily be updated with the same adjustment factor. In any period, the resulting time preference function would be obtained by inserting the composition of all adjustment functions to

Lagrange method.

date into the initial time preference function. Obviously, such an adjustment procedure would allow for much more freedom in fine-tuning time preferences.

Of course, the function that would yield such adjustment functions would have to be a much more complicated construct than those used in previous chapters. Differentiating between adjustments for instantaneous utility levels seems to require that these adjustments would be based on richer data than just excess expenditures. For instance, these richer data could result from keeping track of (recent) histories of the relative sizes of excess expenditures and actual expenditures simultaneously. That is, if there seems to be a correlation between the relative sizes of $\mathcal{E}_t$ (or of $\mathcal{E}_t/(p_t \cdot x_t^*)$, or of $\mathcal{E}_t/m_t$) and the relative sizes of $(p_t \cdot x_t^*)$ or of m_t in the (recent) past, this would allow for distinguishing in what ranges of instantaneous utility levels, expenditures or per-period budgets, the (relative) excess expenditures seem especially bad, and in what ranges adjustments would have to be more pronounced.

The above two alternative ways to make adjustments would widen the margins for initial time preferences that could still lead to convergence towards optimality. However, even then convergence towards optimality is not something that will happen easily.

Under category-based adjustment, convergence towards optimality requires a consistency-inducing vector, rather than a consistency-inducing scalar. In order for such a consistency-inducing vector to exist, the initial time preference function should still have a functional structure that is similar to that of some consistent time preference function. One function being identical to another function, except for N parameter values is still a very restrictive requirement.

In the case of adjusting by inserting functions, instead of inserting scalars, the requirements on initial time preferences would be much widened. However, then in order to reshape an initial time preference function gradually into a very differently shaped consistent time preference function, then adjustments would have to be very sophisticated. This too does not seem completely plausible.

10.4 Explaining suboptimality

Of course, finding that convergence towards optimality will not occur is not necessarily problematic. After all, optimality is a very strong and restrictive property. It seems that in reality choices do not always correspond to optimality. Some people experience serious debt problems, while some elderly people die with very sizeable amounts of savings left, even when they don't have relatives to leave it to. These are a few very simple examples where some people clearly seem to over- or under-spend.

These phenomena could be explained in the ad hoc framework by a reluctance to change or adapt behaviour (substantially) or, in terms of the ad hoc framework, by a (very) slow adjustment process. For example, under value-based adjustment with an exponential adjustment function $\tilde{a}(\mathcal{E}_t) = e^{-\zeta \mathcal{E}_t}$, if the constant $\zeta > 0$ is very small, then all adjustment factors would stay close to one, and the adjustment process would be very slow. Similarly, under min-max adjustment a combination of a $\rho > 1$ and a $\sigma < 1$ (see the definition of min-max adjustment in subsection 6.4.3) that would both be very close to one, would yield an adjustment process that would be very slow (at least in the early stages). If such a slow adjustment process would be combined with an initial time preference function that is rather optimistic (in the sense that it would later turn out that it underestimated the utility of money), then this is likely to result in over-spending. Similarly, combined with a rather pessimistic initial time preference function, a slow adjustment process may result in under-spending.

Also, under min-max adjustment we could characterize a combination of relatively large parameter values of ρ, σ and λ as a somewhat bold adjustment process. Such adjustments seem to reflect being much more inclined to increase rather than decrease the adjustment product (and thus the valuation for instantaneous utility). Instead, a combination of relatively small parameter values for ρ, σ and λ could be characterized as a rather cautious adjustment procedure. Again, a bold adjustment procedure and an optimistic initial time preference function would be likely to result in over-spending. A cautious adjustment procedure and a pessimistic initial time preference function would be likely to result in under-spending.

Note that these implications may even hold in cases where convergence towards optimality *does* occur. The ad hoc framework assumes an infinite amount of learning periods, while people's lives are finite, so even in cases where mathematically convergence towards optimality *would* ultimately occur, the above constellations might still result in serious under- or over-spending within the relevant time horizons.

Also, in section 3.2 we saw that the phenomenon of income tracking was widely observed in econometric studies. This phenomenon is not necessarily contradictory to optimality, but only under some special specifications of consumption/savings models. The ad hoc framework, with value-based adjustment, could perhaps also explain income tracking.

The ad hoc framework, and especially when paired with regular excess expenditure, uses the past and the present as indications of the future. By proposition 7.1.1 we know that if min-max adjustment is used, convergence will always occur. Then, the resulting limit would be mainly determined by what is encountered in early periods. Therefore, min-max adjustment seems most appropriate in models where experiences in early periods could be expected to be comparable to those of later experiences, most notably in (more or less) stationary problems. In situations where this cannot be expected or known, value-based adjustment may seem more reasonable. In those cases, we might expect to find income tracking. While under min-max adjustment information from all previous periods would essentially be treated the same, under value-based adjustment information from the recent past plays a much more prominent role. Thus, under value-based adjustment as dependent on regular excess expenditure, choices would be heavily influenced by the consumer's present situation and by the recent past. Then if experienced conditions do not turn out to be very comparable over a life-time, so that the system does not prove to be very stationary, it seems that income tracking would occur.

10.5 Main shortcomings

As mentioned in the introductory chapter, the present work is still somewhat sketchy, and it should not be regarded as having a very definitive quality. This work consists of theoretical explorations, and we should be careful to draw very firm conclusions about economic reality from this research. In its current form the ad hoc framework is certainly not equipped to give a full account of actual consumers' consumption behaviour. Here we will present some important shortcomings.

Firstly, the ad hoc framework is (almost) entirely based on retrospective viewing and reasoning. Expectations or knowledge about the (economic) future do not play a role in the ad hoc framework, except possibly in the very first period. The ad hoc framework does allow for the possibility that the initial time preferences, from which the learning algorithm starts, would incorporate what is known about the future in the initial period. However, after this first period time preferences are determined by updating previous time preferences, and updating is completely determined by excess expenditure. Still, it seems possible or even likely that a consumer's outlook on the future would change over time. Of course, what happened in periods 1 through t is completely known in period t, which already changes the outlook on the (time-0) future. But moreover, it seems possible that at time t the information about what can be expected from time $t+1$ onwards would have changed with respect to the initial expectations. While the time-t information about periods 1 through t *is* used in determining the period-t time preferences, possible changes between time-0 information and time-t information about periods $t+1$ and later *is not* used in determining period-t time preferences. Therefore an important implication of the ad hoc framework in its current form is that changing expectations will not influence consumption behaviour in any way. Again we see that the ad hoc framework in its current form seems most appropriate in problems that are more or less stationary. The same implication seems to typically hold for other learning models as well.

In section 3.3 we argued, like Gilboa and Schmeidler, that a major drawback of the standard prospective view on consumer choice is that it requires that people know, or imagine, everything that can happen in the future. And although it seems quite unrealistic to assume that people know or can imagine everything that can happen, it seems equally unrealistic to assume that people would not use any information about what can happen, or that they would never learn new information about what can or will happen (i.e. that they face a stationary problem). It will not be very difficult to find empirical evidence that rejects the implication that changes in expectations or knowledge about the future do not influence consumption behaviour.

This probably identifies one of the most important shortcomings of the ad hoc framework. In principle, the ad hoc framework could be extended to incorporate prospective viewing. Here such prospective viewing need not be based on knowing or imagining all that can happen, but it could also be based on one or a few aggregate variables of what the future might bring. For example, such an aggregate variable could reflect the extent to which things can be expected to become better or worse (than the present). Then adjustments would not completely have to be determined by excess expenditure, but could also be influenced by a changing outlook on the future. That is, all other things equal, bad news about the future should result in smaller adjustment factors, and thus in smaller weights for instantaneous utility. Of course, the main difficulty in making such an approach work would be how to quantify changes in a decision-maker's outlook, and the effects that such changes would (or should) have on how to value savings.

And secondly, the ad hoc framework also does not really incorporate the possibility of social learning. Only the determination of initial time preferences could be interpreted to reflect social learning in this framework. It was stated in chapter 6 that these initial time preferences would somehow be given exogenously, so this would allow for the possibility that initial time preferences would be learned from others (e.g. parents). However, after the very first period there is no role for social learning in the ad hoc framework.

In reality social learning might be important as to how people would arrive at their consumption decisions. It seems very well possible that people would learn how to make consumption choices by talking about and observing other (or older) people's choices and their apparent results. There are some limitations to this direct way of social learning as in reality we are (almost) never able to form a more or less complete picture of other persons' consumption patterns; we usually only have some scattered information about such patterns. Also, we don't seem to be able to observe other people's entire income processes, or their preferences.

Still, we might be able to observe some definite don'ts as demonstrated by people getting into serious debt trouble.

However, social learning need not necessarily take the above, direct form. Institutionalized social learning might also be relevant. For instance, the existence of personal finance books, and financial or investment advisors indicate that people do (try to) learn about their consumption behaviour. Moreover, some governments try to teach people how to improve their spending behaviour (e.g. the Dutch government via NIBUD.)

Thus it seems that social learning might play a role in how people come to their consumption decisions. Still, it does not seem clear what role exactly it might play. As is the case with individual learning, social learning could potentially both help explain why people *do* behave optimally, and why people *don't* behave optimally.

It seems possible that social learning could substitute for individual learning, which could help people to improve their consumption behaviour. Some experimental studies (Ballinger et al [3], Chua and Camerer [8]) indicate that this could be the case. However, this seems more likely to work in settings that are quite stationary, only if the past and the present are (likely to be) good predictors of the future. Indeed, the above two papers use settings where all consumers more or less face the same stationary problem, and where all consumers know that they are facing the same problems. If there is important and fundamental uncertainty about the future, then all individuals would face more or less the same uncertainty, which very much limits the possibility for social learning to be very effective. Then, it seems that a large part of learning should consist of learning about what problem is faced, rather than of how to solve the particular problem that is faced. After all, it would seem that what we can learn from our grandparents about the macroeconomic situation that we will face in a few decades from now, is rather limited.

More generally, it is known that social learning is certainly not guaranteed to lead to optimal behaviour. The phenomena of 'herd behaviour' and 'information cascades' (see e.g. Bikhchandani, Hirshleifer and Welch [4]) are examples of collective deviations of optimality. Indeed, it seems that such suboptimalities are more likely to result from social learning in cases where there is important uncertainty about the future. If uncertainty is limited to the type that is faced in expected utility models, then social learning is more likely to focus on how to solve a certain problem instead of on what problem will be faced. In that case social learning may be helpful, rather than detrimental in approaching optimality. Thus it seems that social learning is most likely to facilitate convergence towards optimality in problems that are more or less stationary.

Recall that stationarity was also one of the conditions that were needed for individual learning to be successful. Hence, it seems that in instances where it would be possible that individual learning would lead to (near-)optimality (under stationarity), social learning might improve, or speed up, the learning process. In instances where individual learning is unlikely to lead to optimality, it similarly seems questionable whether social learning could substantially aid the learning process.

Samenvatting

Deze dissertatie is vooral theoretisch van aard. Dit proefschrift houdt zich bezig met een aantal theoretische constructies die gebruikt (kunnen) worden om economische modellen mee op te bouwen, waarmee vervolgens weer getracht wordt de (economische) werkelijkheid te begrijpen. De vragen die in dit werk behandeld worden zijn fundamentele vragen die zich direct bezig houden met enkele bouwstenen zoals deze gebruikt worden in de economische theorie en in het economisch modelleren, in plaats van met de modellen die geconstrueerd (kunnen) worden met deze bouwstenen, of met mogelijke implicaties van deze modellen voor het begrijpen van de economische werkelijkheid.

Dit proefschrift behandelt twee fundamentele onderdelen van de economische theorie, d.w.z. twee van de belangrijkste bouwstenen die tegenwoordig gebruikt worden in het economisch modelleren. Dit werk beschouwt deze standaard bouwstenen kritisch, en vestigt de aandacht op enkele problemen van deze standaard bouwstenen. Vervolgens tracht dit proefschrift een nieuwe, alternatieve bouwsteen te leveren, die niet (zozeer) te kampen heeft met deze problemen. Deze alternatieve bouwsteen zou vervolgens weer gebruikt kunnen worden om nieuwe modellen mee te construeren, waarmee de economische werkelijkheid weer anders bekeken en begrepen zou kunnen worden. En uiteindelijk onderzoekt dit proefschrift de precieze theoretische verbanden die bestaan tussen de standaard benadering en de hier gepresenteerde alternatieve benadering.

Bij de twee bovengenoemde standaard bouwstenen van het economisch modelleren gaat het om de "consumer choice" modellen die worden gebruikt in de neoklassieke micro-economie, en de "consumption/savings" modellen uit de macro-economie. Hierbij is de tweede klasse van modellen in feite een speciaal geval van de eerste klasse, maar omdat deze klasse van consumption/savings modellen op zichzelf al een belangrijke bouwsteen is van het macro-economisch modelleren, worden de twee klassen hier apart behandeld.

Zowel de consumer choice modellen als de consumption/savings modellen beschrijven een keuzeprobleem voor een consument die moet beslissen hoe hij (of zij) zijn inkomen zal spenderen aan alle consumptiemogelijkheden waarmee hij in het heden en de toekomst geconfronteerd zal worden. Zoals gebruikelijk in de standaard economische theorie beschouwen beide klassen van modellen een consument als zijnde "rationeel", of als wat ook wel een "homo economicus" genoemd wordt; de consument wordt geacht al zijn mogelijke opties (in dit geval consumptiepatronen) uitvoerig te beschouwen, hij wordt geacht stabiele voorkeuren over al zijn opties te kunnen bepalen, en uiteindelijk simpelweg zijn meest geprefereerde optie te kiezen. In deze standaard benadering van een consumptieprobleem zijn de keuzeobjecten waar tussen een consument wordt geacht te kiezen allesomvattende consumptiepatronen, ofwel volledige specificaties van alle toekomstige consumptie die nog na het keuzemoment plaats zal vinden. Dat wil zeggen, een beslissing over wat en hoeveel nu te consumeren wordt bepaald als onderdeel van een totale strategie die voor de hele resterende toekomst precies specificeert wat wanneer geconsumeerd zal worden.

De alternatieve bouwsteen die in dit proefschrift wordt gepresenteerd behelst een raamwerk waarin de tijd wordt opgedeeld in periodes, aan de hand waarvan consumptiebeslissingen worden onderverdeeld in een serie van opeenvolgende kleinere beslissingen, die één voor één bepaald zullen worden, en waarin (de resultaten van) eerdere beslissingen kunnen worden gebruikt voor het maken van latere beslissingen. Zo ontstaat een raamwerk waarin het maken van consumptiebeslissingen geleidelijk geleerd kan worden. Dit is in tegenstelling tot de standaard benadering, waarin consumptiebeslissingen worden bepaald in één groot, allesomvattend patroon.

In dit proefschrift modelleren we dus een consument die zijn leven opdeelt in een aantal periodes, en die probeert te leren van wat hij in vorige periodes deed. In tegenstelling tot in het standaard raamwerk nemen we aan dat in elk van deze periodes onze consument alleen een beslissing neemt met betrekking tot de consumptie in die periode, en met betrekking tot hoeveel van het budget -waar hij in die periode over kan beschikken- moet worden gespaard voor volgende periodes. We nemen aan dat binnen elk zo'n periode een beslissing over wat te consumeren en hoeveel te sparen, gemaakt wordt op een gelijksoortige manier als waarop in het standaard raamwerk een allesomvattende consumptiebeslissing gemaakt wordt: met gebruikmaking van preferentierelaties en/of nutsfuncties.

Echter, waar de onderliggende voorkeuren in het standaard raamwerk een absoluut en exogeen gegeven zijn, zijn de voorkeuren dat in het leerraamwerk niet. In elke periode wordt een (impliciete) afruil gemaakt tussen consumeren en sparen. De onderliggende voorkeuren, en in het bijzonder de impliciete waardering van sparen in vergelijking tot die van consumeren, zijn hier niet absoluut maar eerder een schatting of een benadering. Sommige schattingen kunnen daarom beter zijn dan andere, en het idee achter het leerproces is dat deze schattingen aangepast of verbeterd zouden kunnen worden met het verstrijken van de tijd. Dus, het leerraamwerk dat hier gepresenteerd wordt gaat uit van de premisse dat de (impliciete) voorkeuren tussen consumeren en sparen geleerd worden over de tijd.

De gedachte achter hoe dit leren plaats zou vinden, is hier dat als de consument op een bepaald moment zou denken dat hij teveel heeft uitgegeven (en dus te veel geconsumeerd) in het (recente) verleden, hij zijn voorkeuren aan zou passen op zo'n manier dat hij sparen meer zou gaan waarderen. En andersom, indien de consument zou denken dat hij minder zuinig had kunnen zijn, zou hij sparen minder gaan waarderen.

Er is een aantal redenen waarom zo'n leerbenadering interessant zou kunnen zijn.

Ten eerste, er is een aantal (conceptuele) moeilijkheden verbonden aan de standaard benadering van consumptiebeslissingen. Hierbij kunnen we met name denken aan de cognitieve vermogens waar een consument over zou moeten kunnen beschikken om zich zo te gedragen als de standaard modellen suggereren. Deze cognitieve vermogens omvatten het zich voor kunnen stellen van alle toekomstige consumptiemogelijkheden (wat wanneer tegen welke prijs verkrijgbaar zal zijn), en de mogelijkheid om al deze informatie op een zodanige manier te verwerken dat dit inderdaad in een optimale en efficiënte keuze zou resulteren. Dit lijken zeer sterke aannames aangezien het bepalen van en werken met volledige specificaties van alle toekomstige consumptie in werkelijkheid enorm complex zou zijn.

Het nieuwe leerraamwerk heeft niet zozeer te kampen met deze conceptuele moeilijkheden. Hierin hoeven geen consumptiepatronen voor de rest van een leven beschouwd en gekozen te worden. Leren impliceert dat er meerdere, opeenvolgende beslissingen gemaakt worden. Daarom zou in elke periode slechts de consumptie voor die periode gekozen hoeven te worden, waardoor het niet nodig is om een duidelijke voorstelling te hebben van wat nog in het verschiet ligt, of om op een efficiënte manier om te gaan met deze informatie.

Ten tweede, een leerraamwerk voor consumptiebeslissingen kan een nieuwe bouwsteen vormen, waarmee weer nieuwe modellen geconstrueerd zouden kunnen worden, en waarmee de economische werkelijkheid weer anders bekeken en begrepen zou kunnen worden. In tegenstelling tot andere deelgebieden van de economie, zijn er nog (bijna) geen leermodellen op het gebied van consumptiebeslissingen, mogelijk omdat hier conceptuele moeilijkheden mee verbonden zijn. Bijvoorbeeld, een belangrijke moeilijkheid die zich voordoet bij het opzetten van een leermodel in deze context, is dat de consequenties van een bepaalde keuze in een bepaalde periode moeilijk te isoleren zijn van de consequenties van de keuzes die gemaakt werden in andere periodes. Alle periodes zijn nauw met elkaar verbonden, wat nu uitgegeven wordt kan in latere periodes niet meer uitgegeven worden. Omdat alle consequenties van een bepaalde consumptiekeuze in een bepaalde periode pas aan het eind van een "leven" helemaal bekend zijn, is het moeilijk om de resultaten van de keuze uit de vorige periode duidelijk te bepalen. Dit is één van de belangrijkste complicaties die zich voordoen bij het construeren van een leermodel van consumptiebeslissingen, en ons leerraamwerk zal hier op één of andere manier mee om moeten gaan.

En ten derde kunnen we ons afvragen welke (theoretische) verbanden er zouden bestaan tussen het standaard raamwerk en een leerraamwerk. Bijvoorbeeld, zouden optimale of rationele beslissingen (zoals in het standaard raamwerk) op den duur geleerd kunnen worden? Veel economische theoretici vinden theoretische concepten zoals economische evenwichten, en ook optimale of rationele beslissingen, meer plausibel wanneer ze beschouwd worden als een stabiele toestand van een dynamisch systeem, en dus als de uiteindelijke uitkomst in een dynamische setting, dan wanneer ze beschouwd worden als een waarschijnlijke uitkomst in een statische situatie die maar één keer voorkomt. In deze dissertatie worden deze (en andere) verbanden tussen de standaard benadering en de nieuwe, alternatieve benadering uitvoerig behandeld. We zien hier bovendien dat het standaard raamwerk opgevat kan worden als een speciaal geval van het hier gedefinieerde leerraamwerk, of in andere woorden, dat het onderhavige leerraamwerk in feite een uitbreiding is van het standaard raamwerk.

References

[1] Allais, M. (1953), '*Le Comportement de l'Homme Rationnel devant la Risque, Critique des Postulats et Axiomes de l'Ecole Américaine*', Econometrica 21, 503-546.

[2] Allen, T.W., and Carroll, C.D. (2001), '*Individual Learning about Consumption*', NBER Working Paper 8234.

[3] Ballinger, T.P., Palumbo, M.G. and Wilcox, N.T. (2003), '*Precautionary Savings and Social Learning Across Generations: An Experiment*', The Economic Journal 113(October), 920-947.

[4] Bikhchandani, S., Hirshleifer, D. and Welch, I. (1998), '*Learning from the Behavior of Others: Conformity, Fads and informational Cascades*', Journal of Economic Literature, 12(3), 151-170.

[5] Browning, M. and Lusardi, A. (1996), '*Household Saving: Micro Theories and Micro Facts*', Journal of Economic Literature 34(4) (December), 1797-1855.

[6] Camerer, C., Loewenstein, G. and Prelec, D. (2005),'*Neuroeconomics: How neuroscience can inform economics*', Journal of Economic Literature 43(march) 9-64.

[7] Carroll, C.D. and Summers, L.H. (1991), '*Consumption Growth Parallels Income Growth: Some New Evidence*', in Bernheim, B.D. and Shoven, J.B., '*National Saving and Economic Performance*', 305-343, University of Chicago Press.

[8] Chua, Z. and Camerer, C.F. (2004), '*Experiments on Intertemporal Consumption with Habit Formation and Social Learning*', Caltech working paper.

[9] Cochrane, J.H. (1989), '*The Sensitivity of Tests of the Intertemporal Allocation of Consumption to Near-Rational Alternatives*', American Economic Review 79(June) 319-337.

[10] Crevier, D. (1993), '*AI: the Tumultuous History of the Search for Artificial Intelligence*', New York, Basic Books.

[11] Dekel, E., Lipman, B.L. and Rustichini, A. (1998), '*Standard State-Space Models Preclude Unawareness*', Econometrica 66(1), 159-173.

[12] Evans, G.W. and Honkapohja, S. (2001), '*Learning and Expectations in Macroeconomics*', Princeton University Press.

[13] Fehr, E. and Zych, P.K. (1998), '*Do Addicts Behave Rationally?*', Scandinavian Journal of Economics 100(3), 643-662.

[14] Flavin, M.A. (1981), '*The Adjustment of Consumption to Changing Expectations about Future Income*', Journal of Political Economy 89: 974-1009.

[15] Friedman, M.A. (1953), '*Essays in Positive Economics*', University of Chicago Press.

[16] Friedman, M.A. (1957), '*A Theory of the Consumption Function*', Princeton University Press.

[17] Gilboa, I. and Schmeidler, D. (1997), '*Cumulative Utility Consumer Theory*', International Economic Review 38, No. 4(Nov), 737-761.

[18] Gilboa, I. and Schmeidler, D. (2001), '*A Theory of Case-Based Decisions*', Cambridge University Press.
[19] Hey, J.D. and Dardanoni, V. (1988), '*Optimal Consumption under Uncertainty: An Experimental Investigation*', The Economic Journal 98, 105-116.
[20] Johnson, S., Kotlikoff L.J. and Samuelson, W., '*Can People Compute? An Experimental Test of the Life Cycle Consumption Model*', NBER Working Paper 2183.
[21] Kahneman, D. and Tversky, A. (1979), '*Prospect Theory: An Analysis of Decision under Risk*', Econometrica 47, 313-327.
[22] Knight, F.H. (1921), '*Risk, Uncertainty, and Profit*', Houghton Mifflin.
[23] Laibson, D. (1997), '*Golden Eggs and Hyperbolic Discounting*', Quarterly Journal of Economics 112 (May): 443-477.
[24] Lettau, M. (1997), '*Explaining the Facts with Adaptive Agents: The Case of Mutual Fund Flows*', Journal of Economic Dynamics and Control Vol. 21(7), 1117-1147.
[25] Lettau, M. and Uhlig, H. (1999), '*Rules of Thumb versus Dynamic Programming*', American Economic Review 89, 148-174.
[26] Loewenstein, G. and Thaler, R.H. (1989), '*Anomalies: Intertemporal Choice*', Journal of Economic Perspectives 3 (Fall): 181-193.
[27] Lucas, R. (1986), '*Adaptive Behaviour and Economic Theory*', in Hogarth, R. and Reder, M., '*Rational Choice: The contrast between Economics and Psychology*', Chicago, University of Chicago Press.
[28] Luenberger, D.G. (1995), '*Microeconomic Theory*', McGraw-Hill.
[29] Marcet, A. and Sargent, T.J. (1991), '*Convergence of Least Squares Learning Mechanisms in Self-referential Linear Stochastic Models*', Journal of Economic Theory 48 (August): 337-368.
[30] Mas-Colell, A., Whinston, M.D., and Green, J.R. (1995), '*Microeconomic Theory*', Oxford University Press.
[31] Modigliani, F. and Brumberg R. (1954), '*Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data*', in Kurihara, K.K., '*Post-Keynesian Economics*', 388-436, Rutgers University Press.
[32] Morishima, M. (1996), '*Dynamic Economic Theory'*, Cambridge University Press.
[33] von Neuman, J. and Morgenstern, O. (1944), '*Theory of Games and Economic Behaviour*', Princeton University Press.
[34] Noussair, C. and Matheny, K. (2000), '*An Experimental Study of Decisions in Dynamic Optimization Problems*', Economic Theory 15, 389-419.
[35] Patinkin, D. (1948), '*Relative prices, Say's Law and the demand for money'*, Econometrica, 16.
[36] Rader, T. (1963), '*The Existence of a Utility Function to Represent Preferences*', The Review of Economic Studies, Vol. 30, No. 3, 229-232.
[37] Romer, D. (2001), '*Advanced Macroeconomics*', McGraw-Hill.
[38] Savage, L.J. (1954), '*The Foundations of Statistics*', New York: Wiley.
[39] Shefrin H.M. and Thaler, R.H. (1988), '*The Behavioral Life-Cycle Hypothesis*', Economic Enquiry 26: 609-643.
[40] Simon, H.A. (1955), '*A behavioral model of rational choice*', Quarterly Journal of

Economics, 69, 99-118.

[41] Simon, H.A. (1957), '*Models of man: Social and Rational*', New York: Wiley.

[42] Simon, H.A. (1976), '*From substantive to procedural rationality*', In S.J. Latsis (Ed.), '*Method and appraisal in economics*' (pp. 129-148). Cambridge: Cambridge University Press.

[43] Stokey, N.L. and Lucas, R.E. with Prescott, E.C. (1989), '*Recursive Methods in Economic Dynamics*', Harvard University Press.

[44] Timmermann, A.G. (1993), '*How Learning in Financial Markets Generates Excess Volatility and Predictability in Stock Prices*', Quarterly Journal of Economics 45, 1135–1145.

[45] Vaughn, K.I. (1994), '*Austrian Economics in America: The Migration of a Tradition*', Cambridge University Press.

[46] Wolff, E.N. (1998), '*Recent Trends in the Size Distribution of Household Wealth*', Journal of Economic Perspectives 12 (summer):131-150.

[47] Young, H.P. (1997), '*Learning and Evolution in Games*', Princeton University Press.

[48] Zeldes, S.P. (1989a), '*Consumption and Liquidity Constraints: An Empirical Investigation*', Journal of Political Economy 97: 305-346.

[49] Zeldes, S.P. (1989b), '*Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence*', Quarterly Journal of Economics 104, 275-298.